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Biconformal symmetry and static Green functions in the higher-dimensional Reissner-Nordström spacetimes

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We study a static scalar massless field created by a source located near an electrically charged higher-dimensional spherically symmetric black hole. We demonstrate that there exist biconformal transformations relating static field solutions in the metric with different parameters of the mass M and charge Q. Using this symmetry, we obtain the static scalar Green function in the higher-dimensional Reissner-Nordström spacetimes.

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I. INTRODUCTION

In this paper, we continue studying minimally coupled massless scalar fields created by static sources placed in the vicinity of a higher-dimensional static black hole. For this purpose, we use the method of biconformal transformations, which was developed and applied to the case of the Schwarzschild-Tangherlini metrics in our previous paper [1]. This method is based on the following observations.

A scalar massless field Φ in a D-dimensional spacetime with metric $g_{\mu\nu}$ $(\mu,\nu=0,\ldots,D-1)$ obeys the equation

$$\Box\Phi=-4\pi J. \tag{1.1}$$

Let us consider a static scalar field $\Phi(X)$ created by a source $J(X)$ in the static spacetime with the metric

$$\begin{aligned} &ds^2=-\alpha^2dt^2+g_{ab}dx^a dx^b,\\ &X=(t,x^a),\qquad \alpha=\alpha(x),\qquad g_{ab}=g_{ab}(x). \end{aligned}\tag{1.2}$$

Then Eq. (1.1) is reduced and takes the form

$$\hat{F}\Phi=-4\pi J,\qquad \hat{F}=\frac{1}{\alpha\sqrt{g}}\partial_a(\alpha\sqrt{g}g^{ab}\partial_b). \tag{1.3}$$

Here $g=\det(g_{ab})$. The redshift factor α is connected with the norm of the static Killing vector $\boldsymbol{\xi}$ as follows: $\alpha=\sqrt{-\boldsymbol{\xi}^2}=\sqrt{-g_{tt}}$. Equation (1.3) is invariant under the following *biconformal* transformations

$$\Phi=\bar{\Phi},\qquad g_{ab}=\Omega^2\bar{g}_{ab},\qquad \alpha=\Omega^{-n}\bar{\alpha},\qquad J=\Omega^2\bar{J}, \tag{1.4}$$

where $n\equiv D-3$ and Ω is an arbitrary function of spatial coordinates x^a.

This transformation consists of a biconformal map [2,3] of the original background D-dimensional metric $g_{\mu\nu}$

$$\Psi_\Omega:\ \boldsymbol{g}\to\bar{\boldsymbol{g}}, \tag{1.5}$$

accompanied by a properly chosen rescaling of the charge density J. If one starts with a solution of the Einstein equations, a new metric, obtained as a result of this transformation, is not necessarily a solution of the Einstein equations with a physically meaningful stress-energy tensor. However, it may happen that for a specially chosen transformation, this new metric has enhanced symmetries.

An interesting example is a Majumdar-Papapetrou metric, describing the gravitational field of a set of higher-dimensional extremely charged black holes in equilibrium. Under a properly chosen biconformal map, this metric reduces to the higher-dimensional Minkowski metric. This allows one to solve the static scalar field equation in the Majumdar-Papapertou exactly (see [4]).

In the paper [1], we demonstrated that the method of biconformal transformations can be used for solving static equations in spacetimes of static spherically symmetric black holes. The enhanced symmetry of the biconformal metric $\bar{\boldsymbol{g}}$ was used in that paper to obtain the static Green functions for Eq. (1.3) in a higher-dimensional Schwarzschild-Tangherlini spacetime. In this paper, we demonstrate how this method works for the case of a charged higher-dimensional black hole.

There are many possible applications of the proposed result. One of them is an old problem of finding a self-energy and a self-force of charged particles near black holes [5–8]. In four dimensions, the closed form of the exact solution for the field of a point charges in the black hole geometry was obtained earlier [7–12].

