Extra-dimensional generalization of minimum-length deformed QM/QFT and some of its phenomenological consequences

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In contrast to the three-dimensional case, different approaches for deriving the gravitational corrections to the Heisenberg uncertainty relation do not lead to the unique result, whereas additional spatial dimensions are present in the theory. We suggest taking logarithmic corrections to the black hole entropy, which has recently been proved in both string theory and loop quantum gravity to persist in the presence of additional spatial dimensions, as a point of entry for identifying the modified Heisenberg-Weyl algebra. We then use a particular Hilbert-space representation for such a quantum mechanics to construct the correspondingly modified field theory and address some phenomenological issues following from it. Some subtleties arising at the second quantization level are clearly pointed out. Solving the field operator to the first order in the deformation parameter and defining the modified wave function for a free particle, we discuss the possible phenomenological implications for the black hole evaporation. Putting aside modifications arising at the second quantization level, we address the corrections to the gravitational potential due to a modified propagator (backreaction on gravity) and see that a correspondingly modified Schwarzschild-Tangherlini space-time reproduces the disappearance of the horizon and the vanishing of the surface gravity when the black hole mass approaches the quantum gravity scale. This result points to the existence of zero-temperature black hole remnants.

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I. INTRODUCTION

Quantum theory based on a so-called minimum-length deformed uncertainty relation is endowed basically with two new features: 1) the modified dispersion relation and 2) the cutoff on the "standard" momentum [1]. Remarkably enough, this sort of uncertainty relation in three dimensions can uniquely be reached from various *Gedankenexperimente*, which in higher dimensions lead to the ambiguous result. As a guiding principle for identifying the minimum-length deformed quantum theory in higher dimensions, we suggest using logarithmic corrections to the black hole (BH) entropy. The computations made in recent years in the framework of string theory [2] and loop quantum gravity [3] demonstrate that the logarithmic corrections to the BH entropy are universal in arbitrary space-time dimensions ≥4. Taking this fact into account, first we consider a few examples of deriving logarithmic corrections to the BH entropy in the three-dimensional case by using the modified uncertainty relation (MUR). We shall along the way comment on the misleading issues concerning the immediate (heuristic) application of the MUR to BH radiation. The simple physical picture behind this consideration allows one to guess the higher-dimensional generalization of minimum-length deformed quantum mechanics (QM). The deformed QM derived this way disagrees with the result that follows from the well-known arguments [4–6] (and some other closely related arguments [7]) for estimating the gravitational corrections to the uncertainty relation. The rest of the paper is devoted to the discussion of quantum field theory (QFT) in view of the deformed quantization both at the first and second quantization levels. Some phenomenological implications of this study for black hole physics are explored.

II. IDENTIFYING THE PLANCK-LENGTH DEFORMED QM WITH THE USE OF BH ENTROPY CORRECTIONS

A. From MUR to BH entropy corrections: three-dimensional case

We start by pointing out that, in three dimensions, applying the MUR to the BH radiation—either in an immediate heuristic way, or by first finding the corresponding Hilbert-space representation and then using it for the field theory at both first and second quantization levels—uniquely leads to the logarithmic corrections to the BH entropy. The system of units used throughout this paper is $\hbar = c = 1$. The corrections to the BH radiation obtained in a heuristic manner in Ref. [8] can be viewed as a result of the modified dispersion relation. Namely, when applying the MUR

$$\delta X \delta P \simeq 1 + \beta l_P^2 \delta P^2 \tag{1}$$

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to the BH emission, one assumes that δx is set by the horizon radius, and consequently the characteristic momentum for the emitted particle is estimated as

$$\delta P \simeq \frac{\delta X - \sqrt{\delta X^2 - 4\beta l_P^2}}{2\beta l_P^2}. \quad (2)$$

So, in Eq. (2) δP is understood as the momentum of a particle escaping from the BH in the case when the correction term for the uncertainty relation is assumed [Eq. (1)], while δX^{-1} defines the momentum in the standard case. Adopting the notations $P \equiv \delta P$ and $p \equiv \delta X^{-1}$, Eq. (1) can be written as a modified dispersion relation,

$$P = \frac{p^{-1} - \sqrt{p^{-2} - 4\beta l_P^2}}{2\beta l_P^2} = \frac{1 - \sqrt{1 - 4\beta l_P^2 p^2}}{2\beta l_P^2 p}. \quad (3)$$

This relation is qualitatively different from the one that follows from the Hilbert-space representation of the uncertainty relation (1) [9],

$$P = \frac{p}{1 - \beta l_P^2 p^2}, \quad (4)$$

but nevertheless in the low-energy regime $p \ll E_P$ Eqs. (3) and (4) look the same,

$$P = p + \beta l_P^2 p^3 + O(l_P^4 p^5). \quad (5)$$

We notice that in light of Eq. (3) one has to admit that the momentum P is bounded from above by the Planck energy, because when $p > E_P/2\sqrt{\beta}$ it becomes a complex quantity. In light of Eq. (4), the momentum P is not UV bounded, but again p should be restricted to the interval $0 \le p \le E_P/\sqrt{\beta}$ as this is enough to cover the whole momentum space: $0 \le P < \infty$. In both cases P is understood as a physical momentum that might be used for estimating the energy,

$$\varepsilon = \sqrt{P^2 + m^2},$$

while $\mathbf{p}$ is merely a new coordinate in momentum space, which (quantum mechanically) is related to the translations: $\hat{\mathbf{p}} = -i\nabla$.

Yet another dispersion relation—which arises from applying the deformed quantization with respect to Eq. (1) to the field theory—looks like [10]

$$\varepsilon = \sqrt{p^2 + m^2} + \beta l_P^2 \frac{p^2 + m^2}{l_\star}, \quad (6)$$

where $l_* = E^{-1}$ is set by the characteristic energy scale of the problem under consideration. In the context of BH emission, this is just the temperature of the emission; thus, Eq. (6) can be written as

$$\varepsilon = \sqrt{p^2 + m^2} + \beta l_P^2 (p^2 + m^2)^{3/2}. \quad (7)$$

The effect of Eqs. (5) and (7) on the BH emission temperature is that it increases as

$$T \to T + \beta l_P^2 T^3,$$

and correspondingly

$$dS = \frac{dM}{T} \to \frac{dM}{T} - \beta T dM.$$

By taking into account that $T \propto r_g^{-1} \simeq 1/l_P^2 M$, to the first order in β one finds a logarithmic correction to the entropy,

$$S = \pi \left(\frac{r_g}{l_P}\right)^2 - \gamma \ln\left(\frac{r_g}{l_P}\right).$$

It is worth noticing that the $l_P^4 p^5$ term in Eq. (5) results in the inverse-area corrections to the entropy.

