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Magnetofluid dynamics in curved spacetime

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A grand unified field $\mathcal{M}^{\mu\nu}$ is constructed from Maxwell's field tensor and an appropriately modified flow field, both nonminimally coupled to gravity, to analyze the dynamics of hot charged fluids in curved background space-time. With a suitable $3+1$ decomposition, this new formalism of the hot fluid is then applied to investigate the vortical dynamics of the system. Finally, the equilibrium state for plasma with nonminimal coupling through Ricci scalar R to gravity is investigated to derive a double Beltrami equation in curved space-time.

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I. INTRODUCTION

By generalizing the standard minimum coupling prescription, $p_\mu \rightarrow mU_\mu + qA_\mu$, invoked to incorporate the electromagnetic field in charged particle dynamics, an analogous theory to describe the dynamics of a hot charged fluid was developed in [1]. The "hot-fluid" version of the prescription (A_μ is the electromagnetic four potential),

$$p_\mu \rightarrow m\mathcal{G}U_\mu + qA_\mu = P_\mu, \tag{1}$$

combines the kinematic (the mass m, the four momentum p_μ, and the four velocity U_μ) and the statistical (the thermodynamic enthalpy $\mathcal{G}$) attributes of the fluid element. The centerpiece of the formalism was the construction of an antisymmetric, hybrid field tensor,

$$qM^{\mu\nu} = \nabla^\mu P^\nu - \nabla^\nu P^\mu = qF^{\mu\nu} + mS^{\mu\nu}, \tag{2}$$

which is a weighted sum of the electromagnetic field tensor $F^{\mu\nu}$ and the composite (kinematic-statistical) fluid tensor $S^{\mu\nu} = \nabla^\mu \mathcal{G}U_\nu - \nabla^\nu \mathcal{G}U_\mu$; the weighting factors are the electric charge q and the inertial "charge" m. In analogy with electromagnetism, one may associate with $S^{\mu\nu}$ appropriately defined equivalents of the electric and magnetic fields [1,2]. The entire dynamics of the hot fluid is contained in the succinct equation (T is the temperature of the fluid)

$$qU_\mu M^{\mu\nu} = \frac{mn\nabla^\nu \mathcal{G} - \nabla^\nu p}{n} = T\nabla^\nu \sigma, \tag{3}$$

where the right-hand side is the thermodynamic force expressed in terms of the fluid entropy σ using the standard thermodynamic relation between entropy with enthalpy (p is the pressure). It was further shown that the 3-vector part of Eq. (3), after appropriate manipulation, is reducible to the conventional 3D vortex dynamics except that the standard fluid voriticity (and the conserved helicity) is replaced by the hybrid magnetofluid vorticity (magnetofluid helicity). This is a far-reaching consequence because both the methodology and results of the highly investigated nonrelativistic vortex dynamics could, then, be transported to shed light on the much more complicated hot relativistic fluid.

In light of the preceding discussion, Eq. (3) should be taken as a defining equation for the four-dimensional (4D) vortex dynamics.

A very fundamental result of the standard "ideal" vortex dynamics is that it implies a topological invariant, the helicity (the hybrid helicity); i.e., such a dynamics forbids creation or destruction of vorticity (the vorticity could not be created from a state of no vorticity). However, the constraint can be broken by including nonideal behavior; in the standard 3D nonrelativistic system, the ideal behavior corresponds to the right-hand side being a perfect gradient, $T\nabla^\nu \sigma = \nabla^\nu \bar{\sigma}$, which will require an equation of state of the type $\sigma = \sigma(T)$. It must be, however, emphasized that the thermodynamics of the fluid is its intrinsic property and is not dictated by whether the equations of motion can be cast in a "canonical" vortex dynamics form.

The 4D vortex dynamics, though very similar to 3D vortex dynamics, differs from it in one fundamental way: the special and general relativistic effects, through the distortion of space-time, could break the topological invariant even in ideal dynamics ($\sigma = \sigma(T)$). These effects introduce sources and sinks for the relevant generalized helicity (often much more complicated than the 3D fluid helicity) so that the creation and destruction of the generalized vorticity becomes possible in ideal dynamics.

Thus, the relativistic mechanics opens up a new vista and a new mode of analysis. We first try to construct the most general form of 4D vortex dynamics [embodied in Eq. (3)] by manipulating the dynamics so that an appropriately generalized $\mathcal{P}^{\mu\nu}$ (and thus $\mathcal{M}^{\mu\nu}$) can be constructed to satisfy an equation of the type

$$qU_\mu \mathcal{M}^{\mu\nu} = QT\nabla^\nu \sigma, \tag{4}$$

where Q could be a function of space-time geometry. As usual, the components $\mathcal{M}^{ij} = \epsilon^{ijk} X_k$ will define the new generalized vorticity (generalized magnetic field) **X**. And for homogenous thermodynamics ($\nabla^\nu \sigma = 0$), with or without an equation of state, the generalized helicity $\mathcal{H} = \langle \vec{X} \cdot \nabla \times^{-1} \vec{X} \rangle$ will be conserved. Here, $(\nabla \times^{-1} \vec{X})$ is the inverse curl of vortical field $\vec{X}$.