The recent interest in the problem of a self-force is stimulated by a study of the backreaction of the field on the particle moving near black holes [13] in connection with the gravitational wave emission by such particles. More recently, several publications discussed higher-dimensional aspects of this problem (see, e.g., [14,15]). This study was

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stimulated by general interest in spacetimes and brane models with large extra dimensions.

This paper is organized as follows. In Sec. II we discuss biconformal transformations of higher-dimensional spherically symmetric metrics and demonstrate that the Reissner-Nordström metrics are biconformally related to the higher-dimensional Bertotti-Robinson metric. The latter is a product of two-dimensional anti–de Sitter space and a sphere. Using this result, we construct a biconformal map of the Reissner-Nordström metrics with different parameters of mass and charge. In Sec. III we obtain useful representations for static Green functions in a spacetime of static spherically symmetric higher-dimensional charged black holes. Section IV contains example of calculations of the static Green functions for four-, five- and six-dimensional black holes. Section V contains discussion of the obtained results and their possible generalizations.

II. BICONFORMAL MAP AND SYMMETRY ENHANCEMENT OF STATIC SPHERICALLY SYMMETRIC SPACETIMES

A. Symmetry enhancement condition

Let us consider an application of the method of the biconformal maps to the case of a general static spherically symmetric D-dimensional metric. The corresponding metric is

$$ds^2 = -f(r)dt^2 + w^{-1}(r)dr^2 + r^2 d\omega_{n+1}^2, \tag{2.1}$$

where $n = D - 3$ and $d\omega_{n+1}^2$ is the line element on a $(n+1)$-dimensional unit sphere:

$$d\omega_{n+1}^2 = d\theta_n^2 + \sin^2\theta_n d\omega_n^2, \qquad d\omega_0^2 = d\phi^2. \tag{2.2}$$

We denote $\theta_0 \equiv \phi \in [0, 2\pi]$. All other coordinates, $\theta_{i>0} \in [0, \pi]$. This metric is invariant under time translations and spatial rotations. Since ds, t and r have the same dimensionality of the length, the metric (2.1) can be presented in the form $ds^2 = a^2 dS^2$, where the dimensionless metric dS^2 is obtained from (2.1) by substituting $t \to t/a$ and $r \to r/a$, where a is an arbitrary constant parameter with the dimensionality of length.

Let us apply a biconformal transformation (1.4) to this metric with $\Omega = r/a$. This choice guarantees that Ω is dimensionless. After this biconformal transformation, one has

$$d\bar{s}^2 = dh^2 + a^2 d\omega_{n+1}^2,$$
$$dh^2 = -\left(\frac{r}{a}\right)^{2n} f(r)dt^2 + \frac{a^2}{r^2 w(r)} dr^2. \tag{2.3}$$

The scalar curvature of the two-dimensional metric dh^2 is

$$R = -\frac{1}{2a^2 f^2}\{rfw'(2nf + rf') + w[2r^2 f'' - r^2(f')^2 + 2r(2n+1)ff' + 4n^2 f^2]\}. \tag{2.4}$$

Here and later, $(\dots)' = d(\dots)/dr$. The metric dh^2 possesses an enhanced symmetry if its two-dimensional curvature R is constant. We denote its value by

$$R = -\frac{2}{b^2}, \tag{2.5}$$

where b is a constant of the dimensionality of the length. The Eqs. (2.4) and (2.5) can be solved to determine the function w. The result is

$$w = \left(\frac{a^2}{n^2 b^2} + \frac{C}{r^{2n} f}\right)\left(1 + \frac{rf'}{2nf}\right)^{-2}. \tag{2.6}$$

Here C is an integration constant. Let us suppose that function f has the following asymptotic at infinity:

$$f = f_0 + f_1 r^{-\gamma} + \cdots, \qquad \gamma \geq 1. \tag{2.7}$$

Then (2.6) shows that the asymptotic value of w at the infinity is $a^2/(n^2 b^2)$. The spacetime (2.1) does not have a solid angle deficit and is asymptotically flat only if

$$\frac{a}{nb} = 1. \tag{2.8}$$

In what follows, we always assume that this condition is satisfied.