B. From BH entropy corrections to MUR: D > 3

The fact that in higher dimensions one also expects logarithmic corrections to the BH entropy [2,3] can be used to guess the corresponding higher-dimensional generalization of the minimum-length deformed QM. In the higher-dimensional case the gravitational radius (which determines the Hawking temperature, $T \propto r_g^{-1}$) looks like $r_g \simeq (l_F^{2+n} M)^{1/(1+n)}$, where n denotes the number of extra dimensions and l_F stands for the higher-dimensional scale of gravity, $l_F^{2+n} \equiv G_N$. The previous discussion makes it clear that the modified dispersion relation

$$P = p + \beta l_F^\alpha p^{\alpha+1} + \ldots \quad (8)$$

will a reproduce logarithmic correction to the BH entropy if $\alpha = 2 + n$. This suggests a minimum-length deformed QM of the form

$$[\hat{X}, \hat{P}] = i(1 + \beta l_F^{2+n} \hat{P}^{2+n}), \quad (9)$$

which indeed implies the existence of the minimum position uncertainty of the order of [11]

$$\delta X \simeq \left[\int_0^\infty \frac{dP}{1 + \beta l_F^{2+n} P^{2+n}}\right]^{-1} = \frac{\beta^{\frac{1}{2+n}} l_F}{\int_0^\infty \frac{dq}{1+q^{2+n}}}.$$

As in the three-dimensional case [9], the algebra (9) may be written in a somewhat generic form [12],

$$[\hat{X}_i, \hat{X}_j] = 0, \qquad [\hat{P}_i, \hat{P}_j] = 0,$$
$$[\hat{X}_i, \hat{P}_j] = i\{\Xi(\hat{P}^2)\delta_{ij} + \Theta(\hat{P}^2)\hat{P}_i\hat{P}_j\}, \quad (10)$$

where the simplest ansatz for Θ is

$$\Theta(\hat{P}^2) = 2\beta l_F^{2+n}\hat{P}^n.$$

The Hilbert-space representation of Eq. (10) can be constructed in terms of the standard $\hat{\mathbf{x}}, \hat{\mathbf{p}}$ operators as [12]

$$\hat{X}_j = \hat{x}_j, \qquad \hat{P}_j = \frac{\hat{p}_j}{(1 - \frac{2\beta(1+n)}{2+n} l_F^{2+n}\hat{p}^{2+n})^{\frac{1}{1+n}}}, \quad (11)$$

or in the eigenrepresentation of the $\hat{\mathbf{p}}$ operator,

$$\hat{X}_j = i\frac{\partial}{\partial p_j}, \qquad \hat{P}_j = \frac{p_j}{(1 - \frac{2\beta(1+n)}{2+n} l_F^{2+n} p^{2+n})^{\frac{1}{1+n}}}, \quad (12)$$

where the scalar product contains a cutoff on p,

$$\langle\psi_1|\psi_2\rangle = \int_{p^{2+n}<(2+n)/2\beta(1+n)l_F^{2+n}} d^{3+n}p\psi_1^*(\mathbf{p})\psi_2(\mathbf{p}). \quad (13)$$

We note that this construction is a straightforward generalization of the three-dimensional picture described in Ref. [9]. Here the cutoff $p^{2+n} < (2+n)/2\beta(1+n)l_F^{2+n}$ has the same meaning as in Eq. (4). Now the analog of Eq. (4) takes the form

$$P = \frac{p}{(1 - \frac{2\beta(1+n)}{2+n} l_F^{2+n} p^{2+n})^{\frac{1}{1+n}}} = p + \frac{2\beta l_F^{2+n}p^{3+n}}{2+n} + \frac{2\beta^2 l_F^{4+2n}p^{5+2n}}{2+n} + \ldots, \quad (14)$$

where the l_F^{2+n} term reproduces the logarithmic correction to the BH entropy, while the l_F^{4+2n} term is responsible for the inverse-area correction.

C. Comparison with the result following from *Gedankenexperimente* usually used in three dimensions

For the sake of comparison, here we briefly discuss the MUR that follows from the *Gedankenexperimente* that take into account the gravitational effect on the particle's position measurement or some simple dimensional arguments. It is worth noticing that in three dimensions one has a unique picture for various approaches. Let us first look at the dimensional arguments for the gravitational corrections to the Heisenberg uncertainty relation that results in a lower bound on the position uncertainty. For our purposes it will be convenient to choose the system of units $c = 1$; that is, $[\hbar] = \mathrm{g}\cdot\mathrm{cm}$, $[\mathbb{G}_N] = \mathrm{cm}^{n+1}/\mathrm{g}$. Just on dimensional grounds, one can write a somewhat generic expression for the MUR,

$$\delta X\delta P \geq \frac{\hbar}{2} + \beta\hbar^{(\alpha-1)/\alpha}\mathbb{G}_N^{1/\alpha(n+1)}\delta P^{(n+2)/\alpha(n+1)}, \quad (15)$$

where β is a numerical factor of order unity. In order to have a lower bound on the position uncertainty, one should require

$$\alpha \leq \frac{n+2}{n+1}.$$

On the other hand, to allow for the limit $\hbar \to 0$, one has to require $\alpha \geq 1$. It is easy to see that if one picks out the value $\alpha = 1$, then the correction term in Eq. (15) does not depend on $\hbar$ and therefore it survives even in the limit $\hbar \to 0$. By making this specific choice one arrives at the equation

$$\delta X\delta P \geq \frac{\hbar}{2} + \beta\mathbb{G}_N^{\frac{1}{n+1}}\delta P^{\frac{n+2}{n+1}}. \quad (16)$$

From now on we will again adopt the system of units $\hbar = c = 1$ and discuss the correction term in Eq. (16) as a result of certain gravitational effects.

