Vacuum currents induced by a magnetic flux around a cosmic string with finite core

E. R. Bezerra de Mello,[1,*] V. B. Bezerra,[1,†] A. A. Saharian,[2,1,‡] and H. H. Harutyunyan[3]

[1]*Departamento de Física, Universidade Federal da Paraíba, 58.059-970, Caixa Postal 5.008, João Pessoa, Paraíba, Brazil*

[2]*Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 0025 Yerevan, Armenia*

[3]*Department of Mathematics, Armenian State Pedagogical University, 13 Khandjyan Street, 0010 Yerevan, Armenia*

(Received 13 November 2014; published 13 March 2015)

We evaluate the Hadamard function and the vacuum expectation value of the current density for a massive complex scalar field in the generalized geometry of a straight cosmic string with a finite core enclosing an arbitrary distributed magnetic flux along the string axis. For the interior geometry, a general cylindrically symmetric static metric tensor is used with finite support. In the region outside the core, both the Hadamard function and the current density are decomposed into the idealized zero-thickness cosmic string and core-induced contributions. The only nonzero component corresponds to the azimuthal current. The zero-thickness part of the latter is a periodic function of the magnetic flux inside the core, with the period equal to the quantum flux. As a consequence of the direct interaction of the quantum field with the magnetic field inside the penetrable core, the core-induced contribution, in general, is not a periodic function of the flux. In addition, the vacuum current, in general, is not a monotonic function of the distance from the string and may change the sign. For a general model of the core interior, we also evaluate the magnetic fields generated by the vacuum current. As applications of the general results, we have considered an impenetrable core modeled by Robin boundary condition, a core with the Minkowski-like interior and a core with a constant positive curvature space. Various exactly solvable distributions of the magnetic flux are discussed.

DOI: 10.1103/PhysRevD.91.064034 PACS numbers: 04.62.+v, 11.27.+d, 98.80.Cq

I. INTRODUCTION

During the cosmological expansion, the spontaneous breaking of fundamental symmetries in the early Universe leads to phase transitions. In most interesting models of high-energy physics, the formation of a variety of topological defects is predicted as a result of these phase transitions [1]. These topologically stable structures have a number of interesting observable consequences, the detection of which would provide an important link between cosmology and particle physics. Among the various types of topological defects, the cosmic strings are most thoroughly studied due to the importance they may have in cosmology. The early interest in this class of defects was motivated by the scenario in which the strings seed the primordial density perturbations for the formation of the large-scale structures in the Universe. Although the recent observations of the cosmic microwave background radiation (CMB) disfavored this scenario, the cosmic strings are still candidates for the generation of a number interesting physical effects including the generation of gravitational waves, high-energy cosmic rays, and gamma ray bursts. Among the other signatures are the gravitational lensing, the creation of small non-Gaussianities in the CMB and some influence on the corresponding tensor modes. Recently the cosmic strings attracted a renewed interest partly because a variant of their formation mechanism is proposed in the framework of brane inflation [2]. Note that the cosmic string-type conical defects appear also in a number of condensed matter systems such as crystals, liquid crystals and quantum liquids (see, for example, Ref. [3]).

In the simplest theoretical model, the spacetime geometry produced by an infinite straight cosmic string has a conical structure. It is locally flat except on the top of the string where it has a delta-shaped curvature tensor. In quantum field theory, the nontrivial spatial topology induced by the presence of a cosmic string raises a number of interesting physical effects. One of these concerns the effect of a string on the properties of quantum vacuum. The topology change results in the distortion of the zero-point vacuum fluctuations of quantized fields and induces shifts in vacuum expectation values (VEVs) for physical observables. Explicit calculations have been done for the VEVs of the field squared and energy-momentum tensor for scalar, fermion and vector fields (see references given in Ref. [4]). The presence of a magnetic flux along the axis of the string gives rise to additional polarization of the quantum vacuum

*emello@fisica.ufpb.br
†valdir@fisica.ufpb.br
‡saharian@ysu.am

providing another example of a quantum topological interaction [5–10]. Though the gauge field strength vanishes outside the string core, the nonvanishing vector potential leads to Aharonov–Bohm-like effects on the physical characteristics of the vacuum for charged fields. In particular, in Ref. [11] it was demonstrated that the generalized Aharonov–Bohm effect leads to scattering cross sections and to particle production rates that do not go to zero when the geometrical size of the string tends to zero. These types of effects induced by gauge field fluxes have been further investigated in Refs. [12].

The magnetic flux along the cosmic string induces also vacuum current densities. This phenomenon has been investigated for scalar fields in Refs. [13,14]. The analysis of induced fermionic currents in higher-dimensional cosmic string spacetime in the presence of a magnetic flux have been developed in Ref. [15]. In these analyses the authors have shown that induced vacuum current densities along the azimuthal direction appear if the ratio of the magnetic flux by the quantum one has a nonzero fractional part. Moreover, the fermionic current induced by a magnetic flux in a $(2+1)$-dimensional conical spacetime and in the presence of a circular boundary has also been analyzed in Ref. [16]. The VEV of the current density in the geometry of a cosmic string compactified along its axis is investigated in Refs. [17] and [18] for fermionic and scalar fields, respectively. The generalization of the corresponding results for the fermionic case to a cosmic string in background of de Sitter spacetime is given in Ref. [19].

Many of the treatments of quantum fields around a cosmic string deal mainly with the case of the idealized geometry where the string is assumed to have zero thickness. Realistic cosmic strings have internal structure, characterized by the core radius determined by the symmetry breaking scale at which it is formed. Cylindrically symmetric static models of a string where the spacetime curvature is spread over a region of nonzero size are considered in Ref. [20,21]. The vacuum polarization effects due to massless fields in the region outside the core are investigated. In particular, it has been shown that long-range effects can take place due to the nontrivial core structure. As a model of an impenetrable core for a cosmic string, in Ref. [22] a cylindrical boundary is considered with Robin boundary condition on the field operator. The renormalized VEVs of the field squared and the energy-momentum tensor for a massive scalar field with general curvature coupling are investigated. The generalization of the results to the exterior region is given for a general cylindrically symmetric static model of the string core with finite support.

The present paper is devoted to the investigation of the VEV of the current density for a massive charged scalar field in the geometry of a cosmic string with a general cylindrically symmetric core of a finite support. The core encloses a gauge field flux directed along the string axis with an arbitrary radial distribution. The current density is among the most important quantities characterizing the properties of the quantum vacuum. Though the corresponding operator is local, due to the global nature of the vacuum, the corresponding VEV describes the global properties of the bulk and carries information about the structure of the string core. In addition, the VEV of the current density acts as a source in semiclassical Maxwell equations, and, hence, it plays an important role in modeling a self-consistent dynamics involving the electromagnetic field. The results presented below specify the conditions under which the details of the interior structure can be ignored and the effects induced by cosmic strings can be approximated by the idealized model.

The organization of the paper is as follows. In the next section, we describe the geometry at hand and present a complete set of mode functions obeying the matching conditions on the core boundary. By using these mode functions, in Sec. III, the Hadamard function is evaluated for the general case of the core geometry. This function is decomposed into two parts corresponding to an idealized geometry of a zero-thickness cosmic string and to the finite core contribution. The VEV of the current density in the region outside the string core is investigated in Sec. IV. The behavior of the core-induced contribution is discussed in various asymptotic regions of the parameters, and the corresponding magnetic fields are evaluated. Section V is devoted to applications of the general results to special cases of the core interior structure and the magnetic flux distribution. Numerical examples are presented. The main results of the paper are summarized in Sec. VI. In Appendix A we provide an integral representation for series involving the modified Bessel function. This representation is used in the evaluation of the Hadamard function and the current density. In Appendix B a uniform asymptotic expansion is derived, employed in the investigation of the current density near the boundary of the core.

II. GEOMETRY OF THE PROBLEM AND THE MODE FUNCTIONS

In this paper we consider a $(D+1)$-dimensional background geometry corresponding to a generalized straight cosmic string with a core of a finite support having radius a. Outside the core, in the cylindrical coordinate system $(x^1, x^2, \ldots, x^D) = (r, \phi, \mathbf{z})$, with $\mathbf{z} = (z_1, \ldots, z_N)$ and $N = D - 2$, the geometry is given by the line element with planar angle deficit $2\pi - \phi_0$,

$$ds^2 = g_{ik} dx^i dx^k = dt^2 - dr^2 - r^2 d\phi^2 - d\mathbf{z}^2, \qquad (2.1)$$

where $r > a$, $0 \leq \phi \leq \phi_0$, $-\infty < z_i < +\infty$ and the points $(r, \phi, \mathbf{z})$ and $(r, \phi + \phi_0, \mathbf{z})$ are to be identified. Inside the core, $r < a$, the spacetime geometry is described by the line element

$$ds^2 = dt^2 - dr^2 - u^2(r)d\phi^2 - d\mathbf{z}^2. \quad (2.2)$$

By a transformation of the radial coordinate, the line element (2.2) is expressed in the form previously discussed in Ref. [21]. The value of the radial coordinate corresponding to the symmetry axis of the core we shall denote by $r = r_c$ (as it will be seen below, in general, $r_c \neq 0$). If the interior geometry is regular on the axis, then one has

$$u(r) \sim q(r - r_c), \qquad q = 2\pi/\phi_0. \quad (2.3)$$

From the continuity of the metric tensor on the core boundary, we get

$$u(a) = a. \quad (2.4)$$

In general, $u'(r)$ is not continuous at $r = a$. In the interior region, $r < a$, for the Ricci scalar, one has $R = -2u''(r)/u(r)$, and in the exterior region, $R = 0$.

We assume that an additional infinitely thin cylindrical shell located at $r = a$ is present with the surface energy-momentum tensor τ_{ik}. Let us denote by n^i the normal to the shell normalized by the condition $n_i n^i = -1$, assuming that it points into the bulk on both sides. Introducing the induced metric as $h_{ik} = g_{ik} + n_i n_k$, for the extrinsic curvature tensor of the separating boundary, we have $K_{ik} = h_i^l h_k^p \nabla_l n_p$. The latter is related to the surface energy-momentum tensor by the Israel matching condition

$$\{K_{ik} - Kh_{ik}\} = 8\pi G\tau_{ik}, \quad (2.5)$$

where $K = K_i^i$ and the curly brackets denote summation over each side of the shell. In the problem under consideration, for the interior and exterior regions, one has $n_i = \delta_i^1$ and $n_i = -\delta_i^1$, respectively. For the only nonzero component of the extrinsic curvature tensor, we get $K_2^2 = -u'_a/a$ and $K_2^2 = 1/a$, with $u'_a = u'(a)$, for the interior and exterior regions, respectively. Now, the condition (2.5) leads to the surface energy-momentum tensor

$$8\pi G\tau_i^k = \frac{u'_a - 1}{a}\delta_i^k, \qquad i, k = 0, 3, \ldots, D, \quad (2.6)$$

and $\tau_1^1 = \tau_2^2 = 0$.

