PHYSICAL REVIEW D **91,** 064006 (2015)

# First law and anisotropic Cardy formula for three-dimensional Lifshitz black holes

Eloy Ayón-Beato,[1,2,*] Moisés Bravo-Gaete,[3,†] Francisco Correa,[4,5,‡] Mokhtar Hassaïne,[3,§] María Montserrat Juárez-Aubry,[1,2,6,∥] and Julio Oliva[2,¶]

[1]*Departamento de Física, CINVESTAV-IPN, Apartado Postal 14-740, 07000 México D.F., México*

[2]*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Casilla 567, Valdivia, Chile*

[3]*Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile*

[4]*Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*

[5]*Centro de Estudios Científicos (CECs), Casilla 1468, Valdivia, Chile*

[6]*Instituto Tecnológico y de Estudios Superiores de Monterrey, Campus Puebla, Vía Atlixcáyotl No. 2301, Reserva Territorial Atlixcáyotl, Puebla, C.P. 72453 Puebla, México*

(Received 14 January 2015; published 2 March 2015; corrected 2 August 2017)

The aim of this paper is to confirm in new concrete examples that the semiclassical entropy of a three-dimensional Lifshitz black hole can be recovered through an anisotropic generalization of the Cardy formula derived from the growth of the number of states of a boundary nonrelativistic field theory. The role of the ground state in the bulk is played by the corresponding Lifshitz soliton obtained by a double Wick rotation. In order to achieve this task, we consider a scalar field nonminimally coupled to new massive gravity for which we study different classes of Lifshitz black holes as well as their respective solitons, including new solutions for a dynamical exponent $z=3$. The masses of the black holes and solitons are computed using the quasilocal formulation of conserved charges recently proposed by Gim *et al.* and based on the off-shell extension of the ADT formalism. We confirm the anisotropic Cardy formula for each of these examples, providing a stronger base for its general validity. Consistently, the first law of thermodynamics together with a Smarr formula are also verified.

DOI: 10.1103/PhysRevD.91.064006 PACS numbers: 04.70.Bw, 04.50.Kd, 04.70.Dy

## I. INTRODUCTION

During the last decade, there has been an intense activity to promote the ideas underlying the gauge/gravity duality [1] in order to have a better understanding of strongly coupled field theories with anisotropic scaling. These latter are characterized by a scaling symmetry where space and time scale with different weights; their gravity dual metric called the Lifshitz spacetime [2] is given by

$$ds^2=-\frac{r^{2z}}{l^{2z}}dt^2+\frac{l^2}{r^2}dr^2+\frac{r^2}{l^2}d\vec{x}^2. \qquad (1)$$

Here, $z$ is the dynamical critical exponent which reflects the anisotropy of the scaling symmetry

$$t\mapsto\tilde{\lambda}^z t, \qquad r\mapsto\tilde{\lambda}^{-1}r, \qquad \vec{x}\mapsto\tilde{\lambda}\,\vec{x}.$$

Finite temperature effects are intended to be holographically introduced through black holes commonly known as Lifshitz black holes whose asymptotic behaviors match with the spacetime (1). As it is now well known, for $z\neq 1$, in order for the Einstein gravity to accommodate the Lifshitz spacetimes, some extra matter source is required like $p$-form gauge fields, Proca fields [2–4], Brans-Dicke scalars [5], or, eventually, nonlinear electrodynamic theories [6]. There also exists the option of considering higher-order gravity theories for which there are examples of Lifshitz black holes without source; see, e.g., [7–12]. In this paper, we will focus on a combination of these two options in three dimensions by considering a gravity action given by a special combination of quadratic curvature corrections to Einstein gravity known as new massive gravity (NMG) [13], together with a source described by a self-interacting scalar field nonminimally coupled to gravity. We are then interested in the following three-dimensional action

$$S[g,\Phi]=\frac{1}{2\kappa}\int d^3x\sqrt{-g}\left[R-2\lambda-\frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\right)\right]-\int d^3x\sqrt{-g}\left[\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi+\frac{\xi}{2}R\Phi^2+U(\Phi)\right], \qquad (2)$$

where $R$ denotes the scalar curvature, $R_{\mu\nu}$ the Ricci tensor, $\xi$ stands for the nonminimal coupling parameter, and $U(\Phi)$ represents the self-interaction potential. There is a variety of reasons that make this model worth being explored. Among others, it is well known that nonminimally coupled scalar fields are excellent laboratories in order to evade

*ayon-beato-at-fis.cinvestav.mx
†mbravog-at-inst-mat.utalca.cl
‡correa-at-cecs.cl
§hassaine-at-inst-mat.utalca.cl
∥mjuarez-at-fis.cinvestav.mx
¶julio.oliva-at-uach.cl

 

standard "no-hair" theorems. In fact, it is known that NMG supports a vacuum Lifshitz black hole for critical exponent $z = 3$ [7] that is additionally characterized as the only static axisymmetric asymptotically Lifshitz solution that can be written as a Kerr-Schild transformation of the Lifshitz spacetime (1) [14]. Remarkably, it has been recently shown that the spectrum of dynamical critical exponents $z$ supported by new massive gravity at finite temperature becomes largely enriched if one includes self-interacting nonminimally coupled scalar fields as sources of the Lifshitz black holes according to the previous action [15]. Besides, it is known that in standard three-dimensional anti–de Sitter (AdS) gravity supported by scalar fields, solitons play a fundamental role as they can be treated as ground states; the existence of scalar-tensor solitons turns out to be essential for microscopically counting for the black holes' entropy using the Cardy formula [16]. Here, we check that these ideas also hold in the case of asymptotically Lifshitz spacetimes in the presence of scalar fields, by carrying out the microscopical computation of the black hole entropy using the anisotropic generalization of the Cardy formula introduced by Gonzalez *et al.* [17].