In the case $\delta P \ll \mathbb{G}_N^{-1/(2+n)}$ the correction term in Eq. (16) can be considered as a result of the gravitational extension of the wave-packet localization width as compared to the Minkowskian background [7]. Yet, the correction term in Eq. (16) makes sense even for $\delta P \gtrsim \mathbb{G}_N^{-1/(2+n)}$. In this case it is motivated by the fact that at a high center-of-mass-energy scattering, $\sqrt{s} \gtrsim \mathbb{G}_N^{-1/(2+n)}$, the production of the BH dominates all perturbative processes [13–16], and thus the ability to probe short distances is limited. (It is important to note that at high energies, $\sqrt{s} \gg \mathbb{G}_N^{-1/(2+n)}$, the BH production is increasingly a long-distance, semiclassical process). To make the point clearer, the refined measurement of a particle's position requires large energy transfer during a scattering process used for the measurement. But when the gravitational radius associated with this energy transfer $\sim(\mathbb{G}_N\sqrt{s})^{1/(1+n)}$ becomes greater than the impact parameter, the BH will form and all can say about the particle's position is that it was somewhere within the region $\sim(\mathbb{G}_N\sqrt{s})^{1/(1+n)}$. The gravitational radius of the BH formed in the scattering process grows with energy as $r_g \simeq (\mathbb{G}_N\sqrt{s})^{1/(1+n)}$ and thus determines the high-energy behavior of the position uncertainty.

To summarize, in $D > 3$ the deformed QM given by Eq. (9) might be favored over the suggestion made in Ref. [17] as it allows one to reproduce the logarithmic and inverse-area corrections to the BH entropy, which in turn seems to be a sound result irrespective of the number of dimensions [2,3]. We note that the MUR closely related to the deformed QM (9) was suggested in a somewhat different context in Ref. [18].

D. Free field in $3 + n$ dimensions

In this section we recapitulate some textbook material [19] to prepare for our discussion of the minimum-length deformed QFT. Let us consider a neutral scalar field Φ in a finite volume l^{3+n},

$$H = \int_{l^{3+n}} d^{3+n}x \frac{1}{2}[\Pi^2 + \partial_{\mathbf{x}}\Phi\partial_{\mathbf{x}}\Phi + m^2\Phi^2],$$

where $\Pi = \dot{\Phi}$. After using the Fourier expansion for Π and Φ,

$$\Phi(\mathbf{x}) = \frac{1}{l^{3+n}}\sum_{\mathbf{p}_n}\varphi(\mathbf{p}_n)e^{i\mathbf{p}_n\mathbf{x}},$$
$$\Pi(\mathbf{x}) = \frac{1}{l^{3+n}}\sum_{\mathbf{p}_n}\pi(\mathbf{p}_n)e^{i\mathbf{p}_n\mathbf{x}},$$

the Hamiltonian takes the form

$$H = \frac{1}{2l^{3+n}}\sum_{\mathbf{p}_n}[\pi(\mathbf{p}_n)\pi^+(\mathbf{p}_n) + (\mathbf{p}_n^2 + m^2)\varphi(\mathbf{p}_n)\varphi^+(\mathbf{p}_n)].$$

The quantization conditions

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \qquad [\Phi(\mathbf{x}), \Phi(\mathbf{y})] = 0,$$
$$[\Pi(\mathbf{x}), \Pi(\mathbf{y})] = 0$$

for the Fourier modes imply

$$[\varphi(\mathbf{p}_n), \pi(\mathbf{p}_m)] = il^{3+n}\delta_{-\mathbf{p}_n\mathbf{p}_m}, \qquad [\varphi(\mathbf{p}_n), \varphi(\mathbf{p}_m)] = 0,$$
$$[\pi(\mathbf{p}_n), \pi(\mathbf{p}_m)] = 0.$$

Defining

$$a(\mathbf{p}_n) = \frac{1}{\sqrt{2\varepsilon_{\mathbf{p}_n}}}[\varepsilon_{\mathbf{p}_n}\varphi(\mathbf{p}_n) + i\pi(\mathbf{p}_n)],$$
$$a^+(\mathbf{p}_n) = \frac{1}{\sqrt{2\varepsilon_{\mathbf{p}_n}}}[\varepsilon_{\mathbf{p}_n}\varphi(-\mathbf{p}_n) - i\pi(-\mathbf{p}_n)],$$

where $\varepsilon_{\mathbf{p}_n} = \sqrt{\mathbf{p}_n^2 + m^2}$, one finds

$$[a(\mathbf{p}_n), a^+(\mathbf{p}_m)] = l^{3+n}\delta_{\mathbf{p}_n\mathbf{p}_m}, \qquad [a(\mathbf{p}_n), a(\mathbf{p}_m)] = 0,$$
$$[a^+(\mathbf{p}_n), a^+(\mathbf{p}_m)] = 0.$$

So, the field and momentum operators take the form

$$\Phi(\mathbf{x}) = \frac{1}{l^{3+n}}\sum_{\mathbf{p}_n}\frac{1}{\sqrt{2\varepsilon_{\mathbf{p}_n}}}[a(\mathbf{p}_n)e^{i\mathbf{p}_n\mathbf{x}} + a^+(\mathbf{p}_n)e^{-i\mathbf{p}_n\mathbf{x}}],$$
$$\Pi(\mathbf{x}) = \frac{i}{l^{3+n}}\sum_{\mathbf{p}_n}\sqrt{\frac{\varepsilon_{\mathbf{p}_n}}{2}}[a^+(\mathbf{p}_n)e^{-i\mathbf{p}_n\mathbf{x}} - a(\mathbf{p}_n)e^{i\mathbf{p}_n\mathbf{x}}],$$

and the Hamiltonian reduces to

$$H = \frac{1}{2l^{3+n}}\sum_{\mathbf{p}_n}\varepsilon_{\mathbf{p}_n}[a^+(\mathbf{p}_n)a(\mathbf{p}_n) + a(\mathbf{p}_n)a^+(\mathbf{p}_n)].$$

Introducing real variables,

$$Q_{\mathbf{p}_n} = \frac{1}{\sqrt{2\mu l^{3+n}\varepsilon_{\mathbf{p}_n}}}[a(\mathbf{p}_n) + a^+(\mathbf{p}_n)],$$
$$P_{\mathbf{p}_n} = i\sqrt{\frac{\mu\varepsilon_{\mathbf{p}_n}}{2l^{3+n}}}[a^+(\mathbf{p}_n) - a(\mathbf{p}_n)],$$

the Hamiltonian splits into a sum of independent one-dimensional oscillators,

$$H = \sum_{\mathbf{p}_n}\left(\frac{P_{\mathbf{p}_n}^2}{2\mu} + \frac{\mu\varepsilon_{\mathbf{p}_n}^2 Q_{\mathbf{p}_n}^2}{2}\right). \qquad (17)$$

We explicitly introduced an energy scale μ in order to give the variables $Q_{\mathbf{p}_n}$ and $P_{\mathbf{p}_n}$ natural dimensions: $[Q_{\mathbf{p}_n}] =$ cm and $[P_{\mathbf{p}_n}] = \text{cm}^{-1}$. So far, the parameter μ is entirely arbitrary. The basic idea behind explicitly introducing this parameter is a characteristic feature of the minimum-length deformed quantization, i.e., it engenders a mass dependence of the oscillator energy spectrum [20,21], while the standard quantization scheme is free of this feature. Thus, the quantization of the field (suitably altered to respect the effects of a minimal length) necessarily involves some characteristic energy scale μ, in the same vein as an effective QFT. In order to identify the energy scale μ, one may keep in mind that [in view of Eq. (9)] the deviation from the standard quantization becomes appreciable at high energies. This naturally suggests the identification of μ with the characteristic energy scale of the problem under consideration. This sort of reasoning is completely in the spirit of an effective QFT [10].