In this paper we are interested in the VEV of the current density for a charged scalar field $\varphi(x)$ in the region outside the string core. For the field with curvature coupling parameter ξ, the field equation has the form

$$(g^{ik}D_i D_k + m^2 + \xi R)\varphi(x) = 0, \quad (2.7)$$

where $D_k = \nabla_k + ieA_k$, with e being the charge associated with the field, A_k being the vector potential of a gauge field and ∇_i being the covariant derivative operator. The values of the curvature coupling parameter $\xi = 0$ and $\xi = \xi_D \equiv (D-1)/4D$ correspond to the most important special cases of minimally and conformally coupled scalars, respectively. The operator of the current density is given by the expression

$$j_l(x) = ie[\varphi^+(x)D_l\varphi(x) - (D_l\varphi(x))^+\varphi(x)]. \quad (2.8)$$

The corresponding VEV can be expressed in terms of the Hadamard function

$$G(x, x') = \langle 0|\varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x)|0\rangle, \quad (2.9)$$

where $|0\rangle$ stands for the vacuum state. For the VEV one has

$$\langle j_l(x)\rangle = \frac{i}{2}e\lim_{x'\to x}(\partial_l - \partial'_l + 2ieA_l)G(x, x'). \quad (2.10)$$

For the evaluation of the Hadamard function, we shall use the mode summation formula

$$G(x, x') = \sum_\alpha \sum_{s=\pm} \varphi_\alpha^{(s)}(x)\varphi_\alpha^{(s)*}(x'), \quad (2.11)$$

where $\varphi_\alpha^{(\pm)}(x)$ is a complete set of normalized positive- and negative-energy mode functions obeying the field equation (2.7) and specified by the collective index α. First we consider the mode functions for scattering states. In accordance with the symmetry of the problem, these functions can be expressed as

$$\varphi_\alpha^{(\pm)}(x) = f(r)e^{iqn\phi + i\mathbf{kz} \mp i\omega t}, \quad (2.12)$$

where $n = 0, \pm1, \pm2, \ldots$, $\mathbf{k} = (k_1, \ldots, k_N)$, $-\infty < k_j < \infty$. Substituting into the field equation, and assuming the vector potential of the form

$$A_l = (0, 0, A_2(r), 0, \ldots, 0), \quad (2.13)$$

for the radial function in the interior region, we obtain the equation

$$\{u(r)\partial_1[u(r)\partial_1] + u^2(r)\gamma^2 - [qn + eA_2(r)]^2 - \xi u^2(r)R\}f(r) = 0, \quad (2.14)$$

where

$$\gamma = \sqrt{\omega^2 - k^2 - m^2}. \quad (2.15)$$

For the interior region, $R = -2u''/u$. In the exterior region, $r > a$, the corresponding equation is obtained taking in Eq. (2.14) $u(r) = r$ and $R = 0$. Note that in Eq. (2.14) $A_2(r)$ is the covariant component of the vector potential. For the physical component, one has $A_\phi(r) = -A_2(r)/u(r)$ for $r < a$ and $A_\phi(r) = -A_2(r)/r$ for $r > a$. The only nonzero components of the field tensor F_{il} corresponding to Eq. (2.13) are given by $F_{12} = -F_{21} = \partial_r A_2(r)$. In $D = 3$ models this corresponds to a magnetic field directed along the z-axis with the strength $B_z = -r^{-1}\partial_r A_2(r)$.

The radial functions in the interior and exterior regions are connected by the matching conditions at the boundary. The radial function is continuous at the boundary: $f(a-) = f(a+)$. To find the jump condition for its first derivative, we note that the Ricci scalar contains a delta function term $2(u_a' - 1)\delta(r-a)/a$ located on the separating boundary. By taking into account this term and integrating the radial equation near the boundary, we find the following condition:

$$f'(a+) - f'(a-) = 2\xi(u_a' - 1)f(a)/a. \tag{2.16}$$

In deriving this condition, we have assumed that the vector potential contains no Dirac delta-type terms.

In what follows we shall consider the case when the magnetic field in the exterior region vanishes, corresponding to $A_2(r) = A_2 = \mathrm{const}$ for $r > a$. In this case, for the exterior region, the general solution of the radial equation is a linear combination of the Bessel and Neumann functions $J_{\beta_n}(\gamma r)$ and $Y_{\beta_n}(\gamma r)$ of the order

$$\beta_n = q|n + \beta|, \qquad \beta = eA_2/q. \tag{2.17}$$

Note that, if the function $A_2(r)$ is regular, the parameter β is expressed in terms of the magnetic flux inside the core as $\beta = -\Phi/\Phi_0$, with $\Phi_0 = 2\pi/e$ being the quantum flux. Let us denote by $R_n(r,\gamma)$ the solution of the radial equation (2.14) inside the core, regular on the axis, namely at $r = r_c$. By taking into account that the parameter γ enters in the equation in the form γ^2, without loss of generality, we can take this function to be real and obeying the condition $R_n(r, \gamma e^{\pi i}) = \mathrm{const}\, R_n(r,\gamma)$. By the same reason, if $A_2(r) = 0$ for $r < a$, we can also assume that $R_{-n}(r,\gamma) = \mathrm{const}\, R_n(r,\gamma)$.

Now, the radial part of the eigenfunctions can be written in the form

$$f(r) = \begin{cases} R(r,\gamma), & r < a, \\ AJ_{\beta_n}(\gamma r) + BY_{\beta_n}(\gamma r), & r > a. \end{cases} \tag{2.18}$$

The coefficients A and B in Eq. (2.18) are determined from the condition of the continuity of the radial functions and from the jump condition (2.16) at $r = a$,

$$A = \frac{\pi}{2} R_n(a,\gamma) Y_{\beta_n}(\gamma a, p_n(\gamma)),$$
$$B = -\frac{\pi}{2} R_n(a,\gamma) J_{\beta_n}(\gamma a, p_n(\gamma)), \tag{2.19}$$

where

$$p_n(\gamma) = a\frac{R_n'(a,\gamma)}{R_n(a,\gamma)} + 2\xi(u_a' - 1), \tag{2.20}$$

with $R_n'(a,\gamma) = \partial_r R_n(r,\gamma)|_{r=a}$. Here and in what follows, for a given function $F(x)$, we use the notation

$$F(x,y) = xF'(x) - yF(x). \tag{2.21}$$

Note that, because of our choice of the function $R_n(r,\gamma)$, we have the property

$$p_n(\gamma e^{\pi i}) = p_n(\gamma). \tag{2.22}$$

In addition, if $A_2(r) = 0$ for $r < a$, one has $p_{-n}(\gamma) = p_n(\gamma)$.

From the normalization condition for the mode functions, one has

$$\int d^D x u(r) \varphi_\alpha^{(\lambda)}(x) \varphi_{\alpha'}^{(\lambda')*}(x) = \frac{1}{2\omega}\delta_{\alpha\alpha'}\delta_{\lambda\lambda'}, \tag{2.23}$$

where the symbol $\delta_{\alpha\alpha'}$ is understood as Kronecker delta for discrete components of the collective index α and as the Dirac delta function for continuous ones. In Eq. (2.23), the integration over the radial coordinate goes over $r_c \le r < \infty$ and $u(r) = r$ in the region $r > a$. For the radial functions, we get

$$(2\pi)^N \phi_0 \int_{r_c}^{\infty} dr u(r) f(r,\gamma) f^*(r,\gamma') = \frac{1}{2\omega}\delta(\gamma - \gamma'), \tag{2.24}$$

where $f(r,\gamma)$ is defined by the right-hand side of Eq. (2.18). The integral over the interior region is finite, and, hence, the dominant contribution in Eq. (2.24) for $\gamma = \gamma'$ comes from the integration over the exterior region. By using the standard integrals involving the cylinder functions, we find the relation

$$(\pi/2)^2 R^2(a,\gamma) = \frac{(2\pi)^{-N}\gamma}{2\phi_0\omega}[J_{\beta_n}^2(\gamma a, p_n(\gamma)) + Y_{\beta_n}^2(\gamma a, p_n(\gamma))]^{-1}. \tag{2.25}$$

This relation determines the normalization of the interior radial function. With this normalization, the radial function in the exterior region, $r > a$, is expressed as

$$f(r) = \left[\frac{(2\pi)^{-N}\gamma}{2\phi_0\omega}\right]^{1/2} \frac{g_{\beta_n}(\gamma r, \gamma a, p_n(\gamma))}{[J_{\beta_n}^2(\gamma a, p_n(\gamma)) + Y_{\beta_n}^2(\gamma a, p_n(\gamma))]^{1/2}}, \tag{2.26}$$

where we use the notation

$$g_{\beta_n}(z,x,y) = Y_{\beta_n}(x,y)J_{\beta_n}(z) - J_{\beta_n}(x,y)Y_{\beta_n}(z). \tag{2.27}$$

In addition to the scattering states, one can have bound states for which γ is purely imaginary, $\gamma = i\chi$, $\chi > 0$. To have a stable vacuum, we assume that $\chi < m$; otherwise, the modes with imaginary energy would be present. For the bound states, the exterior radial function is expressed in terms of the MacDonald function:

$$f(r) = A_b K_{\beta_n}(\chi r), \qquad r > a. \tag{2.28}$$

From the continuity condition for the radial function at $r = a$, we find

$$A_b = R_n(a, i\chi)/K_{\beta_n}(\chi a). \tag{2.29}$$

The jump condition for the radial derivative gives

$$K_{\beta_n}(\chi a, p_n(i\chi)) = 0, \tag{2.30}$$

with the notation defined by Eq. (2.21). The solutions of Eq. (2.30) for χ determine the possible bound states.

The normalization condition for the radial functions of the bound states is given by Eq. (2.24), where in the right-hand side we should take the Kronecker delta $\delta_{\chi\chi'}$ and the energy is given by $\omega = \sqrt{k^2 + m^2 - \chi^2}$. Now, both the interior and exterior regions contribute to the normalization integral, and one needs the formula for the integrals involving the function $R_n(r, \gamma)$. From Eq. (2.14) it follows that for the two solutions $f_i(r, \gamma)$ $(i = 1, 2)$ of that equation one has

$$\int dr u(r) f_1(r,\gamma) f_2(r,\gamma') = \frac{u(r)}{\gamma^2 - \gamma'^2}[f_1(r,\gamma)\partial_r f_2(r,\gamma') - f_2(r,\gamma')\partial_r f_1(r,\gamma)]. \tag{2.31}$$

From here we get

$$\int_{r_c}^{a} dr u(r) R_n^2(r,\gamma) = \frac{a}{2\gamma}[(\partial_\gamma R_n(a,\gamma))R'_n(a,\gamma) - R_n(a,\gamma)\partial_\gamma R'_n(a,\gamma)]. \tag{2.32}$$

For the integral over the exterior region, we use the standard integral involving the square of the MacDonald function [23]. By taking into account Eq. (2.30) for the bound states, the normalization integral can be presented as

$$\int_{r_c}^{a} dr u(r) R_n^2(r, i\chi) + A_b^2 \int_a^\infty dr r K_{\beta_n}^2(\chi r) = -\frac{A_b^2}{2\chi} K_{\beta_n}(\chi a)\partial_\chi K_{\beta_n}(\chi a, p_n(i\chi)). \tag{2.33}$$

Now, from the normalization condition, one gets

$$R_n^2(a, i\chi) = -\frac{q\chi K_{\beta_n}(\chi a)}{(2\pi)^{D-1}\omega\partial_\chi K_{\beta_n}(\chi a, p_n(i\chi))}. \tag{2.34}$$

The coefficient A_b in Eq. (2.28) is found from Eq. (2.29).

III. HADAMARD FUNCTION

After the construction of the mode functions, we turn to the evaluation of the Hadamard function in the exterior region. First we consider the case when the bound states are absent. Plugging the functions (2.12) with (2.26) into the mode sum (2.11), we find the expression for the Hadamard function,

$$G(x, x') = \frac{1}{\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} \sum_{n=-\infty}^{+\infty} e^{i\mathbf{k}\Delta\mathbf{z} + iqn\Delta\phi} \int_0^\infty d\gamma \frac{\cos(\omega\Delta t)}{\omega} \times \frac{\gamma g_{\beta_n}(\gamma r, \gamma a, p_n(\gamma)) g_{\beta_n}(\gamma r', \gamma a, p_n(\gamma))}{J_{\beta_n}^2(\gamma a, p_n(\gamma)) + Y_{\beta_n}^2(\gamma a, p_n(\gamma))}, \tag{3.1}$$

where $\Delta\mathbf{z} = \mathbf{z} - \mathbf{z}'$, $\Delta\phi = \phi - \phi'$, $\Delta t = t - t'$ and $\omega = \sqrt{\gamma^2 + k^2 + m^2}$. For the further transformation of this formula, we use the identity

$$\frac{g_{\beta_n}(\gamma r, \gamma a, p_n(\gamma)) g_{\beta_n}(\gamma r', \gamma a, p_n(\gamma))}{J_{\beta_n}^2(\gamma a, p_n(\gamma)) + Y_{\beta_n}^2(\gamma a, p_n(\gamma))} = J_{\beta_n}(\gamma r) J_{\beta_n}(\gamma r') - \frac{1}{2}\sum_{s=1}^{2} \frac{J_{\beta_n}(\gamma a, p_n(\gamma))}{H_{\beta_n}^{(s)}(\gamma a, p_n(\gamma))} H_{\beta_n}^{(s)}(\gamma r) H_{\beta_n}^{(s)}(\gamma r'), \tag{3.2}$$

where $H_{\beta_n}^{(s)}(z)$, $s = 1, 2$, are the Hankel functions and, again, we use the notation (2.21). Substituting Eq. (3.2) into Eq. (3.1), in the part with the second term in the right-hand side of Eq. (3.2), in the complex plane γ, we rotate the integration contour by the angle $\pi/2$ for $s = 1$ and by the angle $-\pi/2$ for $s = 2$. By using the property (2.22), one can see that the integrals over the segments $(0, i\sqrt{k^2 + m^2})$ and $(0, -i\sqrt{k^2 + m^2})$ are canceled. In the remaining integral over the imaginary axis, we introduce the modified Bessel functions. As a result, the Hadamard function is presented in the decomposed form

$$G(x, x') = G_0(x, x') + G_c(x, x'), \tag{3.3}$$

where

$$G_0(x, x') = \frac{1}{\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} \sum_{n=-\infty}^{+\infty} e^{i\mathbf{k}\Delta\mathbf{z} + iqn\Delta\phi} \int_0^\infty d\gamma \times \frac{\gamma}{\omega} J_{\beta_n}(\gamma r) J_{\beta_n}(\gamma r') \cos(\omega\Delta t), \tag{3.4}$$

and

$$G_c(x, x') = -\frac{2}{\pi\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} \sum_{n=-\infty}^{+\infty} e^{i\mathbf{k}\Delta\mathbf{z} + iqn\Delta\phi} \int_0^\infty dx \times \cosh(x\Delta t) K_{\beta_n}(zr) K_{\beta_n}(zr') \frac{I_{\beta_n}(za, p_n(iz))}{K_{\beta_n}(za, p_n(iz))}, \tag{3.5}$$

with $z = \sqrt{x^2 + k^2 + m^2}$. In Eq. (3.5), we have used the notation (2.21) with $F(x) = I_{\beta_n}(x)$ being the modified Bessel function.

