Energy decomposition within Einstein-Born-Infeld black holes

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We analyze the consequences of the recently found generalization of the Christodoulou-Ruffini black hole mass decomposition for Einstein-Born-Infeld black holes [characterized by the parameters (Q, M, b), where $M = M(M_{\rm irr}, Q, b)$, b scale field, Q charge, $M_{\rm irr}$ "irreducible mass," physically meaning the energy of a black hole when its charge is null] and their interactions. We show in this context that their description is largely simplified and can basically be split into two families depending upon the parameter $b|Q|$. If $b|Q| \leq 1/2$, then black holes could have even zero irreducible masses and they always exhibit single nondegenerated horizons. If $b|Q| > 1/2$, then an associated black hole must have a minimum irreducible mass (related to its minimum energy) and has two horizons up to a transitional irreducible mass. For larger irreducible masses, single horizon structures raise again. By assuming that black holes emit thermal uncharged scalar particles, we further show in light of the black hole mass decomposition that one satisfying $b|Q| > 1/2$ takes an infinite amount of time to reach the zero temperature, settling down exactly at its minimum energy. Finally, we argue that depending on the fundamental parameter b, the radiation (electromagnetic and gravitational) coming from Einstein-Born-Infeld black holes could differ significantly from Einstein-Maxwell ones. Hence, it could be used to assess such a parameter.

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I. INTRODUCTION

Although solving Einstein equations for a classical charged black hole (BH) (Reissner-Nordström one) is a relatively simple task [1], such an approach does not make evident the relationship between its two parameters, namely its mass (M) and charge (Q). Intuitively, this relation must exist since electromagnetic energies have their origin in charges, and it can be found in a variety of ways. An interesting notably physical manner was put forward by Christodoulou [2] and Christodoulou and Ruffini [3], by introducing the concept of BH reversible transformations [2]. Such transformations are the only ones that could bring back the BH parameters to their original values after any transformation processed by a test particle with parameters m and q (where $M \gg m$ and $Q \gg q$). Another known approach was due to Bardeen *et al.* [4], which takes advantage of the spacetime symmetries.

It has been recently shown [5], in the context of spherically symmetric spacetimes, that reversible transformations are fully equivalent to the constancy of the event horizon upon such changes for any nonlinear theory of the electromagnetism $L(F)$ that leads to asymptotically flat solutions. Due to the generality of the analysis, such a constant must be $2M_{\rm irr}$, where $M_{\rm irr}$ is the irreducible BH mass given by the total mass energy of the system in the uncharged case, namely when $Q = 0$. Due to this fact, $M_{\rm irr}$ must be always positive. The aforementioned equivalence allows us to exchange the problem of solving nonlinear differential equations for nonlinear theories by the problem of solving algebraic equations. This procedure works only for the cases where event horizons are present. We recall that after the seminal work of Bekenstein [6], it is known that the entropy of a black hole is equivalent to its $M_{\rm irr}$. Nevertheless, it is more appealing to our reasoning to make use of the original concept of irreducible mass, $M_{\rm irr}$.

The aim of this work is to elaborate on the consequences of the mass-energy decomposition for nonlinear BHs and their interactions. In order to do it, we use the specific nonlinear theory of electromagnetism due to Born and Infeld (BI) [7]. Such a theory has regained interest due to its analogous emergence as an effective theory to string theory [8]. It was constructed with the purpose of remedying the singular behavior in terms of energy of a pointlike charged particle. The theory introduces a parameter b identified with the absolute upper limit of the electric field of a system when just electric aspects are present. Born and Infeld fixed this parameter by imposing that in the Minkowski spacetime the associated electromagnetic energy coming from a pointlike electron equals its rest mass (unitarian viewpoint [7]). Nevertheless, the dualistic viewpoint [7] could equally well have been assumed and the parameter b should be determined by a theory relying on it, such as quantum

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mechanics [7]. Actually, the BI theory has been applied to the description of the hydrogen atom, both the nonrelativistic and relativistic one [9,10], and their numerical analyses show that b must be much larger than the value initially proposed by BI. Notwithstanding, a definite value has not been obtained.

Rasheed [8] has analyzed mathematically the validity of the zeroth and first laws of black hole mechanics and concluded that they do hold for any nonlinear Lagrangian of the electromagnetism. Although Rasheed concluded that the black hole mass formula for such a case does not keep the same simple functional form as for the Maxwellian Lagrangian, a further scrutiny of the consequences of this fact was not performed. Following our results in Ref. [5], we instead shall analyze in this work some consequences of the black hole mass formula in the case of Einstein-Born-Infeld black holes, and their interactions. Since such a relation establishes a constraint for the parameters of the theory, physically based on conservation laws, the description is expected to be greatly simplified, as it will turn out to be exactly the case. To the best of our knowledge, this has not been done before.

The article is organized as follows. In the next section, the mathematical approach for reversible transformations is briefly elaborated and the mass decomposition for $L(F)$ theories in the spherically symmetric case is exhibited. In Sec. III, we revisit some aspects of the Einstein-Born-Infeld black hole solution and exhibit the black hole mass decomposition for this theory. In Sec. IV, we analyze some properties of the above-mentioned mass decomposition and show that when b is finite, there are always intrinsic nonclassical islands of black hole solutions where each member has a single, nondegenerated horizon. Section V is devoted to the study of the consequences of assuming that Einstein-Born-Infeld black holes evaporate within the framework of the mass decomposition. In Sec. VI we analyze the radiation emitted by two interacting Einstein-Born-Infeld black holes and show by means of a toy model that in principle there are alternative ways to infer the constant b even from astrophysical scenarios. Section VII closes the paper with an analysis of the main points raised.