Then, the next step is to investigate relativistic mechanisms that, in combination with inhomogeneous thermodynamics, will create sources and sinks for $\mathcal{H}$ and **X**. Finding the origin of seed vorticity (of the magnetic field, for instance), which could be amplified in an ideal dynamo-like mechanism, is one of the most fascinating problems of theoretical astrophysics and cosmology, and several recent papers have advanced the effort by making the special relativistic model of Refs. [1,3,4] generally covariant [5–7], i.e., by including gravity.

The general relativistic formulation in Ref. [5] was attempted within the framework of minimal coupling to gravity in the spherically symmetric and static space-time. Later, the work was extended to a rotating black hole using the ψ-N (pseudo-Newtonian formalism) framework [7]. In this paper, we further generalize the recent work by including nonminimal coupling between gravity and plasma in a general background space-time.

Since our investigation facilitates a generalization to $f(R)$ gravity (not just Einstein's general relativity) [8–12], we begin our analysis with an action functional for $f(R)$ gravity with a perfect fluid and Maxwell's fields nonminimally coupled to gravity to derive the equation of motion for a new hybrid magnetofluid. Also, the vortical dynamics of the magnetofluid is explored by deriving the generalized Faraday's law from dual tensor $\mathcal{M}^{*\mu\nu}$, and its static spherically symmetric limits are explored with their possible astrophysical applications. Finally, source-free plasma equilibrium states in curved background space-time are investigated. In this paper, the calculation for the generalized equation of motion of a new hybrid magnetofluid in curved background space-time is presented in the first section. Next, the Arnowitt-Deser-Misner (ADM) formalism of electrodynamics [13–16] is applied to this new formulation of magnetofluids. Equations obtained after $3+1$ decomposition are later cast into the vorticity evolution equation. Two limiting cases of the equation are shown later. Finally, the source-free vorticity equation for the limiting cases with $f_m(R) = R$ is used to derive the equilibrium state of the system.

II. PLASMA DYNAMICS

The dynamics of an ideal plasma in curved background space-time can be investigated with the extremization of the action functional in the convention $G = c = 1$,

$$S = S_g + S_{pfg} + S_M + S_{NM},$$

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2} f_g(R) - (1 + \lambda f_m(R))\rho(n, \tilde{\sigma}) - \frac{1}{16\pi}(g^{\mu\alpha} g^{\nu\beta} + \mathcal{R}^{\mu\nu\alpha\beta}) F_{\mu\nu} F_{\alpha\beta}\right], \tag{5}$$

where S_g, S_{pfg}, S_M, and S_{NM} represent the corresponding action functionals for pure gravity, a perfect fluid, and Maxwell's field minimally as well as nonminimally coupled to gravity. Since our analysis does not change in the context of modified gravity, we include, for generality, the functions of Ricci scalar $f_g(R)$ and $f_m(R)$ in the above action for pure gravity and coupling to matter, respectively. Here, $g_{\mu\nu}$, g, $F_{\mu\nu}$, R, and $R^{\mu\nu\alpha\beta}$ represent the metric tensor, the determinant of the metric tensor, the Maxwell field tensor, the Ricci scalar, and the Riemann tensor, respectively. The quantity $\rho(n, \tilde{\sigma})$, the energy density, is a function of the number density, n, and entropy density, $\tilde{\sigma}$, of the plasma. Also, λ is a phenomenological parameter that represents the coupling strength of the plasma to its background geometry; thus, it can be treated as a coupling constant with a dimension of length^2 (see Ref. [12] for a detailed discussion). The nonminimal three-parameter tensor $\mathcal{R}^{\mu\nu\alpha\beta}$ introduced in Ref. [9] has the form

$$\mathcal{R}^{\mu\nu\alpha\beta} = q_1 R g^{\mu\nu\alpha\beta} + q_2 \mathfrak{R}^{\mu\nu\alpha\beta} + q_3 R^{\mu\nu\alpha\beta},$$

where two auxiliary tensors

$$\mathfrak{R}^{\mu\nu\alpha\beta} = \frac{1}{2}(R^{\mu\alpha} g^{\nu\beta} - R^{\mu\beta} g^{\nu\alpha} + R^{\nu\beta} g^{\mu\alpha} - R^{\nu\alpha} g^{\mu\beta})$$

and

$$g^{\mu\nu\alpha\beta} = \frac{1}{2}(g^{\mu\alpha} g^{\nu\beta} - g^{\nu\alpha} g^{\mu\beta})$$

are introduced. The parameters q_1, q_2, and q_3 are used to describe nonminimal linear coupling of the electromagnetic tensor $F_{\mu\nu}$ to the curvature. In general, these parameters are arbitrary and hence have to be chosen to satisfy some desired phenomenological or other constraints. For example, the Lagrangian with $q_1 = q_2 = 0$ and $q_3 = -\lambda_1$ has been used by Prasanna for phenomenological study of the nonminimal modifications of the electrodynamics [17–19]. Other suitable constraints for q_1, q_2, and q_3 have been chosen to befit the desired phenomenological or other results. Such a nonminimal coupling of the electromagnetic field to gravity, with dimensional coupling constants q_1, q_2, and q_3 as a natural choice of action, has been discussed by Balakin [9] and Horndeski [20]. Here, we keep all q_1, q_2, and q_3 without any constraint for the purpose of completeness and generality. One can choose a more generalized form of coupling of the electromagnetic field to gravity through nonlinear Riemann curvature, but we limit our analysis to the above action functional for simplicity.