As previously emphasized, the action (2) allows different classes of Lifshitz black hole configurations with different values of the parameters and dynamical exponent. With these solutions at hand, it is tempting to compute their respective masses. However, the task of identifying the conserved charges is a highly nontrivial problem whose difficulty is increased in our case because of two main reasons. Indeed, we are dealing with Lifshitz black holes which have a rather nonstandard asymptotic behavior (1), and even more, the gravity theory we consider (2) contains quadratic corrections. In order to tackle this problem, we will test the quasilocal formulation of conserved charges recently proposed in [18,19] and based on the off-shell extension of the Abbott-Deser-Tekin (ADT) formalism [20]. In the ADT formalism, which is a covariant generalization of the ADM method [21], the metric $g_{\mu\nu}$ is linearized around the zero mass spacetime with metric $\bar{g}_{\mu\nu}$ as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. However, in this approach, the perturbed metric $h_{\mu\nu}$ in the case of Lifshitz black holes may not satisfy the correct assumptions concerning the falloff boundary conditions, and consequently yields an expression for the mass that does not satisfy the first law of thermodynamics as shown in [22]. This problem has been recently circumvented by the authors of [18,19] who have proposed a quasilocal generalization of the ADT formalism that can be used even with slow falloff conditions. In order to be as self-contained as possible and even if our work is focused in the three-dimensional case, we briefly present the main ingredients of the quasilocal ADT method in arbitrary dimension $D$ following the notations and results of [18,19]. One of the interesting aspects of this formalism lies in the fact that the computations are done only on the basis of the full Lagrangian defining the theory

$$S[g,\Phi] = \int d^D x \sqrt{-g}\mathcal{L},$$

and the Killing vectors $\xi^\mu$ associated to the conserved charges without explicitly using the linearization of the field equations. Here, $\Phi$ collectively denotes any matter content. The main result of Ref. [18] is the following prescription for the off-shell ADT potential

$$\sqrt{-g}\mathcal{Q}^{\mu\nu}_{\text{ADT}} = \frac{1}{2}\delta K^{\mu\nu} - \xi^{[\mu}\Theta^{\nu]} \tag{3}$$

in terms of the surface term $\Theta^\mu$ arising from the variation of the action

$$\delta S = \int d^D x[\sqrt{-g}(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \delta_\Phi \mathcal{L}) + \partial_\mu \Theta^\mu],$$

and the off-shell Noether potential $K^{\mu\nu}$ associated to the identical conservation of the off-shell Noether current

$$J^\mu = \sqrt{-g}(\mathcal{L}g^{\mu\nu} + 2\mathcal{E}^{\mu\nu})\xi_\nu - \Theta^\mu = \partial_\nu K^{\mu\nu}.$$

It is important to remark that the exclusive use of off-shell conserved currents in the derivation makes that the background metric is not required to satisfy the field equations. Another important fact is that the standard linearization method is only compatible at the asymptotic regime. Hence, in order to circumvent this problem and to construct quasilocal charges, the authors in [18,19] consider linearizing along a one-parameter family of configurations as has been advocated, for example, in [23]. More concretely, for the configurations under study, a parameter $0 \le s \le 1$ can be introduced which allows the interpolation with the asymptotic solution at $s = 0$. In doing so, the quasilocal conserved charge reads

$$Q(\xi) = \int_{\mathcal{B}} d^{D-2}x_{\mu\nu}\left(\Delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\int_0^1 ds \Theta^{\nu]}(\xi|s)\right), \tag{4}$$

where $\Delta K^{\mu\nu}(\xi) \equiv K^{\mu\nu}_{s=1}(\xi) - K^{\mu\nu}_{s=0}(\xi)$ denotes the difference of the Noether potential between the interpolated solutions, and $d^{D-2}x_{\mu\nu}$ represents the integration over the codimension-two boundary $\mathcal{B}$. In our case (2), the involved quantities are given by

$$\Theta^\mu = 2\sqrt{-g}\left(P^{\mu\alpha\beta\gamma}\nabla_\gamma \delta g_{\alpha\beta} - \delta g_{\alpha\beta}\nabla_\gamma P^{\mu\alpha\beta\gamma} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi\right), \tag{5}$$

$$K^{\mu\nu} = \sqrt{-g}(2P^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma - 4\xi_\sigma \nabla_\rho P^{\mu\nu\rho\sigma}), \tag{6}$$

with $P^{\mu\nu\sigma\rho} \equiv \partial\mathcal{L}/\partial R_{\mu\nu\sigma\rho}$. Notice that we have added the contribution to the surface term coming from the scalar field, which was not included in the original of

Refs. [18,19] since they dealt with the vacuum case. It is worth to note that generic matter contributions to the quasilocal ADT formalism have been investigated in [24].

After computing the masses of our solutions, we subject the quasilocal ADT method to two nontrivial tests. First, we will verify that the first law of thermodynamics holds in each case by computing the Wald formula [25] for the entropy as

$$\mathcal{S}_{\rm W} = -2\pi\Omega_{D-2}\left(\frac{r_h}{l}\right)^{D-2}[P^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}]_{r=r_h}, \qquad (7)$$

together with the Hawking temperature $T$; here, $r_h$ stands for the location of the horizon. Independently, in three dimensions, a Lifshitz black hole characterized by a dynamical exponent $z$ and mass $\mathcal{M}$ has a corresponding soliton with dynamical exponent $z^{-1}$ and mass $\mathcal{M}_{\rm sol}$ obtained by operating a double Wick rotation [17]. We will confirm that our mass expressions are compatible with the anisotropic generalization of the Cardy formula proposed in [17]

$$\mathcal{S}_{\rm C} = 2\pi l(z+1)\left[\left(-\frac{\mathcal{M}_{\rm sol}}{z}\right)^{z}\mathcal{M}\right]^{\frac{1}{z+1}}. \qquad (8)$$

This formula is obtained starting from a boundary nonrelativistic field theory and computing the asymptotic growth of the number of states with fixed energy, assuming the role of ground state in the bulk is to be played by the soliton. Notice that for AdS asymptotics, $z=1$, this formula becomes exactly the Cardy formula used in [16] for the case of standard gravity supported by scalar fields. Remarkably, both tests will be checked satisfactorily by the quasilocal ADT method.

