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Constructing black hole entropy from gravitational collapse

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Based on a recent proposal for the gravitational entropy of free gravitational fields, we investigate the thermodynamic properties of black hole formation through gravitational collapse in the framework of the semitetrad $1 + 1 + 2$ covariant formalism. In the simplest case of an Oppenheimer–Snyder–Datt collapse, we prove that the change in gravitational entropy outside a collapsing body is related to the variation of the surface area of the body itself, even before the formation of horizons. As a result, we are able to relate the Bekenstein–Hawking entropy of the black hole end state to the variation of the vacuum gravitational entropy outside the collapsing body.

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I. INTRODUCTION

Black hole entropy in the case of eternal black holes (the maximally extended Schwarzschild vacuum solution) is a very well-understood subject since the pioneering work of Bekenstein [1] and of Bardeen *et al.*; see Ref. [2] for a review. However, astrophysical black holes form in a dynamic way. Entropy is not so well understood in that context.

In the context of astrophysical formation of black holes, a key question arises. We know that astrophysical black holes are not eternal in the past: they are created by the continual gravitational collapse of massive stars. Therefore, the question is as follows: should black hole entropy be only a property of the black hole event horizon, manifesting suddenly as the horizon forms, or should it be an artifact of a time-varying gravitational field due to gravitational collapse, with gravitational entropy changing smoothly from initial values to the canonical value $S_{\rm BH} = A/4$ as an event horizon comes into being when the stellar surface area r crosses the value $r = 2M$?

The difficulty in working this out is that we need a definition of gravitational entropy for a generic gravitational field, not only for a black hole; but until recently, we have not had such a definition. Penrose [3] has suggested such a definition should be based in properties of the Weyl tensor but gave no specific formula. The important idea behind this proposal is as follows: we know that for any general relativistic spacetime all the information about the spacetime curvature, and hence the gravitational field, is encoded in the Riemann curvature tensor [4]. However the trace of this tensor, namely, the Ricci tensor, is related pointwise to the energy-momentum tensor of the matter fields via the Einstein field equations. Hence, the information on entropy encoded in the Ricci tensor is the same as the entropy of the matter fields. Therefore, to characterize the entropy of the free gravitational field (or the pure gravity) apart from that encoded in the matter terms, one must use the Weyl tensor, which is the trace-free part of the Riemann curvature tensor [4–6].

Recently, a thermodynamically motivated measure of gravitational entropy based on this idea was proposed by Clifton *et al.* [7]. A strong candidate for providing a measure of the gravitational entropy of an arbitrary gravitational field is the Bel–Robinson tensor [8], which is constructed from the Weyl tensor and its dual. It has been shown in Ref. [9] that this tensor is the unique Maxwellian tensor that can be constructed from the Weyl tensor and acts like an effective energy-momentum tensor of the free gravitational field, albeit having the dimension L^{-4} rather than L^{-2}. The proposed measure of gravitational entropy in Ref. [7], therefore, uses the square root of the Bel–Robinson tensor, which was shown to be unique for spacetimes that are of Petrov type D or N.

This measure of gravitational entropy for a free gravitational field has all the important requirements that a measure of entropy should have. It is strictly non-negative and vanishes only for conformally flat spacetimes in which the Weyl tensor is zero. It measures the local anisotropy of the free gravitational field and increases monotonically as structures form in the early Universe. Most importantly, this measure reproduces the Bekenstein–Hawking entropy of a black hole, which is the famous theorem that states that the

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black hole entropy at any time slice is proportional to the surface area of the black hole, which is the two-dimensional intersection of the black hole horizon and a constant time slice [1,10,11]. Through this definition of black hole entropy, one can naturally develop the concepts of black hole thermodynamics in both classical and semiclassical regimes, leading to quantum particle creations and Hawking radiation [12].

To investigate the question stated above, in light of the gravitational entropy proposal of Ref. [7], in this paper, we consider the simplest example of black hole formation by Oppenheimer–Snyder–Datt [13,14] collapse, which describes the gravitational collapse of a spherical dustlike star immersed in a Schwarzschild vacuum. Since the exterior of the star is of Petrov type D, we can uniquely determine the entropy of the free gravitational field for a static observer even when no event horizon exists. We explicitly prove that the Bekenstein–Hawking entropy of the black hole, which is formed after an infinite time for the static observer, can be linked to the net monotonic increase in the entropy of the free gravitational field during this dynamic gravitational collapse. This result relates the time-varying gravitational field during the continuous gravitational collapse to the thermodynamic property of the final state, the black hole, in which gravitational entropy is well understood [2].