Now, the extremization of the above action functional, i.e., varying it with respect to the metric $g_{\mu\nu}$, will result in the modified Einstein equation with the total stress-energy tensor $T^{\mu\nu}_{\rm total}$ for the perfect fluid and Maxwell's field in curved space-time,

$$F_g(R)R^{\mu\nu} - \frac{1}{2}f_g(R)g^{\mu\nu} - (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)F_g(R) = 8\pi T^{\mu\nu}_{\rm total}, \tag{6}$$

where $F_g = f'_g(R)$ (differentiation with respect to R) and $T^{\mu\nu}_{\rm total}$ is the total stress-energy tensor computed from $T^{\mu\nu} = (2/\sqrt{-g})(\delta S/\delta g_{\mu\nu})$. It is straightforward to show that Einstein's equation for general relativity is recovered by setting $f_g(R) = R$ and $F_g(R) = 1$, and the divergence of the modified Einstein equation (6) produces the equation of motion of the plasma in curved background space-time since the vanishing divergence of the left-hand side will not contribute to the equation of motion for the plasma.

The stress-energy tensor for perfect fluid and Maxwell's field can be computed from $T^{\mu\nu} = (2/\sqrt{-g})(\delta S/\delta g_{\mu\nu})$ with the corresponding action functional. Varying the action functional S_{pfg}, we obtain the stress tensor for perfect fluid coupled to gravity [8,21–23]

$$T^{\mu\nu}_{pfg} = (1+\lambda f_m(R))T^{\mu\nu}_{pf} + 2\lambda\rho F_m(R)R^{\mu\nu} - 2\lambda(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\rho F_m(R), \tag{7}$$

where $T^{\mu\nu}_{pf}$ is the stress-energy tensor for noncoupled perfect fluid (See Appendix A for details): $T^{\mu\nu}_{pf} = (p+\rho)U^\mu U^\nu + pg^{\mu\nu}$ with $U^\mu = dx^\mu/d\tau$ being the 4-velocity of the plasma particles, and $F_m(R) = f'_m(R)$. The quantity $p+\rho = h$ is known to be the enthalpy density of the plasma, which can be expressed by introducing an auxiliary function $\mathcal{G} = h/mn$ with m and n being the mass and number density, respectively.

Using the identity $\nabla_\mu(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\rho = R^{\mu\nu}\nabla_\mu\rho$, the expression for the divergence of the stress tensor reduces to [8,10,11,21–24]

$$\nabla_\mu T^{\mu\nu}_{pfg} = (1+\lambda f_m(R))\nabla_\mu T^{\mu\nu}_{pf} + \lambda F(T^{\mu\nu}_{pf} + g^{\mu\nu}\rho)\nabla_\mu R. \tag{8}$$

On the other hand, the variation of the action functional S_M gives the usual electromagnetic stress tensor

$$T^{\mu\nu}_M = \frac{1}{4\pi}\left(F^{\nu\beta}F^\mu{}_\beta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{9}$$

and, with the Bianchi identity, the divergence of $T^{\mu\nu}_M$ takes the following form:

$$4\pi\nabla_\mu T^{\mu\nu}_M = -F^\nu{}_\beta\nabla_\alpha F^{\beta\alpha}. \tag{10}$$

Next, owing to the linear nature of nonminimal action, we can assume the total nonminimal stress-energy tensor to be [9]

$$T^{\mu\nu}_{NM} = q_1T^{\mu\nu}_1 + q_2T^{\mu\nu}_2 + q_3T^{\mu\nu}_3.$$

Varying the action for nonminimal coupling, we obtain the following set of three stress-energy tensors:

$$T^{\mu\nu}_1 = \frac{1}{4\pi}\left[-\frac{1}{2}(\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)F_{\alpha\beta}F^{\alpha\beta} + RF^{\mu\beta}F^\nu{}_\beta\right], \tag{11}$$

$$\begin{aligned}T^{\mu\nu}_2 = \frac{1}{4\pi}g^{\mu\alpha}g^{\nu\beta}\bigg[&\frac{1}{2}g_{\alpha\beta}(\nabla_\gamma\nabla_\theta(F^{\gamma\sigma}F^\theta{}_\sigma) - R^{\gamma\sigma}F_{\gamma\theta}F^\theta{}_\sigma)\\ &+ F^{\gamma\sigma}(R_{\gamma\beta}F_{\alpha\sigma} + R_{\gamma\alpha}F_{\beta\sigma}) + \frac{1}{2}\Box(F_{\beta\sigma}F_\alpha{}^\sigma)\\ &-\frac{1}{2}\nabla_\gamma[\nabla_\alpha(F_{\beta\sigma}F^{\gamma\sigma}) + \nabla_\beta(F_{\alpha\sigma}F^{\gamma\sigma})] + R^{\gamma\sigma}F_{\gamma\alpha}F_{\sigma\beta}\bigg],\end{aligned} \tag{12}$$

$$\begin{aligned}T^{\mu\nu}_3 = \frac{1}{4\pi}\bigg[g^{\mu\alpha}g^{\nu\beta}\bigg(&-\frac{1}{4}g_{\alpha\beta}R^{\gamma\theta\sigma\rho}F_{\gamma\theta}F_{\sigma\rho}\\ &+\frac{3}{4}F^{\sigma\rho}(F_\alpha{}^\theta R_{\beta\theta\sigma\rho} + F_\beta{}^\theta R_{\alpha\theta\sigma\rho})\\ &+\frac{1}{2}\nabla_\gamma\nabla_\theta[F_\alpha{}^\gamma F_\beta{}^\theta + F_\beta{}^\gamma F_\alpha{}^\theta]\bigg)\bigg].\end{aligned} \tag{13}$$