The rest of the paper is organized as follows. In the next section, we will study different classes of Lifshitz black hole solutions for which we will obtain their mass using the quasilocal ADT formalism. We explicitly verify the fulfillment of the first law in all these cases. Moreover, for each Lifshitz black hole solution, we obtain their corresponding soliton counterpart, compute their mass, and spotlight the validity of the anisotropic Cardy formula (8). Finally, the last section is devoted to our conclusions where we also anticipate the generalization of some of these results to higher dimensions.

## II. NONMINIMALLY DRESSED LIFSHITZ BLACK HOLES

The field equations obtained by varying the action (2) read

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (9a)$$

$$\Box\Phi - \xi R\Phi = \frac{dU(\Phi)}{d\Phi}, \qquad (9b)$$

where the higher-order contribution $K_{\mu\nu}$ and the nonminimally coupled energy-momentum tensor $T_{\mu\nu}$ are defined by

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}(g_{\mu\nu}\Box + \nabla_\mu\nabla_\nu - 9R_{\mu\nu})R - 8R_{\mu\alpha}R^{\alpha}{}_{\nu} + g_{\mu\nu}\left(3R^{\alpha\beta}R_{\alpha\beta} - \frac{13}{8}R^2\right), \qquad (9c)$$

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - g_{\mu\nu}\left(\frac{1}{2}\nabla_\sigma\Phi\nabla^\sigma\Phi + U(\Phi)\right) + \xi(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\Phi^2. \qquad (9d)$$

In what follows, we will consider Lifshitz black holes within the following ansatz

$$ds^2 = -\frac{r^{2z}}{l^{2z}}f(r)dt^2 + \frac{l^2}{r^2}\frac{dr^2}{f(r)} + \frac{r^2}{l^2}d\varphi^2, \qquad (10)$$

with coordinates ranges defined by $-\infty < t < \infty$, $0 < r < \infty$, and the angular variable $0 \le \varphi < 2\pi l$. We start by reanalyzing one of the solutions obtained by some of the authors in Ref. [15], the single one characterized by having a nonvanishing Wald entropy and, therefore, suitable for our analysis. For the exponent $z=3$, the configurations of Ref. [15] only contain the vacuum black hole of NMG [7]; for this reason, we later concentrate on this value of the exponent and exhibit nonminimally dressed Lifshitz black holes also in this case.

### A. Nonminimally dressed black holes for generic $z$

The following family of Lifshitz black hole solutions was found in Ref. [15]

$$ds^2 = -\frac{r^{2z}}{l^{2z}}\left[1 - M\left(\frac{l}{r}\right)^{\frac{z+1}{2}}\right]dt^2 + \frac{l^2}{r^2}\left[1 - M\left(\frac{l}{r}\right)^{\frac{z+1}{2}}\right]^{-1}dr^2 + \frac{r^2}{l^2}d\varphi^2, \qquad (11a)$$

$$\Phi(r) = \sqrt{\frac{(z-3)(9z^2-12z+11)M}{2\kappa(z-1)(z^2-3z+1)}}\left(\frac{l}{r}\right)^{\frac{z+1}{4}}, \qquad (11b)$$

where the scalar field has the self-interaction

$$U(\Phi) = \frac{(z-1)(21z^3-13z^2+31z-15)}{32l^2(9z^2-12z+11)}\Phi^2 - \frac{(z-1)^3(z^2-3z+1)(9z^2-12z+19)\kappa}{32l^2(z-3)(9z^2-12z+11)^2}\Phi^4, \qquad (11c)$$

and the coupling constants are parametrized by

$$m^2 = -\frac{z^2 - 3z + 1}{2l^2},$$
$$\lambda = -\frac{z^2 + z + 1}{2l^2},$$
$$\xi = \frac{3z^2 - 4z + 3}{2(9z^2 - 12z + 11)}. \tag{11d}$$

Before proceeding with the computations of the Noether potential and the surface term in order to derive the mass, we would like to emphasize some aspects of this solution. First of all, because of the expression of the scalar field, this solution has no AdS limit $z = 1$. The other special value of the dynamical exponent is given by $z = 3$, for which the scalar field as well as the potential vanish identically and one ends with the vacuum black hole of NMG [7]. We notice from now that for this class of solution, as well as for the two other solutions derived below, the allowed potential always involves a mass term.

It is easy to see that this class of solution has a nonvanishing Wald entropy and a Hawking temperature given by

$$\mathcal{S}_{\mathrm{W}} = -\frac{\pi^2 (z+1)^2 (3z-5) r_h}{2\kappa (z-1)(z^2 - 3z + 1)}, \tag{12}$$

$$T = \frac{(z+1){r_h}^z}{8\pi l^{z+1}}, \qquad r_h = l M^{\frac{2}{z+1}}. \tag{13}$$

Let us now compute the mass through the quasilocal ADT formalism. For the timelike Killing vector $\xi^t = (1,0,0)$, and after some tedious but straightforward computations, the expressions for the Noether potential and the surface term are given by

$$\int_0^M dM \Theta^r = \frac{(z+1)(9z^3 - 31z^2 + 31z - 25)M}{16\kappa l (z-1)(z^2 - 3z + 1)} \left(\frac{r}{l}\right)^{\frac{z+1}{2}} - \frac{(z+1)(15z^3 - 60z^2 + 67z - 50)M^2}{32\kappa l (z-1)(z^2 - 3z + 1)},$$
$$K^{rt} = -\frac{(z+1)(9z^3 - 31z^2 + 31z - 25)M}{16\kappa l (z-1)(z^2 - 3z + 1)} \left(\frac{r}{l}\right)^{\frac{z+1}{2}} + \frac{3(z+1)(z-3)(5z^2 - 6z + 5)M^2}{32\kappa l (z-1)(z^2 - 3z + 1)}.$$