The Heisenberg equation of motion reads

$$\dot{a}(\mathbf{p}_n) = i[H, a(\mathbf{p}_n)] = -i\varepsilon_{\mathbf{p}_n}a(\mathbf{p}_n),$$

which can be solved as

$$a(t, \mathbf{p}_n) = a(t = 0, \mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}t}.$$

The field and momentum operators take the forms

$$\Phi(t,\mathbf{x}) = \frac{1}{l^{3+n}}\sum_{\mathbf{p}_n}\frac{1}{\sqrt{2\varepsilon_{\mathbf{p}_n}}}[a(0,\mathbf{p}_n)e^{i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)} + a^+(0,\mathbf{p}_n)e^{-i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)}],$$
$$\Pi(t,\mathbf{x}) = \frac{i}{l^{3+n}}\sum_{\mathbf{p}_n}\sqrt{\frac{\varepsilon_{\mathbf{p}_n}}{2}}[a^+(0,\mathbf{p}_n)e^{-i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)} - a(0,\mathbf{p}_n)e^{i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)}].$$

Then, we write $a(\mathbf{p}_n)$ for $a(0,\mathbf{p}_n)$ and $a^+(\mathbf{p}_n)$ for $a^+(0,\mathbf{p}_n)$ in the field theory and call these quantities the annihilation and creation operators, respectively.

III. MINIMUM-LENGTH DEFORMED QFT

As long as we are restricting ourselves to the leading-order corrections due to minimum-length deformed quantum theory, the corrections arising at the first and second quantization levels do not interfere and can be considered separately.

A. Corrections arising at the first quantization level

The modified field theory

$$\mathcal{W}[\Phi] = -\int d^{4+n}x(\Phi\partial_t^2\Phi + \Phi\hat{\mathbf{P}}^2\Phi + m^2\Phi^2) \tag{18}$$

leads to the equation of motion

$$\partial_t^2\Phi + \hat{\mathbf{P}}^2\Phi + m^2\Phi = 0, \tag{19}$$

which in turn admits the plane-wave solution $\sim\exp(i\mathbf{px})$ with a modified dispersion relation,

$$\varepsilon^2 = \mathbf{P}^2 + m^2 = \frac{p^2}{(1-\frac{2\beta(1+n)l_F^{2+n}p^{2+n}}{2+n})^{\frac{2}{1+n}}} + m^2. \tag{20}$$

This dispersion relation implies the superluminal motion; namely, by taking $m=0$ one finds

$$\frac{d\varepsilon}{dp} = \frac{2+n+2\beta l_F^{2+n}p^{2+n}}{2+n-2\beta(1+n)l_F^{2+n}p^{2+n}} > 1. \tag{21}$$

B. Corrections arising at the second quantization level

The corrections at the second quantization level are obtained by quantizing the field Hamiltonian with respect to Eq. (9). In effect, the appearance of the energy scale μ together with l_F^{-1} lends the possibility of introducing a dimensionless parameter $(\mu l_F)^{2+n}$ that measures the deviation from the standard picture in accordance with Eq. (9). For each oscillator entering Eq. (17) now we have

$$[Q_{\mathbf{p}_n}, P_{\mathbf{p}_m}] = i\delta_{\mathbf{p}_n\mathbf{p}_m}(1 + ßP^{2+n}), \tag{22}$$

where we have used the notation

$$\beta l_F^{2+n} \equiv ß. \tag{23}$$

To the first order in β, from Eq. (9) one finds

$$\hat{X} = \hat{x}, \qquad \hat{P} = \hat{p} + \frac{ß\hat{p}^{3+n}}{3+n} + O(ß^2). \tag{24}$$

Therefore, the Hamiltonian

$$H = \frac{P^2}{2\mu} + \frac{\mu\varepsilon^2Q^2}{2}$$

to the first order in β takes the form

$$H = \frac{p^2}{2\mu} + \frac{\mu\varepsilon^2q^2}{2} + \frac{ßp^{4+n}}{\mu(3+n)} = \varepsilon\left(b^+b + \frac{1}{2}\right) + \frac{ßi^{4+n}}{\mu(3+n)}\left(\frac{\mu\varepsilon}{2}\right)^{\frac{4+n}{2}}(b^+ - b)^{4+n},$$

where

$$b = \frac{1}{\sqrt{2\mu\varepsilon}}(\mu\varepsilon q + ip), \qquad b^+ = \frac{1}{\sqrt{2\mu\varepsilon}}(\mu\varepsilon q - ip).$$

Using this Hamiltonian, from the Heisenberg equation $\dot{b} = i[H,b]$ one finds

$$\dot{b} = -i\varepsilon b - \frac{(4+n)i^{5+n}ß}{\mu(3+n)}\left(\frac{\mu\varepsilon}{2}\right)^{\frac{4+n}{2}}(b^+ - b)^{3+n}. \tag{25}$$

By writing the operator b to the first order in ß in the form

$$b = f + ßg,$$

Eq. (25) takes the form

$$\dot{f} + ß\dot{g} = -i\varepsilon(f + ßg) - ß\aleph(f^+ - f)^{3+n}, \tag{26}$$

where we have used the notation

$$\aleph = \frac{(4+n)i^{1+n}}{\mu(3+n)}\left(\frac{\mu\varepsilon}{2}\right)^{\frac{4+n}{2}}. \tag{27}$$