The equation (3.4) corresponds to the Hadamard function in the geometry of an idealized cosmic string with zero-thickness core and constant vector potential with the regularity condition imposed at the origin. By using the transformations described in Appendix A, this function is presented in the form

$$G_0(x,x') = q\frac{(rr')^{-(D-1)/2}}{(2\pi)^{(D+1)/2}}\int_0^\infty dx x^{(D-3)/2} \times e^{-m^2rr'/(2x)-x(r^2+r'^2+|\Delta\mathbf{z}|^2-(\Delta t)^2)/(2rr')} \times \mathcal{I}_q(\beta,\Delta\phi,x), \tag{3.6}$$

where we have introduced the function

$$\mathcal{I}_q(\beta,\Delta\phi,x) = \sum_{n=-\infty}^{+\infty} e^{iqn\Delta\phi} I_{\beta_n}(x). \tag{3.7}$$

An alternative expression for the Hadamard function $G_0(x,x')$ is obtained by using the integral representation (A10) for the function (3.7). After the integration over x, we get

$$G_0(x,x') = \frac{2m^{D-1}}{(2\pi)^{(D+1)/2}}\left\{\sum_n e^{i\beta(2\pi n - q\Delta\phi)} f_{(D-1)/2}(ms_n(x,x')) - \frac{qe^{-iqn_0\Delta\phi}}{2\pi i}\sum_{j=\pm1} je^{ji\pi q\beta_f}\int_0^\infty dy \frac{\cosh[qy(1-\beta_f)] - \cosh(q\beta_f y)e^{-iq(\Delta\phi+j\pi)}}{\cosh(qy)-\cos[q(\Delta\phi+j\pi)]} f_{(D-1)/2}(mc(y,x,x'))\right\}, \tag{3.8}$$

where the summation over n goes under the same conditions as in the right-hand side of Eq. (A7),

$$s_n(x,x') = [(\Delta r)^2 + |\Delta\mathbf{z}|^2 - (\Delta t)^2 + 4rr'\sin^2(\pi n/q - \Delta\phi/2)]^{1/2},$$
$$c(y,x,x') = [(\Delta r)^2 + |\Delta\mathbf{z}|^2 - (\Delta t)^2 + 4rr'\cosh^2(y/2)]^{1/2}, \tag{3.9}$$

and we have defined the function

$$f_\nu(x) = K_\nu(x)/x^\nu. \tag{3.10}$$

Note that the $n=0$ term in Eq. (3.8), up to the factor $e^{-i\beta q\Delta\phi}$, corresponds to the Hadamard function in Minkowski spacetime in the absence of the magnetic flux. The divergences in the coincidence limit of the arguments are contained in this term only. The remaining part is induced by the planar angle deficit and by the magnetic flux. Note that the Euclidean Green function for a massive scalar field in the geometry of a zero-thickness cosmic string has been considered in Ref. [6] in the special case $q<2$ and in the subspace $-1+q/2 < \Delta\phi/\phi_0 < 1-q/2$. Under these conditions, the only contribution in the series over n in Eq. (3.8) comes from the term $n=0$.

Now, let us discuss the contribution to the Hadamard function coming from possible bound states. The latter are solutions of Eq. (2.30), and the corresponding mode functions are given by Eq. (2.28). The contribution of the bound states to the Hadamard function is given by

$$G_b(x,x') = -\frac{2}{\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}\sum_{n=-\infty}^{+\infty} e^{i\mathbf{k}\Delta\mathbf{z}+iqn\Delta\phi}\sum_\chi\frac{\chi}{\omega} \times \frac{\cos(\omega\Delta t)K_{\beta_n}(\eta r)K_{\beta_n}(\eta r')}{K_{\beta_n}(\chi a)\partial_\chi K_{\beta_n}(\chi a, p_n(i\chi))}, \tag{3.11}$$

where $\sum_\chi$ stands for the summation over the solutions of Eq. (2.30) and $\omega = \sqrt{k^2+m^2-\chi^2}$. In the presence of the bound states, the evaluation of the part in the Hadamard function corresponding to scattering modes is similar to what we have described before. The only difference appears in the part where we rotate the integration contour in the contribution coming from the second term in the right-hand side of Eq. (3.2). Now, the integrand has simple poles $\gamma = \pm i\chi$ on the imaginary axis corresponding to the zeros of the function $H^{(s)}_{\beta_n}(\gamma a, p_n(\gamma))$. We escape these poles by semicircles of small radius in the right half-plane. The contributions from the semicircles in the upper and lower half-planes combine into the residue of the term with $s=1$ at $i\chi$ multiplied by $2\pi i$. As a result, an additional term comes from the poles, given by

$$G_{\rm poles}(x,x') = \frac{2}{\phi_0}\int\frac{d^N\mathbf{k}}{(2\pi)^N}\sum_{n=-\infty}^{+\infty} e^{i\mathbf{k}\Delta\mathbf{z}+iqn\Delta\phi} \times \sum_\chi\frac{\chi}{\omega}\frac{\cos(\omega\Delta t)I_{\beta_n}(\chi a, p_n(i\chi))}{\partial_\chi K_{\beta_n}(\chi a, p_n(i\chi))} \times K_{\beta_n}(\chi r)K_{\beta_n}(\chi r'). \tag{3.12}$$

From the Wronskian relation for the modified Bessel functions, by using Eq. (2.30), one can show that

$$I_{\beta_n}(\chi a, p(i\chi)) = 1/K_{\beta_n}(\chi a). \tag{3.13}$$

With this relation, we see that the contribution coming from the poles on the imaginary axis cancels the part (3.11) corresponding to the bound states. As a result, the expression for the Hadamard function (3.5) is valid in the presence of bound states as well.

IV. CURRENT DENSITY

With a given Hadamard function, we can evaluate the VEV of the current density by using Eq. (2.10). The current density is decomposed as

$$\langle j_l \rangle = \langle j_l \rangle_0 + \langle j_l \rangle_c, \tag{4.1}$$

where the part $\langle j_l \rangle_0$ is the VEV in the geometry of the zero-thickness cosmic string and the contribution $\langle j_l \rangle_c$ is induced by the nontrivial structure of the core. In the region outside the core, $r > a$, the local geometry is the same as that for an idealized cosmic string. Consequently, possible divergences in the coincidence limit are contained in the part $\langle j_l \rangle_0$ only. In this way, with the decomposition (4.1), the renormalization is reduced to the one in the geometry of a zero-thickness cosmic string. In the corresponding Hadamard function, given by Eq. (3.8), the only divergence in the coincidence limit comes from the $n = 0$ term in the right-hand side. The contribution of this term is expressed in terms of the VEV of the current density in Minkowski spacetime in the absence of the magnetic flux. The latter is renormalized to zero.

In the problem under consideration, the only nonzero component of the current density $\langle j_l \rangle$ corresponds to the azimuthal current with $l = 2$. The vanishing of the potential divergent contribution to this component can also be seen by using the representation (3.6). The divergence can come only from the $n = 0$ term in the integral representation (A10) for the function $\mathcal{I}_q(\beta, \Delta\phi, z)$. The action of the operator in the right-hand side of (2.10), with $l = 2$, gives the expression $-2z \sin(\Delta\phi) e^{z\cos(\Delta\phi) - i\beta q \Delta\phi}/q$ which vanishes in the limit $\Delta\phi \to 0$.

A. Azimuthal current around a zero-thickness cosmic string

First we consider the current density for the geometry of an idealized cosmic string. By using the Hadamard function given by Eq. (3.6), for the corresponding physical component $\langle j_\phi \rangle_0 = -\langle j_2 \rangle_0 / r$, one finds

$$\langle j_\phi \rangle_0 = \frac{eq^2 r^{-D}}{(2\pi)^{(D+1)/2}} \int_0^\infty dx x^{(D-3)/2} e^{-x - m^2 r^2/(2x)} \times \sum_{n=-\infty}^{+\infty} (n+\beta) I_{\beta_n}(x). \tag{4.2}$$

As it is seen, in the geometry of a zero-thickness cosmic string, the current density is a periodic function of the parameter β with period 1. This corresponds to the periodicity of the current as a function of the magnetic flux with the period equal to the quantum flux. So, if we present the parameter β in the form

$$\beta = n_0 + \beta_f, \qquad 0 \le \beta_f < 1, \tag{4.3}$$

with n_0 being an integer, then the VEV, $\langle j_\phi \rangle_0$, is a function of β_f alone. For $\beta_f = 0$ this VEV vanishes.

An equivalent expression for the current density is obtained with the help of the integral representation (A14), given in the Appendix,

$$\langle j_\phi \rangle_0 = \frac{4em^{D+1} r}{(2\pi)^{(D+1)/2}} \left\{ \sum_{n=1}^{[q/2]} \sin(2\pi n\beta) \sin(2\pi n/q) f_{(D+1)/2}(2mr \sin(\pi n/q)) + \frac{q}{2\pi} \int_0^\infty dy f_{(D+1)/2}(2mr\cosh(y/2)) \frac{\sinh y f(q, \beta_f, y)}{\cosh(qy) - \cos(\pi q)} \right\}, \tag{4.4}$$

where the function $f(q, \beta, y)$ is defined by the relation

$$f(q, \beta_f, y) = \sin(\pi q \beta_f) \sinh[q(1-\beta_f)y] - \sin[\pi q(1-\beta_f)] \sinh(q\beta_f y). \tag{4.5}$$

In the special case $1 \le q \le 2$, the sum over n is absent, and Eq. (4.4) is reduced to the expression given in Ref. [14] (see also Ref. [18]). Note that the current density for an idealized geometry of a zero-thickness cosmic string does not depend on the curvature coupling parameter. At large distances from the string, $mr \gg 1$, and for $q > 2$, the dominant contribution comes from the term with $n = 1$, and the current density decays as $e^{-2mr\sin(\pi/q)}/(mr)^{D/2}$. For $q \le 2$ the suppression of the VEV at large distances is stronger, by the factor $e^{-2mr}/(mr)^{(D+3)/2}$.