This implies that the mass $\mathcal{M}$ of the Lifshitz black hole solution (11) turns out to be

$$\mathcal{M} = -\frac{\pi (z+1)^2 (3z-5)}{16\kappa (z-1)(z^2 - 3z + 1)} \left(\frac{r_h}{l}\right)^{z+1}. \tag{14}$$

It is simple to verify that the black hole entropy (12) and the mass (14) satisfy the first law of black hole thermodynamics $d\mathcal{M} = T d\mathcal{S}_{\mathrm{W}}$. In fact, they satisfy an anisotropic version of the Smarr formula

$$\mathcal{M} = \frac{T}{z+1} \mathcal{S}_{\mathrm{W}}, \tag{15}$$

which stipulates that the mass $\mathcal{M}$ as a function of the entropy $\mathcal{S}_{\mathrm{W}}$ is a homogeneous function of degree $z + 1$ [26]. Notice that these nice properties are satisfied independently of the sign of the mass.

TABLE I. Range of possibilities for the dynamical exponent $z$ allowing positive mass black holes, $\mathcal{M} > 0$.

| $\kappa$ | Range of $z$ |
|---|---|
| $\kappa > 0$ | $1.7 \approx 5/3 < z < (3 + \sqrt{5})/2 \approx 2.6$ |
| $\kappa < 0$ | $2.6 \approx (3 + \sqrt{5})/2 < z \leq 3$ |

It is also interesting to note that for the vacuum case $z = 3$ [7], this expression of the mass coincides with the one derived in different papers using others formalisms [8,17,27,28], provided that the Einstein constant is taken negative $\kappa = -8\pi G$, i.e., by choosing the so-called "wrong" sign of NMG which turns tensor ghosts at the linearized level into unitary Fierz-Pauli massive excitations on maximally symmetric vacua [13]. Clearly, the range of the dynamical exponent $z$ which ensures a positive mass strongly depends on the sign of the coupling constant $\kappa$. Imposing the positivity of the mass and the reality of the scalar field, the critical exponent $z$ must be restricted according to Table I.

We now derive the corresponding soliton solution which exists for the same range of parameters and self-interacting potential than those of the black hole solution (11). This soliton solution turns out to have a dynamical exponent $z^{-1}$ and a characteristic scale $lz^{-1}$ which is a consequence of the two-dimensional isomorphism between the Lifshitz Lie algebras with dynamical exponents $z$ and $z^{-1}$ obtained by swapping the role of the Hamiltonian with the momentum generator [17]. We will present in detail the different steps in this case and only report the main results in the other two solutions. We first consider the Euclidean version of the Lifshitz black hole (11) obtained by the Wick rotation $t = i\tau$,

$$ds^2 = \frac{r^{2z}}{l^{2z}} f(r) d\tau^2 + \frac{l^2}{r^2 f(r)} dr^2 + \frac{r^2}{l^2} d\varphi^2, \tag{16}$$

where the metric function $f(r)$ can be read from Eq. (11), and the static scalar field remains the same. In order to avoid conical singularities, the Euclidean time must be periodic with period $\beta = T^{-1}$, that is, $0 \leq \tau < \beta$ and the angle keeps being identified as $0 \leq \varphi < 2\pi l$. Under the Euclidean diffeomorphism defined by

$$(\tau, r, \varphi) \mapsto \left(\bar{\tau} = \left(\frac{2\pi l}{\beta}\right)^{\frac{1}{z}} \varphi, \bar{r} = \frac{\beta}{2\pi z}\left(\frac{r}{l}\right)^{z}, \bar{\varphi} = \frac{2\pi l}{\beta}\tau\right), \tag{17}$$

the line element (16) becomes

$$ds^2 = \left(\frac{z\bar{r}}{l}\right)^{\frac{2}{z}} d\bar{\tau}^2 + \frac{l^2}{z^2\bar{r}^2 F(\bar{r})} d\bar{r}^2 + \frac{z^2\bar{r}^2}{l^2} F(\bar{r}) d\bar{\varphi}^2,$$
$$F(\bar{r}) = 1 - \bar{M}\left(\frac{l}{z\bar{r}}\right)^{\frac{z+1}{2z}}, \qquad \bar{M} = \left(\frac{4}{z+1}\right)^{\frac{z+1}{2z}}; \tag{18}$$

i.e., the Euclidean Lifshitz black hole is diffeomorphic to another asymptotically Lifshitz solution with dynamical exponent $z^{-1}$, scale $lz^{-1}$, and temperature

$$\bar{\beta} = (2\pi l)^{1+\frac{1}{z}} \beta^{-\frac{1}{z}}. \tag{19}$$

It is, in fact, a soliton; its regular character is not manifest in these coordinates which have the advantage of exposing the Lifshitz asymptotic behavior with exponent $z^{-1}$ and scale $lz^{-1}$. Finally, the corresponding Lorentzian soliton obtained through $\bar{\tau} = i\bar{t}$ reads

$$ds^2 = -\left(\frac{z\bar{r}}{l}\right)^{\frac{2}{z}} d\bar{t}^2 + \frac{l^2}{z^2\bar{r}^2 F(\bar{r})} d\bar{r}^2 + \frac{z^2\bar{r}^2}{l^2} F(\bar{r}) d\bar{\varphi}^2, \tag{20a}$$

$$\Phi(\bar{r}) = \sqrt{\frac{(z-3)(9z^2-12z+11)\bar{M}}{2\kappa(z-1)(z^2-3z+1)}} \left(\frac{l}{z\bar{r}}\right)^{\frac{z+1}{4z}}. \tag{20b}$$