Units are such that $c = G = 1$ and the signature of the spacetime is -2.

II. BLACK HOLE MASS DECOMPOSITION FOR ANY NONLINEAR THEORY

In the context of spherically symmetric solutions to general relativity minimally coupled to nonlinear Lagrangians of the electromagnetism, it can be shown that the general solution to the metric is [11]

$$ds^2 = e^{\nu(r)}dt^2 - e^{-\nu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where [5]

$$e^{\nu(r)} = 1 - \frac{2M}{r} + \frac{8\pi}{r}\int_r^\infty r'^2 T^0{}_0(r')dr' = 1 - \frac{2M}{r} + \frac{2QA_0}{r} - \frac{2\mathcal{N}}{r}, \quad (2)$$

$$E_r \doteq -\frac{\partial A_0}{\partial r}, \quad T^\mu{}_\nu = \frac{4L_F F^{\mu\beta}F_{\nu\beta} - L\delta^\mu{}_\nu}{4\pi}, \quad \frac{\partial \mathcal{N}}{\partial r} \doteq -Lr^2. \quad (3)$$

We are assuming that the Lagrangian describing the electromagnetic interactions is $L = L(F)$, $F \doteq F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic field tensor [1,12]. Besides, L_F was defined as the derivative of $L(F)$ with respect to the invariant F and $T^\mu{}_\nu$ is the energy-momentum tensor of the matter fields [1,12], here the electromagnetic fields described by $L(F)$. In the above expressions, E_r is the radial component of the electric field and A_0 is its associated potential. In the expressions for A_0 and $\mathcal{N}$, it has been chosen a gauge where they are null at infinity. We stress that for obtaining $A_0(r)$ and $\mathcal{N}(r)$ from given $E_r(r)$ and $L(F)$, it is tacit one has to integrate from an arbitrary r to infinity, since we are interested in black hole solutions [13]. The radial electric field satisfies the equation

$$L_F E_r r^2 = -\frac{Q}{4} \quad \text{or} \quad \frac{\partial L}{\partial E_r} = \frac{Q}{r^2}. \quad (4)$$

In a spherically symmetric spacetime, infinitesimal reversible transformations are defined by

$$\delta M = \delta Q A_0(r_+), \quad (5)$$

where r_+ is the outermost horizon from a given black hole theory, defined as the largest zero of Eq. (3). For a general transformation, one has the formal replacement "$=\to\geq$" in the above equation.

The customary approach for obtaining the mass formula (energy decomposition) would be integrating Eq. (5), given the outer horizon in terms of the parameters coming from the electromagnetic theory under interest and the spacetime. In general, it turns out to be impossible to work analytically for $L(F)$ theories in such a case. Since one knows that there is a correlation between black holes and thermodynamics [4,14], one would suspect that Eq. (5) (thermodynamics) is somehow inside the equations of general relativity (or vice versa). It can be shown easily that this is indeed the case, provided that the outer horizon keeps constant under reversible transformations [5]. Since it is so, it follows that the outer horizon must be identified with its associated Schwarzschild horizon (where $Q = 0$), and it will be denoted by $r_+ = 2M_{\rm irr}$.

For the nonlinear theories where the electric potential A_0 is independent of the parameter M, it follows from the above reasoning and Eq. (3) that

$$M = M_{\rm irr} + QA_0|_{r=2M_{\rm irr}} - \mathcal{N}|_{r=2M_{\rm irr}} = M_{\rm irr} + 4\pi \int_{2M_{\rm irr}}^{\infty} r'^2 T^0{}_0(r')dr'. \tag{6}$$

The above equation is the way of decomposing the total energy in terms of intrinsic ($M_{\rm irr}$) and extractable quantities ($M - M_{\rm irr}$). It can be shown with ease [5] that it implies the so-called generalized first law of black hole mechanics for nonlinear electrodynamics [8], thus superseding it. Notice from the above equation that one could not associate all $M_{\rm irr}$ (given M and $T^0{}_0$) with the outer horizon. The reason for this is simple: Eq. (6) was defined by $e^{\nu(2M_{\rm irr})} = 0$, which encompasses also $M_{\rm irr}$ related to the inner horizon. Nevertheless, it is uncomplicated to single out the set of $M_{\rm irr}$ corresponding to the outer horizon. One knows that the condition that leads to the degeneracy of the horizons is the common solution to $e^{\nu(2M_{\rm irr})} = 0$ and $de^{\nu}/dr|_{r=2M_{\rm irr}} = 0$. These requirements and Eq. (6) imply that the horizons of black holes are degenerated at the critical points of M as a function of $M_{\rm irr}$. Hence, since outer horizons are larger than inner ones, it follows that the set of irreducible masses relevant in our analysis is the one that always gives $dM/dM_{\rm irr} \geq 0$. In the mass decomposition approach the region inside the outer horizon is not of physical relevance.

III. BORN-INFELD LAGRANGIAN

The Born-Infeld Lagrangian $L_{\rm BI}$ can be written as (compatible with our previous definitions)

$$L_{\rm BI} = b^2\left(1 - \sqrt{1 + \frac{F}{2b^2}}\right), \tag{7}$$

where b is the fundamental parameter of the theory and counts for the maximum electric field exhibited by an electrically charged and at rest particle in flat spacetime [7]. This parameter naturally defines a scale to the Born-Infeld theory.