By using the relation (2.6), one finds such functions $\{f(r), w(r)\}$, for which the biconformal transformation of the metric (2.1) has an enhanced symmetry. The corresponding metric,

$$d\bar{s}^2 = dh^2 + a^2 d\omega_{n+1}^2, \tag{2.9}$$

is a direct sum of the two-dimensional anti–de Sitter metric dh^2 and the metric $a^2 d\omega_{n+1}^2$ on $(n+1)$-dimensional sphere. The ratio of the curvature radii for these two metrics is fixed by the condition (2.8). This metric describes a particular Bertotti-Robinson spacetime and can be written in the following canonical form,

$$d\bar{s}^2 = a^2\left[\frac{1}{n^2}\left(-(\rho^2 - 1)d\bar{\sigma}^2 + \frac{1}{\rho^2 - 1}d\rho^2\right) + d\omega_{n+1}^2\right], \tag{2.10}$$

where

$$\rho = \cosh\left(n\int_{r_g}^{r}\frac{dr}{r\sqrt{w(r)}}\right),$$
$$\bar{\sigma} = nr^n a^{-1-n}\frac{\sqrt{f}}{\sqrt{\rho^2-1}}\Big|_{r=r_g} t. \qquad (2.11)$$

Let us emphasize that the parameter a has dimensionality of the length and is arbitrary.

B. Biconformal map of Reissner-Nordström metric to the Bertotti-Robinson space

Let us consider a special case of the metric (2.1) with an extra condition:

$$w = f. \qquad (2.12)$$

For this choice, the relation (2.6) becomes an equation which allows one to obtain the function f. The ordinary differential equation (2.6) with $w = f$ is of the first order. Hence, its solution besides the constant C contains another arbitrary integration constants C_1. It is possible to show that one can choose these constants so that the solution takes the form

$$f = 1 - \frac{2M}{r^n} + \frac{Q^2}{r^{2n}}. \qquad (2.13)$$

For real positive M and real Q, which satisfies the condition $|Q| \le M$, the metric (2.1) with (2.12) and (2.13) is the metric of a higher-dimensional spherically symmetric electrically charged black hole. The parameters M and Q are proportional to the Arnowitt-Deser-Misner mass and charge of the black hole, respectively. The coefficients of proportionality (see, e.g., [16]) depend on the dimensionality of the spacetime and on the choice of units.

In order to rewrite the metric ds^2, obtained as a result of the biconformal map (2.3), in the standard (canonical) form (2.10) it is sufficient to make the following coordinate transformations:

$$r^n = M + \mu\rho, \quad t = \frac{a^{n+1}}{n\mu}\bar{\sigma}, \quad \mu = \sqrt{M^2 - Q^2}. \qquad (2.14)$$

We denote this biconformal map as follows:

$$\Psi_\Omega\colon \boldsymbol{g}_{M,Q} \to \bar{\boldsymbol{g}}_{\mathrm{BR}}, \qquad \Omega = r/a. \qquad (2.15)$$

C. Biconformal transformations within the Reissner-Nordström family of solutions

The method of biconformal maps was used in our paper, Ref. [1], to obtain static Green functions in the background of the higher-dimensional Schwarzschild-Tangherlini spacetimes. For this purpose, one uses at first the enhanced symmetry of a related Bertotti-Robinson space to find the D-dimensional Green function in this space, and after this one obtains the static Green function by means of the dimensional reduction. One can apply the same method for finding static Green functions in the Reissner-Nordström geometry. However, there exists another much simpler way. One can generate the corresponding static Green function in the spacetime of charged black holes by using the already known Green function for the Schwarzschild-Tangherlini spacetime.