Now promoting the fixed constant to a variable one, $\bar{M} \mapsto s\bar{M}$, we obtain a one-parameter family of local solutions which facilitates the computation of the mass along the same lines as before. Before proceeding with the computation of the mass, let us analyze this last point carefully. For the black hole solution, since the constant $M$ is an integration constant, a parameter $s$ with range $s \in [0, 1]$ could have been introduced in the solution via the change $M \mapsto sM$. This change is useful only for computing the mass, since in this case, the variation will be operated with the parameter $s$, and the surface term will be rather integrated as $\int_0^1 ds \Theta^r$. Nevertheless, the result is the same if one promotes the constant $M$ as the moving parameter and integrating the surface term from 0 to $M$ as we did previously. Now, for the counterpart soliton, one can start with the black hole solution parametrized in terms of $s$ and operate the same diffeomorphism (17). The resulting soliton solution will correspond to the solution (20) with $\bar{M} \mapsto s\bar{M}$, and the integration of the surface term will be given by $\int_0^1 ds \Theta^{\bar{r}}$. However, as in the black hole case, the new variable constant $\bar{M}$ can be used as the moving parameter, and the result is exactly the same. Hence, choosing the Killing vector as $\xi^{\bar{t}} = (1, 0, 0)$, the Noether potential and the surface term take the following form

$$\int_0^{\bar{M}} d\bar{M}\Theta^{\bar{r}} = \frac{(z+1)(9z^3-31z^2+31z-25)\bar{M}}{16\kappa l(z-1)(z^2-3z+1)} \left(\frac{z\bar{r}}{l}\right)^{\frac{z+1}{2z}} - \frac{(z+1)(15z^3-60z^2+67z-50)\bar{M}^2}{32\kappa l(z-1)(z^2-3z+1)},$$
$$K^{\bar{r}\,\bar{t}} = \frac{(z+1)(9z^3-31z^2+31z-25)\bar{M}}{16\kappa l(z-1)(z^2-3z+1)} \times \left[\bar{M} - \left(\frac{z\bar{r}}{l}\right)^{\frac{z+1}{2z}}\right],$$

giving a unique value for the mass of the Lifshitz soliton, independent of any integration constant, as expected,

$$\mathcal{M}_{\mathrm{sol}} = \frac{\pi z(3z-5)}{\kappa(z-1)(z^2-3z+1)} \left(\frac{z+1}{4}\right)^{\frac{z-1}{z}}. \tag{21}$$

It is straightforward to check that the mass of the soliton and the mass of the black hole have opposite signs (14), as expected. As long as the mass of the soliton is negative $\mathcal{M}_{\mathrm{sol}} < 0$ (see Table I), the holographic picture unveiled in Ref. [17] applies: the semiclassical entropy of the Lifshitz black hole (11) can be understood from the asymptotic growth of the number of states of a $1+1$ nonrelativistic field theory with the ground state corresponding in the bulk to the soliton (20). This gives rise to the anisotropic generalization of the Cardy formula (8), which after evaluation, perfectly coincides with the Wald formula (12)

$$\mathcal{S}_{\mathrm{W}} = \mathcal{S}_{\mathrm{C}}. \tag{22}$$

We would like to emphasize that the bulk semiclassical derivation of Ref. [17] is also applicable to negative mass black holes (positive mass solitons), which as we already show are compatible with the first law. In this case, the anisotropic formula involves the absolute values of the masses and consequently with the first law produces a negative entropy; i.e., a general formula would be

$$\mathcal{S}_{\mathrm{C}} = \epsilon 2\pi l(z+1)\left[\left(\frac{|\mathcal{M}_{\mathrm{sol}}|}{z}\right)^{z} |\mathcal{M}|\right]^{\frac{1}{z+1}}, \tag{23}$$

where $\epsilon = \pm 1$ corresponds to the sign of the black hole mass. Obviously, for $\epsilon = -1$, a holographic interpretation makes no sense; even the mere existence of a thermodynamical one can be challenged.

Moreover, for the vacuum dynamical exponent $z = 3$ and with $\kappa = -8\pi G$, the soliton mass (21) becomes $\mathcal{M}_{\mathrm{sol}} = -3/(4G)$, which precisely corresponds to the mass of the vacuum gravitational soliton found in [17]. In the following sections, we exhibit new nonminimally dressed Lifshitz solutions for the same exponent $z = 3$, and since the involved steps are similar, only the important results are reported.

### B. Nonminimally dressed black holes for $z = 3$

For dynamical exponent $z = 3$, a new family of Lifshitz black hole solutions is presented:

$$ds^2 = -\frac{r^6}{l^6}\left(1-\frac{Ml^4}{r^4}\right)dt^2 + \frac{l^2}{r^2}\left(1-\frac{Ml^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{l^2}d\varphi^2, \tag{24a}$$

$$\Phi(r) = \sqrt{\frac{M}{\kappa(2-13\xi)}\frac{l^2}{r^2}}. \tag{24b}$$

This configuration is supported by the self-interacting potential

$$U(\Phi) = -\frac{2-13\xi}{2l^2}[2\Phi^2 + (2-\xi)\kappa\Phi^4], \tag{24c}$$

where the nonminimal coupling parameter $\xi$ is not restricted *a priori*, and the remaining coupling constants are related as in vacuum

$$\frac{\lambda}{13} = m^2 = -\frac{1}{2l^2}. \tag{24d}$$

As before, choosing the Killing vector $\xi^t = (1,0,0)$, the Noether potential and the surface term are calculated as

$$\int_0^M dM\Theta^r = -\frac{2(4\xi-1)M^2l^3}{\kappa(13\xi-2)r^4} + \frac{(104\xi-17)M}{\kappa l(13\xi-2)},$$
$$K^{rt} = \frac{2(4\xi-1)M^2l^3}{\kappa(13\xi-2)r^4} - \frac{2(68\xi-11)M}{\kappa l(13\xi-2)},$$

from which we obtain the mass of the Lifshitz black hole (24) as

$$\mathcal{M} = -\frac{2\pi(32\xi-5)}{\kappa(13\xi-2)}\left(\frac{r_h}{l}\right)^4. \tag{25}$$

It is easy to see that this expression for the mass satisfies the first law, since the related Wald entropy and temperature are expressed by

$$\mathcal{S}_{\mathrm{W}} = -\frac{8\pi^2(32\xi-5)r_h}{\kappa(13\xi-2)}, \tag{26}$$

$$T = \frac{{r_h}^3}{\pi l^4}, \qquad r_h = lM^{1/4}, \tag{27}$$

which is also compatible with the Smarr formula (15) for $z = 3$. The possible values for the nonminimal coupling parameter warranting the existence of the solution and from them those allowing positive mass are all summarized in Table II.