In this paper, we work on spherically symmetric black holes and use the semitetrad $1+1+2$ covariant formalism for a slightly more general class of locally rotationally symmetric class (LRS) II spacetimes [15] (of which spherical symmetry is a subclass). We discuss this covariant formalism and its usefulness in describing LRS-II spacetimes briefly in the next two sections. We then recast the equations of the gravitational entropy in this formalism in Sec. IV and finally prove the proposition relating the net increase in entropy to the change in the surface area of the collapsing star in Sec. V.

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this paper, and Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative, and ∂ corresponds to partial differentiation. We use the $(-,+,+,+)$ signature, and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}. \tag{1}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd}R_{cadb}. \tag{2}$$

The Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}[R - 2\Lambda - 2\mathcal{L}_m], \tag{3}$$

thr variation of which gives the Einstein field equations as

$$G_{ab} + \Lambda g_{ab} = T_{ab}. \tag{4}$$

II. SEMITETRAD COVARIANT FORMALISMS

Spacetimes can be described using tetrad formalisms or metric- (or coordinate-)based approaches [6]. The key idea behind the semitetrad formalisms is to identify preferred directions in spacetime and to project all the geometrical quantities describing the spacetime along these preferred directions and onto the spaces perpendicular to them. Among the most used semitetrad methods are $3+1$ Arnwitt, Deser and Misner formalism (which uses a global foliation of the spacetime, and hence the spacetime has to be globally hyperbolic) and the $1+3$ and $1+1+2$ covariant formalisms (which use a local decomposition, and hence there is no constraint on the global structure). Below, we briefly describe the last two formalisms.

A. $1+3$ covariant formalism

The $1+3$ formalism developed by Ehlers, Kristian and Sachs, and Trümper, and summarised by Ellis [5,6], is based on a local decomposition of the spacetime manifold by choosing a preferred timelike vector: all vectors and tensors are projected either along that timelike direction or on the instantaneous rest 3-space of the observers moving along the timelike direction. We define a timelike congruence with a unit tangent vector u^a. The natural choice of such a vector is the tangent to the matter flow lines. Any vector X^a in the manifold can then be projected on the tangent 3-space by the projection tensor $h^a{}_b = g^a{}_b + u^a u_b$. We can similarly decompose the full covariant derivative of any tensorial quantity in two parts. The dot derivative ($u^a\nabla_a$) is the derivative along the timelike vector u^a, and the spacial derivative D is the projected derivative onto the 3-space, where the projection is done on all indices by the tensor $h^a{}_b$. The covariant derivative of the timelike vector u^a can now be decomposed into irreducible parts as $\nabla_a u_b = -A_b u_a + \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c$, where $A_a = \dot{u}_a$ is the acceleration, $\Theta = D_a u^a$ is the expansion scalar, $\sigma_{ab} = D_{\langle a}u_{b\rangle}$ is the shear tensor, and $w^a = \epsilon^{abc}D_b u_c$ is the vorticity vector.

Similarly the Weyl curvature tensor can be decomposed irreducibly into the gravitoelectric and gravitomagnetic parts as $E_{ab} = C_{acbd}u^c u^d$ and $H_{ab} = \frac{1}{2}\epsilon_{acd}C^{cd}{}_{be}u^e$. The energy-momentum tensor for a general matter field can be also decomposed as $T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}$ where $\mu = T_{ab}u^a u^b$ is the energy density, $p = (1/3)h^{ab}T_{ab}$ is the isotropic pressure, $q_a = q_{\langle a\rangle} = -h^c{}_a T_{cd}u^d$ is the heat flux 3-vector, and $\pi_{ab} = \pi_{\langle ab\rangle}$ is the anisotropic stress.

B. $1+1+2$ covariant formalism

As an extension to the $1+3$ formalism to spacetimes having a preferred spatial direction, Clarkson and Barrett developed a $1+1+2$ formalism that has been applied to spherically symmetric spacetimes [16–18]. A choice of a second preferred vector along the spatial direction e^a orthogonal to u^a produces another split of the spacetime: this allows any 3-vector to be irreducibly split into a scalar and a vector. The projection tensor $N^a{}_b \equiv h^a{}_b - e^a e_b$ projects the quantities onto tangent 2-surfaces, which we will refer as *sheets*. We can now introduce two new derivatives for any object $\psi_{a\ldots b}{}^{c\ldots d}$ as natural result of the new splitting of the 3-space (for detailed discussions, see Ref. [18]):

$$\hat{\psi}_{a\ldots b}{}^{c\ldots d} \equiv e^f D_f \psi_{a\ldots b}{}^{c\ldots d}, \tag{5}$$

$$\delta_f \psi_{a\ldots b}{}^{c\ldots d} \equiv N_a{}^p \ldots N_b{}^q N_r{}^c \ldots N_s{}^d N_f{}^j D_j \psi_{p\ldots q}{}^{r\ldots s}. \tag{6}$$