The calculation of the divergences of the above three stress tensors requires electrodynamic equations corresponding to the total action functional, i.e., adding an interaction or source term in the action. Therefore, we take the variation of the action functional with the source term with respect to the field variable A^μ and obtain [9]

$$\nabla_\alpha H^{\alpha\beta} = -4\pi nqU^\beta, \tag{14}$$

where

$$H^{\alpha\beta} = F^{\alpha\beta} + \mathcal{R}^{\mu\nu\alpha\beta}F_{\mu\nu} \tag{15}$$

may be considered as a generalized Faraday tensor in curved background space-time and nqU^β is the Lorentz 4-current associated with the charged fluid. Equations (14) and (15) can be regarded as the constitutive relations of the unified system to preserve Maxwell's equations in curved background space-time.

Invoking the fact the total stress-energy tensor should be divergence free, i.e.,

$$\nabla_\mu[T^{\mu\nu}_{pfg} + T^{\mu\nu}_M + T^{\mu\nu}_{NM}] = 0, \tag{16}$$

with $T^{\mu\nu}_{NM} = q_1 T^{\mu\nu}_1 + q_2 T^{\mu\nu}_2 + q_2 T^{\mu\nu}_2$, and using Eqs. (10), (15), and (16), we obtain the divergences of the three constituents of the nonminimal stress energy tensor $T^{\mu\nu}_{NM}$:

$$4\pi\nabla_\mu T^{\mu\nu}_1 = -F^\nu{}_\beta \nabla_\alpha (RF^{\alpha\beta}), \tag{17}$$

$$4\pi\nabla_\mu T^{\mu\nu}_2 = -F^\nu{}_\beta \nabla_\mu (R^{\mu\gamma} F_\gamma{}^\beta + R^{\gamma\beta} F^\mu \gamma), \tag{18}$$

$$4\pi\nabla_\mu T^{\mu\nu}_3 = -F^\nu{}_\beta \nabla_\mu (R^{\mu\beta\gamma\theta} F_{\gamma\theta}). \tag{19}$$

Substituting these expressions for divergence in Eq. (16), we obtain

$$\begin{aligned}(1+\lambda f_m(R))\nabla_\mu T^{\mu\nu}_{pf} \\ = [qnF^\nu{}_\beta U^\beta - \lambda F_m(R)(T^{\mu\nu}_{pf} + g^{\mu\nu}\rho)\nabla_\mu R].\end{aligned} \tag{20}$$

It is interesting to note that the coupling between gravity and electromagnetic field is now explicitly manifest in Eq. (15) through the generalized Faraday tensor and is implicit in the equation of motion (20) for plasma in curved background space-time through the current since, unlike standard electromagnetic field, U^β is governed by a hybrid field $H^{\alpha\beta}$.

Until now, we mainly followed the standard approach to derive the covariant equation of motion for the plasma in curved background space-time from the action functional (5). The main result is Eq. (20) that captures the nonminimally gravity-coupled plasma dynamics. To advance in our program of unifying the electromagnetic field with an appropriately weighted flow field, we next derive a generalized expression for the corresponding unified magnetofluid.

A. Magnetofluid unification

Following the prescription presented in Ref. [1], we substitute the expression for the stress tensor $T^{\mu\nu}_{pf} = (p+\rho)U^\mu U^\nu + pg^{\mu\nu}$ for the perfect fluid in Eq. (20) and invoke the continuity equation $\nabla_\mu(nU^\mu) = 0$ to obtain.

$$\begin{aligned}&(1+\lambda f_m(R) - \lambda RF_m(R))mnU^\mu \nabla_\mu(\mathcal{G}U^\nu) \\ &\quad + (1+\lambda f_m(R) - \lambda RF_m(R))\nabla^\nu p \\ &\quad + \lambda mnF_m(R)U^\mu\nabla_\mu(R\mathcal{G}U^\nu) + \lambda F_m(R)\nabla^\nu(pR) \\ &= qnF^{\nu\beta}U_\beta - \lambda F_m(R)\rho\nabla^\nu R.\end{aligned} \tag{21}$$

In terms of the standard perfect fluid flow tensor $S^{\mu\nu} = \nabla^\mu(\mathcal{G}U^\nu) - \nabla^\nu(\mathcal{G}U^\mu)$, and a new curvature-coupled weighted antisymmetric flow field tensors $K^{\mu\nu} = \nabla^\mu(R\mathcal{G}U^\nu) - \nabla^\nu(R\mathcal{G}U^\mu)$, we can manipulate Eq. (21) to obtain ($\mathcal{G} = h/mn$)

$$(1+\lambda f_m(R))\left[\frac{\nabla^\nu p}{n} - m\nabla^\nu \mathcal{G}\right] = qU_\mu \mathcal{M}^{\nu\mu}, \tag{22}$$

where the new grand vorticity tensor has the canonical form ([1])

$$\mathcal{M}^{\nu\mu} = F^{\nu\mu} + \frac{m}{q} D^{\nu\mu} \tag{23}$$

with

$$D^{\nu\mu} = (1+\lambda f_m(R) - \lambda RF_m)S^{\nu\mu} + \frac{m}{q}\lambda F_m K^{\nu\mu}. \tag{24}$$