TABLE II. Range of possibilities for the nonminimal coupling parameter $\xi$.

| $\kappa$ | Range of $\xi$ | $\mathcal{M} > 0$ |
|---|---|---|
| $\kappa > 0$ | $\xi < 2/13 \approx 0.154$ | $\emptyset$ |
| $\kappa < 0$ | $\xi > 2/13 \approx 0.154$ | $\xi > 5/32 \approx 0.156$ |

As in the previous case, operating the same diffeomorphism (17) with $z = 3$ on the Euclidean version of the solution (24), we obtain a Lifshitz soliton whose Lorentzian counterpart is

$$ds^2 = -\left(\frac{3\bar{r}}{l}\right)^{2/3}d\bar{t}^2 + \frac{l^2}{9\bar{r}^2}\left[1-\bar{M}\left(\frac{l}{3\bar{r}}\right)^{4/3}\right]^{-1}d\bar{r}^2 + \frac{9\bar{r}^2}{l^2}\left[1-\bar{M}\left(\frac{l}{3\bar{r}}\right)^{4/3}\right]d\bar{\varphi}^2, \tag{28a}$$

$$\Phi(\bar{r}) = \sqrt{\frac{\bar{M}}{\kappa(2-13\xi)}\left(\frac{l}{3\bar{r}}\right)^{2/3}}, \qquad \bar{M} = 2^{-4/3}. \tag{28b}$$

Once again, introducing a one-parameter family of locally equivalent solutions via $\bar{M} \mapsto s\bar{M}$, the Noether potential and the surface term are obtained as

$$\int_0^{\bar{M}} d\bar{M}\Theta^{\bar{r}} = -\frac{2(4\xi-1)\bar{M}^2}{\kappa l(13\xi-2)}\left(\frac{l}{3\bar{r}}\right)^{4/3} + \frac{(104\xi-17)\bar{M}}{\kappa l(13\xi-2)},$$
$$K^{\bar{r}\bar{t}} = \frac{2(4\xi-1)\bar{M}^2}{\kappa l(13\xi-2)}\left(\frac{l}{3\bar{r}}\right)^{4/3} - \frac{2(4\xi-1)\bar{M}}{\kappa l(13\xi-2)},$$

which, in turn, implies the following fixed mass for the soliton (28):

$$\mathcal{M}_{\mathrm{sol}} = \frac{3\pi(32\xi-5)}{2^{1/3}\kappa(13\xi-2)}. \tag{29}$$

For $\xi > 5/32$ and $\kappa < 0$, it is straightforward to check that the generalized Cardy formula (8) fits perfectly with the expressions of the masses of the Lifshitz black hole (25), its soliton counterpart (29), and the Wald entropy (26).

Here again, the cases with negative black hole masses, that can be inferred from Table II, i.e., $\xi < 2/13$ for $\kappa > 0$ and $2/13 < \xi < 5/32$ for $\kappa < 0$, are compatible with the first law, the Smarr formula (15), and its entropy can be rewritten *à la* Cardy according to the general formula (23) without further interpretation. In fact, for one of these nonminimal couplings, namely, $\xi = 3/20 < 2/13$ with $\kappa > 0$, the solution can be improved by generalizing the self-interaction with the addition of a cubic contribution. The result is a sort of rigid dressing of the vacuum $z = 3$ black hole [7].

### C. Dressing the vacuum $z = 3$ black hole for $\xi = 3/20$

In the scenario where the nonminimal coupling takes the value $\xi = 3/20$ and $\kappa > 0$, the solution (24) is improved to

$$ds^2 = -\frac{r^6}{l^6}\left(1 - \frac{\alpha\sqrt{M}l^2}{r^2} - \frac{Ml^4}{r^4}\right)dt^2 + \frac{l^2}{r^2}\left(1 - \frac{\alpha\sqrt{M}l^2}{r^2} - \frac{Ml^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{l^2}d\varphi^2, \tag{30a}$$

$$\Phi(r) = \sqrt{\frac{20M}{\kappa}\frac{l^2}{r^2}}, \tag{30b}$$

provided that the self-interaction potential is generalized as

$$U(\Phi) = -\frac{1}{20l^2}\Phi^2 - \frac{\alpha\sqrt{\kappa}}{5\sqrt{5}l^2}\Phi^3 - \frac{37\kappa}{800l^2}\Phi^4, \tag{30c}$$

with no restrictions in the cubic coupling constant $\alpha$ and the remaining coupling constants fixed as in (24d). Additionally to the $\alpha = 0$ limit, where we consistently recover the black hole solution (24) for $\xi = 3/20$, this solution allows another nontrivial limit: for $M \to 0$ and $\alpha \to \infty$ keeping fixed the quantity $M_v = \alpha\sqrt{M}$ this solution becomes just the vacuum $z = 3$ black hole [7] with integration constant $M_v$. This solution can be interpreted as a sort of rigid dressing of the vacuum $z = 3$ black hole by a self-interacting scalar field with nonminimal coupling $\xi = 3/20$.