We can easily see that the *hat derivative* is the projection of the spatial derivative D along the preferred spacelike direction, and the δ *derivative* is the projection on the 2-sheets. The $1+3$ kinematical and Weyl quantities can be decomposed as follows: setting $\{\theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}$ [18], we have

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \tag{7}$$

$$\omega^a = \Omega e^a + \Omega^a, \tag{8}$$

$$\sigma_{ab} = \Sigma\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}, \tag{9}$$

$$E_{ab} = \mathcal{E}\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \tag{10}$$

$$H_{ab} = \mathcal{H}\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}. \tag{11}$$

Similarly, we may split the fluid variables q^a and π_{ab},

$$q^a = Q e^a + Q^a, \tag{12}$$

$$\pi_{ab} = \Pi\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\Pi_{(a} e_{b)} + \Pi_{ab}. \tag{13}$$

We are now able to decompose the covariant derivative of e^a on the instantaneous rest 3-space of the observers moving along u^a into its irreducible parts, giving

$$\mathrm{D}_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi \epsilon_{ab} + \zeta_{ab}, \tag{14}$$

where

$$a_a \equiv e^c \mathrm{D}_c e_a = \hat{e}_a, \tag{15}$$

$$\phi \equiv \delta_a e^a, \tag{16}$$

$$\xi \equiv \frac{1}{2} \epsilon^{ab} \delta_a e_b, \tag{17}$$

$$\zeta_{ab} \equiv \delta_{\{a} e_{b\}}. \tag{18}$$

We see that on the 3-space, moving along the preferred vector e^a, ϕ represents the *spatial expansion of the sheet*, ζ_{ab} is the *spatial shear of* e^a (i.e., the distortion of the sheet), and a^a is its *acceleration*. We can also interpret ξ as the *spatial vorticity* associated with e^a so that it is a representation of the "twisting" or rotation of the sheet.

III. LRS-II SPACETIMES

A spacetime manifold $(\mathcal{M}, g)$ is called *locally isotropic* if every point $p \in (\mathcal{M}, g)$ has a continuous nontrivial isotropy group. When this group consists of spatial rotations, the spacetime is called *locally rotationally symmetric* [15]. The variables that uniquely describe an LRS spacetime are $\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}$. The LRS-II class contains all spherically symmetric solutions, which are free of rotation and are described by the variables $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}$, since Ω, ξ, and $\mathcal{H}$ all vanish. These spacetimes include Schwarzschild, Robertson–Walker, Lemaître–Tolman–Bondi (LTB), and Kottler spacetimes.

The most general form of the metric that describes LRS-II can be written as [19]

$$ds^2 = -A^{-2}(t,r)dt^2 + B^2(t,r)dr^2 + C^2(t,r)[dy^2 + D^2(y,k)dz^2], \tag{19}$$

where t and r are the affine parameters along the integral curves of u^a and e^a, respectively, and $k = (1, 0, -1)$ specifies the closed, flat, or open geometry of the 2-sheets, respectively. Since we are concentrating on spherically symmetric spacetimes, henceforth, we will only consider $k = 1$.

A. Field equations

The field equations that describe the propagation and the evolution of the geometrical covariant variables can now be found using the Ricci identities of the vectors u^a and e^a and the doubly contracted Bianchi identities. These are as follows:

(i) Propagation:

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{2}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi, \tag{20}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} = -\frac{3}{2}\phi\Sigma - Q, \tag{21}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)Q. \tag{22}$$

(ii) Evolution:

$$\dot{\phi} = -\left(\Sigma - \frac{2}{3}\Theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + Q, \tag{23}$$

$$\dot{\Sigma} - \frac{2}{3}\dot{\Theta} = -\mathcal{A}\phi + 2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 + \frac{1}{3}(\mu + 3p - 2\Lambda) - \mathcal{E} + \frac{1}{2}\Pi, \tag{24}$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot{\mu} + \frac{1}{2}\dot{\Pi} = +\left(\frac{3}{2}\Sigma - \Theta\right)\mathcal{E} + \frac{1}{4}\left(\Sigma - \frac{2}{3}\Theta\right)\Pi + \frac{1}{2}\phi Q - \frac{1}{2}(\mu + p)\left(\Sigma - \frac{2}{3}\Theta\right). \tag{25}$$