For this purpose, let us notice that the canonical form (2.10) is *universal* in the following sense: It is the same for any Reissner-Nordström metric, and it does not depend on its parameters M and Q. This observation opens an interesting possibility to relate metrics with different parameters. Let us introduce new coordinates $\hat{t}$ and $\hat{r}$,

$$\hat{r}^n = \hat{M} + \hat{\mu}\rho, \quad \hat{t} = \frac{a^{n+1}}{n\hat{\mu}}\bar{\sigma}, \quad \hat{\mu} = \sqrt{\hat{M}^2 - \hat{Q}^2}, \qquad (2.16)$$

and denote

$$\hat{\Omega} = \hat{r}/a. \qquad (2.17)$$

Then one has the following biconformal map of the Reissner-Nordström metric with parameters $\hat{M}$ and $\hat{Q}$ to the canonical Bertotti-Robinson metric

$$\Psi_{\hat{\Omega}}\colon \boldsymbol{g}_{\hat{M},\hat{Q}} \to \bar{\boldsymbol{g}}_{\mathrm{BR}}. \qquad (2.18)$$

Combining the direct biconformal map (2.15) with the biconformal map, inverse to (2.18), one obtains a biconformal map:

$$\Psi = \Psi_{\hat{\Omega}}^{-1}\circ\Psi_\Omega\colon \boldsymbol{g}_{M,Q} \to \boldsymbol{g}_{\hat{M},\hat{Q}}. \qquad (2.19)$$

This biconformal map is a transformation of the original Reissner-Nordström metric with parameters M and Q to a similar metric with different parameters $\hat{M}$ and $\hat{Q}$. The static equation (1.3) is invariant under such a transformation, provided one, in addition, properly transforms the source term $J \to \hat{J}$.

In other words, the solutions for the static field Φ in the original background space are simply related to solutions in a spacetime with modified parameters of the mass and the charge. In particular, if one knows the static Green function in the spacetime of an uncharged black hole, one can obtain the static Green function for the charged black hole by using the above described transformations. In the next section we demonstrate how this method works in more detail.

III. STATIC GREEN FUNCTIONS

A. Biconformal map of static Green functions

Following the paper [1], we define a static Green function $G(x, x')$ as follows:

$$G(x,x') = \int_{-\infty}^{\infty} dt \mathbb{G}_{\mathrm{Ret}}(t,x;0,x'). \tag{3.1}$$

Here $\mathbb{G}_{\mathrm{Ret}}(t,x;0,x')$ is a retarded Green function in the D-dimensional spacetime. This static Green function satisfies the equation

$$\hat{F}G(x,x') = -\frac{\delta(x-x')}{\alpha\sqrt{g}}. \tag{3.2}$$

In what follows, we assume that this Green function is decreasing when one of its parameters x tends to infinity and remains regular at the horizon (for more details see [1]).

The static Green function is simply related to the expression for a scalar field created by a point charge. The current of a static point charge q positioned at the point y reads

$$J(x) = q\frac{\delta(x-y)}{\sqrt{g}}. \tag{3.3}$$

In this case, the scalar field at the point x takes the form [17]

$$\Phi(x) = 4\pi q\alpha(y)G(x,y). \tag{3.4}$$

The field of a distributed source $q(y)$ can be easily obtained by integration over y of the right-hand side of this relation.

It is convenient to introduce a new radial variable ρ related to the radial coordinate r as follows (2.14):

$$\rho = \frac{r^n - M}{\mu}. \tag{3.5}$$

The Reissner-Nordström metric takes the form

$$ds^2 = -\frac{\mu^2(\rho^2-1)}{(M+\mu\rho)^2}dt^2 + (M+\mu\rho)^{2/n}\left[\frac{1}{n^2(\rho^2-1)}d\rho^2 + d\omega_{n+1}^2\right]. \tag{3.6}$$

The horizon corresponds to $\rho = 1$. In the metric (2.1), (2.12), (2.13), the gravitational radius r_{g} can be defined by the condition $f(r_{\mathrm{g}}) = 0$ and is given by the expression $r_{\mathrm{g}}^n = M + \mu$. The surface gravity at the horizon is

$$\kappa = \frac{n\mu}{r_{\mathrm{g}}^{n+1}}. \tag{3.7}$$

In the Reissner-Nordström metric (3.6), the equation for the static Green function takes the form

$$[n^2(\rho^2-1)\partial_\rho^2 + 2n^2\rho\partial_\rho + \triangle_\omega^{n+1}]G(x,x') = -\frac{n}{\mu}\delta(\rho-\rho')\delta(\omega,\omega'). \tag{3.8}$$