Equating the coefficients of like powers of ß from Eq. (26), one finds

$$\dot{f} = -i\varepsilon f, \qquad \dot{g} = -i\varepsilon g - \aleph(f^+ - f)^{3+n},$$

which admits the following analytic solution:

$$f(t) = f(0)e^{-i\varepsilon t},$$
$$\dot{g} = -i\varepsilon g - \aleph[f^+(0)e^{i\varepsilon t} - f(0)e^{-i\varepsilon t}]^{3+n},$$
$$g(t) = e^{-i\varepsilon t}\left[g(0) - \aleph\int_0^t d\tau e^{i\varepsilon\tau}\{f^+(0)e^{i\varepsilon\tau} - f(0)e^{-i\varepsilon\tau}\}^{3+n}\right]. \tag{28}$$

Using Eq. (28) to the first order in ß, one can write

$$b(t) = b(0)e^{-i\varepsilon t} - ß\aleph e^{-i\varepsilon t}\int_0^t d\tau e^{i\varepsilon\tau}\{b^+(0)e^{i\varepsilon\tau} - b(0)e^{-i\varepsilon\tau}\}^{3+n}.$$

Thus, the corrected field operator takes the form

$$\Phi(t,\mathbf{x}) = \frac{1}{l^{3+n}}\sum_{\mathbf{p}_n}\frac{1}{\sqrt{2\varepsilon_{\mathbf{p}_n}}}\left[\left(b(\mathbf{p}_n) - ß\aleph\int_0^t d\tau e^{i\varepsilon_{\mathbf{p}_n}\tau}[b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau} - b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau}]^{3+n}\right)e^{i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)} + \left(b^+(\mathbf{p}_n) - ß\aleph^*\int_0^t d\tau e^{-i\varepsilon_{\mathbf{p}_n}\tau}[b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau} - b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau}]^{3+n}\right)e^{-i(\mathbf{p}_n\mathbf{x}-\varepsilon_{\mathbf{p}_n}t)}\right]. \tag{29}$$

Keeping in mind that at a fundamental level the notion of a particle (quantum) comes from the quantized field, we define the free-particle wave function by means of the matrix element $\langle 0|\Phi(t,\mathbf{x})|\mathbf{p}_i\rangle$, which in the standard case gives just the de Broglie wave. Following this definition and using Eq. (29), we estimate corrections to the free-particle wave function due to minimum-length deformed QM to the first order in the deformation parameter ß. One immediately sees that if n is odd, then the matrix element $\langle 0|\Phi(t,\mathbf{x})|\mathbf{p}_i\rangle \propto e^{i(\mathbf{p}_i\mathbf{x}-\varepsilon_{\mathbf{p}_i}t)}$. Let us assume n is an even number. For simplicity we take $n = 2$. The terms from

$$[b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau} - b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau}]^5 \tag{30}$$

contributing to the matrix element $\langle 0|\Phi(t,\mathbf{x})|\mathbf{p}_i\rangle$ are

$$-e^{-i\varepsilon_{\mathbf{p}_n}\tau}[b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n) + b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n) + b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n) + b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)]. \tag{31}$$

Analogously, one finds that the terms form

$$[b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau} - b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau}]^5 \tag{32}$$

contributing to the matrix element $\langle 0|\Phi(t,\mathbf{x})|\mathbf{p}_i\rangle$ are

$$e^{-i\varepsilon_{\mathbf{p}_n}\tau}[b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n) + b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n) + b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n) + b(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)b(\mathbf{p}_n)b^+(\mathbf{p}_n)]. \tag{33}$$

Hence, one finds

$$\langle 0|[b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau} - b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau}]^5|\mathbf{p}_i\rangle = -9\delta_{in}e^{-i\varepsilon_{\mathbf{p}_n}\tau},$$
$$\langle 0|[b(\mathbf{p}_n)e^{-i\varepsilon_{\mathbf{p}_n}\tau} - b^+(\mathbf{p}_n)e^{i\varepsilon_{\mathbf{p}_n}\tau}]^5|\mathbf{p}_i\rangle = 9\delta_{in}e^{-i\varepsilon_{\mathbf{p}_n}\tau},$$

and correspondingly

$$\langle 0|\Phi(t,\mathbf{x})|\mathbf{p}_i\rangle \propto e^{i(\mathbf{p}_i\mathbf{x}-\varepsilon_{\mathbf{p}_i}t)}\left(1 - i\frac{ß\mu^2 27\varepsilon^3_{\mathbf{p}_i}}{20}t\right) - \frac{ß\mu^2 27\varepsilon^2_{\mathbf{p}_i}}{40}e^{i(\mathbf{p}_i\mathbf{x}+\varepsilon_{\mathbf{p}_i}t)} + \frac{ß\mu^2 27\varepsilon^2_{\mathbf{p}_i}}{40}e^{-i(\mathbf{p}_i\mathbf{x}-\varepsilon_{\mathbf{p}_i}t)}. \tag{34}$$

IV. CORRECTIONS TO THE BH EMISSION

As was discussed in Sec. II, if we subject the particles emitted by the BH to the modified dispersion relation (14) and retain in this equation only leading and subleading terms, then the BH entropy acquires logarithmic and inverse-area corrections. The minimum-length deformed prescription applied at the second quantization level leads essentially to the same sort of corrections to the BH entropy [10]. Let us address this question in some detail.

The first term in the wave function of a free particle [Eq. (34)],

$$e^{i(\mathbf{p}_i\mathbf{x}-\varepsilon_{\mathbf{p}_i}t)}\left(1 - i\frac{27ß\mu^2\varepsilon^3_{\mathbf{p}_i}}{20}t\right) \approx e^{i(\mathbf{p}_i\mathbf{x}-[\varepsilon_{\mathbf{p}_i}+1.35ß\mu^2\varepsilon^3_{\mathbf{p}_i}]t)}, \tag{35}$$

(to the first order in ß) gives just the modified dispersion relation

$$\varepsilon_{\mathbf{p}_i} \to \varepsilon_{\mathbf{p}_i} + 1.35ß\mu^2\varepsilon^3_{\mathbf{p}_i},$$

where the energy scale μ is set by the BH emission temperature: $\mu = T$ [10]. It results in the logarithmic correction to the BH entropy (see Sec. II).

The second term in Eq. (34) represents a reflected wave. In the context of the BH emission, it indicates the existence of the backscattered flux, the rate of which is proportional to $|ß\mu\varepsilon(\mathbf{p})^2|^2$, that is, to $l_F^8 T^8$. This flux increases as the BH evaporates and thus tries to compensate the emission. It reproduces the inverse-area correction to the entropy [10]. Namely, the standard Hawking temperature (for $n=2$) is defined as $T \propto (G_N M)^{-1/3}$. Hence, during the evaporation the BH mass changes as $dM \propto -dT/G_N T^4$, and for the mass increment due to backscattered flux one finds $dM_+ \propto |l_F^8 T^8 dM| \propto dT l_F^4 T^4$. Using this equation and the formula $dS = dM/T$, one finds the entropy correction $\propto (l_F/r_g)^4$.