(iii) Propagation/evolution:

$$\hat{\mathcal{A}} - \dot{\Theta} = -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 + \frac{1}{2}(\mu + 3p - 2\Lambda), \tag{26}$$

$$\dot{\mu} + \hat{Q} = -\Theta(\mu + p) - (\phi + 2\mathcal{A})Q - \frac{3}{2}\Sigma\Pi, \tag{27}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\mu + p)\mathcal{A}. \tag{28}$$

The 3-Ricci scalar of the spacelike 3-space orthogonal to u^a can be expressed as

$${}^3R = -2\left[\hat{\phi} + \frac{3}{4}\phi^2 - K\right], \tag{29}$$

where K is the Gaussian curvature of the 2-sheet and it is defined by ${}^2R_{ab} = KN_{ab}$. In terms of the covariant scalars, we can write the Gaussian curvature K as

$$K = \frac{1}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2 - \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2. \tag{30}$$

Finally, the evolution and propagation equations for the Gaussian curvature K are

$$\dot{K} = -\left(\frac{2}{3}\Theta - \Sigma\right)K, \tag{31}$$

$$\hat{K} = -\phi K. \tag{32}$$

B. Misner–Sharp mass for spherically symmetric spacetimes

In this section, we derive the Misner–Sharp [20] mass equation for LRS-II spacetimes in terms of the $1 + 1 + 2$ kinematical quantities. This quantity represents the mass inside a 2-sphere of radius r at time t in terms of geometric properties on that sphere.

In terms of the metric quantities (19), the Misner–Sharp mass is defined as (assuming $k = 1$ for spherical symmetry) [21]

$$\mathcal{M}_{\rm ms}(r, t) = \frac{1}{2\sqrt{K}}(1 - \nabla_a C\nabla^a C), \tag{33}$$

where C represents the area radius of the spherical 2-sheets. In terms of the Gaussian curvature of the 2-sheets, we obtain

$$\mathcal{M}_{\rm ms}(r, t) = \frac{1}{2\sqrt{K}}\left(1 - \frac{1}{4K^3}\nabla_a K\nabla^a K\right). \tag{34}$$

Using the $1 + 1 + 2$ decomposition of the covariant derivative for LRS-II together with Eqs. (30), (31), and (32), the Misner–Sharp mass takes the form

$$\mathcal{M}_{\rm ms}(r, t) = \frac{1}{2K^{3/2}}\left(\frac{1}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi\right). \tag{35}$$

We can see from the above expression that both matter and the Weyl tensor contribute to the Misner–Sharp mass. Hence, even in the absence of matter (as in vacuum Schwarzschild spacetime), we have nonzero gravitational mass sourced by the Weyl curvature.

IV. THERMODYNAMICS OF A GRAVITATIONAL FIELD

We use a thermodynamically motivated expression of the gravitational entropy measure $S_{\rm grav}$ given in Ref. [7] and based on the Bel–Robinson tensor [8], which has a natural interpretation as super-energy-momentum tensor [9] for the gravitational field. To be well defined, the gravitational entropy has to (i) be non-negative, (ii) vanish if and only if $C_{abcd} = 0$, (iii) measure the local anisotropy of free gravitational field, (iv) reduce to the Bekenstein–Hawking entropy for a Schwarzschild black hole, and (v) increase as structures (inhomogeneities) form in the Universe. All these conditions are met by the definition given in Ref. [7] in the cases of Coulomb-like or wavelike gravitational fields; in the following, we will be interested in the former case. This definition of gravitational entropy has been explored also in Ref. [22], along with other proposals, in the context of LTB dust models.

Following Ref. [7], in order to define a thermodynamically motivated gravitational entropy, one has first to assume validity of the second law of thermodynamics for a generic gravitational field; that is,

$$T_{\rm grav}dS_{\rm grav} = dU_{\rm grav} + p_{\rm grav}dV > 0, \tag{36}$$

where $T_{\rm grav}$, $U_{\rm grav}$, and $p_{\rm grav}$ represent the effective temperature, internal energy, and isotropic pressure of the free gravitational field, respectively, and where V is the spatial volume. Another key ingredient is the equation of energy-momentum conservation, which in $1+3$ decomposition is

$$(\rho v)^{\cdot} + p\dot{v} = v[-u_a T^{ab}_{;b} - q^b_{;b} - \dot{u}^a q_a - \sigma_{ab}\pi^{ab}], \tag{37}$$

where v is a spatial volume element and the dot represents the derivative with respect to time. By comparison between the right-hand side of Eq. (36) and the left-hand side of Eq. (37), one can define an effective thermodynamic equation

$$T_{\rm grav}\dot{s}_{\rm grav} = (\mu_{\rm grav}v)^{\cdot} + p_{\rm grav}\dot{v}, \tag{38}$$