Here $\triangle_\omega^{n+1}$ and $\delta(\omega,\omega')$ are the Laplace operator and a covariant delta function on the unit $(n+1)$-dimensional sphere, respectively,

$$\begin{aligned}\triangle_\omega^{n+1} &= \partial_{\theta_n}^2 + n\frac{\cos\theta_n}{\sin\theta_n}\partial_{\theta_n} + \frac{1}{\sin^2\theta_n}\triangle_\omega^n,\\ \triangle_\omega^1 &= \partial_\phi^2,\\ \delta^{n+1}(\omega,\omega') &= \frac{\delta(\theta_n-\theta_n')}{\sin^n\theta_n}\delta^n(\omega,\omega'),\\ \delta^1(\omega,\omega') &= \delta(\phi-\phi').\end{aligned} \tag{3.9}$$

Because of the spherical symmetry of background geometry, the resulting static Green functions are the functions of radial coordinates of the observer ρ, the source ρ', as well as the angular distance $\gamma \equiv \gamma_{n+1}$ between the source and the observational point:

$$G(x,x') = G(\rho,\rho';\gamma). \tag{3.10}$$

The angular distance on the $(n+1)$-dimensional sphere can be written explicitly in terms of the angular coordinates (2.2):

$$\begin{aligned}\cos\gamma_{n+1} &= \cos\theta_n\cos\theta_n' + \sin\theta_n\sin\theta_n'\cos\gamma_n,\\ \gamma_0 &= \phi-\phi'.\end{aligned} \tag{3.11}$$

The canonical Bertotti-Robinson spacetime (2.10) is homogeneous. In Ref. [1], we used the knowledge of heat kernels on homogeneous spaces to derive the static Green functions. Now, using the biconformal symmetry of the static operator (1.3), we can use these results to derive the static Green functions in the Reissner-Nordström spacetime with arbitrary parameters of the mass M and the charge Q.

One can see that the biconformal transformation (1.4) with

$$\Omega = \frac{r}{a} = \frac{(M+\mu\rho)^{1/n}}{a} \tag{3.12}$$

leads to the Bertotti-Robinson canonical metric (2.10) if $\bar{\sigma}$ is identified with the rescaled Reissner-Nordström time coordinate t,

$$\bar{\sigma} = \bar{\kappa}t, \qquad \bar{\kappa} = \frac{n\mu}{a^{n+1}} = \left(\frac{r_{\mathrm{g}}}{a}\right)^{n+1}\kappa. \tag{3.13}$$

Here κ is given by (3.7) and $\bar{\kappa}$ is the surface gravity of the horizon $\rho = 1$ in the Bertotti-Robinson spacetime, normalized according to the Killing vector $\bar{\xi}^\mu = \delta_t^\mu$:

$$\bar{\kappa}^2 = -\frac{1}{2}\bar{\xi}^{\alpha;\beta}\bar{\xi}_{\alpha;\beta}|_{\rho=1}. \tag{3.14}$$

One can define the static Green function in the canonical metric (2.10) as the integral over the dimensionless time coordinate $\bar{\sigma}$,

$$\bar{G}(x,x') = \int_{-\infty}^{\infty} d\bar{\sigma}\bar{\mathbb{G}}_{\mathrm{Ret}}(\bar{\sigma},x;0,x'), \tag{3.15}$$

where $\bar{\mathbb{G}}_{\mathrm{Ret}}(\bar{\sigma},x;\bar{\sigma}',x')$ is a retarded Green function in the canonical Bertotti-Robinson spacetime (2.10). It satisfies the equation

$$[n^2(\rho^2-1)\partial_\rho^2 + 2n^2\rho\partial_\rho + \triangle_\omega^{n+1}]\bar{G}(x,x') = -\frac{n^2}{a^{n+1}}\delta(\rho-\rho')\delta(\omega,\omega'). \tag{3.16}$$