Putting Eq. (7) into Eqs. (3) and (3) and performing the integral from a given arbitrary radial coordinate r up to infinity, one gets (see for instance Ref. [13])

$$e^{\nu(r)} = 1 - \frac{2M}{r} - \frac{2}{3}b^2y^2 + \frac{2Q^2}{3\sqrt{|\beta|}r}\mathcal{F}\left[x(r), \frac{1}{\sqrt{2}}\right], \tag{8}$$

where we have defined

$$x(r) \doteq \arccos\left(\frac{r^2 - |\beta|}{r^2 + |\beta|}\right), \qquad y^2 \doteq \sqrt{r^4 + \beta^2} - r^2, \tag{9}$$

$$\beta^2 \doteq \frac{Q^2}{b^2}, \qquad \mathcal{F}\left[x(r), \frac{1}{\sqrt{2}}\right] = 2\int_{\frac{r}{\sqrt{|\beta|}}}^{\infty} \frac{du}{\sqrt{1+u^4}}, \tag{10}$$

where $\mathcal{F}[x(r), 1/\sqrt{2}]$ is the elliptic function of first kind [15].

The modulus of the radial electric field and its scalar potential in this case, as given by the first term of Eqs. (3) and (4), are

$$E_r(r) = \frac{Q}{\sqrt{r^4 + \beta^2}}, \qquad A_0(r) = \frac{Q}{2\sqrt{|\beta|}}\mathcal{F}\left[x(r), \frac{1}{\sqrt{2}}\right]. \tag{11}$$

As it is clear from Eq. (11), the electric field of a pointlike charged particle is always finite, as well as its associated scalar potential and they are positive monotonically decreasing functions of the radial coordinate. Hence, from Eq. (5), it implies that the necessary and sufficient condition for extracting energy from an Einstein-Born-Infeld black hole is to use test particles with an opposite charge to the hole.

IV. ANALYSIS OF THE EINSTEIN-BORN-INFELD MASS FORMULA

The metric given by Eqs. (8), (9) and (10) has been studied in detail in Ref. [13]. It has been pointed out there that the dimensionless quantities $\tilde{M} \doteq bM$, $\alpha \doteq Q/M$ and $u \doteq r/M$ are convenient to scrutinize the properties of such a metric. Nevertheless, apparently some interesting properties of Eq. (8) have not been stressed. Under the above definitions, Eq. (8) may be written as

$$e^{\nu(u)} = 1 - \frac{2}{u} + \frac{2}{3}\tilde{M}^2u^2\left(1 - \sqrt{1 + \frac{\alpha^2}{\tilde{M}^2u^4}}\right) + \frac{2\alpha^2}{3u}\sqrt{\frac{\tilde{M}}{|\alpha|}}\mathcal{F}\left[\arccos\left(\frac{\tilde{M}u^2 - |\alpha|}{\tilde{M}u^2 + |\alpha|}\right), \frac{1}{\sqrt{2}}\right]. \tag{12}$$

The horizons are obtained as the zeros of the above equation. As a result, one can verify that Eq. (12) has no minimum, and hence it is a monotonic function iff

$$b < \frac{9M^2}{|Q|^3\mathcal{F}^2[\pi, \frac{1}{\sqrt{2}}]} \approx \frac{0.654M^2}{|Q|^3}, \tag{13}$$

which can also be cast as

$$M > M_0, \qquad M_0 \doteq \frac{\sqrt{b|Q|^3}}{3}\mathcal{F}\left[\pi, \frac{1}{\sqrt{2}}\right]. \tag{14}$$

As the limit of u going to zero in Eq. (12) shows us, Eq. (13) also guarantees that the associated spacetime will always exhibit just one horizon (not degenerated). The above inequality has no classical counterpart, since it can be formally obtained by taking the limit of b going to infinity. Equation (13) sets a fundamental inequality concerning the parameters Q, b and M. Whenever it is not verified, it does automatically imply the existence of a minimum. A simple analysis shows us that such a requirement can be cast as

$$u_+ \leq \frac{\sqrt{4\tilde{M}^2\alpha^2 - 1}}{2\tilde{M}}, \qquad \frac{d}{du}(e^{\nu})|_{u=u_+} = 0, \tag{15}$$

which is just the consequence of imposing that $e^{\nu(u_+)} \leq 0$, u_+ being the critical point of e^{ν}, thus guaranteeing the existence of an outer horizon. Just as a reference, in the limit when $\tilde{M}$ goes to infinity, the above condition reduces to $|\alpha| \leq 1$, as it is well known from the Reissner-Nordström solution for assuring the existence of horizons. As the above inequality suggests, the term $(4b^2Q^2 - 1)$ plays a fundamental role into the horizon description. We shall see that this is also the case in the approach related to the energy decomposition. Specialized to the Born-Infeld Lagrangian, Eq. (7), the total mass [see Eq. (6)] of an Einstein-Born-Infeld black hole can be decomposed as

$$M = M_{\rm irr} - \frac{8}{3}b^2 M_{\rm irr}^3\left(\sqrt{1+\frac{\beta^2}{16M_{\rm irr}^4}} - 1\right) + \frac{\sqrt{b|Q|^3}}{3}\mathcal{F}\left[\arccos\left(\frac{4M_{\rm irr}^2 - |\beta|}{4M_{\rm irr}^2 + |\beta|}\right), \frac{1}{\sqrt{2}}\right]. \tag{16}$$

From now on we shall assume that Eq. (16) is a valid decomposition to the total energy of a Einstein-Born-Infeld black hole. A simple analysis tells us that whenever

$$2b|Q| > 1 \tag{17}$$

is valid for the parameter Q, given b, Eq. (16) does have a minimum with respect to $M_{\rm irr}$, associated with the critical irreducible mass

$$M_{\rm irr}^c \equiv M_{\rm irr}^{\rm min} = \frac{\sqrt{4b^2Q^2 - 1}}{4b}. \tag{18}$$