For a massless field, Eq. (4.4) is reduced to

$$\langle j_\phi \rangle_0 = \frac{4e\Gamma((D+1)/2)}{(4\pi)^{(D+1)/2} r^D} \left\{ \sum_{n=1}^{[q/2]} \frac{\sin(2\pi n\beta)\cos(\pi n/q)}{\sin^D(\pi n/q)} + \frac{q}{\pi} \int_0^\infty dy \frac{\sinh y}{\cosh^D y} \frac{f(q, \beta_f, 2y)}{\cosh(2qy) - \cos(\pi q)} \right\}. \tag{4.6}$$

For a massive field, the expression in the right-hand side of Eq. (4.6) gives the leading term in the asymptotic

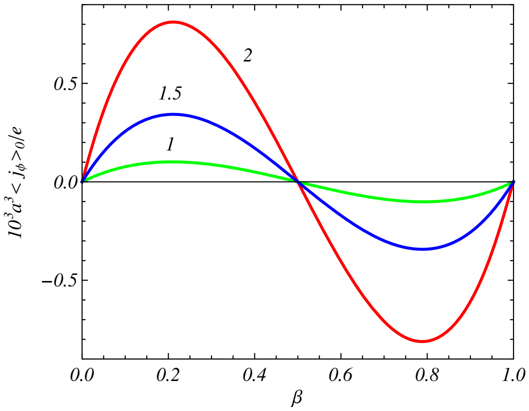

FIG. 1 (color online). The azimuthal current density in the geometry of a zero-thickness cosmic string, as a function of the parameter β, for a $D=3$ massless scalar field. The graphs are plotted for $r/a=2$, and the numbers near the curves correspond to the values of q.

expansion of $\langle j_\phi\rangle_0$ for points close to the string, $mr \ll 1$. In particular, the current density diverges on the string as $1/r^D$. Another special case corresponds to the absence of the planar angle deficit, $q=1$. From Eq. (4.4) we get

$$\langle j_\phi\rangle_0 = \frac{16em^{D+1}r}{(2\pi)^{(D+3)/2}}\sin(\pi\beta_f)\int_0^\infty dy\sinh y\sinh[(1-2\beta_f)y] \times f_{(D+1)/2}(2mr\cosh y). \quad (4.7)$$

In this case the azimuthal current density is positive for $0<\beta_f<1/2$ and negative for $1/2<\beta_f<1$. After the change of the integration variable, it can be seen that Eq. (4.7) coincides with the result obtained in Ref. [24].

In Fig. 1, the VEV of the current density, measured in units of a, $a^3\langle j_\phi\rangle_0$, is displayed as a function of the parameter β for a massless scalar field in a three-dimensional space ($D=3$). The graphs are plotted for a fixed value of the distance from the string, corresponding to $r/a=2$, and the numbers near the curves correspond to the values of the parameter q.

B. Current density induced by a finite core

For the core-induced contribution in Eq. (4.1), by using Eq. (3.5) for the corresponding part of the Hadamard function, we get

$$\langle j_\phi\rangle_c = -8eq^2\frac{(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)r}\sum_{n=-\infty}^{+\infty}(n+\beta) \times\int_m^\infty dxx(x^2-m^2)^{(D-3)/2}\frac{I_{\beta_n}(ax,p_n(ix))}{K_{\beta_n}(ax,p_n(ix))}K_{\beta_n}^2(rx). \quad (4.8)$$

If $A_2(r)=0$ for $r<a$, one has $p_{-n}(\gamma)=p_n(\gamma)$, and this contribution vanishes for $\beta_f=0$. Note that the function $p_n(ix)$, in general, depends on the parameter β through the boundary condition for the vector potential at the core boundary. As a consequence of this, the core-induced contribution to the current density, in general, is not a periodic function of the magnetic flux enclosed by the core. The physical reason for this is the interaction of the quantum field with the magnetic field inside the core if the latter is penetrable. Substituting Eq. (4.3) into Eq. (4.8) and redefining the summation variable, the expression is written in an alternative form,

$$\langle j_\phi\rangle_c = -8eq^2\frac{(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)r}\sum_{n=-\infty}^{+\infty}(n+\beta_f)\int_m^\infty dxx \times(x^2-m^2)^{(D-3)/2}\frac{I_{\beta_{fn}}(ax,p_{n-n_0}(ix))}{K_{\beta_{fn}}(ax,p_{n-n_0}(ix))}K_{\beta_{fn}}^2(rx), \quad (4.9)$$

where $\beta_{fn}=q|n+\beta_f|$. The only dependence on n_0 appears in the function $p_{n-n_0}(ix)$. Note that the dependence of the function $p_{n-n_0}(ix)$ on n_0 can also come through the parameter β. For $\beta_f=0$ the part $\langle j_\phi\rangle_0$ vanishes, and the appearance of the nonzero current density is a purely finite-core effect.

Let us consider the behavior of the core-induced part at large distances from the core. For a massive field and for $mr\gg 1$, the dominant contribution into Eq. (4.8) comes from the region near the lower limit of integration. To the leading order, we get

$$\langle j_\phi\rangle_c \approx -\frac{eq^2m^D}{2(4\pi)^{(D-1)/2}}\frac{e^{-2rm}}{(mr)^{(D+3)/2}} \times\sum_{n=-\infty}^{+\infty}(n+\beta)\frac{I_{\beta_n}(am,p_n(im))}{K_{\beta_n}(am,p_n(im))}, \quad (4.10)$$

and the core-induced part is suppressed exponentially. Comparing with the behavior of the part $\langle j_\phi\rangle_0$, we see that for $q>2$ the latter dominates at large distances, and the relative contribution of the finite core-induced effects are suppressed by the factor $e^{-2mr[1-\sin(\pi/q)]}/(mr)^{3/2}$. For $q\le 2$ the contributions of $\langle j_\phi\rangle_0$ and $\langle j_\phi\rangle_c$ to the total VEV, at large distances, are of the same order if $am\sim 1$.

For a massless field and at large distances from the core boundary, introducing in Eq. (4.9) a new integration variable, $y=rx$, we expand the integrand in powers of a/r. For $\beta_f\neq 0$, the dominant contribution comes from the term with $n=n_\beta$, where $n_\beta=0$ for $0<\beta_f<1/2$ and $n_\beta=-1$ for $1/2<\beta_f<1$. To the leading order, we get

$$\langle j_\phi\rangle_c \approx \mathrm{sgn}(1/2-\beta_f)\frac{2eq\Gamma(2\sigma_\beta+(D-1)/2)}{(4\pi)^{D/2}r^D(2r/a)^{2\sigma_\beta}} \times\frac{\sigma_\beta-p_{n_\beta-n_0}(0)}{\sigma_\beta+p_{n_\beta-n_0}(0)}\frac{\Gamma(\sigma_\beta+(D-1)/2)}{\Gamma^2(\sigma_\beta)\Gamma(\sigma_\beta+D/2)}, \quad (4.11)$$

with the notation

$$\sigma_\beta = \begin{cases} q\beta_f, & 0 < \beta_f < 1/2, \\ q(1-\beta_f), & 1/2 < \beta_f < 1. \end{cases} \tag{4.12}$$

In this case the relative contribution of the core-induced effects is suppressed by the factor $(a/r)^{2\sigma_\beta}$. For $\beta_f = 0$ the dominant contribution comes from the terms with $n = \pm 1$:

$$\langle j_\phi \rangle_c \approx \frac{2eq\Gamma(2q + (D-1)/2)}{(4\pi)^{D/2} r^D (2r/a)^{2q}} \times \sum_{n=\pm 1} n \frac{q - p_{n-n_0}(0)}{q + p_{n-n_0}(0)} \frac{\Gamma(q + (D-1)/2)}{\Gamma^2(q)\Gamma(q + D/2)}. \tag{4.13}$$

In this case $\langle j_\phi \rangle_0 = 0$, and the only effect comes from the nontrivial core structure.

Now, let us consider the behavior of the core-induced part in the current density near the core boundary. At the boundary, in general, the current density diverges. For points near the boundary, the dominant contribution in Eq. (4.8) comes from large values of $|n|$. To find the leading term in the asymptotic expansion over the distance from the core boundary, it is convenient to introduce in Eq. (4.8) a new integration variable $y = x/\beta_n$. In what follows we shall need the uniform asymptotic expansion of the function $p_n(i\beta_n y)$ for large values of $|n|$. The first two leading terms are found in Appendix B, and they are given by Eq. (B6). By using that expression and the uniform asymptotic expansions for the modified Bessel functions for large values of the order [25], we can see that, to the leading order,

$$\frac{I_{\beta_n}(\beta_n ay, p(i\beta_n y))}{K_{\beta_n}(\beta_n ay, p(i\beta_n y))} K^2_{\beta_n}(\beta_n ry) \approx \frac{e^{-2\beta_n \sqrt{a^2y^2+1}(r/a-1)}}{4\beta_n^2(a^2y^2+1)} V(ay), \tag{4.14}$$

where

$$V(x) = \left(2\xi - \frac{x^2/2}{x^2+1}\right)(u'_a - 1) + \mathrm{sgn}(n) \frac{eA_{2a} - q\beta}{\sqrt{x^2+1}}. \tag{4.15}$$

Note that the leading term in the expansion of the function $I_{\beta_n}(\beta_n ay, p(i\beta_n y))$ vanishes due to the cancelation of the leading terms for the functions $\beta_n ay I'_{\beta_n}(\beta_n ay)$ and $p(i\beta_n y) I_{\beta_n}(\beta_n ay)$.

Substituting Eq. (4.14) into Eq. (4.8) (with x replaced by $\beta_n y$), we first evaluate the series over n (to the leading order over $r - a$). Then, the integral over y is expressed in terms of the gamma function. Two cases should be considered separately. In the case $eA_{2a} \neq q\beta$, to the leading order, one gets

$$\langle j_\phi \rangle_c \approx e \frac{q\beta - eA_{2a}}{(4\pi)^{(D+1)/2}} \frac{\Gamma((D-1)/2)}{Da(r-a)^{D-1}}. \tag{4.16}$$

The condition $eA_{2a} \neq q\beta$ means that the vector potential is not continuous at the core boundary and the corresponding magnetic field contains the part with the delta-type distribution located on the cylindrical shell $r = a$. For $eA_{2a} = q\beta$ the vector potential is continuous, and the leading term vanishes. The next-to-the-leading term in the expansion of the core-induced contribution behaves as $(r/a - 1)^{2-D}$. The corresponding coefficient depends on $A'_2(a)$ and $u''(a)$.

The analysis given above shows that, in the models of the core for which the core-induced contribution in the current density diverges on the boundary, the latter dominates for points near the boundary, whereas at large distances the contribution of the part $\langle j_\phi \rangle_0$ dominates. Depending on the core model, these two contributions may have different signs, and as a result the total VEV is not a monotonic function of the radial coordinate and changes the sign for some value of the latter. An example of this type of behavior will be given below in the flower-pot model for the core geometry.

It is of interest to investigate the asymptotic behavior of the core-induced part in the current density for large values of the magnetic flux. To this aim the form of the current density (4.9) is more convenient. From Eq. (B6) it follows that for a fixed value of x and for large $|n|$ one has $p_n(ix) \approx q|n|$. Hence, in models where $p_n(ix)$ does not depend on β (for an example, see below), for fixed x and n, to the leading order, we get $p_{n-n_0}(ix) \approx q|n_0|$. By taking into account that the dominant contribution to the series and integral in Eq. (4.9) comes from the regions $|n| \lesssim 1/(r/a - 1)$ and $x \lesssim 1/(r - a)$, we see that the latter estimate is valid for $|n_0| \gg 1/(r/a - 1)$. Under this condition, in Eq. (4.9) we can use the following replacement:

$$\frac{I_{\beta_{fn}}(ax, p_{n-n_0}(ix))}{K_{\beta_{fn}}(ax, p_{n-n_0}(ix))} \approx \frac{I_{\beta_{fn}}(ax)}{K_{\beta_{fn}}(ax)}. \tag{4.17}$$

In this case, for large values of the magnetic flux, the core-induced contribution in the current density coincides, to the leading order, with the corresponding result for a cylindrical shell with Dirichlet boundary condition (see below). In particular, it is a periodic function of the magnetic flux. In models where the function $p_n(ix)$ depends on β, an additional dependence on n_0 appears in the function $p_{n-n_0}(ix)$. As a result of this, a leading term proportional to $|n_0|$ may vanish (an example of this type of situation will be given below). The next-to-the-leading term does not depend on n_0, and, again, the corresponding contribution to the current density is periodic in the magnetic flux.

Now, we consider the core-induced contribution in the current density in the limit $a \to 0$ for a fixed value of β. The latter means that the magnetic flux inside the core is fixed.

The dominant contribution to Eq. (4.9) comes from the term with $n = 0$ in the case $0 < \beta_f < 1/2$ and from the term $n = -1$ in the case $1/2 < \beta_f < 1$. To the leading order, we find

$$\langle j_\phi \rangle_c \approx 16\mathrm{sgn}(1/2 - \beta_f) e q^2 \frac{(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2) r^D} \frac{(a/2r)^{2\sigma_\beta}}{\Gamma^2(\sigma_\beta)} \times \int_{mr}^\infty dx x^{2\sigma_\beta+1} (x^2 - m^2 r^2)^{(D-3)/2} \times \frac{\sigma_\beta - p^{(0)}_{n-n_0}(ix/r)}{\sigma_\beta + p^{(0)}_{n-n_0}(ix/r)} K^2_{\sigma_\beta}(r), \tag{4.18}$$

where $p^{(0)}_{n-n_0}(ix) = \lim_{a\to 0} p_{n-n_0}(ix)$ and σ_β is defined by Eq. (4.12). In models where the limiting value $p^{(0)}_{n-n_0}(ix) = p^{(0)}_{n-n_0}$ does not depend on x, for a massless field, the leading term is given by Eqs. (4.11) and (4.13). Note that for integer values of β the term with $\beta_n = 0$ does not contribute to the core-induced contribution in the current, and, hence, the long-range effects of the core, discussed in Ref. [21] for the VEV of the field squared, are absent for the current density.