where s is the entropy density. The quantities on the right-hand side of Eq. (38) are calculated through contractions of the effective gravitational energy-momentum tensor, defined as the square root of the Bel–Robinson tensor. It was shown in Ref. [7] that the gravitational pressure vanishes in a Coulomb-like field ($p_{\rm grav} = 0$) and the gravitational energy density is given by

$$8\pi\mu_{\rm grav} = 2\alpha\sqrt{\frac{2W}{3}} = \alpha|\mathcal{E}|, \tag{39}$$

where α is constant introduced by the definition of the gravitational energy-momentum tensor. Using Eq. (35), the gravitational energy density can be expressed as

$$\mu_{\rm grav} = \frac{\alpha}{8\pi}\left|\left(\frac{1}{3}(\mu+\Lambda) - 2\mathcal{M}_{\rm ms}(r,t)K^{3/2} - \frac{1}{2}\Pi\right)\right|. \tag{40}$$

Equation (38) can then be written as [7]

$$T_{\rm grav}\dot{s}_{\rm grav} = (\mu_{\rm grav}v)^{\cdot} = -v\sigma_{ab}\left(\Pi^{ab}_{\rm grav} + \frac{4\pi(p+\mu)}{\sqrt{3}\alpha\sqrt{2W}}\right). \tag{41}$$

This expression in $1+1+2$ decomposition for LRS-II reads

$$T_{\rm grav}\dot{s}_{\rm grav} = (\mu_{\rm grav}v)^{\cdot} = -v\Sigma\left(\Pi_{\rm grav} + \frac{8\pi(p+\mu)}{3\alpha|\mathcal{E}|}\right). \tag{42}$$

If we want to obtain the entropy of a specific gravitational configuration, we can write Eq. (38) with $p_{\rm grav} = 0$ as

$$\delta s_{\rm grav} = \frac{\delta(\mu_{\rm grav}v)}{T_{\rm grav}} \tag{43}$$

and integrate over a spacelike hypersurface. The last ingredient needed is a definition for the temperature of the gravitational field.

A. Temperature of the gravitational field

In determining the temperature of a thermodynamic system, lacking a definition based in microscopic degrees of freedom underlying the macroscopic, coarse-grained physics, we need a phenomenological definition. We follow the proposal of Ref. [7], which, in $1+3$ decomposition, is given by

$$T_{\rm grav} \equiv \frac{|u_{a;b}l^a k^b|}{\pi} = \frac{|\dot{u}_a e^a + \frac{1}{3}\Theta + \sigma_{ab}e^a e^b|}{2\pi}, \tag{44}$$

where $l^a = \frac{u^a - e^a}{\sqrt{2}}$, $k^a = \frac{u^a + e^a}{\sqrt{2}}$, and e^a is a spacelike unit vector aligned with the Weyl principal tetrad [23]. The temperature in Eq. (44) can be represented in the $1+1+2$ decomposition as

$$T_{\rm grav} = \frac{|\mathcal{A} + \frac{1}{3}\Theta + \Sigma|}{2\pi}. \tag{45}$$

This definition is clearly inspired by the results obtained by quantum field theory in curved spacetimes and black hole thermodynamics, but we can not directly interpret it as a generalization of Hawking and Unruh [24] temperatures, which are all tightly related to (and describe features of) horizons. Instead, we identify it with the temperature of a gravitational field as measured locally at a point of the spacetime, associated with the symmetry 2-sphere through that point in a LRS spacetime. The Hawking temperature for a Schwarzschild black hole as evaluated at infinity is $(8\pi M)^{-1}$, while the definition given in Eq. (45) goes to zero for the static observer at infinity, the latter being related to the curvature of the free gravitational field and not to some kind of emission from the horizon. Such an emission does not take any part in the scenario we consider here of the formation of a black hole, in which Hawking radiation is suppressed by cosmic background radiation in the expanding Universe [25].

B. Gravitational entropy and structure formation

We have already stated before that the square root of the Bel–Robinson tensor being the measure of gravitational entropy enables structure formation naturally as the entropy increases as the structure (or inhomogeneities) forms in the Universe. In this subsection, we explicitly show this in terms of the covariant variables and the Misner–Sharp mass. From Eq. (35), we get the temporal and spatial evolution of the Misner–Sharp mass as

$$\hat{\mathcal{M}}_{\rm ms} = \frac{1}{4K^{3/2}}\left(\phi(\mu+\Lambda) - \left(\Sigma - \frac{2}{3}\Theta\right)Q\right) \quad (46)$$

$$\dot{\mathcal{M}}_{\rm ms} = \frac{1}{4K^{3/2}}\left(\left(\frac{2}{3}\Theta - \Sigma\right)(\Lambda - p) - Q\phi\right). \quad (47)$$