Note that $M_{\rm irr}^c$ is always related to the case where the horizons are degenerated (extreme black holes), as we have pointed out in Sec. II, and it is always smaller than its classical counterpart, $|Q|/2$ (where $M = |Q|$). From our previous discussions, the relevant irreducible masses to the analysis for reversible transformations for black holes are $M_{\rm irr} \geq M_{\rm irr}^c$. Substituting the above critical irreducible mass into Eq. (16), one has that its associated minimum total energy is

$$M_{\rm min} = \frac{\sqrt{4b^2Q^2 - 1}}{6b} + \frac{\sqrt{b|Q|^3}}{3}\mathcal{F}\left[x\left(\frac{\sqrt{4b^2Q^2 - 1}}{2b}\right), \frac{1}{\sqrt{2}}\right], \tag{19}$$

which is naturally positive and it can be verified to be smaller than M_0 defined by Eq. (14). For the case $2b|Q| > 1$, one can check that an immediate solution to $M = M_0$ is $M_{\rm irr} = 0$ (not of relevance for us for the present case). There also is a nontrivial solution that cannot be expressed analytically in general, that we shall denote by $M_{\rm irr}^t$. This solution is very important since it will delimit the transition from spacelike singularities to timelike ones with respect to the radial coordinate. This signifies that the range of irreducible masses that generalizes Reissner-Nordström black holes (with two horizons) is $M_{\rm irr}^{\rm min} \leq M_{\rm irr} < M_{\rm irr}^t$. An arbitrary black hole with $M_{\rm irr} \geq M_{\rm irr}^t$ shall present a sole horizon and hence when test particles have crossed it, their fate is unavoidably its associated singularity. Note that Reissner-Nordström black holes are such that $M_{\rm irr}^t \to \infty$ and the existence of M_{rr}^t for Einstein-Born-Infeld black holes is only due to the finiteness of b. Figure 1 exemplifies the analysis from the previous sentences for a selected value of the parameter $b|Q|$ for the case $2b|Q| > 1$.

We consider now the case where Eq. (17) is violated. In this case, M, as given by Eq. (16), is a monotonic function of $M_{\rm irr}$. Since it is given by Eq. (14) when $M_{\rm irr} = 0$ and it is monotonic, we conclude that Eq. (14) is always satisfied and therefore the associated singularity is unavoidable for test particles. Just for completeness, Fig. 2 compactifies the above-mentioned properties for a selected value of the parameter $b|Q|$ such that $2b|Q| \leq 1$. Besides, in Fig. 3 we depicted all the different classes associated with the parameter $b|Q|$, assuming in all cases it is fixed.

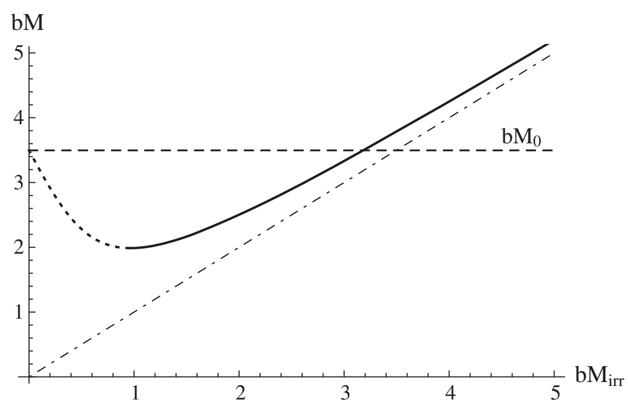

FIG. 1. Mass formula (thick plus dotted curves), Eq. (16), when the parameter $b|Q|$ satisfies Eq. (17), chosen here as 2. The dashed curve represents bM_0, as given by Eq. (14). The dot-dashed curve is the asymptote to M, $M_{\rm irr}$. Besides, bM exhibits a minimum at the critical point $M_{\rm irr}^c b \approx 0.97$ (where the horizons become degenerated) and for $M_{\rm irr}^c b \leq M_{\rm irr} b < M_{\rm irr}^t b \approx 3.18$, we have the range of irreducible masses that generalize Reissner-Nordström black holes. For $M_{\rm irr} \geq M_{\rm irr}^t$, there is a sole horizon (not degenerated), whose radial coordinate inside of it is always spacelike. The irreducible masses associated with the outer horizon are $M_{\rm irr} \geq M_{\rm irr}^c$. The dotted curve is related to the inner horizon solutions (for given configurations) and is not relevant to the analyses concerning the black hole mass decomposition.

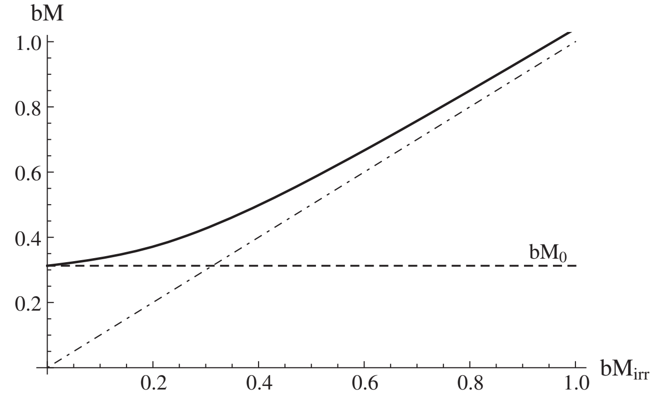

FIG. 2. Mass decomposition when the parameter $b|Q|$ does not satisfy Eq. (17) and is chosen to be 0.4. The curves have the same meaning as the ones in Fig. 1. From the solid curve we see that M is a monotonic function and always larger than M_0. This means that such a case characterizes a scenario where there is always a sole event horizon and there is no classical analogue to it.