The field strength for the electromagnetic field, F_{ik}, generated by the vacuum current density is found from the semiclassical Maxwell equation $\nabla_k F^{ik} = -4\pi \langle j^i \rangle$. In the problem under consideration, the only nonzero components correspond to $F^{12} = -F^{21}$. After the simple integration, from the Maxwell equation for the contravariant component, we get

$$F^{21} = \frac{4\pi}{r} \int_r^\infty dr \langle j_\phi \rangle. \tag{4.19}$$

In the model with $D = 3$, the corresponding magnetic field $\mathbf{B}$ is directed along the string axis, and $B_z = rF^{21}$. Similar to the current density, the field strength is decomposed into the zero-thickness cosmic string and core-induced contributions:

$$F^{21} = F^{21}_{(0)} + F^{21}_{(c)}. \tag{4.20}$$

By using the integration formula $\int_r^\infty dr r f_\nu(br) = b^{-2} f_{\nu-1}(br)$, for the idealized cosmic string part, we get

$$F^{21}_{(0)} = \frac{4em^{D-1}}{(2\pi)^{(D-1)/2} r} \left\{ \sum_{n=1}^{[q/2]} \sin(2\pi n\beta) \cot(\pi n/q) f_{(D-1)/2}(2mr \sin(\pi n/q)) + \frac{q}{\pi} \int_0^\infty dy \frac{\tanh y f(q, \beta_f, 2y)}{\cosh(2qy) - \cos(\pi q)} f_{(D-1)/2}(2mr \cosh y) \right\}. \tag{4.21}$$

In the case $1 \le q < 2$, the corresponding magnetic field has been investigated in Ref. [14].

For the core-induced contribution to the field strength, by using Eq. (4.9), one obtains

$$F^{21}_{(c)} = -4eq \frac{(4\pi)^{-(D-1)/2}}{\Gamma((D-1)/2)} \sum_{n=-\infty}^{+\infty} \mathrm{sgn}(n + \beta_f) \int_m^\infty dx x^2 \times (x^2 - m^2)^{(D-3)/2} \frac{I_{\beta_{fn}}(ax, p_{n-n_0}(ix))}{K_{\beta_{fn}}(ax, p_{n-n_0}(ix))} G_{\beta_{fn}}(xr), \tag{4.22}$$

where we have introduced the function

$$G_\nu(y) = K'_\nu(y) \partial_\nu K_\nu(y) - K_\nu(y) \partial_\nu K'_\nu(y). \tag{4.23}$$

In deriving Eq. (4.22), we have used the formula from Ref. [23] for an indefinite integral involving the product of the MacDonald functions and applied the l'Hôpital's rule.

V. EXAMPLES OF THE CORE MODEL

In the discussion above, we have considered a general model specified by the functions $u(r)$ and $A_2(r)$ in the region $r < a$. In this section we discuss exactly solvable examples of these functions. Note that, in general, the radii of the string core and of the gauge field flux can be different. For example, in the Abelian Higgs model, these radii are determined by the Higgs and vector particle masses (see, for instance, Ref. [1]). Though our analysis described above is valid for the general case, in the examples below, for simplicity we assume that the radii of the flux tube and string core coincide. In particular, the latter is the case for Bogomol'nyi-Prasad-Sommerfeld cosmic strings. The VEVs of the field squared and energy-momentum tensor for a charged scalar field in the geometry of an idealized cosmic string with a zero-thickness core and in the presence of a finite radius magnetic flux have been considered in Ref. [7].

A. Cylindrical shell with Robin boundary condition

First we consider a model with an impenetrable core, on the boundary of which the field obeys the Robin boundary condition,

$$(\partial_r - \sigma)\varphi = 0, \qquad r = a, \tag{5.1}$$

with a constant parameter σ. In this case the VEVs in the exterior region do not depend on the interior geometry and are completely determined by the total flux, enclosed by the boundary, through the parameter β and by the Robin coefficient σ. For a neutral scalar field, the corresponding VEVs of the field squared and the energy-momentum tensor inside and outside a cylindrical shell with Robin boundary condition have been investigated in Ref. [22] (for boundary-induced quantum vacuum effects in the geometry of a cosmic string, see Ref. [26]). For a charged field and in the presence of a magnetic flux, the mode functions in the exterior region have the form (2.12) with the radial function $f(r)$ being the linear combination of the functions $J_{\beta_n}(\gamma r)$ and $Y_{\beta_n}(\gamma r)$. The relative coefficient is determined from the boundary condition (5.1), and for the mode functions, one has

$$\varphi_\alpha^{(\pm)}(x) = \beta_\alpha g_{\beta_n}(\gamma r, \gamma a, \sigma a) e^{iqn\phi + i\mathbf{kz} \mp i\omega t}, \qquad (5.2)$$

where the function $g_{\beta_n}(z, x, y)$ is defined in Eq. (2.27).

The normalization coefficient is determined from the condition (2.23) where now the integration goes over the exterior region. This gives

$$\beta_\alpha^2 = \frac{q(2\pi)^{1-D}\gamma/(2\omega)}{J_{\beta_n}^2(\gamma a, \sigma a) + Y_{\beta_n}^2(\gamma a, \sigma a)}. \qquad (5.3)$$

We see that for the Robin shell the mode functions in the exterior region are obtained from the corresponding functions for the general model of the core taking

$$p_n(\gamma) = \sigma a. \qquad (5.4)$$

Note that, similar to the case of the zero-thickness cosmic string, here the current density does not depend on the curvature coupling parameter.

The expressions for the Hadamard function and for the VEV of the current density are obtained from the formulas given above by the substitution (5.4). Now, the function $p_n(ix)$ in Eq. (4.8) does not depend on n and β. From here it follows that the core-induced contribution and the total VEV of the current density are periodic functions of the magnetic flux with the period equal to the quantum flux. The equation (2.30) for the bound states is reduced to $yK'_{\beta_n}(y)/K_{\beta_n}(y) = \sigma a$ with $y = \chi a$. By taking into account that $yK'_{\beta_n}(y)/K_{\beta_n}(y) < -\beta_n$, we conclude that there are no bound states for $\sigma a \geq -q \min(\beta_f, 1 - \beta_f)$. For points near the boundary, the leading term in the asymptotic expansion for the current density is obtained in a way similar to that we have described above in the general case. By using the uniform asymptotic expansion of the modified Bessel functions [25], we can see that the coefficient of the leading term vanishes.

Figure 2 displays the dependence of the total current density on the radial coordinate for $D = 3$ massless scalar field with Dirichlet, Neumann, and Robin boundary conditions. For the Robin boundary condition, we have taken $\sigma a = 1$, and in all cases the graphs are plotted for fixed values $\beta = 0.2$, $q = 1.5$. The dashed line presents the current density for a zero-thickness cosmic string, namely, $10^3 a^3 \langle j_\phi \rangle_0 / e$. For the corresponding value on the boundary, one has $\langle j_\phi \rangle_0 \approx 2.7 \times 10^{-3} e/a^3$. Numerical results show that the subleading term in the expansion of the core-induced current density over the distance from the boundary vanishes as well and the current is finite on the boundary.

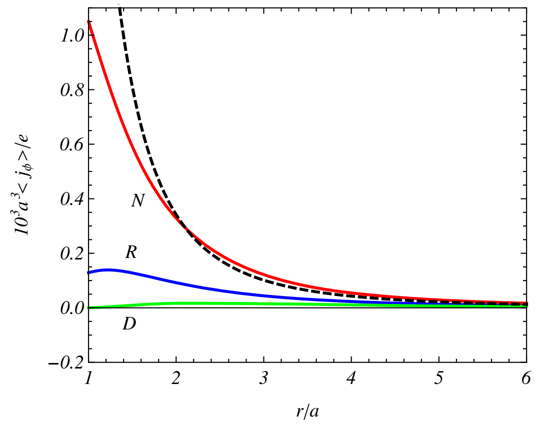

FIG. 2 (color online). The current density for a $D = 3$ massless scalar field with Dirichlet, Neumann, and Robin boundary conditions on the core boundary, as a function of r/a, for fixed values $\beta = 0.2$ and $q = 1.5$. For the Robin boundary condition, $\sigma a = 1$. The dashed line presents the current density in the geometry of a zero-thickness cosmic string.

In Fig. 3, for $D = 3$ massless scalar field and for $r/a = 2$, we have plotted the core-induced contribution to the VEV of the current density as a function of β for different values of the parameter q (numbers near the curves). The left and right panels correspond to Dirichlet and Neumann boundary conditions on the core boundary.

B. Flower-pot model

In the flower-pot model, discussed in Ref. [21], the interior geometry is Minkowskian. Introducing a new angular coordinate ϕ, $0 \leq \phi \leq 2\pi/q$, related to the Minkowskian one by $\phi_M = q\phi$, the interior line element is written as

$$ds_M^2 = dt^2 - dr_M^2 - q^2 r_M^2 d\phi^2 - d\mathbf{z}^2, \qquad (5.5)$$

with $0 \leq r_M \leq r_{Ma}$, where r_{Ma} is the corresponding value at the core boundary (see below). From the continuity of the radial component of the metric tensor at the boundary of the core, we have $r_M = r + \text{const}$. From the continuity of the g_{22} component, one gets $\text{const} = a(1/q - 1)$. Hence, for the function $u(r)$ in Eq. (2.2), we obtain

$$r_M = r - r_c, \qquad u(r) = q(r - r_c), \qquad r_c = a(1 - 1/q), \qquad (5.6)$$

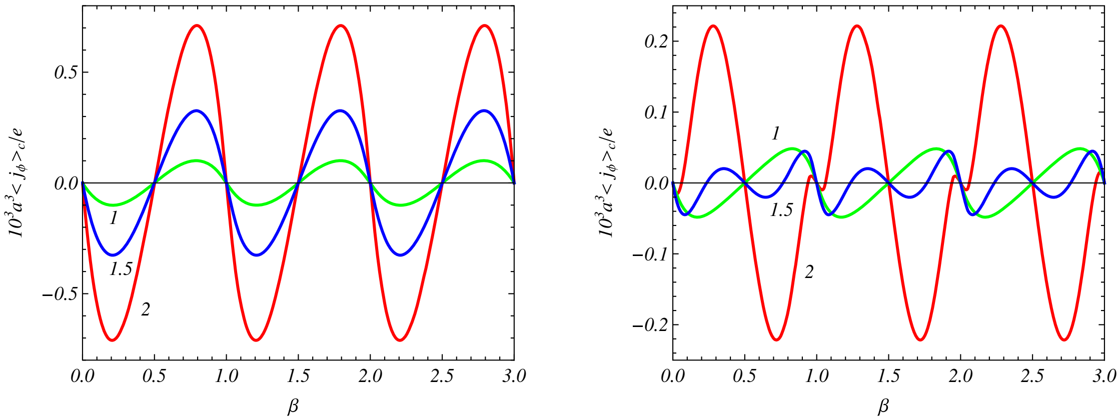

FIG. 3 (color online). The core-induced contribution in the current density for a $D = 3$ massless scalar field vs the parameter β. The left and right panels correspond to the model of impenetrable core with Dirichlet and Neumann boundary conditions, respectively. The numbers near the curves present the corresponding values of q, and the graphs are plotted for $r/a = 2$.

with $r \geq r_c$. Note that $r_{Ma} = r_M|_{r=a} = a/q$; i.e., in terms of the radial coordinate r_M, the radius of the core is equal to a/q. For the nonzero components of the surface energy-momentum tensor, one has

$$8\pi G \tau_i^k = \frac{q-1}{a}\delta_i^k, \qquad i,k = 0,3,\ldots,D, \tag{5.7}$$

and the corresponding energy density is positive for $q > 1$. We should also specify the function $A_2(r)$ inside the core. Here we consider examples where the equation for the radial parts of the mode functions is exactly integrable.