The new fluid tensor $D^{\mu\nu}$ displays, explicitly, the coupling of the flow field to gravity. Using the thermodynamic identity

$$\nabla^\nu \sigma = \frac{mn\nabla^\nu \mathcal{G} - \nabla^\nu p}{nT}, \tag{25}$$

we cast Eq. (22), governing the dynamics of a hot fluid system in curved background space-time, into the canonical 4D vortex form

$$qU_\mu \mathcal{M}^{\mu\nu} = (1+\lambda f_m(R))T\nabla^\nu \sigma. \tag{26}$$

Here, it must be noted that invoked thermodynamic relation is contingent upon an appropriately well-defined local concept of temperature in curved space-time. The above thermodynamic identity (25) can be derived from the first law of thermodynamics expressed in the form of exact differential $dH = md\mathcal{G} = Td\sigma + Vdp = Td\sigma + dp/n$, with H and V being the enthalpy and the volume of the fluid element, respectively, which in turn can be cast into the form $m\nabla^\mu \mathcal{G} = T\nabla^\mu \sigma + (1/n)\nabla^\mu p$ for a fluid element moving along the worldline with 4-velocity U^μ [15].

Notice that, when $\lambda = 0$, $\mathcal{M}^{\mu\nu}$ reduces to its minimally coupled counterpart, the tensor $M^{\mu\nu}$ defined in Refs. [1,5].

Equation (26) is the main result of this formalism; we have just shown that a charged relativistic fluid, coupled nonminimally to gravity, obeys a 4D vortex dynamics like its gravity free, and minimally coupled to gravity, counterparts. The new grand vorticity tensor subsumes earlier limiting cases in a transparent manner.

We will now apply the above covariant formulation to spell out and investigate the more advanced vortical structures contained in this system. To do calculations in terms of familiar quantities, we will begin with a 3 + 1 decomposition.

III. 3 + 1 DYNAMICS OF GRAVITOMAGNETOFLUID

The 3 + 1 decomposition of the 4D vortex dynamics will help us, *inter alia*, to find 1) a generalized electric and magnetic field from $\mathcal{M}^{\mu\nu}$ and 2) the energy and the continuity equation rewritten in terms of generalized electric and magnetic fields.

The approach chosen for the 3 + 1 splitting selects a family of foliated fiducial 3D hypersurfaces (slices of simultaneity) Σ_t labelled by a parameter $t =$ constant in terms of a time function on the manifold. Furthermore, we let t^μ be a timeline vector of which the integral curves intersect each leaf Σ_t of the foliation precisely once and that is normalized such that $t^\mu \nabla_\mu t = 1$. This t^μ is the "evolution vector field" along the orbits of which different points on all $\Sigma_t \equiv \Sigma$ can be identified. This allows us to write all space-time fields in terms of t-dependent components defined on the spatial manifold Σ_t. Lie derivatives of the space-time field along t^μ are identified with "time derivatives" of the spatial fields since Lie derivatives reduce to a partial time derivative for an adapted coordinate system $t^\mu = (1, 0, 0, 0)$.

Moreover, since we are using the Lorentzian signature, the vector field t^μ is required to be future directed. Let us decompose t^μ into normal and tangential parts with respect to Σ_t by defining the lapse function α and the shift vector β^μ as $t^\mu = \alpha n^\mu + \beta^\mu$ with $\beta^\mu n_\mu = 0$, where n^μ is the future directed unit normal vector field to the hypersurfaces Σ_t. More precisely, the natural timelike covector $n_\mu = (-\alpha, 0, 0, 0) = -\alpha \nabla_\mu t$ is defined to obtain $n^\mu = (1/\alpha, -\beta^\mu/\alpha)$, which satisfies the normalization condition $n^\mu n_\mu = -1$. Then, the space-time metric $g_{\mu\nu}$ induces a spatial metric $\gamma_{\mu\nu}$ by the formula $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Finally, the $3+1$ decomposition is usually carried out with the projection operator $\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$, which satisfies the condition $n^\mu \gamma_{\mu\nu} = 0$. Also, the acceleration is defined as $a_\mu = n^\nu \nabla_\nu n_\mu$.

Now, with the above foliation of space-time, the space-time metric takes the canonical form [15]

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{27}$$

and it immediately follows that, with respect to an Eulerian observer, the Lorentz factor turns out to be

$$\Gamma = [\alpha^2 - \gamma_{ij}(\beta^i \beta^j + 2\beta^i v^j + v^i v^j)]^{-1/2}, \tag{28}$$

satisfying $d\tau = dt/\Gamma$, where v^i is the ith component of fluid velocity $\vec{v} = d\vec{x}/dt$. Then, the decomposition for the 4-velocity is [5]

$$U^\mu = \alpha \Gamma n^\mu + \Gamma \gamma^\mu{}_\nu v^\nu, \tag{29}$$

with $n_\mu U^\mu = -\alpha\Gamma$.