Calculating the Wald entropy and temperature of this black hole gives

$$\mathcal{S}_{\mathrm{W}} = -\frac{32\pi^2\sqrt{\alpha^2+4}\,r_h}{\kappa(\alpha + \sqrt{\alpha^2+4})}, \tag{31}$$

$$T = \frac{\sqrt{\alpha^2+4}\,{r_h}^3}{\pi l^4(\alpha + \sqrt{\alpha^2+4})}, \qquad {r_h}^2 = \frac{l^2\sqrt{M}}{2}(\alpha + \sqrt{\alpha^2+4}). \tag{32}$$

For the same timelike Killing vector, the expressions of the Noether potential and the surface term read

$$\int_0^M dM\Theta^r = \frac{4\alpha\sqrt{M}r^2}{\kappa l^3} - \frac{(\alpha^2-28)M}{\kappa l} - \frac{20l\alpha M^{3/2}}{\kappa r^2} - \frac{16l^3M^2}{\kappa r^4},$$
$$K^{rt} = \frac{16l^3M^2}{\kappa r^4} + \frac{20l\alpha M^{3/2}}{\kappa r^2} - \frac{32M}{\kappa l} - \frac{4\alpha\sqrt{M}r^2}{\kappa l^3},$$

giving the mass

$$\mathcal{M} = -\frac{8\pi(\alpha^2+4)}{\kappa(\alpha + \sqrt{\alpha^2+4})^2}\left(\frac{r_h}{l}\right)^4. \tag{33}$$

As in the previous examples, the first law is satisfied as well as the Smarr formula. Notice that the cubic interaction does not enhance the sign of the mass, which remains negative as in the case with $\alpha = 0$.

Following the same lines of the two previous examples, the corresponding soliton reads

$$ds^2 = -\left(\frac{3\bar{r}}{l}\right)^{2/3}d\bar{t}^2 + \frac{l^2}{9\bar{r}^2}\frac{d\bar{r}^2}{F(\bar{r})} + \frac{9\bar{r}^2}{l^2}F(\bar{r})d\bar{\varphi}^2, \tag{34a}$$

$$\Phi(\bar{r}) = \sqrt{\frac{20\bar{M}}{\kappa}\left(\frac{l}{3\bar{r}}\right)^{2/3}}, \tag{34b}$$

$$F(\bar{r}) = 1 - \alpha\sqrt{\bar{M}}\left(\frac{l}{3\bar{r}}\right)^{2/3} - \bar{M}\left(\frac{l}{3\bar{r}}\right)^{4/3}, \tag{34c}$$

$$\bar{M} = \left(\frac{2}{(\alpha^2+4)(\alpha + \sqrt{\alpha^2+4})}\right)^{2/3}. \tag{34d}$$

In this case, the Noether potential and the surface term yield

$$\int_0^{\bar{M}} d\bar{M}\Theta^r = \frac{4\alpha\bar{M}^{1/2}}{\kappa l}\left(\frac{3\bar{r}}{l}\right)^{2/3} - \frac{(\alpha^2-28)\bar{M}}{\kappa l} - \frac{20\alpha\bar{M}^{3/2}}{\kappa l}\left(\frac{l}{3\bar{r}}\right)^{2/3} - \frac{16\bar{M}^2}{\kappa l}\left(\frac{l}{3\bar{r}}\right)^{4/3},$$
$$K^{rt} = -\frac{4\alpha\bar{M}^{1/2}}{\kappa l}\left(\frac{3\bar{r}}{l}\right)^{2/3} + \frac{4(\alpha^2-4)\bar{M}}{\kappa l} + \frac{20\alpha\bar{M}^{3/2}}{\kappa l}\left(\frac{l}{3\bar{r}}\right)^{2/3} + \frac{16\bar{M}^2}{\kappa l}\left(\frac{l}{3\bar{r}}\right)^{4/3}.$$

Finally, the rigid mass of the soliton is given by

$$\mathcal{M}_{\mathrm{sol}} = \frac{12\pi(\alpha^2+4)^{1/3}}{2^{1/3}\kappa(\alpha + \sqrt{\alpha^2+4})^{2/3}}, \tag{35}$$

which again allows us to rewrite the entropy (31) *à la* Cardy, providing an additional realization of the general formula (23).