The left-hand side of this equation coincides with that of (3.8). The right-hand sides of these equations differ only by a constant factor related to the rescaling of the time coordinate (3.13). Thus, the static Green functions in these spaces also differ only by a constant factor

$$G(\rho,\rho';\gamma) = \frac{1}{\bar{\kappa}}\bar{G}(\rho,\rho';\gamma). \tag{3.17}$$

To construct the biconformal map (2.19) relating Reissner-Nordström with different parameters M and Q, one proceeds as follows. Let us define two radial coordinates r and $\hat{r}$ by the relation

$$\frac{r^n - M}{\mu} = \frac{\hat{r}^n - \hat{M}}{\hat{\mu}} \equiv \rho. \tag{3.18}$$

Here,

$$\mu = \sqrt{M^2 - Q^2}, \qquad \hat{\mu} = \sqrt{\hat{M}^2 - \hat{Q}^2}. \tag{3.19}$$

This allows one to express the new coordinate $\hat{r}$ in terms of the original radial coordinate r. The time coordinates are related as follows:

$$\hat{t} = \frac{\mu}{\hat{\mu}}t. \tag{3.20}$$

Then the biconformal transformation with

$$\Omega = \left[\frac{M+\mu\rho}{\hat{M}+\hat{\mu}\rho}\right]^{1/n} \tag{3.21}$$

relates two arbitrary Reissner-Nordström metrics (3.6) characterized by the parameters M, Q and $\hat{M}, \hat{Q}$, correspondingly.

Because of the time rescaling, the relation between the static Green functions in these Reissner-Nordström spacetimes becomes

$$\mu G(r,r';\gamma) = \hat{\mu}\,\hat{G}(\hat{r},\hat{r}';\gamma). \tag{3.22}$$

Note that though the static Green function depends on the time rescaling, this dependence is dropped out of the expression for the scalar field Φ. The resulting Φ is invariant with respect to the time rescaling. One can say that the static scalar potentials Φ for all Reissner-Nordström geometries are given by the same function of ρ. In terms of the radial coordinates r and $\hat{r}$, they are related by the coordinate transformation (3.18). Therefore, as soon as we know the static scalar Green function for a particular choice of the charge of a black hole, for example, for a neutral one, the identity (3.22) makes it possible to generate the solution for the scalar field near the Reissner-Nordström black hole with an arbitrary mass and charge.

Since the cases of even- and odd-dimensional spacetimes differ, we shall treat them separately.

B. Even dimensions

Using the property (3.22) and the result of the paper [1] (see Eqs. (6.23)–(6.24)), where we have calculated the static Green function in the Tangherlini spacetime, we obtain the Green function for arbitrary parameters M and Q. In even dimensions the exact static Green function can be represented in the form of the integral:

$$G(x,x') = \frac{1}{n\mu}\frac{1}{2(2\pi)^{\frac{n+3}{2}}}\left(\frac{\partial}{\partial\cos\gamma}\right)^{(n+1)/2}\int_0^{2\pi} d\sigma A_n. \tag{3.23}$$

Here $n = D-3$ and $\mu = \sqrt{M^2-Q^2}$. When $n \geq 2$, the functions $A_n(\sigma,\rho,\rho';\gamma)$ are given by the integral

$$A_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh(y)-\cosh(\chi)}}\frac{\sinh(\frac{y}{n})}{\sqrt{\cosh(\frac{y}{n})-\cos(\gamma)}}, \tag{3.24}$$

where

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2-1}\sqrt{\rho'^2-1}\cos\sigma. \tag{3.25}$$

At large values of y, the integrand in (3.24) behaves like $\exp[-y(n-1)/(2n)]$. Therefore, (3.24) is convergent for any $n \geq 2$. In the case of the four-dimensional spacetime $(n=1)$, the integrand has to be modified to guarantee convergence of the integral. For example, one can subtract the asymptotic of the integrand, which does not depend on γ. Since (3.23) contains the derivative of A_n over γ, the resulting Green function does not depend on the particular form of the subtracted γ-independent asymptotic. Thus, for $n=1$ one can choose

$$A_1 = \int_\chi^\infty dy \frac{\sinh(y)}{\sqrt{\cosh(y)-\cosh(\chi)}} \times\left[\frac{1}{\sqrt{\cosh(y)-\cos(\gamma)}} - \frac{1}{\sqrt{\cosh(y)+1}}\right]. \tag{3.26}$$

Substitution

$$\rho = \frac{r^n - M}{\mu} = \frac{r^n - M}{\sqrt{M^2 - Q^2}} \tag{3.27}$$

into (3.23) gives the static Green function of a scalar charge near the Reissner-Nordström black hole (3.6) in terms of the radial coordinate r.