Now, since our unified antisymmetric field tensor $\mathcal{M}^{\mu\nu}$ is constructed from the antisymmetric tensors $F^{\mu\nu}$ and $D^{\mu\nu}$, we apply the ADM formalism of electrodynamics presented in Refs. [13–16] to define the generalized electric and magnetic field, respectively, as

$$\xi^\mu = n_\nu \mathcal{M}^{\mu\nu}; \qquad X^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} \mathcal{M}_{\sigma\tau}, \tag{30}$$

and thus express the unified field tensor

$$\mathcal{M}^{\mu\nu} = n^\mu \xi^\nu - n^\nu \xi^\mu - \epsilon^{\mu\nu\rho\sigma} X_\rho n_\sigma. \tag{31}$$

We remind the reader that the generalized magnetic field and the generalized vorticity are essentially synonymous. Using the definition of the unified field tensor $\mathcal{M}^{\mu\nu}$, the expressions of the 3D generalized electric and magnetic field turn out to be

$$\begin{aligned}\vec{\xi} = \vec{E} &- \frac{m}{q}(1 + \lambda f_m(R) - \lambda R F_m(R))\vec{\nabla}(\alpha \mathcal{G}\Gamma) \\ &- \frac{m}{q}\lambda F_m(R)\vec{\nabla}(\alpha \mathcal{G} R \Gamma) \\ &- \frac{m}{q}(1 + \lambda f_m(R))\left[2\underline{\underline{\sigma}} \cdot (\mathcal{G}\Gamma\vec{v}) + \frac{2}{3}\theta \mathcal{G}\Gamma\vec{v}\right] \\ &- \frac{m}{q\alpha}(1 + \lambda f_m(R) - \lambda R F_m(R))(\mathcal{L}_t(\mathcal{G}\Gamma\vec{v}) - \mathcal{L}_{\vec{\beta}}(\mathcal{G}\Gamma\vec{v})) \\ &- \frac{m}{q\alpha}\lambda F_m(R)(\mathcal{L}_t(\mathcal{G}R\Gamma\vec{v}) - \mathcal{L}_{\vec{\beta}}(\mathcal{G}R\Gamma\vec{v}));\end{aligned} \tag{32}$$

$$\begin{aligned}\vec{X} = \vec{B} &+ \frac{m}{q}(1 + \lambda f_m(R) - \lambda R F_m(R))\vec{\nabla} \times (\mathcal{G}\Gamma\vec{v}) \\ &+ \lambda F_m(R)\frac{m}{q}\vec{\nabla} \times (R\mathcal{G}\Gamma\vec{v}),\end{aligned} \tag{33}$$

where $\underline{\underline{\sigma}} = \sigma^\nu_\mu$ and θ are, respectively, the shear and expansion of the congruence, defined as $\sigma_{\alpha\beta} = \gamma^\mu_\alpha \gamma^\nu_\beta \nabla_{(\mu} n_{\nu)} - \frac{1}{3}\theta\gamma_{\mu\nu}$ and $\theta = \nabla_\mu n^\mu$. We have also used the relation $\nabla_\mu n_\nu = -a_\nu n_\mu + \sigma_{\alpha\beta} + \frac{1}{3}\theta\gamma_{\mu\nu}$ to derive Eq. (32).

Finally, the $\gamma^\beta{}_\mu$ projection of the unified field equation of motion (26) gives us the momentum evolution equation

$$\alpha q \Gamma \vec{\xi} + q\Gamma(\vec{v} \times \vec{X}) = -(1 + \lambda f_m(R)) T \vec{\nabla}\sigma, \tag{34}$$

whereas the n_μ projection gives the equation of energy conservation

$$\alpha q \Gamma \vec{v} \cdot \vec{\xi} = T(1 + \lambda f_m(R))(\mathcal{L}_t \sigma - \vec{\beta} \cdot \vec{\nabla}\sigma). \tag{35}$$

A. Vortical dynamics

Understanding the full extent of this formalism is, perhaps, a very long-term project. We can, however, begin to appreciate its rich content by exploring some aspects of the new vortical dynamics. Sources responsible for magnetic field generation, in particular, the sources that are gravity driven, can be derived by finding the generalized vorticity evolution equation (which is really the generalized Faraday law) by manipulating Eq. (34).

Since $\mathcal{M}^{\mu\nu}$ is an antisymmetric tensor, the divergence of its dual is zero, i.e., $\nabla_\mu \mathcal{M}^{*\mu\nu} = 0$. Taking the $\gamma^\beta{}_\mu$ projection of the preceding identity, we derive

$$\mathcal{L}_t \vec{X} = \mathcal{L}_{\vec{\beta}} \vec{X} - \vec{\nabla} \times (\alpha\vec{\xi}) - \alpha\theta\vec{X}, \tag{36}$$

where $\mathcal{L}$ denotes the Lie derivatives with $\mathcal{L}_t = \partial_t$ along t^μ and $\mathcal{L}_{\vec{\beta}}\vec{X} = [\vec{\beta}, \vec{X}]$.

It should be noted that, even in the absence of nonminimal coupling to gravity ($\lambda = 0$), minimal coupling to gravity is always present. Equation (36), in conjunction with Eq. (34), gives us the vorticity evolution equation of the system

$$\mathcal{L}_t\vec{X} - \vec{\nabla}\times(\vec{v}\times\vec{X}) - \mathcal{L}_{\vec{\beta}}\vec{X} + \alpha\theta\vec{X} = \vec{\nabla}\times\left(\frac{T}{q\Gamma}(1+\lambda f_m(R))\vec{\nabla}\sigma\right). \quad (37)$$

All terms on the left-hand side operate on the vorticity 3-vector $\vec{X}$, while the right-hand side provides, just as in the conventional picture, possible sources for vorticity generation. The left-hand side, however, has lot more structure than the conventional 3D vortex dynamics; the first two terms are like the standard Helmholtz, while $\alpha\theta\vec{X}$ and $\mathcal{L}_{\vec{\beta}}\vec{X}$ are nontrivial gravity modifications. Thus, the gravity coupling does, fundamentally, modify the projected 3D vortex dynamics, in spite of the fact that the 4D vortex equations had exactly the same form.