An important general remark is here in order, especially for astrophysical analyses. Assume that $b|Q| = C_1$ and $M/|Q| = C_2$, where C_1 and C_2 are given constants. This means that $Mb = C_1 C_2$ is also known. Assuming that $0 \leq C_1 < \infty$ and from the fact that $bM \geq (bM)_{\rm min} \geq 0$, we first conclude that C_2 cannot be any, but $C_2 \geq (bM)_{\rm min}/C_1$. This means that $|\alpha| \doteq |Q|/M \leq C_1/(bM)_{\rm min}$ and this is the condition that guarantees the presence of an outer horizon in an Einstein-Born-Infeld black hole. In the classical case for instance, where $(bM)_{\rm min} = b|Q|$ [see Eq. (19) in the limit $b \to \infty$], the previous inequality means $|\alpha| \leq 1$, as it is already known. Finally, after one chooses arbitrarily another parameter to be M or $|Q|$ or b, all the remaining ones are automatically fixed, which could be assessed by the aforesaid choice. It is not complicated to see that when $M_{\rm irr}/|Q|$ is given instead of $M/|Q|$, a similar reasoning as the above one also ensues.

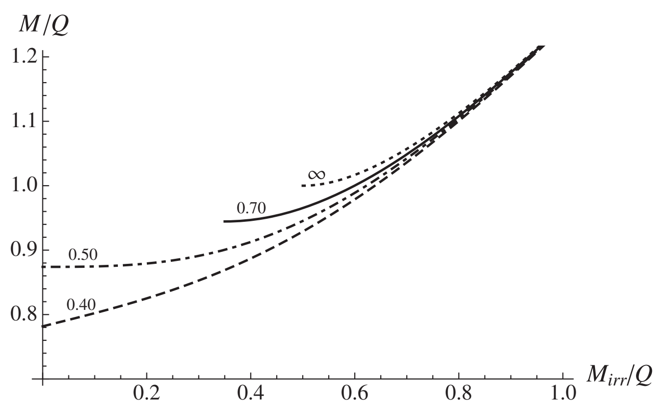

FIG. 3. Mass formula for selected values of the parameter $b|Q|$ (numbers on the curves) that encompasses all physically distinct classes of black holes for the Born-Infeld Lagrangian. The dotted curve represents the mass formula for the Maxwell Lagrangian. The dot-dashed curve demarcates the transition from two horizon solutions (as given by the thick curve) to a single one (as given by the dashed curve), where its associated inner horizon is null. The branches related to the inner horizons were removed.

V. HAWKING RADIATION FROM EINSTEIN-BORN-INFELD BLACK HOLES

Subsequent to the work of Hawking on the semiclassical quantization of scalar fields in some curved spacetimes [16], it is widely accepted that black holes radiate thermally, although this view has still some criticisms [17,18]. Motivated by the first law of black hole thermodynamics, which is a direct consequence of the mass decomposition expression given by Eq. (6) [5], and the results from the aforesaid semiclassical quantization, we shall now study the consequences of conjecturing that clothed black holes should behave like blackbodies to observers at infinity (no backreaction effects are considered here), radiating at temperatures proportional to their surface gravity [16]. In the spherically symmetric case, such a quantity is proportional to $de^{\nu}/dr|_{r=r_+}$ [14,19]. From Eq. (12) and preceding definitions, one has

$$T \propto \frac{1 + 8b^2 M_{\rm irr}^2 - 2b\sqrt{16 b^2 M_{\rm irr}^4 + Q^2}}{M_{\rm irr}}. \tag{20}$$

We notice some particularities of the insertion of the parameter b into the description of the electromagnetic fields. As in the classical case, $b \to \infty$, it is possible to attain $T = 0$, but now as far as

$$M_{\rm irr}^{(T=0)} = \frac{\sqrt{4b^2 Q^2 - 1}}{4b}. \tag{21}$$

Notice that $M_{\rm irr}^c = M_{\rm irr}^{(T=0)}$. This is not surprising, since from our previous comments, the condition for null temperature of a black hole with charge Q occurs exactly at the critical points of the energy with respect to its irreducible mass. When Eq. (17) holds, one sees that the temperatures of the associated clothed black holes must decrease with the decrease of their irreducible masses until they eventually reach zero, for $M_{\rm irr} = M_{\rm irr}^{(T=0)}$. This would mean that black holes where Eq. (17) is valid should radiate off finite amounts of energy, namely $M(M_{\rm irr}) - M(M_{\rm irr}^{(T=0)})$. Besides, from the analyses of the energy decomposition, black holes could never have negative temperatures. For the case Eq. (17) does not hold, it is impossible to have $T = 0$ and the temperature increases with the decrease of the irreducible mass. Figure 4 compactifies the dependence of the temperature upon the irreducible mass for selected values of $b|Q|$.