Let us, for simplicity, consider the Universe is filled with perfect fluid with $Q = \Pi = 0$. Then using Eq. (46) in Eq. (40), we get

$$\mu_{\rm grav} = \frac{\alpha K^{3/2}}{6\pi\phi}\left|\left(\hat{\mathcal{M}}_{\rm ms} - \frac{3}{2}\phi\mathcal{M}_{\rm ms}\right)\right|. \quad (48)$$

Now, from LRS-II field equations, we can easily see that for a homogeneous distribution of perfect fluid with $\hat{\mu} = \hat{p} = 0$ we have $\hat{\mathcal{M}}_{\rm ms} - \frac{3}{2}\phi\mathcal{M}_{\rm ms} = 0$ on every constant time slice and hence $\delta s_{\rm grav} = 0$. However, as discussed in Ref. [7], if we start with an inhomogeneous distribution of collapsing matter (as it happens during structure formation), we have $\hat{\mathcal{M}}_{\rm ms} - \frac{3}{2}\phi\mathcal{M}_{\rm ms} \neq 0$. This will then make $d\mu_{\rm grav} > 0$ and hence $dS_{\rm grav} > 0$. Thus, the thermodynamics of free gravity naturally favors structure formation, in contrast with the thermodynamics of standard matter that favors dispersion. In light of above discussion, we can predict that the vacuum gravitational entropy outside a collapsing star (integrated over each constant time slice) will increase with time, favoring the process of continual gravitational collapse. This we prove explicitly in the next section.

V. GRAVITATIONAL ENTROPY OF THE VACUUM AROUND A COLLAPSING STAR

Having all the ingredients at hand, we want to look now at the variation of gravitational entropy outside a body that is collapsing to form a black hole in the simplest scenario, the Oppenheimer–Snyder–Datt [13,14] dust collapse model, represented schematically in Fig. 1. The main features of this collapsing scenario are as follows:

(1) The interior of the collapsing star is described by a Friedmann-Lema-Robertson-Walker spacetime with coordinates (t, r, θ, ϕ) matched with the exterior vacuum solution represented by Schwarzschild spacetime with coordinates (τ, R, ψ, Φ).

(2) These two spacetimes (internal and external) are matched via the usual Israel–Darmois matching conditions [26,27], where we match the first and second fundamental forms across the boundary layer of the collapsing star, which is given by $r = r_b$ for the interior spacetime and $R = R(\tau)$ for the exterior spacetime. Since the stellar interior has dustlike matter with vanishing pressure and heat flux, from Eq. (47), we can easily see that $\dot{\mathcal{M}}_{\rm ms} = 0$ (for $\Lambda = 0$). This implies that the Misner–Sharp mass in the interior is only a function of the comoving shell labelling coordinate r. Matching the second fundamental form across the boundary, we then get $M_{\rm Schw} = \mathcal{M}_{\rm ms}(r_b)$; that is, the Schwarzschild mass in the exterior region is equal to the Misner–Sharp mass of the interior, calculated on the comoving boundary layer, and this remains constant during the collapse.

(3) Though in general the entropy of the spacetime has contributions coming both from the matter and the gravitational field, in the interior spacetime, the matter entropy only contributes, since $\mathcal{E} = 0$ for Friedmann-Lema-Robertson-Walker spacetimes, while in the exterior vacuum, only the gravitational part of the entropy does not vanish.

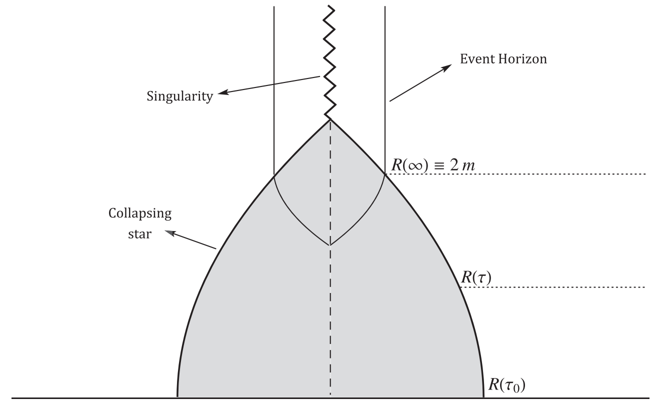

FIG. 1. Oppenheimer–Snyder dust collapse of a star (shaded). In the reference frame of a static external observer, the crossing of the star's surface with the horizon at radius $2m$ occurs at $\tau \to \infty$.