Until now, our analysis has been very general with the assumption that the space-time structure satisfies the modified Einstein equation (6) and it permits the $3+1$ foliation adopted above. Further investigation is better done after specifying the precise structure of space-time. Without the knowledge of the structure of space-time, it may not be possible, even, to specify conditions under which the helicity, a topological invariant of the system, is conserved [1,3,25]. Since it is beyond the scope of our current endeavor to find the solutions of the modified Einstein equation (6) to explore the vortical dynamics, we instead briefly discuss the vortical dynamics in the context of a couple of well-known space-time solutions to the original Einstein equation.

First, for a minimal coupling ($\lambda = 0$) and a spherically symmetric and static space-time like the Schwarzchild solution, the above vortical evolution equation (37) reduces to the one presented in Ref. [5], i.e., $\mathcal{L}_t\vec{X} - \vec{\nabla}\times(\vec{v}\times\vec{X}) = \vec{\nabla}\times((T/q\Gamma)\vec{\nabla}\sigma)$. Since the spherically symmetric and static space-time can be foliated without the shift function $\vec{\beta}$, and the foliation obeys the time translation symmetry leading to a vanishing extrinsic curvature, the two new terms on the left-hand side disappear. Thus, the structure is precisely like the 3D vortex dynamics. The simplified vortical evolution equation can be used to approximately compute the weak field seed generation in the hot fluid system in the accretion disk of the Schwarzchild black hole [5].

Second, for a nonminimal coupling and a spherically symmetric and static space-time, the above vortical evolution equation (37) again reduces to the 3D-like vortex dynamics , i.e., $\mathcal{L}_t\vec{X} - \vec{\nabla}\times(\vec{v}\times\vec{X}) = \vec{\nabla}\times((T/q\Gamma)(1+\lambda f_m(R))\vec{\nabla}\sigma)$. Again, spherical symmetry and nonrotating space-time demand that the two terms on the left-hand side corresponding to the shift function and the expansion factor disappear and thus render the vortical evolution equation to be applied for computing seed generation in massive astrophysical objects, especially with $f_m(R) = R$. The equilibrium plasma state in this space-time will be discussed a little later.

Finally, for $f_m(R) = R$ and a stationary and axially symmetric space-time such as the Kerr black hole, the entire vortical evolution equation (37) with appropriate modifications can be used to explore a number of astrophysical applications including gamma ray bursts and seed generation, which will be explored in the near future. Therefore, the study of plasma dynamics in curved background space-time in light of the newly constructed grand unified field tensor $\mathcal{M}^{\mu\nu}$ provides much new useful insight that can be used to explore some astrophysical phenomena.

B. Equilibrium state

For $f_m(R) = R$, we can explore the source-free vorticity evolution equation with nonzero λ, $\mathcal{L}_t\vec{X} - \vec{\nabla}\times(\vec{v}\times\vec{X}) = 0$. The trivial solution for this equation is $\vec{X} = \vec{B} + (mc/q)\vec{\nabla}\times((1+\lambda R)\mathcal{G}\Gamma\vec{v}) = 0$. Using the general relativity modified Ampere law $(\vec{\nabla}\times\alpha\vec{B}) = (4\pi/c)(\alpha qn\Gamma\vec{v})$, we obtain the equation for the equilibrium state,

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = -\frac{1}{\mathcal{Q}\lambda_L^2}\vec{B} - \vec{\nabla}\ln(\mathcal{Q})\times(\vec{a}\times\vec{B}) - \vec{\nabla}\times(\vec{a}\times\vec{B}) - \vec{\nabla}\ln(\mathcal{Q})\times(\vec{\nabla}\times\vec{B}), \quad (38)$$

where $\mathcal{Q} = \mathcal{G}(1+\lambda R)$, skin depth $\lambda_L^2 = c^2/\omega_p^2$, plasma frequency $\omega_p^2 = 4\pi nq^2/m$, and acceleration $\vec{a} = \vec{\nabla}\ln\alpha$. Without gravity terms, we see that Eq. (38) becomes the familiar London equation, i.e., $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = -(1/\lambda_L^2)\vec{B}$. However, with gravity entering the system with minimal and nonminimal coupling, we observe that the equilibrium state is more restrictive than the corresponding classical system. Terms with α and λR on the right-hand side of the above equation (38) show contributions from minimal and nonminimal coupling, respectively. The acceleration $\vec{a}$ in the equation stands for the gravitational force felt by plasma as it goes from one hypersurface to the next. It is interesting to notice the explicit appearance of the interaction between Maxwell's field and gravity through the equilibrium state that was previously hidden in the definition of Lorentz 4-current nqU^β mentioned in Sec. II.