First we consider the gauge field configuration corresponding to Dirac delta function-type field strength located on the boundary of the core. The Aharonov–Bohm scattering of particles on this type of flux distribution has been discussed in Refs. [27–29]. The corresponding vector potential is given by the expression

$$A_2(r) = A_2\theta(r-a). \tag{5.8}$$

In $D = 3$ models, this corresponds to the magnetic field strength $B_z = -A_2\delta(r-a)/a$. The regular solution in the interior region is given by

$$R_n(r,\gamma) = CJ_n(\gamma r_M), \tag{5.9}$$

and the coefficient C is determined from the relation (2.25). For the function in the expression (4.8) of the current density, one gets

$$p_n(ix) = ax\frac{I_n'(xa/q)}{I_n(xa/q)} + 2\xi(q-1). \tag{5.10}$$

This function does not depend on the parameter β. In accordance with the asymptotic analysis presented in the previous section, for large values of the magnetic flux, $|n_0| \gg 1/(r/a-1)$, the leading term in the expansion of the core-induced VEV coincides with that for a cylindrical shell with Dirichlet boundary condition. This term is periodic in the magnetic flux with the period equal to the quantum flux. From the relation $yI_n'(y)/I_n(y) \geq |n|$, it follows that the bound states are absent if the condition $2\xi(1-1/q) \geq -\beta_f$ is satisfied. In particular, this is the case for minimally and conformally coupled fields.

The second example we want to consider corresponds to a homogeneous magnetic field inside the core with the vector potential

$$A_2(r) = -qA^{(2)}r_M^2. \tag{5.11}$$

This model of the magnetic flux tube in the context of the Aharonov–Bohm effect has been considered in Refs. [28,30]. From the continuity at the core boundary, one gets $A^{(2)} = -qA_2/a^2$ or, in terms of the parameter β, $eA^{(2)} = -\beta q^2/a^2$. In the Minkowskian coordinates $(t, r_M, \phi_M, \mathbf{z})$, the nonzero component of the vector potential is given by $A_2(r)/q$. In the $D = 3$ model, the strength of the corresponding magnetic field is expressed as $B_z = -2\beta q^2/(ea^2)$. For the regular solution inside the core, one has

$$R_n(r,\gamma) = \frac{C_1 r_M^{|n|}}{e^{|\beta|(r_M q/a)^2/2}} M\left(\frac{1+|n|}{2} - \frac{(\gamma a/q)^2 - 2n\beta}{4|\beta|},\right.$$
$$\left. 1+|n|, |\beta|(r_M q/a)^2\right), \tag{5.12}$$

where $M(a,b,x) = {}_1F_1(a;b;x)$ is the confluent hypergeometric function [25] and C_1 is determined from Eq. (2.25). For the function $p(ix)$ in Eq. (4.8), this gives

$$p_n(ix) = q|n| - q|\beta| + 2\xi(q-1) + q|\beta|\frac{1+|n|+2\kappa(x)}{1+|n|}$$
$$\times \frac{M((3+|n|)/2+\kappa(x), 2+|n|, |\beta|)}{M((1+|n|)/2+\kappa(x), 1+|n|, |\beta|)}, \tag{5.13}$$

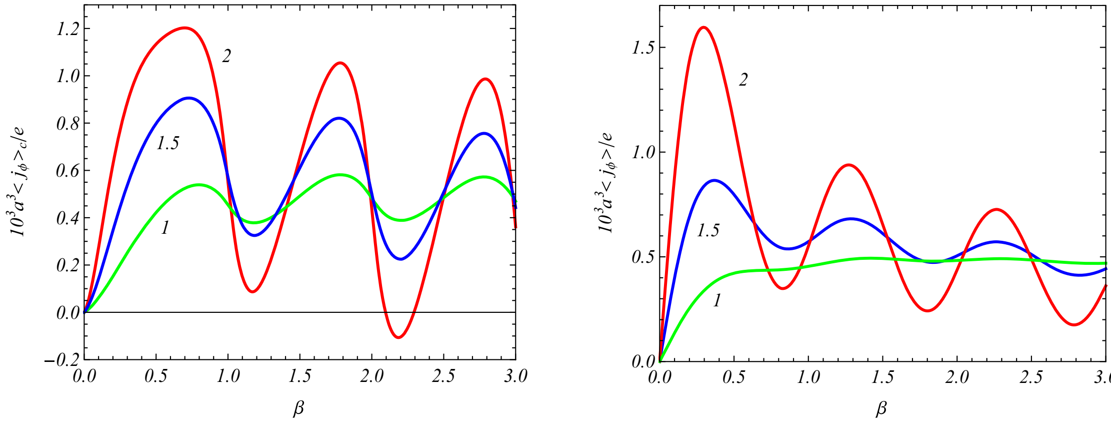

FIG. 4 (color online). The core-induced contribution (left panel) and the total VEV (right panel) of the current density for a $D = 3$ minimally coupled massless scalar field in the flower-pot model with an interior homogeneous magnetic field, as a function of the parameter β. The numbers near the curves are the corresponding values of q, and the graphs are plotted for $r/a = 2$.

where we have defined

$$\kappa(x) = \frac{(ax/q)^2 + 2n\beta}{4|\beta|}. \tag{5.14}$$

For large values of the magnetic flux, $|n_0| \gg 1/(r/a - 1)$, the leading term in the expansion of Eq. (5.13), which is proportional to $|n_0|$, vanishes. The next-to-the-leading term does not depend on n_0, and, to the leading order, the current density is a periodic function of the magnetic flux.

In Fig. 4, the core-induced part (left panel) and the total VEV of the current density (right panel) are depicted as functions of β for $D = 3$ minimally coupled massless scalar field. The graphs are plotted for fixed value $r/a = 2$ in the flower-pot model with the vector potential distribution given by Eq. (5.11), and the numbers near the curves are the corresponding values of the parameter q.

For the model (5.11) and for $D = 3$, we have also numerically investigated the behavior of the current density as a function of the radial distance for $q = 1.5$ and for two values of the magnetic flux corresponding to $\beta = 0.25$ and $\beta = 1.75$. For both these values, the ratio $\langle j_\phi \rangle_c/e$ is positive and monotonically decreasing with increasing r. It diverges on the boundary inversely proportional to the distance from the boundary, which is in agreement with the general analysis of the previous section. The zero-thickness cosmic string part, $\langle j_\phi \rangle_0/e$, is positive for $\beta = 0.25$ and negative for $\beta = 1.75$, being monotonic in both cases. As a result, in the case $\beta = 0.25$, the ratio $\langle j_\phi \rangle/e$ for the total current is a positive monotonically decreasing function of r. For $\beta = 1.75$ the total VEV is positive near the boundary and negative at large distances. It vanishes at $r/a \approx 3.15$ and takes its minimum value at $r/a \approx 4$.

The third exactly solvable model corresponds to the vector potential

$$A_2(r) = -qA_\phi r_M, \qquad r < a, \tag{5.15}$$

where $A_\phi = \text{const}$ is the corresponding physical component. For the $D = 3$ model and in the interior Minkowski coordinates (t, r_M, ϕ_M, z), the nonzero component of the magnetic field is given by $B_z = A_\phi/r_M$. A charged particle with magnetic moment in this type of magnetic field has been considered in Ref. [28]. From the continuity condition for the vector potential at the core boundary, for the covariant component in the exterior region, we get $A_2 = -aA_\phi$. In terms of the parameter β, one has $eA_\phi = -q\beta/a$. With the function (5.15), the regular solution of the radial equation (2.14) is expressed as

$$R_n(r,\gamma) = C_2 r_M^{|n|} e^{-hr_M/a} M(|n| + nq\beta/h + 1/2, 2|n| + 1, 2hr_M/a), \tag{5.16}$$

where $h = (q^2\beta^2 - a^2\gamma^2)^{1/2}$ and the coefficient C_2 is defined from Eq. (2.25). For the function in the expression (4.8) of the core-induced part of the current density, we get

$$p_n(ix) = q|n| + 2\xi(q-1) - q\lambda(x) + q\left[\lambda(x) + \frac{n\beta}{|n| + 1/2}\right] \times \frac{M(|n| + n\beta/\lambda(x) + 3/2, 2|n| + 2, 2\lambda(x))}{M(|n| + n\beta/\lambda(x) + 1/2, 2|n| + 1, 2\lambda(x))}, \tag{5.17}$$

with the notation

$$\lambda(x) = \sqrt{(ax/q)^2 + \beta^2}. \tag{5.18}$$

Similarly to the previous case, for large magnetic fluxes, $|n_0| \gg 1/(r/a - 1)$, the leading term in the expansion of Eq. (5.17) vanishes, and the next-to-the-leading term does not depend on n_0. As a result, to the leading order, the current density is a periodic function of the magnetic flux.

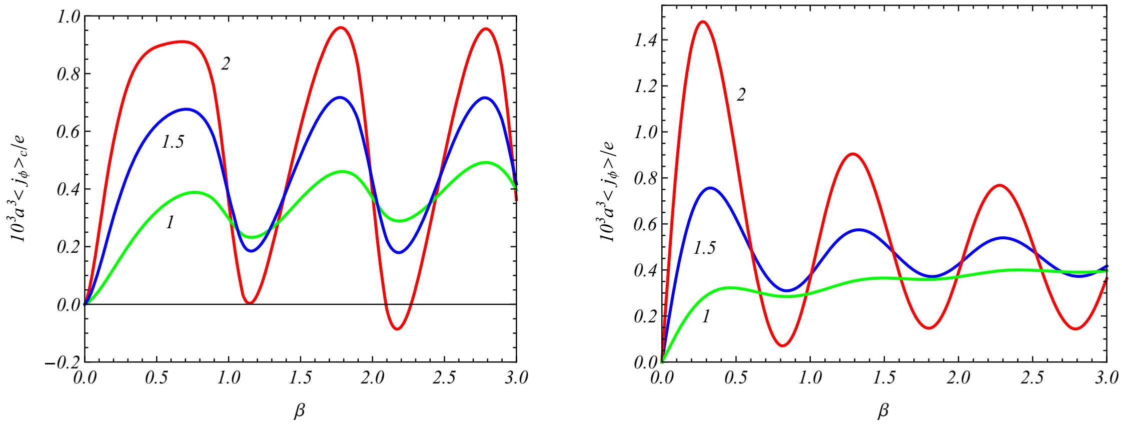

FIG. 5 (color online). The same as in Fig. 4 for the flower-pot model with the gauge field distribution given by Eq. (5.15).

In Fig. 5, the core-induced part (left panel) and the total VEV (right panel) of the current density are depicted as functions of β for the $D = 3$ minimally coupled massless scalar field. The graphs are plotted for fixed value $r/a = 2$ in the flower-pot model with the gauge field distribution given by Eq. (5.15). As before, the numbers near the curves are the corresponding values of the parameter q.

C. Ballpoint-pen model

For this model the interior geometry is described by a positive constant curvature space. In this case we have

$$u(r) = qb\sin(r_i/b), \tag{5.19}$$

where r_i is the interior radial coordinate. From the continuity of the metric tensor component g_{11}, we find $r_i = r - r_c$. From the condition (2.4) one gets two possible values of the constant $r_c = r_{cj}$, $j = 1, 2$, which are

$$\begin{aligned} r_{c1} &= a - b\arcsin[a/(qb)], \\ r_{c2} &= a - b[\pi - \arcsin(a/(qb))]. \end{aligned} \tag{5.20}$$

For the corresponding surface energy-momentum tensor, we find

$$8\pi G\tau_i^k = \frac{\pm\sqrt{q^2 - a^2/b^2} - 1}{a}\delta_i^k, \qquad i, k = 0, 3, \ldots, D, \tag{5.21}$$

where the upper and lower signs correspond to r_{c1} and r_{c2}, respectively. For the upper sign and in the special case

$$a/b = \sqrt{q^2 - 1}, \tag{5.22}$$

the surface energy-momentum tensor vanishes. In this case the first derivatives of the metric tensor are continuous as well. The corresponding model has been discussed in Ref. [20] (see also Refs. [21,31]). Note that, in the case $D = 2$, the geometry under consideration models a graphitic cone with a spherical cup, long wavelength properties of which are well described by a $(2+1)$-dimensional field-theoretical model. The corresponding opening angle is given by $\phi_0 = 2\pi(1 - N_c/6)$, where $N_c = 1, 2, \ldots, 5$ is the number of sectors removed from the planar graphene sheet. All these angles have been experimentally observed [32].