We elaborate now on the temperature evolution of evaporating blackbodies. For an arbitrary black hole case where $2b|Q| > 1$, as we know, the temperature decreases as

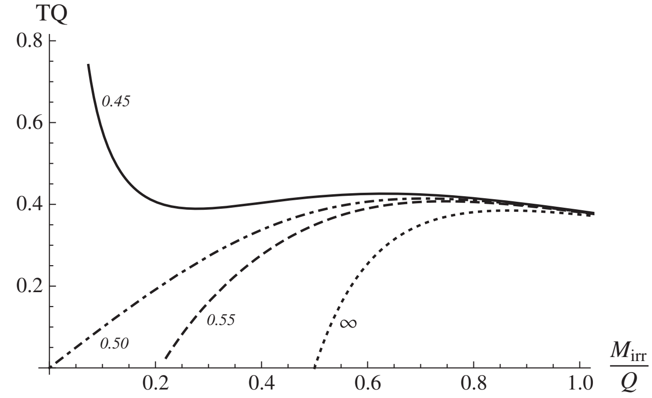

FIG. 4. Einstein-Born-Infeld black hole temperature as a function of the irreducible mass for selected values of the parameter $b|Q|$. The temperature goes to infinity as the irreducible mass tends to zero whenever $2b|Q| \leq 1$ (thick curve). Whenever $2b|Q| > 1$ (dashed curve), it decreases with the decrease of the irreducible mass (keeping the charge constant), always being null for a finite value of the latter. The temperature experiences a transitional behavior for $2b|Q| = 1$ (dot-dashed curve), being null just when the irreducible mass of the system is so [see Eq. (20)], albeit it cannot be seen directly from this plot. Finally, $b|Q| \to \infty$ (dotted curve) corresponds to the Reissner-Nordström case.

the irreducible mass of the system does so (see Fig. 4). Hence, it would allow us to conceive a situation where just the emission of uncharged scalar particles are present. For this simplified case, the charge of a hole would remain constant. Given that the black holes would behave like blackbodies for observers located at infinity (where there is a meaning to talk about the total energy of a black hole), their energy loss could be estimated by Stefan's law [20]

$$\frac{dM}{d\lambda} = -M_{\rm irr}^2 T^4, \tag{22}$$

where λ is proportional to the observer's time receiving the radiation. For the emission of uncharged scalar particles, the above equation and Eq. (16) imply that

$$\frac{d\tilde{M}_{\rm irr}}{d\lambda} \propto -\frac{\left(1 + 8\tilde{M}_{\rm irr}^2 - 2\sqrt{16\tilde{M}_{\rm irr}^4 + \tilde{Q}^2}\right)^3}{\tilde{M}_{\rm irr}^2}. \tag{23}$$

In the above equation, for an arbitrary quantity A, $\tilde{A} \doteq bA$. We show now that for this case the temperature never reaches the absolute zero. Since the irreducible mass can decrease until $M_{\rm irr}^{\rm min}$, after a convenient transient time interval, the right-hand side of Eq. (23) can always be expanded about $M_{\rm irr}^{\rm min}$, leading to

$$\frac{d\tilde{M}_{\rm irr}}{d\bar{\lambda}} = -\left(\tilde{M}_{\rm irr} - \frac{1}{4}\sqrt{4\tilde{Q}^2 - 1}\right)^3, \tag{24}$$

where $\bar{\lambda}$ is proportional to λ and other terms that are constants and not important to our analysis. The above equation has an analytic solution and when the limit of $\bar{\lambda}$ going to infinity is taken, one obtains $\tilde{M}_{\rm irr}(\infty) = \tilde{M}_{\rm irr}^{\rm min}$. This means an associated black hole never reaches the absolute zero and tends asymptotically to have just one horizon. Our analyses in light of the energy decomposition give the same known mathematical results for the thermodynamics for Reissner-Nordström black holes [20,21], but in a simpler way.

For an arbitrary black hole satisfying $2b|Q| \leq 1$, it seems that a juncture shall arrive where its thermal energy will be sufficient to create pairs that could even neutralize the hole. This would happen since in this case the thermal energy of a black hole would augment with the diminution of its irreducible mass (see Fig 4). Hence its description would be much more elaborated than the former one. Black holes with $2b|Q| \leq 1$ are expected to evaporate after finite amounts of time, as corroborated by numerical analyses from Eq. (23). We shall not pursue further into these issues in this work.

VI. ENERGY LOSS OF INTERACTING EINSTEIN-BORN-INFELD BLACK HOLES

In this section we shall make use of the energy decomposition given by Eq. (16) to find the imprint the parameter b has on the energy radiated off by two interacting Einstein-Born-Infeld black holes. For accomplishing such a goal, we shall also utilize the second law of black hole mechanics [1,4]. Such a theorem implies that the area of the resultant black hole can never be smaller than the sum of the areas of the initially (far away) interacting black holes [1,4]. For simplifying the reasoning, we will assume that all the black holes involved are spherically symmetric Einstein-Born-Infeld ones. This problem can easily be solved for Einstein-Maxwell black holes (Einstein theory minimally coupled to the Maxwell Lagrangian), because their outer horizons are analytical. For nonlinear black holes, in general just numerical solutions are possible. In the mass decomposition approach, it is possible to carry out the analytical investigations further. The key for this is that whenever the mass formula is taken into account, the outer horizon must be always proportional to its associated irreducible mass for any theory.

Assume that the two initially interacting black holes have irreducible masses M_{i1} and M_{i2}, respectively, giving rise to another (final) one of the same kind with irreducible mass M_{if}. Concerning its final charge, if one assumes that just radiation is allowed to leave the system (carried away by neutral particles), it must be the sum of the charges of the two initial black holes [1]. Since the irreducible masses are proportional to the horizon areas, Hawking's theorem (or the second law of black hole mechanics) implies that

$$M_{if}^2 \geq M_{i1}^2 + M_{i2}^2. \tag{25}$$

Invoking the first law of black hole mechanics for an isolated system [1], the final energy of the two interacting black holes M_f can never be larger than $M_1 + M_2$. The difference in the energy balance is due to the emission of radiation (here gravitational and electromagnetic), hence, $W_{\rm rad} = M_1 + M_2 - M_f \geq 0$. By the cognizance of the minimum final energy of the system, it is even possible to obtain its maximum energy radiated off, a point we shall not pursue here.