It must be emphasized that Eq. (38) is a grand generalization of the London equation (the canonical vorticity being zero) and, thus, acquires much more structure due to gravity—a kind of generalized superconductivity in which the grand vorticity is expelled from the interior. Such states, belonging to the well-known category of relaxed states obtained by satisfying the constraint $\vec{\nabla}\cdot\vec{B} = 0$ and by imposing appropriate boundary conditions that are dependent on the specific geometry of the system, may be used to model the equilibria of plasmas coupled to strongly gravitating sources. We could, for example, seek localized solutions ($\vec{B} = 0$ as $r \to \infty$) for a black hole accretion disk, an example that is discussed briefly in the context of vorticity generation. A complete analysis of this equation will depend on many aspects: spacetime geometry, temperature profile, and, most importantly, a profile for plasma frequency as the number density in the accretion disk is a complicated function of distance that also varies in different regions of the disk [26]. Our future work will explore a complete analysis for the equilibrium state that will require substantial numerical analysis.

IV. CONCLUSION

As a next step to the unified theory of electromagnetic fields and flow fields ([1,5]), we have constructed a formalism describing the dynamics of hot charged relativistic fluids, nonminimally coupled to gravity. It is shown that, even with nonminimal coupling, the dynamics obeys the 4D vortical structure, first exposed in Eq. (26) [1]. The new vorticity tensor $\mathcal{M}^{\mu\nu}$ represents a grand synthesis of fluid, electromagnetic, and modified gravity fields with the nonminimal gravity coupling appearing, explicitly, in its definition. The current formalism, when expressed in $3+1$ decomposition incorporates shear and expansion of the congruences. Consequently, the equations for the generalized electric and magnetic fields (generalized vorticity), which are but the appropriate projections of tensor $\mathcal{M}^{\mu\nu}$, turn out to be considerably more involved than previous studies. We have briefly discussed these evolution equations for a couple of specified geometries. In the process, we derived a relaxed-state equilibrium for a plasma coupled to a strongly gravitating source. Gravity gives much more structure to what would have been a London-like state.

The generally covariant formulation provides a basic framework for investigating the charged fluid dynamics in the presence of strongly gravitating sources when spacetime curvature might play a significant/dominant role. Our basic equations could be used, for example, to extend the scope and content of numerical simulations of the seed magnetic field generation ([27,28]).

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APPENDIX: STRESS TENSOR FOR PERFECT FLUID

The action for perfect fluid is [29,30]

$$S_{pf}(g_{\mu\nu}, n, \tilde{\sigma}) = \int d^4x(-\sqrt{-g}\rho(n, \tilde{\sigma})), \tag{A1}$$

where entropy density $\tilde{\sigma} = ns$ and n and s are the number density and entropy per particle, respectively. The variation of the above action is

$$\begin{aligned}\delta S &= \delta(-\sqrt{-g}\rho(n,\tilde{\sigma}))\\ &= -\delta(\sqrt{-g})\rho(n,\tilde{\sigma}) - \sqrt{-g}\left(\frac{\partial\rho}{\partial n}\delta n + \frac{\partial\rho}{\partial\tilde{\sigma}}\delta\tilde{\sigma}\right)\\ &= -\frac{1}{2}\sqrt{-g}\rho(n,\tilde{\sigma})g^{\mu\nu}\delta g_{\mu\nu}\\ &\quad -\frac{1}{2}\sqrt{-g}\left(\frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma}\right)(u^\mu u^\nu - g^{\mu\nu})\delta g_{\mu\nu}.\end{aligned} \tag{A2}$$

Here, we have used the conservation laws, $\nabla_\mu(nu^\mu) = 0$ and $\nabla_\mu(\tilde{\sigma}u^\mu) = 0$, with the 4-velocity $u^\mu = \frac{dx^\mu}{ds}$. Now, we can write the two following expressions for variations of n and $\tilde{\sigma}$ [10,31]:

$$\begin{aligned}\delta n &= \frac{1}{2}n(u^\mu u^\nu - g^{\mu\nu})\delta g_{\mu\nu};\\ \delta\tilde{\sigma} &= \frac{1}{2}\tilde{\sigma}(u^\mu u^\nu - g^{\mu\nu})\delta g_{\mu\nu}.\end{aligned} \tag{A3}$$

Therefore, we can simplify Eq. (A2) using Eq. (A3) and get

$$\begin{aligned}\delta S = -\bigg[&\frac{1}{2}\sqrt{-g}\left(\frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma}\right)u^\mu u^\nu\\ &+\frac{1}{2}\sqrt{-g}\left(\frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma} - \rho\right)g^{\mu\nu}\bigg]\delta g_{\mu\nu}.\end{aligned} \tag{A4}$$

Now, using the definition $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$, we obtain the following expression:

$$T^{\mu\nu} = -\left(\frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma}\right)u^\mu u^\nu + \left(\frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma} - \rho\right)g^{\mu\nu}. \tag{A5}$$

After Legendre transformation [32], we define the pressure p to be

$$p = \frac{\partial\rho}{\partial n}n + \frac{\partial\rho}{\partial\tilde{\sigma}}\tilde{\sigma} - \rho. \tag{A6}$$

This finally gives us the expressions of the stress-energy tensor for a perfect fluid,

$$\begin{aligned}T^{\mu\nu} &= -(p+\rho)u^\mu u^\nu + pg^{\mu\nu},\\ T^{\mu\nu} &= (p+\rho)u^\mu u^\nu + pg^{\mu\nu},\end{aligned} \tag{A7}$$

where we have the 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ with $ds^2 = -d\tau^2$ and $c = 1$. Alternate expression for the stress tensor can be written by defining enthalpy density $h = p + \rho$ as follows:

$$T^{\mu\nu} = hu^\mu u^\nu + pg^{\mu\nu}. \tag{A8}$$

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