The third term in Eq. (34) could be interpreted as indicating the possibility that a particle will transition into an antiparticle. The discussion concerning this term can be found in Ref. [10].

So far we have confined the application of the minimum-length deformed QM to the matter fields. But what if gravity (the graviton field) is also subject to this sort of modification? Putting aside corrections arising at the second quantization level, one can address this question by estimating the gravitational potential with the use of the modified propagator that follows from Eq. (18). The spherically symmetric gravitational field in $4+n$ space-time dimensions is described by the Schwarzschild-Tangherlini solution [22,23]

$$ds^2 = \left[1 - \left(\frac{r_g}{r}\right)^{n+1}\right]dt^2 - \left[1 - \left(\frac{r_g}{r}\right)^{n+1}\right]^{-1}dr^2 - r^2 d\Omega_{n+2}^2, \tag{36}$$

where $d\Omega_{n+2}^2$ is a line element of a $2+n$-dimensional unit sphere and the gravitational radius reads

$$r_g(M) = (\mathbb{G}_N M)^{\frac{1}{n+1}}\left[\frac{16\pi}{(n+2)\mathrm{Vol}(S^{n+2})}\right]^{\frac{1}{n+1}}. \tag{37}$$

Let us consider a modified Schwarzschild-Tangherlini space-time,

$$ds^2 = [1 - r_g^{n+1}V(r)]dt^2 - [1 - r_g^{n+1}V(r)]^{-1}dr^2 - r^2 d\Omega_{n+2}^2, \tag{38}$$

where r_g is given by Eq. (37) and $V(r)$ is calculated by the modified propagator with respect to Eq. (18),

$$V(r) = \frac{\mathrm{Vol}(S^{2+n})}{(2\pi)^{3+n}}\int_{k^{2+n}<ß^{-1}} d^{3+n}k\,\frac{(1-ßk^{2+n})^{\frac{2}{1+n}}}{k^2}e^{i\mathbf{kr}}, \tag{39}$$

where now [not to be confused with Eq. (23)] ß stands for

$$\frac{2\beta(1+n)l_F^{2+n}}{2+n} \equiv ß. \tag{40}$$

The potential $V(r)$ has the following generic properties. It is a monotonically decreasing function that is finite at the origin with a vanishing derivative at this point (see the Appendix). Its asymptotic behavior when $r \to 0$ looks like

$$V(r) = \mathcal{A} - \mathcal{B}r^2 + O(r^4), \tag{41}$$

where $\mathcal{A}$ and $\mathcal{B}$ are positive quantities (see the Appendix). Now the equation for the horizon looks like

$$\frac{1}{r_g^{n+1}(M)} = V(r). \tag{42}$$

As $V(0)$ is a maximum of the potential, this equation does not have any solution for $M < M_{\mathrm{remnant}}$,

$$\frac{1}{r_g^{n+1}(M_{\mathrm{remnant}})} = \frac{(n+2)\Gamma(\frac{3+n}{2})}{32\pi^{\frac{n+5}{2}} l_F^{2+n} M_{\mathrm{remnant}}} = V(0) = \frac{\int_0^1 dq q^n(1-q^{2+n})^{2/(1+n)} \int_0^1 dt(1-t^2)^{n/2}}{2^n\sqrt{\pi}\Gamma(\frac{2+n}{2})\Gamma(\frac{3+n}{2}) l_F^{1+n}} \times \left(\frac{2+n}{2\beta(1+n)}\right)^{\frac{1+n}{2+n}}, \tag{43}$$

where we have used Eqs. (37), (40), and (42). So what we see is that as the BH evaporates down to M_{remnant} its horizon disappears, and at the same time its surface gravity vanishes: $V'(0) = 0$ [see Eq. (41)]. That is, the Hawking temperature, which is proportional to the surface gravity, becomes zero.

V. SUMMARY

Let us briefly summarize the basic points of our discussion.

(1) In deriving the higher-dimensional minimum-length modified QM we were guided by two recent papers [2,3], which demonstrated (in the framework of string theory and loop quantum gravity) that the logarithmic corrections to the BH entropy are universal in arbitrary space-time dimensions. The MUR we have found disagrees with the relation obtained in Ref. [17], but coincides with the one derived in Ref. [18] in a somewhat different context.

(2) Using the Hilbert-space representation for a relatively broad class of Planck-length deformed QM [12], we considered minimum-length deformed QFT that follows from the higher-dimensional minimum-length deformed QM mentioned above.

(3) In discussing the minimum-length deformed QFT (both at the first and second quantization levels), we restricted ourselves to the first order in the deformation parameter (in this limit the corrections arising at the first and second quantization levels decouple). From the standpoint of the Einstein equations, up to this point we only

considered corrections to the matter fields. These corrections result in the logarithmic and inverse-area corrections to the BH entropy, thus providing a self-contained picture.

(4.1) General relativity viewed as a field theory in the Minkowskian background acquires corrections with respect to the minimum-length deformed QFT. Putting aside the corrections arising at the second quantization level, one can study the modified Schwarzschild-Tangherlini space-time by using the modified gravitational potential that comes from the minimum-length deformed QFT propagator. In this way, one finds a regular (de Sitter-like) geometry near the origin. Indeed, the modified Schwarzschild-Tangherlini space-time is free of the curvature singularity at the origin because now the metric (as well as its first and second derivatives) do not diverge when $\mathbf{r} \to \mathbf{0}$ [see Eq. (41)]. On the other hand, the Schwarzschild-Tangherlini space-time modified in this way produces the zero-temperature BH remnants. The behavior of the potential and Hawking temperature are plotted in Fig. 1 and Fig. 2, respectively. The typical behavior of the emission temperature as a function of the BH mass is shown in Fig. 3. It should be remarked that the emission temperature in Fig. 2 vanishes when the BH horizon approaches zero.