For fixing ideas, let us analyze first the classical case, namely two Reissner-Nordström black holes interacting in a way to lead to another Reissner-Nordström black hole. We know that the total energy of each black hole can be written as [3]

$$M_a = M_{ia} + \frac{Q_a^2}{4M_{ia}}, \tag{26}$$

where we have defined Q_a as the charge of the ath black hole. It is easy to see that just $M_{if}^- \leq M_{if} \leq M_{if}^+$ with

$$M_{if}^{\pm} = \frac{M_1 + M_2 \pm \sqrt{(M_1+M_2)^2 - (Q_1+Q_2)^2}}{2} \tag{27}$$

is in agreement with the above-mentioned positivity of $W_{\rm rad}$. Naturally, choices for M_{if} must satisfy simultaneously Eqs. (25) and (27). When nonlinear theories are present, it is clear that in general the above range of final irreducible masses will not agree with the classical (Einstein-Maxwell black holes) case. It means that many possible classical situations will not exist in the nonlinear case and vice versa even in the simple case of symmetry conserved binary interactions. This could possibly lead to significant deviations for the amounts of radiation emitted by some systems when they are treated classically or not.

In the Einstein-Born-Infeld theory, the physical interval for M_{if} cannot be determined (numerically) unless the fundamental parameter b is given. What is known [9] is that $b > b_0 \approx 10^{-9}\ {\rm cm}^{-1}$, where b_0 is the value for the scale field determined by Born and Infeld using the unitarian viewpoint [7].

Let us take a closer look at the Einstein-Born-Infeld black holes when compared to their classical counterparts. Assume just for simplicity that $M_{i1} = M_{i2}$ and $Q_1 = Q_2 \equiv Q > 0$. For this choice, Eq. (27) gives us $-\sqrt{1/\alpha^2 - 1} \leq M_{if}/Q - 1/\alpha \leq \sqrt{1/\alpha^2 - 1}$, where α is here defined as the charge-to-mass ratio of the initially interacting black holes. Let us choose, just for simpleness, $M_{if}/Q = 1/\alpha$. From the Einstein-Maxwell case, one can check easily that for the above analysis $W_{\rm rad(clas)}/Q = (1-\alpha^2)/\alpha$. For the above choice of parameters, one can show that Eq. (25) is just satisfied if $\alpha \geq \sqrt{2(\sqrt{2}-1)} \approx 0.91$. Such cases are of theoretical interest since they would evidence the departures of the Born-Infeld theory from the Maxwell theory. For investigating smaller values of α, one should select different final irreducible masses for the black holes.

Figure 5 compactifies the possibilities for the above chosen M_{if} for $\alpha = 0.95$, due to miscellaneous values of bQ. One sees in this case that nonlinear and linear black holes may radiate off very different amounts of energy. Besides, the energy released for interacting Born-Infeld black holes is always larger than its Maxwellian counterpart. Notice finally that $Q = \alpha M$, M being the mass of any of the black holes when they are far apart, which would also allow one to compare the energies radiated off by the black holes during their process of interaction with the total initial energy of the system.

Some simple estimates can be done here assessing astrophysical scenarios where Fig. 5 could be of relevance. As we stressed before, from the hydrogen atom one knows that $b \gg b_0 \simeq 10^{-9}\ {\rm cm}^{-1} \simeq 10^{15}$ eletronstatic unit. We also commented at the end of Sec. IV that with fixed $M_{\rm irr}/|Q|$ or $M/|Q|$ and $b|Q|$, one still has freedom to choose arbitrarily another parameter, such as M, even having already taken into account the mass formula. Let us choose, as it is reasonable under the point of view of black hole interactions coming from neutron stars, $M \simeq M_\odot \simeq 1.48 \times 10^5$ cm, where $M_\odot$ is the mass of the Sun. Let us focus our attention at a given value of $b|Q|$ such that the associated radiated energy may differ considerably from its classical counterpart. As a simple inspection in Fig. 5 reveals, one could take as a

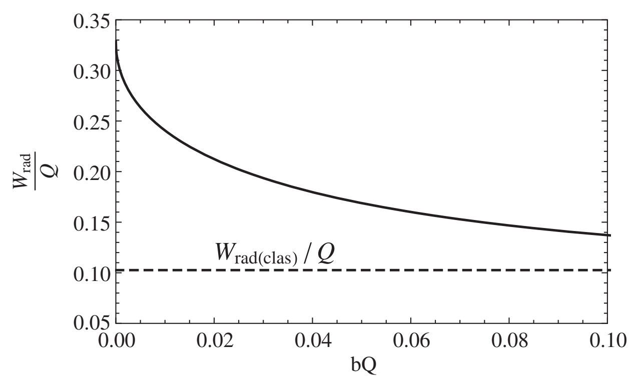

FIG. 5. Total radiation (gravitational plus electromagnetic) $W_{\rm rad}/Q$ released in the process of coalescence of two identical Einstein-Born-Infeld black holes with $\alpha = 0.95$ under the assumption it leads to another one of the same type with the same parameters as their classical counterparts. The thick curve represents such a case. The dashed curve stands for the radiation encountered in the Einstein-Maxwell theory, $W_{\rm rad(clas)}/Q$. The associated radiation tends to its classical counterpart when bQ goes to infinity. The energy released in the case of nonlinear black hole interaction is always larger than the one coming from its classical counterpart, for a given charge Q.

good example of this case $b|Q| = 0.1$, where the energy radiated off by Born-Infeld black holes is around 30% more than Maxwellian ones with identical parameters. Besides, we recall that we have already chosen $\alpha = 0.95$ for plotting Fig. 5. From this case, we have $bM_0 \simeq 4 \times 10^{-2}$ [see Eq. (14)], which shows that $|\alpha| = 0.95$ is a perfectly good candidate for the case $2b|Q| \leq 1$, the one we are interested in here. For this case we know that $Q = M/C_2 = 1.4 \times 10^5$ cm $= 1.6 \times 10^{20}$ C and finally $b = C_1C_2/M = 7.1 \times 10^{-7}$ cm^{-1}, which is about 1000 times larger than b_0 and hence in agreement with the bound given by the hydrogen atom, the only remaining physical constraint. Therefore, the above example suggests that the radiation coming from coalescing astrophysical black holes could be a good tool to access and discriminate their electrodynamical properties.