To have an exactly solvable problem, for the gauge field configuration in the interior region, we shall take the model given by Eq. (5.8). The regular solution of the radial equation (2.14) in the interior region is expressed in terms of the associated Legendre function

$$R_n(r,\gamma) = CP^{|n|}_{\nu(\gamma)-1/2}(\cos(r_i/b)), \tag{5.23}$$

where

$$\nu(\gamma) = \sqrt{b^2\gamma^2 - 2\xi + 1/4}. \tag{5.24}$$

For the function $p_n(ix)$ in (4.8), this gives

$$p_n(ix) = 2\xi\left(\pm\sqrt{q^2 - a^2/b^2} - 1\right) - \frac{a^2}{qb^2}\frac{P^{|n|\prime}_{\nu(ix)-1/2}(\pm\sqrt{1 - a^2/q^2b^2})}{P^{|n|}_{\nu(ix)-1/2}(\pm\sqrt{1 - a^2/q^2b^2})}, \tag{5.25}$$

where the upper and lower signs correspond to the models with $r_c = r_{c1}$ and $r_c = r_{c2}$, respectively. If the derivative of the metric tensor is continuous as well, then the expression is simplified to

$$p_n(ix) = -\frac{q^2 - 1}{q}\frac{P^{|n|\prime}_{\nu(ix)-1/2}(1/q)}{P^{|n|}_{\nu(ix)-1/2}(1/q)}. \tag{5.26}$$

Note that, though the function $\nu(ix)$ can be either real or purely imaginary, the function $P^{|n|}_{\nu(ix)-1/2}(1/q)$ is always real because of the property $P^{|n|}_{\nu(ix)-1/2}(1/q) = P^{|n|}_{-\nu(ix)-1/2}(1/q)$ for the associated Legendre function.

VI. CONCLUSION

In the present paper, we have investigated the finite core effects on the vacuum current, induced by magnetic fluxes, for a massive scalar field in the geometry of a straight cosmic string. For the interior structure of the core, we have considered a general static cylindrically symmetric geometry with an arbitrary distributed gauge field flux along the string axis. For generality we have also assumed the presence of the surface energy-momentum tensor on the core boundary. In the region outside the core, the geometry is described by the standard conical line element with a planar angle deficit, and for the gauge field configuration, we have taken the vector potential given by Eq. (2.13) with $A_2(r) = \text{const}$. Though the corresponding field strength vanishes, the magnetic field inside the core induces a nonzero VEV for the azimuthal current in the exterior region. This current is a consequence of two types of effects: the Aharonov–Bohm-like effect and the direct interaction of the quantum field with the magnetic field inside the core. In models with an impenetrable core, the first effect is present only, and the induced current is a periodic function of the magnetic flux inside the core with the period equal to the quantum flux. For penetrable cores both types of effects contribute, and the current density, in general, is not a periodic function of the magnetic flux.

In the region outside the core, all the properties of the quantum vacuum can be deduced from the corresponding two-point function. As such we have considered the Hadamard function. In particular, the VEV of the current density is obtained by using Eq. (2.10). For the evaluation of the Hadamard function, we have used the direct summation over a complete set of modes for a scalar field. In cylindrical coordinates and for scattering states, the radial parts in the mode functions are given by Eq. (2.18), where the coefficients are determined from the matching conditions at the core boundary. The normalization condition determines the value of the interior radial function at the boundary [see Eq. (2.25)]. In addition to the scattering modes, there can be also bound states for which the radial function at large distances from the core decreases exponentially. The possible bound states are determined by Eq. (2.30), and the normalization of the corresponding interior function is given by Eq. (2.34). We have explicitly decomposed the Hadamard function into two contributions. The first one, given by Eq. (3.4), corresponds to the geometry of a zero-thickness cosmic string and magnetic flux, and the second one is induced by the nontrivial core structure. For the first contribution, a closed expression (3.8) is provided which is convenient for the investigation of the expectation values of various local characteristics of the vacuum state (field squared, energy-momentum tensor, current density). This expression generalizes various special cases previously discussed in the literature. The core-induced effects are encoded in the part of the Hadamard function given by Eq. (3.5). It contains both the contributions from the scattering and bound states. The specific properties of the core appear through the function $p_n(iz)$.

The VEV of the current density is decomposed into the zero-thickness cosmic string part, Eq. (4.4), and the contribution coming from the finite core, given by Eq. (4.8). The only nonzero component corresponds to the azimuthal current. The zero-thickness part in the current density is a periodic function of the magnetic flux inside the core with the period equal to the quantum flux. For penetrable cores, the core-induced contribution, in general, is not periodic. The physical reason for this is the direct interaction of the quantum field with the magnetic field inside the penetrable core. For a massive field, at large distances from the string, $mr \gg 1$, both parts in the current density are exponentially small. For $q > 2$ the idealized string part dominates at large distances, and the relative contribution of the finite core effects are suppressed by the factor $e^{-2mr[1-\sin(\pi/q)]}/(mr)^{3/2}$. In the case $q \leq 2$ and for a massive field, the contribution of the finite core effects is of the same order as that coming from $\langle j_\phi \rangle_0$ if $am \sim 1$. For a massless field, the part corresponding to the geometry of the zero-thickness cosmic string behaves as $1/r^D$, and the relative contribution of the finite core effects at large distances is suppressed by the factor $(a/r)^{2\sigma_f}$, where σ_f is defined by Eq. (4.12). On the core boundary, the VEV of the current density, in general, diverges. To find the leading term in the asymptotic expansion of the current density over the distance from the core boundary, we have provided the leading terms in the uniform asymptotic expansion of the function $p_n(ix)$, for large values of $|n|$. The coefficients in the asymptotic expansion are found by using Eq. (B2), which is obtained from the equation for the interior radial function. If the vector potential has discontinuity at the boundary, then the leading term in the expansion of the current density near the boundary is given by Eq. (4.16), and the current density diverges as $1/(r-a)^{D-1}$. If the vector potential is continuous at the boundary, the leading term vanishes, and the asymptotic expansion starts with the term of the order $1/(r-a)^{D-2}$. In special cases of the interior geometry, the coefficient of the latter may vanish as well. The part $\langle j_\phi \rangle_0$ is finite on the boundary, and, hence, near the boundary the current density, in general, is dominated by the core-induced contribution. In models where the function $p_n(ix)$ does not depend on β, for large values of the magnetic flux, the core-induced contribution in the current density, to the leading order, coincides with the corresponding result for an impenetrable cylindrical shell with Dirichlet boundary condition, and the leading term is periodic with the period equal to the quantum flux. The VEV of the total current density, in general, is not a monotonic function of the distance from the core boundary. By using the Maxwell semiclassical equation, we have also investigated the magnetic field generated by the vacuum currents. In particular, the core-induced contribution in the field strength is given by Eq. (4.22).

As applications of the general results, we have considered four examples of the core structure and the distribution of the gauge field. In the first example, the core is impenetrable for the quantum field under consideration and is modeled by a cylindrical surface on which the field operator obeys the Robin boundary condition. The core-induced contribution to the vacuum current density is obtained from the general expression (4.8) with the function $p_n(ix) = \sigma a$. In this case the current density is a periodic function of the magnetic flux with the period equal to the quantum flux. For the second example, we have considered the interior geometry being Minkowskian (flower-pot model). The corresponding surface energy-momentum tensor is obtained from the matching condition at the core boundary and is given by Eq. (5.7). For the gauge field distribution, we have discussed three exactly integrable models. For $D = 3$ they correspond to the Dirac delta function-type magnetic field located on a cylindrical shell, to a homogeneous magnetic field inside the core, and to a magnetic field proportional to $1/r$. The corresponding functions are given by Eqs. (5.10), (5.13), and (5.17), respectively. In these examples, as a consequence of the direct interaction of the quantum field with the interior magnetic field, the current density is not a periodic function of the magnetic flux. However, for large fluxes the leading terms in the corresponding asymptotic expansions are periodic with period equal to the quantum flux. As a model for the interior geometry, we have also considered the ballpoint-pen model with a constant curvature space. In this case two possible matching conditions are obtained with the surface energy-momentum tensors given by Eq. (5.21). In the special case (5.22), the derivatives of the metric tensor are continuous at the core boundary, and the surface energy-momentum tensor vanishes. In the ballpoint-pen model, in order to have an exactly solvable problem, we have considered the Dirac delta-type distribution of the magnetic field on the core boundary. The corresponding function in the expression for the core-induced contribution to the current density is given by Eq. (5.25).

ACKNOWLEDGMENTS

The authors thank Conselho Nacional de Desenvolvimento Científico e Tecnológico for the financial support. A. A. S. was supported by the State Committee of Science of the Ministry of Education and Science RA, within the frame of Grant No. SCS 13-1C040.

APPENDIX A: INTEGRAL REPRESENTATIONS

For the transformation of the Hadamard function (3.4) in the geometry of a zero-thickness cosmic string, we use the integral representation [33]

$$\frac{\cos(\omega\Delta t)}{\omega} = -\frac{1}{2\sqrt{\pi}}\int_C \frac{ds}{s^{1/2}} e^{-\omega^2 s + (\Delta t)^2/(4s)}, \tag{A1}$$

with the contour of the integration depicted in Fig. 6. The integral can also be presented in the form $\int_C ds = \int_{c_\rho} ds - 2\int_\rho^\infty ds$, where c_ρ is a circle of radius ρ with the center at the origin of the complex s-plane and having the counterclockwise direction.

After the substitution of Eq. (A1) into Eq. (3.4), the integrals are evaluated by using the formulas

$$\int \frac{d^N\mathbf{k}}{(2\pi)^N} e^{i\mathbf{k}\cdot\Delta\mathbf{z} - sk^2} = \frac{e^{-|\Delta\mathbf{z}|^2/(4s)}}{(4\pi)^{N/2} s^{N/2}} \tag{A2}$$

and [23]

$$\int_0^\infty d\gamma\gamma J_{\beta_n}(\gamma r) J_{\beta_n}(\gamma r') e^{-s\gamma^2} = \frac{1}{2s}\exp\left(-\frac{r^2 + r'^2}{4s}\right) I_{\beta_n}(rr'/2s). \tag{A3}$$

In the remaining integral over the contour C, under the condition $|\Delta\mathbf{z}|^2 + r^2 + r'^2 > (\Delta t)^2$, the part corresponding to the integral over the circle c_ρ vanishes in the limit $\rho \to 0$. Taking this limit and introducing a new integration variable $x = rr'/s$, one gets Eq. (3.6).

Next we derive an integral representation for the function $\mathcal{I}_q(\beta, \Delta\phi, z)$ defined by Eq. (3.7). By taking into account Eq. (4.3), we obtain

$$\mathcal{I}_q(\beta, \Delta\phi, x) = e^{-iqn_0\Delta\phi} \mathcal{I}_q(\beta_f, \Delta\phi, x). \tag{A4}$$

From here it follows that, without loss of generality, in the evaluation below, we can assume that $0 \le \beta < 1$. We use the integral representation for the modified Bessel function [25]:

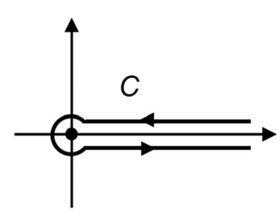

FIG. 6. The contour of the integration in the representation (A1).

$$I_{\beta_n}(z) = \frac{1}{\pi}\int_0^{\pi} dy \cos(\beta_n y) e^{z\cos y} - \frac{\sin(\pi\beta_n)}{\pi}\int_0^{\infty} dy e^{-z\cosh y - \beta_n y}. \tag{A5}$$

Substituting this into the right-hand side of Eq. (3.7), for the part with the first integral in Eq. (A5), we use the formula

$$\sum_{n=-\infty}^{+\infty} e^{ibn} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(b - 2\pi n). \tag{A6}$$

This gives

$$\frac{1}{\pi}\int_0^{\pi} dy e^{z\cos y} \sum_{n=-\infty}^{+\infty} e^{iqn\Delta\phi}\cos(\beta_n y) = \frac{1}{q}\sum_n e^{z\cos(2\pi n/q - \Delta\phi)} e^{iq\beta(2\pi n/q - \Delta\phi)}, \tag{A7}$$

where in the right-hand side the summation goes under the condition

$$-q/2 + \Delta\phi/\phi_0 \leq n \leq q/2 + \Delta\phi/\phi_0. \tag{A8}$$

If $-q/2 + \Delta\phi/\phi_0$ or $q/2 + \Delta\phi/\phi_0$ are integers, then the corresponding terms in the right-hand side of Eq. (A7) should be taken with the coefficient $1/2$.