C. Odd dimensions

Using the results of the paper [1] [see Eq. (6.8)], in odd-dimensional spacetimes we obtain

$$G(x,x') = \frac{1}{n\mu}\frac{1}{\sqrt{2}(2\pi)^{\frac{n+4}{2}}}\left(\frac{\partial}{\partial\cos\gamma}\right)^{n/2}\int_0^{2\pi} d\sigma B_n, \tag{3.28}$$

$$B_n = \int_\chi^\infty dy \frac{1}{\sqrt{\cosh y - \cosh\chi}}\frac{\sinh(\frac{y}{n})}{\cosh(\frac{y}{n}) - \cos\gamma}, \tag{3.29}$$

where $n = D-3$, $\mu = \sqrt{M^2 - Q^2}$, and χ is given by (3.25). Similarly to the even dimensions, the Green function in the radial coordinates r can be obtained after the substitution (3.27).

IV. CLOSED FORM OF THE GREEN FUNCTION: EXAMPLES

A. Four dimensions $D = 4$

In four dimensions ($n = 1$), the integral (3.26) can be done and one obtains

$$A_1 = \ln\left(\frac{\cosh(\chi) + 1}{\cosh(\chi) - \cos(\gamma)}\right). \tag{4.1}$$

The integral over σ can be taken explicitly, and we obtain the closed form for the static Green function:

$$G(x,x') = \frac{1}{4\pi\mu}\frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\gamma - 1 + \cos^2\gamma}},$$
$$\rho = \frac{r - M}{\mu} = \frac{r - M}{\sqrt{M^2 - Q^2}}. \tag{4.2}$$

When written in terms of the radial coordinate r, it reads

$$G(x,x') = \frac{1}{4\pi\mathcal{R}}, \tag{4.3}$$

where

$$\mathcal{R}^2 = (r - M)^2 + (r' - M)^2 - 2(r - M)(r' - M)\cos\gamma - (M^2 - Q^2)\sin^2\gamma. \tag{4.4}$$

This formula exactly reproduces the closed form of the well-known results [8,11,18] for the scalar Green function in four-dimensional Schwarzschild and Reissner-Nordström geometries. It is easy to check that using the biconformal symmetry (3.22), this solution could be generated from that of the Schwarzschild case ($Q = 0$).

In the limit of the extremally charged black hole $Q = M$, the obtained solution (4.2) reproduces the result [4] for the four-dimensional Majumdar-Papapetrou geometry.

B. Five dimensions $D = 5$

The other case when there exists a closed form for the static Green function is the five-dimensional ($n = 2$) Reissner-Nordström black hole. One can generate this solution using the biconformal symmetry (3.22) from that of the Tangherlini black hole [1] or, equivalently, just make the substitution (3.27) in the expression for the five-dimensional Green function (see Eq. (6.14) of [1]),

$$G(x,x') = \frac{1}{8\pi^2\mu}\frac{1}{(\rho^2 - 1)^{1/4}(\rho'^2 - 1)^{1/4}} \times\frac{\partial}{\partial\cos\gamma}\{\varkappa[\mathbf{F}(\psi,\varkappa) + \mathbf{K}(\varkappa)]\}, \tag{4.5}$$

where $\mathbf{F}$ and $\mathbf{K}$ are the elliptic functions,

$$\rho = \frac{r^2 - M}{\mu}, \tag{4.6}$$

and

$$\sin\psi = \cos\gamma\frac{\sqrt{2}}{\sqrt{\rho\rho' - \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1} + 1}},$$
$$\varkappa = \frac{\sqrt{2}(\rho^2 - 1)^{1/4}(\rho'^2 - 1)^{1/4}}{\sqrt{\rho\rho' + \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1} + 1 - 2\cos^2\gamma}}. \tag{4.7}$$

To the best of our knowledge, this closed form for the static Green function in a five-dimensional Reissner-Nordström black hole is new.