VII. DISCUSSION

Foremost, it is clear that the approach of analyzing a given black hole solution just from its metric and the one from its metric and energy decomposition expression must be consistent since both approaches use intrinsic properties of the spacetime. Nevertheless, the latter approach is much more restrictive than the former one. It must be stressed that the energy decomposition (black hole thermodynamics) is mandatory for the proper description of any (clothed) black hole phenomenon, since it is in accord with conservation laws. Such a constraint equation (energy decomposition) in turn automatically evidences the physically relevant cases in black hole physics, hence leading to a pellucid description of them.

The energy decomposition analysis within Einstein-Born-Infeld black holes leads us to their split into two fundamental families of black holes. Whenever $2b|Q| \leq 1$, independent of their irreducible masses, one is led to an associated black hole whose singularity cannot be forestalled after test particles cross its sole nondegenerated horizon. Besides, the previous inequality naturally leads to an absolute upper limit to the charge of approximately 10^8 cm $\approx 10^3 M_\odot \simeq 10^{23}$ C, given that $b > 10^{-9}$ cm^{-1} [9]. Finally, we notice that for this class of black holes, the extractable energy could be up to 100%, since black holes with $2b|Q| \leq 1$ could even have $M_{\rm irr} = 0$. We stress that the previous conclusions are strictly nonclassical consequences of the finiteness of b.

The second family of black holes is defined by those satisfying $2b|Q| > 1$, where $M_{\rm irr} \geq M_{\rm irr}^{\rm min}$ [see Eqs. (18) and (19)] for each black hole associated. It constitutes the family that generalizes Einstein-Reissner-Nordström black holes for irreducible masses smaller than transitional values, the nontrivial solutions of $M = M_0$, and larger than $M_{\rm irr}^{\rm min}$ (related to their minimum energies), whose associated energies (masses) are always smaller than M_0. Above such transitional irreducible masses, again due to the finiteness of b, nonclassical black holes with single horizons also rise, all of them having masses larger than M_0. The total amount of energy that could be extracted [$M - M_{\rm irr}$, see Eq. (16)] in this case is always inferior to half of the total energy of the hole (as it occurs for Reissner-Nordström black holes, see [3]), here due to the self-interactions present.

Black holes satisfying $2b|Q| > 1$ should radiate off (suppose by emitting uncharged scalar particles) until their temperatures reach $T = 0$, taking for doing so an infinite amount of time, settling down exactly at their lowest energy state, as one would intuitively expect and here as a direct consequence of the mass formula. Further energy could be extracted from them (obviously by means of other processes rather than the emission of uncharged scalar particles) even when $T = 0$, since they still have an ergosphere. For the case $2b|Q| \leq 1$, it is impossible to have $T = 0$ and they are expected to keep radiating, with a much more complex dynamics, until their total evaporation likely after a finite amount of time as measured by the observer who receives the radiation. Whenever charged scalar fields are taken into account, the phenomenon of superradiance could also take place, rendering their dynamics even more cumbersome. Superradiance is of interest for charged nonlinear black holes, since it is another energy extraction mechanism for them and would couple to the nonlinearities of the electromagnetic field. We let more precise analyses of this case to be done elsewhere.

Concerning the issue of energies radiated off due to the interaction of black holes, as we showed here with a toy model, the changes imprinted by the Einstein-Born-Infeld black holes with respect to their classical counterparts may be significant, depending on α for a range of values of the fundamental parameter $b|Q|$. This could be important for gravitational wave detectors calibrated based on classical results. Besides, if it is possible to identify sources of radiation, then measurements upon such a quantity could give us information about electromagnetic interactions. We analyzed the radiated energies due to charged black hole interactions. This means that also electromagnetic radiation is always present in such processes. Identifying and analyzing this part of the radiation would give direct information about astrophysical electrodynamical processes.

We further point out that all the above conclusions remain valid even in the case where the systems present a slow rotation (when the rotational parameter $a \doteq J/M$, J being the total angular momentum of the system as seen by distant observers, is much smaller than the outer horizon area or the mass of the hole). This is the case since the energy decomposition must be an even power of a, due to invariance requirements. Thereby, the previous analyses are in a sense stable against rotational perturbations.

Summing up, in this work we tried to emphasize the need of also taking into account the mass decomposition of a charged black hole for talking about the physical aspects it could display. Conceptually speaking this is of relevance since it could give us acumen of where and how to search experimentally for charged black holes and their interactions. In this regard, it would be also of interest to investigate the aspects of the electromagnetic radiation coming from the coalescence of charged black holes; because it could be much more easily observed, it would give us direct information about electromagnetic phenomena and of the coalescence process itself. It also seems that quasi periodic oscillations could also shed a light on the illation of black hole charges and the role played by the nonlinearities of the electromagnetism in the astrophysical scope, since they talk about phenomena that take place in the innermost regions of black holes (see [22] and references therein). We let this issue be elaborated elsewhere.

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