(4.2) In effect, the approach we have pursued starts from the modified Poisson equation $\hat{\mathbf{P}}^2 V(r) \propto \delta_{ß}^{(3+n)}(\mathbf{r})$, where the source energy density is given by the smeared-out ($3+n$-dimensional) δ function (in the limit $ß \to 0$, one recovers the point-like source),

$$\delta_{ß}^{(3+n)}(\mathbf{r}) = \int_{k^{2+n}<ß^{-1}} d^{3+n}k e^{i\mathbf{kr}} \propto \frac{1}{r^{(n+1)/2}} \int_0^{ß^{-1/(2+n)}} dk k^{(n+3)/2} J_{(n+1)/2}(kr).$$

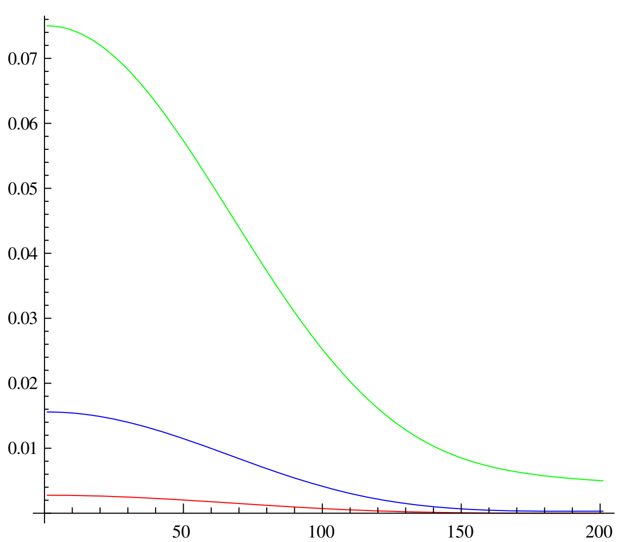

FIG. 1 (color online). $V(r)ß^{1/(2+n)}$ (vertical axis) vs distance in units of $ß^{1/(2+n)}$ (horizontal axis) for $n=1$ (green line), $n=2$ (blue line), and $n=3$ (red line).

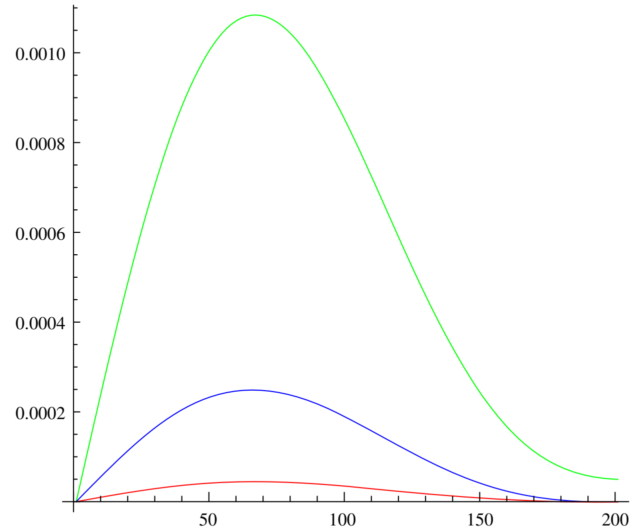

FIG. 2 (color online). $Tß^{1/(2+n)}/r_g^{n+1}$ (vertical axis) vs distance in units of $ß^{1/(2+n)}$ (horizontal axis) for $n=1$ (green line), $n=2$ (blue line), and $n=3$ (red line).

Thus, the BH we have discussed is surrounded by the matter. This sort of BH is known as "dirty" [24]. For our discussion, we did not need to address the generic picture of dirty BHs.

We note that one arrives at the regular BHs by considering the minimum-length deformed matter sector, when the smeared-out source is taken in the framework of the standard theory of gravity [25–29].

(4.3) Special attention has to be paid to the validity conditions of the approximation assumed throughout the

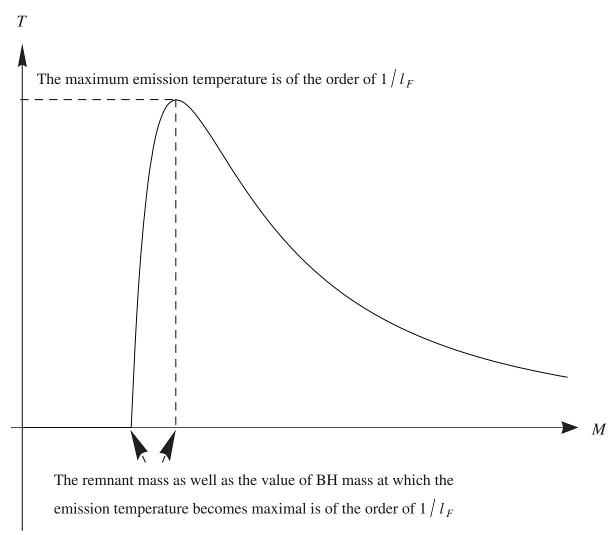

FIG. 3. Typical behavior of the emission temperature as a function of the BH mass. The emission temperature reaches its maximum (of the order of l_F^{-1}) when the BH evaporates down to the Planck mass, and then swiftly drops to zero at M_{remnant} (which is also of the order of l_F^{-1}).

above discussion. We have taken the gravitational field on an equal footing with the matter fields, that is, the QFT picture for gravity is taken as a starting point. This means that the graviton field is defined as the difference between the full metric and its Minkowski background value. The calculations show that the gravity behaves as an asymptotically free interaction and, correspondingly, the radiative corrections close to the Planck scale can be safely ignored in this case [30–32].

(5) Finally, let us remark that one can speculate about the possible observations of this sort of BH remnants in the context of the large extra-dimensional models with a low quantum gravity scale [14,33–40].

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APPENDIX

The integral determining the gravitational potential

$$\int_{k^{2+n}<ß^{-1}} d^{3+n}k\, \frac{(1-ßk^{2+n})^{\frac{2}{1+n}}}{k^2} e^{i\mathbf{kr}} \qquad \text{(A1)}$$

for large values of r is dominated by the wave modes $k \ll ß^{-1/(2+n)}$, that is, in this limit the term $ßk$ in the numerator can be neglected and one recovers the standard result. To estimate its behavior for small values of r, let us choose the axis x_{3+n} along $\mathbf{k}$ and introduce spherical coordinates in the momentum space,

$$k_1 = k\sin\varphi \prod_{j=1}^{n+1}\sin\theta_j, \qquad k_2 = k\cos\varphi \prod_{j=1}^{n+1}\sin\theta_j,$$

$$k_{i+2} = k\cos\theta_i \prod_{j=i}^{n+1}\sin\theta_j, \qquad k_{3+n} = k\cos\theta_{n+1},$$

where $i = 1, \ldots, n$, $k \geq 0$, $0 \leq \varphi < 2\pi$, and $0 \leq \theta_j \leq \pi$. Thus, we get $\mathbf{kr} = kx_{3+n}\cos\theta_{n+1}, d^{3+n}k = k^{2+n}dkd\varphi\prod_{j=1}^{n+1}\sin^j\theta_j d\theta_j$, and the integral (A1) reduces to

$$\mathrm{Vol}(S^{n+1}) \int_0^{ß^{-1/(2+n)}} dkk^n (1-ßk^{2+n})^{\frac{2}{1+n}} \times \int_0^\pi d\theta_{n+1}\sin^{n+1}\theta_{n+1} e^{ikx_{3+n}\cos\theta_{n+1}}. \qquad \text{(A2)}$$