Being interested in the variation of gravitational entropy from the point of view of an external static observer, we choose to integrate Eq. (43) over a spacelike hypersurface *outside* the collapsing body. However, here we need to make clear what is the domain of our interest where the thermodynamic equations will hold. In many situations in physics, one needs a cutoff of an integral in order to obtain physically meaningful results, and that is the case here, where the entropy defined will diverge if we extend the integral to infinity. We wish to consider collapse to form a black hole in a realistic physical context, that is, when the collapsing object is imbedded in an expanding Universe, and to regularize the integral in that context. In that case, information about the local system is restricted to a finite radius, and outside that radius, the geometry and physics are dominated by other objects in the expanding Universe that have nothing to do with the collapsing star. The result we wish to prove is a local result, applicable to the locally restricted nature of real physical systems.

Let us define a *finite-infinity* $\mathcal{F}$ as a two sphere surrounding the star at any given instant of time, far enough away that spacetime is asymptotically flat to a very good approximation; this will represent infinity for all practical purposes [28,29]. Farther out, matter and radiation are irrelevant to the development of the collapsing star. Hence, if $R(\tau_0)$ denotes the radius of the stellar surface at a given instant of time and $R_{\mathcal{F}}$ the denotes radius of the "finite infinity," then the domain $\mathcal{D}$ defined by $R(\tau_0) < R < R_{\mathcal{F}}$ is the local domain where our results will apply. In the case of the solar system, $R_{\mathcal{F}}$ can be taken to be about a light year, and this is defined as the heliopause, beyond which the effect of the sun nearly ceases to exist as far as spacetime curvature is concerned and the dynamics of galactic dust and other stars takes over. We would like to emphasize here that the dynamics of the stellar collapse do not affect the radius $R_{\mathcal{F}}$ (which can be taken as a constant), as this radius only depends on the Misner–Sharp mass of the interior, calculated on the comoving boundary layer, and this remains constant during the stellar collapse.

Based on the assumptions stated above, in this scenario, we are able to show the following:

Proposition 1.—The increase in the instantaneous gravitational entropy outside a collapsing star during a given interval of time is proportional to the change in the surface area of the star during that interval.

Proof.—Outside the collapsing star, the spacetime is Schwarzschild. Therefore, by using Eqs. (39) and (45) in eq. (43) and integrating over a 3-volume of the exterior region at fixed time, the total entropy at a given time can be expressed as

$$S_{\rm grav} \equiv \int_\sigma \delta s_{\rm grav} = \pi\alpha \int_{R(\tau_0)}^{R_{\mathcal{F}}} \frac{|\mathcal{E}|}{\mathcal{A}} \frac{\bar{R}^2}{\sqrt{1-\frac{2m}{\bar{R}}}} d\bar{R}, \tag{49}$$

where we have used $v = u^a \eta_{abcd} dx^b dx^c dx^d$, and the timelike vector u^a is given by $u^a = (1/\sqrt{|1-\frac{2m}{R}|}, 0, 0, 0)$, $R(\tau_0)$ is the radius of the collapsing star at time τ_0, and m is the total mass of the star. We know that for Schwarzschild spacetime $\mathcal{E}$ and $\mathcal{A}$ are given by [17]

$$|\mathcal{E}| = \frac{2m}{R^3}, \tag{50}$$

$$\mathcal{A} = \frac{m}{R^2}\left(1-\frac{2m}{R}\right)^{-1/2}. \tag{51}$$

Using the last two expressions in Eq. (49), the gravitational entropy is then given by

$$S_{\rm grav} = 2\pi\alpha \int_{R(\tau_0)}^{R_{\mathcal{F}}} \bar{R} d\bar{R}. \tag{52}$$

As we have already discussed, the local dynamics of the stellar collapse do not affect the upper limit of the integration, which is the radius of the finite-infinity 2-sphere surrounding the collapsing star. This radius can be taken as a constant, typically of the order $R_{\mathcal{F}} \simeq 1$ light year for a solar mass star. However, the key point we emphasise here is that we can take this radius to be any other finite constant multiple of one light year, and it will add a constant finite number to the magnitude of the entropy at any given epoch. When we are considering the entropy change between two epochs, this additive constant will cancel out, and only the lower limit of the integration will determine the result. Thus, the specific value chosen for $R_{\mathcal{F}}$ does not matter.