In the limit of the extremally charged black hole, when $Q = M$, the expression (4.5) leads to

$$G(x,x') = \frac{1}{4\pi^2\mathcal{R}^2}, \tag{4.8}$$

where

$$\mathcal{R}^2 = (r^2 - M) + (r'^2 - M) - 2\cos\gamma\sqrt{r^2 - M}\sqrt{r'^2 - M}.$$

It exactly reproduces the result [4] for the five-dimensional Majumdar-Papapetrou geometry in the case of a single extremal black hole of the mass M.

C. Six dimensions $D = 6$

Application of the (3.23) to a six-dimensional ($n = 3$) Reissner-Nordström black hole leads to

$$A_3 = \int_\chi^\infty dy \frac{1}{(\cosh y - \cosh \chi)^{1/2}} \frac{\sinh(\frac{y}{3})}{\sqrt{\cosh(\frac{y}{3}) - \cos(\gamma)}}$$
$$= 3 \int_{\cosh(\chi/3)}^\infty dz \frac{1}{\sqrt{4z^3 - 3z - \cosh \chi}} \frac{1}{\sqrt{z - \cos \gamma}}. \quad (4.9)$$

This integral can be expressed in terms of the elliptic function **F**,

$$A_3 = \frac{6}{\sqrt{v(w-u)}} \mathbf{F}\left(\arcsin\sqrt{\frac{w-u}{w}}, \frac{w(v-u)}{v(w-u)}\right), \quad (4.10)$$

where

$$p = \cosh(\chi/3), \qquad w = 2(p - \cos \gamma),$$
$$u = 3p - i\sqrt{3p^2 - 3}, \qquad v = 3p + i\sqrt{3p^2 - 3}. \quad (4.11)$$

Note that A_3 is real in spite of the complexity of the functions u and v. Thus, the static Green function in the six-dimensional Reissner-Nordström spacetime is given by the integral

$$G(x, x') = \frac{1}{48\pi^3 \mu} \left(\frac{\partial}{\partial \cos \gamma}\right)^2 \int_0^{2\pi} d\sigma A_3. \quad (4.12)$$

It is problematic to obtain an answer for the Green functions in a closed form for $D \geq 6$. However, a rather simple integral representation is possible in all higher dimensions. For some applications, like computing of the self-force and self-energy of scalar charges, this integral representation is sufficient to obtain the final results in a closed form.

V. DISCUSSION

In this paper, we demonstrated that there exist biconformal transformations relating static solutions of the minimally coupled massless field equation in the Reissner-Nordström spacetimes with different values of the parameters of the mass M and the charge Q. We used this symmetry to generate expressions for the static Green functions in such space, starting from similar Green functions for the neutral (uncharged) higher-dimensional black holes, which have been obtained earlier [1]. To check the obtained results, we considered the limit of higher-dimensional extreme black holes with $|Q| = M$. This is a special case of the Majumdar-Papapetrou metrics related by means of a biconformal map to the flat spacetime. It is possible to show that the obtained static Green functions in a generic Reissner-Nordström spacetime obey a correct flat spacetime limit.

The results obtained in our earlier publication [1] and in this paper have natural application to the study of the problem of the self-energy and self-force of the point scalar charged in the background of higher-dimensional static black holes. Especially interesting is the origin of near-horizon logarithmic terms in these expressions in odd-dimensional black holes [14,15] and the relation of these terms with the biconformal anomalies (see discussion in [19–21]). It is interesting also to test the method of the biconformal transformations in application to electric fields of static sources in the static black hole backgrounds. Another interesting question: Is it possible to generalize the method of biconformal transformations to the case of fields from stationary sources in a spacetime of rotating black holes? We are going to address these questions in our further work.

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