Let us first consider the specific case $n = 1$. Changing the variable $\cos\theta_2 = t$, one finds

$$\int_0^\pi d\theta_2 \sin^2\theta_2 e^{ikx_4\cos\theta_2} = \int_{-1}^1 dt\sqrt{1-t^2}\cos(kx_4 t) = 2\int_0^1 dt\sqrt{1-t^2}\cos(kx_4 t),$$

and, correspondingly, Eq. (A2) takes the form

$$8\pi \int_0^1 dt\sqrt{1-t^2} \int_0^{ß^{-1/3}} dkk(1-ßk^3)\cos(kx_4 t). \qquad \text{(A3)}$$

Performing the integrals

$$\int_0^{ß^{-1/3}} dkk\cos(kx_4t) = \frac{d}{d(x_4t)}\int_0^{ß^{-1/3}} dk\sin(kx_4t) = \frac{d}{d(x_4t)}\frac{1-\cos(x_4t/ß^{1/3})}{x_4t} = \frac{\sin(x_4t/ß^{1/(2+n)})}{x_4tß^{1/3}} - \frac{1-\cos(x_4t/ß^{1/3})}{x_4^2t^2},$$

$$\int_0^{ß^{-1/3}} dkk^4\cos(kx_4t) = \frac{d^4}{d(x_4t)^4}\int_0^{ß^{-1/3}} dk\cos(kx_4t) = \frac{d^4}{d(x_4t)^4}\frac{\sin(x_4t/ß^{1/3})}{x_4t} = \frac{\sin(x_4t/ß^{1/3})}{ß^{4/3}x_4t} - \frac{4\sin(x_4t/ß^{1/3})}{ß^{2/3}(x_4t)^3} - \frac{24\sin(x_4t/ß^{1/3})}{ß^{1/3}(x_4t)^4} + \frac{24\sin(x_4t/ß^{1/3})}{(x_4t)^5},$$

the final result reads

$$V(r) = \frac{1}{2\pi}\int_{-1}^{1} dt\sqrt{1-t^2}\left[\frac{4\beta^{1/3}\sin(rt/ß^{1/3})}{r^3t^3} + \frac{24\beta^{2/3}\sin(rt/ß^{1/3})}{r^4t^4} - \frac{1-\cos(rt/ß^{1/3})}{r^2t^2} - \frac{24\beta\sin(rt/ß^{1/3})}{r^5t^5}\right]. \tag{A4}$$

To find the asymptotic behavior of the potential for $r \to 0$, one can immediately use Eq. (A3),

$$V(r) = \frac{3}{20\pi ß^{2/3}}\int_{-1}^{1} dt\sqrt{1-t^2} - \frac{3r^2}{112\pi ß^{4/3}}\int_{-1}^{1} dt t^2\sqrt{1-t^2} + O(r^4). \tag{A5}$$

From this expression it is immediately seen that $V(0)$ is finite and $V'(0) = 0$. Now let us show that $V(r)$ is a monotonically decreasing function, that is, $V'(r) < 0$. From Eq. (A3) one finds

$$\begin{aligned}\frac{d}{dr}\int_0^1 dt\sqrt{1-t^2}\int_0^{ß^{-1/3}} dk k(1-ßk^3)\cos(krt) &= -\int_0^1 dt t\sqrt{1-t^2}\int_0^{ß^{-1/3}} dk k^2(1-ßk^3)\sin(krt) \\ &= -\frac{1}{r^3}\int_0^1 dt t\sqrt{1-t^2}\int_0^{r/ß^{1/3}} d\tilde{k}\tilde{k}^2\left(1-\frac{ß\tilde{k}^3}{r^3}\right)\sin(\tilde{k}t) < 0;\end{aligned}$$

then, from the statement

$$\int_0^a f(z)\sin(z) > 0,$$

whenever $f(z)$ is a positive and monotonically decreasing function we see that $V(r)$ is indeed a monotonically decreasing function.

Now let us address the general case. Denoting $\cos\theta_{n+1} = t$, one finds

$$\int_0^{\pi} d\theta_{n+1}\sin^{n+1}\theta_{n+1}e^{ikx_{3+n}\cos\theta_{n+1}} = 2\int_0^1 dt(1-t^2)^{n/2}\cos(kx_{3+n}t),$$

and correspondingly the potential takes the form

$$V(r) = \frac{2\mathrm{Vol}(S^{n+1})\mathrm{Vol}(S^{n+2})}{(2\pi)^{3+n}}\int_0^{ß^{-1/(2+n)}} dk k^n(1-ßk^{2+n})^{\frac{2}{1+n}}\int_0^1 dt(1-t^2)^{n/2}\cos(krt). \tag{A6}$$

Its asymptotic behavior for $r \to 0$ can readily be found by expanding the $\cos(krt)$ term into a Taylor series,

$$\begin{aligned}V(r) = \frac{2\mathrm{Vol}(S^{n+1})\mathrm{Vol}(S^{n+2})}{(2\pi)^{3+n}}&\left[\int_0^{ß^{-1/(2+n)}} dk k^n\times(1-ßk^{2+n})^{\frac{2}{1+n}}\int_0^1 dt(1-t^2)^{n/2} - \frac{r^2}{2}\int_0^{ß^{-1/(2+n)}} dk\right. \\ &\left.\times k^{n+2}(1-ßk^{2+n})^{\frac{2}{1+n}}\int_0^1 dt t^2(1-t^2)^{n/2} + O(r^4)\right].\end{aligned} \tag{A7}$$

It is evident from this expression that $V(0)$ is finite, $V'(0) = 0$, and $V'(r) < 0$ for $r \to 0$. In general, the statement $V'(r) < 0$ for $r > 0$ can be proved much in the same way as it was done for $n = 1$.

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