Now, if we calculate the change in the gravitational entropy in a time interval $(\tau - \tau_0)$, we obtain

$$\delta S_{\rm grav}|_{(\tau-\tau_0)} = \frac{\alpha}{4}(A(\tau_0) - A(\tau)), \tag{53}$$

where $A(\tau)$ is the surface area of the star at any time $\tau > \tau_0$. ∎

A. Constraining α

To constrain the value of the parameter α, we can consider the variation of entropy between a configuration with $R(\tau_\epsilon) = 2m + \epsilon$ (with $\epsilon \ll 2m$) and the asymptotic black hole state with $R(\infty) = 2m$. The time elapsed between these spatially neighboring states is actually infinite because the formation of a black hole as a result of the collapse from the point of view of a static external observer is a process that takes an infinite amount of time. Equation (53) gives

$$\delta S_{\rm grav}|_{(\infty-\tau_\epsilon)} = \frac{\alpha}{4}(A(\tau_\epsilon) - A_H) \tag{54}$$

$$\simeq \alpha 4\pi m\epsilon, \tag{55}$$

where $\simeq$ means that we are neglecting higher orders in ϵ. The energy/mass supplied by this final stage of collapse to form the black hole is $dU \simeq \epsilon/2$, so Eq. (55) can be rewritten as

$$dS \simeq \alpha(8\pi m) dU. \tag{56}$$

The term in round brackets is the Hawking temperature $T_H = (8\pi m)^{-1}$, and hence if $\alpha = 1$, we recover the second law of black hole thermodynamics [30]. The same value for α was found in Ref. [7] by calculating the instantaneous gravitational entropy of a black hole and comparing the result with the known Bekenstein–Hawking value.

B. Building up the Bekenstein–Hawking entropy

As a consequence of Proposition 1, it is possible to show that the Bekenstein–Hawking entropy of a Schwarzschild black hole can be related to the process of collapse that leads to its formation, as stated in the following corollary

Corollary 1.—The Bekenstein–Hawking entropy of a black hole, formed as an end state of a spherically symmetric collapse of a massive star with Schwarzschild spacetime as the exterior, is the difference between one-fourth of the initial area of the collapsing star and the net increase in the vacuum entropy in infinite collapsing time.

Proof.—In the reference frame of an external static observer, the crossing between the collapsing star's surface and the horizion (and hence the formation of the black hole) will take an infinite time. Assuming that for the asymptotic black hole end state the Bekenstein–Hawking relation $S_{\mathrm{BH}} = \frac{1}{4}A_H$ holds, where A_H is the area of the event horizon, then from Proposition 1 with $\alpha = 1$, we have

$$S_{\mathrm{BH}} = \frac{1}{4}A(\tau_0) - \delta S_{\mathrm{grav}}|_{(\infty-\tau_0)}. \tag{57}$$

■

We note that this result is applicable to any static observer outside the collapsing star, for whom the stellar surface will take infinite time to cross the horizon. However, if we want to deal with the entropy of the region between the surface of the star and the horizon, where there is no notion of a static observer, we can use the spatially homogeneous nonstatic metric of the Schwarzschild manifold and calculate the corresponding gravitational temperature and entropy. This calculation is done in Ref. [7] to derive the Bekenstein–Hawking entropy of a static black hole.

The above result may have important consequences on the holographic principle [31,32], which was inspired by black hole thermodynamics. From our result above, we can easily see that the entropy being related to the surface area is not an *exclusive* property of horizons (black hole or cosmological); rather, this property is common to other 2-surfaces enclosing a 3-volume (such as the boundary of the collapsing star). Hence, this result may expand the scope of applicability of holographic principles, which can be viewed as a manifestation of the boundary value problem of the thermodynamical properties of any closed domain.

VI. PUTTING IT ALL TOGETHER

The standard story of gravitational entropy relates only to black holes; it does not show how that entropy behaves as a black hole forms. But black holes form in the context of the expanding Universe. The major paradox is that any standard text tells you that the second law of thermodynamics implies that entropy increases, and that in turn is taken to show that disorder increases at microscales while order increases at macro scales [33]. No structure can form spontaneously. But, in fact, order does indeed spontaneously form on large scales as the Universe expands—an apparent contradiction with the second law [34]. To resolve this, one needs a good definition of gravitational entropy.

The definition given in Ref. [7], in which (following Penrose's suggestions) gravitational entropy is based in the properties of the Weyl tensor, resolves this issue as far as the growth of perturbations in the expanding Universe, due to gravitational attraction, is concerned (see Eqs. (54) and (55) in Ref. [7]). The present paper has shown that initial growth of gravitational entropy, taking place in conjunction with the initial formation of structure in the expanding Universe, can be smoothly joined onto the formation of black holes. The famous black hole entropy does not suddenly appear when the event horizon is formed; it grows steadily as gravitational attraction causes ever more concentrated objects to form, eventually leading to the existence of black holes with the standard gravitational entropy.

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