In the part with the second integral in the right-hand side of Eq. (A5), we use the formula

$$\sum_{n=-\infty}^{+\infty} e^{iqn\Delta\phi - \beta_n y}\sin(\pi\beta_n) = \frac{1}{2i}\sum_{j=+,-} j e^{j\pi iq\beta} \frac{\cosh[qy(1-\beta)] - \cosh(q\beta y) e^{-iq(\Delta\phi + j\pi)}}{\cosh(qy) - \cos(q(\Delta\phi + j\pi))}. \tag{A9}$$

Combining Eqs. (A4), (A7), and (A9), for the general case of β, we obtain the following integral representation:

$$\mathcal{I}_q(\beta, \Delta\phi, z) = \frac{1}{q}\sum_n e^{z\cos(2\pi n/q - \Delta\phi)} e^{i\beta(2\pi n - q\Delta\phi)} - \frac{e^{-iqn_0\Delta\phi}}{2\pi i}\sum_{j=+,-} j e^{ji\pi q\beta_f}\int_0^{\infty} dy \frac{\cosh[qy(1-\beta_f)] - \cosh(q\beta_f y) e^{-iq(\Delta\phi + j\pi)}}{e^{z\cosh y}[\cosh(qy) - \cos(q(\Delta\phi + j\pi))]}. \tag{A10}$$

For the special case $\beta = 0$, this formula reduced to the one given in Ref. [34].

Taking $\Delta\phi = 0$, from (A10) one gets

$$\sum_{n=-\infty}^{+\infty} I_{\beta_n}(z) = \frac{2}{q}\sum_{n=0}^{[q/2]}{}' \cos(2\pi n\beta) e^{z\cos(2\pi n/q)} - \frac{1}{\pi}\int_0^{\infty} dy \frac{\sin(\pi q\beta_f)\cosh[qy(1-\beta_f)] + \sin(\pi q(1-\beta_f))\cosh(q\beta_f y)}{e^{z\cosh y}[\cosh(qy) - \cos(\pi q)]}, \tag{A11}$$

where the prime on the sign of the sum means that the terms $n = 0$ and $n = q/2$ (if q is an even integer) should be taken with the coefficient $1/2$.

A similar formula for the series in Eq. (4.2) can be obtained by using Eq. (A11) and the relation

$$q(n+\beta) I_{\beta_n}(z) = -z\partial_z I_{\beta_n}(z) + z I_{q|n+\beta-1/q|}(z), \tag{A12}$$

valid for $n \neq 0$ and $0 < \beta < 1$. For $\beta > 1/q$ this gives

$$q\sum_{n=-\infty}^{+\infty}(n+\beta) I_{\beta_n}(z) = -z\partial_z \sum_{n=-\infty}^{+\infty} I_{\beta_n}(z) + z\sum_{n=-\infty}^{+\infty} I_{q|n+\beta-1/q|}(z). \tag{A13}$$

Applying the formula (A11) [with the replacement $\beta \to \beta - 1/q$ for the last series in (A13)], after some transformations we get

$$\sum_{n=-\infty}^{+\infty}(n+\beta) I_{\beta_n}(z) = \frac{2z}{q^2}\sum_{n=1}^{[q/2]}\sin(2\pi n/q)\sin(2\pi n\beta) e^{z\cos(2\pi n/q)} + \frac{z}{\pi q}\int_0^{\infty} dy \frac{\sinh y e^{-z\cosh y} f(q, \beta_f, y)}{\cosh(qy) - \cos(\pi q)}, \tag{A14}$$

with the function $f(q, \beta_f, y)$ defined by Eq. (4.5). Note that for q being an even integer the contribution of the term $n = q/2$, in Eq. (A14), vanishes.

For $\beta < 1/q$ one has the relation

$$q\sum_{n=-\infty}^{+\infty}(n+\beta)I_{\beta_n}(z) = \frac{2}{\pi}\sin[\pi(1-q\beta)]zK_{1-q\beta}(z) - z\partial_z\sum_{n=-\infty}^{+\infty}I_{\beta_n}(z) + z\sum_{n=-\infty}^{+\infty}I_{q|n+\beta-1/q+1|}(z). \quad \text{(A15)}$$

Again, by using Eq. (A11) and the integral representation

$$K_{1-q\beta}(z) = \int_0^\infty dy e^{-z\cosh y}\cosh[(1-q\beta)y], \quad \text{(A16)}$$

we obtain the same representation (A14). Hence, the formula (A14), with β_f defined by Eq. (4.3), is valid for general values of β.

APPENDIX B: UNIFORM ASYMPTOTIC EXPANSION FOR THE FUNCTION $p_n(i\beta_n y)$

In this section we derive first two terms in the uniform asymptotic expansion for the function $p_n(i\beta_n y)$, defined by Eq. (2.20). They are used in Sec. IV, in order to find the leading term in the asymptotic expansion of the current density over the distance from the core boundary. First we consider the corresponding expansion for the function

$$y_n(r,x) = \frac{R'_n(r,ix)}{R_n(r,ix)}. \quad \text{(B1)}$$

By using Eq. (2.14) for the interior radial function, the following equation is obtained:

$$\frac{1}{u(r)}[u(r)y_n(r,x)]' + y_n^2(r,x) - x^2 - \frac{[qn+eA_2(r)]^2}{u^2(r)} + 2\xi\frac{u''(r)}{u(r)} = 0, \quad \text{(B2)}$$

where the prime stands for the derivative over r. Substituting $x = \beta_n y$, to the leading order over $|n|$, one finds from Eq. (B2) the result

$$y_n(r,\beta_n y) = \pm\beta_n\sqrt{y^2 + 1/u^2(r)}. \quad \text{(B3)}$$

For the regular solution, the upper sign should be taken. Putting Eq. (B3) in the first term of Eq. (B2) (with $x = \beta_n y$), we can find the next-to-the-leading term which is given by

$$ay_n(a,\beta_n y) \approx \beta_n\sqrt{1+a^2y^2}\left[1 + \frac{U(ay)}{\beta_n} + \cdots\right], \quad \text{(B4)}$$

where

$$U(x) = \mathrm{sgn}(n)\frac{eA_{2a} - q\beta}{x^2+1} - \frac{x^2u'_a}{2(x^2+1)^{3/2}}, \quad \text{(B5)}$$

and $A_{2a} = \lim_{r\to a-}A_2(r)$. The corresponding expansion for the function $p_n(i\beta_n y)$ is obtained from Eq. (2.20):

$$p_n(i\beta_n y) \approx \sqrt{1+a^2y^2}[\beta_n + U(ay)] + 2\xi(u'_a - 1) + \cdots. \quad \text{(B6)}$$

The next terms in the expansion over the powers of $1/\beta_n$ are found in a similar way.

[1] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).

[2] S. Sarangi and S. H. H. Tye, Phys. Lett. B **536**, 185 (2002); E. J. Copeland, R. C. Myers, and J. Polchinski, J. High Energy Phys. 06 (2004) 013; G. Dvali and A. Vilenkin, J. Cosmol. Astropart. Phys. 03 (2004) 010.

[3] D. R. Nelson, *Defects and Geometry in Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2002); G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon, Oxford, 2003).

[4] S. Bellucci, E. R. Bezerra de Mello, A. de Padua, and A. A. Saharian, Eur. Phys. J. C **74**, 2688 (2014).

[5] J. S. Dowker, Phys. Rev. D **36**, 3742 (1987).

[6] M. E. X. Guimarães and B. Linet, Commun. Math. Phys. **165**, 297 (1994); M. E. X. Guimarães, Classical Quantum Gravity **12**, 1705 (1995); B. Linet, Classical Quantum Gravity **13**, 97 (1996).

[7] J. Spinelly and E. R. Bezerra de Mello, Classical Quantum Gravity **20**, 873 (2003).

[8] J. Spinelly and E. R. Bezerra de Mello, Int. J. Mod. Phys. D **13**, 607 (2004); Nucl Phys. B, Proc. Suppl. **127**, 77 (2004).

[9] J. Spinelly and E. R. Bezerra de Mello, J. High Energy Phys. 09 (2008) 005.

[10] Yu. A. Sitenko and N. D. Vlasii, Classical Quantum Gravity **29**, 095002 (2012).

[11] M. G. Alford and F. Wilczek, Phys. Rev. Lett. **62**, 1071 (1989).

[12] K. Jones-Smith, H. Mathur, and T. Vachaspati, Phys. Rev. D **81**, 043503 (2010); Y.-Z. Chu, H. Mathur, and T. Vachaspati, Phys. Rev. D **82**, 063515 (2010); D. A. Steer and T. Vachaspati, Phys. Rev. D **83**, 043528 (2011).

[13] L. Sriramkumar, Classical Quantum Gravity **18**, 1015 (2001).
[14] Yu. A. Sitenko and N. D. Vlasii, Classical Quantum Gravity **26**, 195009 (2009).
[15] E. R. Bezerra de Mello, Classical Quantum Gravity **27**, 095017 (2010).
[16] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and V. M. Bardeghyan, Phys. Rev. D **82**, 085033 (2010).
[17] E. R. Bezerra de Mello and A. A. Saharian, Eur. Phys. J. C **73**, 2532 (2013).
[18] E. A. F. Bragança, H. F. Santana Mota, and E. R. Bezerra de Mello, arXiv:1410.1511.
[19] A. Mohammadi, E. R. Bezerra de Mello, and A. A. Saharian, arXiv:1407.8095.
[20] W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985); J. R. Gott, Astrophys. J. **288**, 422 (1985).
[21] B. Allen and A. C. Ottewill, Phys. Rev. D **42**, 2669 (1990); B. Allen, J. G. Mc Laughlin, and A. C. Ottewill, Phys. Rev. D **45**, 4486 (1992); B. Allen, B. S. Kay, and A. C. Ottewill, Phys. Rev. D **53**, 6829 (1996).
[22] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, Phys. Rev. D **74**, 025017 (2006).
[23] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.
[24] Yu. A. Sitenko and A. Yu. Babansky, Phys. At. Nucl. **61**, 1594 (1998).
[25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972).
[26] I. Brevik and T. Toverud, Classical Quantum Gravity **12**, 1229 (1995); E. R. Bezerra de Mello, V. B. Bezerra, and A. A. Saharian, Phys. Lett. B **645**, 245 (2007); E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, Phys. Rev. D **78**, 105007 (2008); E. R. Bezerra de Mello and A. A. Saharian, Classical Quantum Gravity **28**, 145008 (2011); G. Fucci and K. Kirsten, J. High Energy Phys. 03 (2011) 016; S. Bellucci, E. R. Bezerra de Mello, and A. A. Saharian, Phys. Rev. D **83**, 085017 (2011); V. V. Nesterenko and I. G. Pirozhenko, Classical Quantum Gravity **28**, 175020 (2011); A. A. Saharian and A. S. Kotanjyan, Eur. Phys. J. C **71**, 1765 (2011); G. Fucci and K. Kirsten, J. Phys. A **44**, 295403 (2011); E. R. Bezerra de Mello, A. A. Saharian, and A. Kh. Grigoryan, J. Phys. A **45**, 374011 (2012); E. R. Bezerra de Mello, A. A. Saharian, and S. V. Abajyan, Classical Quantum Gravity **30**, 015002 (2013).
[27] C. R. Hagen, Phys. Rev. Lett. **64**, 503 (1990); C. R. Hagen and S. Ramaswamy, Phys. Rev. D **42**, 3524 (1990).
[28] M. Bordag and S. Voropaev, J. Phys. A **26**, 7637 (1993); S. A. Voropaev and M. Bordag, J. Exp. Theor. Phys. **78**, 127 (1994).
[29] F. M. Andrade, E. O. Silva, and M. Pereira, Phys. Rev. D **85**, 041701(R) (2012).
[30] E. M. Serebryanyi, Theor. Math. Phys. **64**, 846 (1985).
[31] N. R. Khusnutdinov and M. Bordag, Phys. Rev. D **59**, 064017 (1999).
[32] A. Krishnan, E. Dujardin, M. M. J. Treacy, J. Hugdahl, S. Lynum, and T. W. Ebbesen *et al.*, Nature (London) **388**, 451 (1997); S. N. Naess, A. Elgsaeter, G. Helgesen, and K. D. Knudsen, Sci. Tech. Adv. Mater. **10**, 065002 (2009).
[33] E. R. Bezerra de Mello, A. A. Saharian, and V. Vardanyan, Phys. Lett. B **741**, 155 (2015).
[34] E. R. Bezerra de Mello and A. A. Saharian, Classical Quantum Gravity **29**, 035006 (2012).