## III. CONCLUSIONS

In this paper, we confirm diverse general results concerning Lifshitz black holes in new concrete examples. More precisely, we are interested in identifying the mass of any reasonable asymptotically Lifshitz configuration. For this, we successfully test the quasilocal formulation of conserved charges recently proposed in [18,19] and based on the off-shell extension of the ADT formalism [20]. We focus our attention on the three-dimensional case where the advantage lies in the fact that our expressions for the masses can be checked, on one hand, by using the first law of black hole thermodynamics and, on the other hand, by independently verifying the anisotropic generalization of the Cardy formula [17]. In order to achieve this task, we supplement the action of new massive gravity [13], which already supports a Lifshitz black hole in vacuum [7], with the one of a self-interacting scalar field nonminimally coupled to gravity. Some of the authors have proven that this is a useful strategy to enlarge the zoo of three-dimensional Lifshitz black holes [15]. We start by studying a family of solutions formerly found in Ref. [15] for the generic dynamical exponent $z$ and characterized by a nonvanishing Wald entropy. Later, we concentrate in the exponent $z = 3$, relevant for the vacuum [7], but excluded from the nontrivial configurations exhibited in [15]. We find a new family of Lifshitz black holes for a generic value of the nonminimal coupling parameter $\xi$. Both families allow massive and quartic contributions in their self-interactions; however, the last solution is enhanced for $\xi = 3/20$ by turning on also a cubic contribution. We derive the black hole mass of each of these solutions through the generalization of the ADT formalism. The advantages of this method lie essentially in the fact that the expression for the mass can be obtained without assuming *a priori* any asymptotic conditions, without linearizing the equations of motion, and uniquely requiring to work out with the Lagrangian and the appropriate Killing vector. We compute the Wald entropy and check that the first law of black hole thermodynamics is valid in these three cases for the obtained mass. In fact, all of them satisfy an anisotropic version of the Smarr formula saying the mass is a homogeneous function of the entropy with degree $z + 1$. We operate a completely independent verification of the results of the method in three steps. We first derive the corresponding soliton solution for the three different classes of solutions; second, we compute their respective mass using the same quasilocal formulation of conserved charges. In order to apply the method in these cases, we use a one-parameter family of solutions locally equivalent to the solitons that properly have no integration constants and consistently obtain a fixed value for their masses. Finally, we confirm the validity of the anisotropic generalization of the Cardy formula (8) obtained from holographic arguments under the assumption that the soliton plays in the bulk the role of the ground state of a nonrelativistic boundary theory [17]. The family (11) was originally derived in Ref. [15] together with other two classes of Lifshitz black hole solutions. It is simple to verify that the two remaining classes have a zero Wald entropy. In the interest of performing a cross-check of the efficiency of the quasilocal method, we verify that the generalized ADT formalism yields to a zero mass in these cases, which again fits consistently with the first law and the other tested formulas. In addition to the zero mass Lifshitz black holes produced in the studied theory, the three examples analyzed in the paper contain Lifshitz black holes with negative mass. It is important to emphasize that our checking of the first law is performed independently of the sign of the mass. The same happens for the Smarr formula. Regarding the anisotropic Cardy formula, we point out a subtlety for these cases, where, additionally, the mass of the corresponding soliton is positive. This invalidates the holographic interpretation provided in Ref. [15]; however, their semiclassical arguments still apply, which allow a general writing of their formula involving the absolute values of the masses and a sign correction in the entropy compatible with the first law. All the cases under study are compatible with this general formula (23). Returning to the Smarr formula, it would be interesting to derive the expression (15) by exploiting a scaling symmetry of the field equations and to derive the corresponding Noether conserved current. Evaluating this latter at infinity and at the horizon and equating these two expressions, the expectation is to obtain the anisotropic Smarr formula, generalizing the results of [29]. Finally, we would like to stress that it is worth to pursue testing of this method by obtaining the mass of higher-dimensional Lifshitz black holes. This task has been started in [19] for special cases of the vacuum configurations with square gravity corrections found in [9], and for which it is known that the naive extension of the ADT formalism does not work [22]. See, also, [30] for a different perspective based on the first law.

## ACKNOWLEDGMENTS

This work has been partially funded by FONDECYT Grants No. 1121031, No. 1130423, No. 1141073, No. 11090281, No. 11121651, CONACYT Grants No. 175993 and No. 178346, and CONICYT Grants No. 80130051 and No. 79112034. This project was partially funded by Proyectos CONICYT, Research Council UK (RCUK) Grant No. DPI20140053. E. A.-B. is supported by the "Programa Atracción de Capital Humano Avanzado del Extranjero, MEC" from CONICYT. M. B.-G. is supported by the BECA DOCTORAL CONICYT Grant No. 21120271. F. C. is also supported by the Alexander von Humboldt Foundation. M. H. is also supported by the Departamento de Relaciones Internacionales "Programa Regional MATHAMSUD 13 MATH-05." M. M. J.-A. received financial support from the "Programa de Becas Nacionales" of CONACYT, "Plataforma de Movilidad Estudiantil Alianza del Pacífico" of the Agencia de Cooperación Internacional de Chile, and "Apoyo para obtención de grado" of CINVESTAV. CECs is funded by the Chilean Government through the Centers of Excellence Base Financing Program of CONICYT.

[1] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999).
[2] S. Kachru, X. Liu, and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).
[3] M. Taylor, arXiv:0812.0530.
[4] D.-W. Pang, J. High Energy Phys. 01 (2010) 116.
[5] H. Maeda and G. Giribet, J. High Energy Phys. 11 (2011) 015.
[6] A. Alvarez, E. Ayón-Beato, H. A. González, and M. Hassaïne, J. High Energy Phys. 06 (2014) 041.
[7] E. Ayon-Beato, A. Garbarz, G. Giribet, and M. Hassaine, Phys. Rev. D **80**, 104029 (2009).
[8] R. G. Cai, Y. Liu, and Y. W. Sun, J. High Energy Phys. 10 (2009) 080.
[9] E. Ayon-Beato, A. Garbarz, G. Giribet, and M. Hassaine, J. High Energy Phys. 04 (2010) 030.
[10] J. Matulich and R. Troncoso, J. High Energy Phys. 10 (2011) 118.
[11] J. Oliva and S. Ray, Phys. Rev. D **86**, 084014 (2012).
[12] A. Giacomini, G. Giribet, M. Leston, J. Oliva, and S. Ray, Phys. Rev. D **85**, 124001 (2012).
[13] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[14] E. Ayón-Beato, M. Hassaïne, and M. M. Juárez-Aubry, Phys. Rev. D **90**, 044026 (2014).
[15] F. Correa, M. Hassaine, and J. Oliva, Phys. Rev. D **89**, 124005 (2014).
[16] F. Correa, C. Martinez, and R. Troncoso, J. High Energy Phys. 01 (2011) 034; 02 (2012) 136; F. Correa, A. Faundez, and C. Martinez, Phys. Rev. D **87**, 027502 (2013).
[17] H. A. Gonzalez, D. Tempo, and R. Troncoso, J. High Energy Phys. 11 (2011) 066.
[18] W. Kim, S. Kulkarni, and S. H. Yi, Phys. Rev. Lett. **111**, 081101 (2013).
[19] Y. Gim, W. Kim, and S. H. Yi, J. High Energy Phys. 07 (2014) 002.
[20] L. F. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982); Phys. Lett. **116B**, 259 (1982); S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002); Phys. Rev. D **67**, 084009 (2003); C. Senturk, T. C. Sisman, and B. Tekin, Phys. Rev. D **86**, 124030 (2012).
[21] R. L. Arnowitt, S. Deser, and C. W. Misner, Gen. Relativ. Gravit. **40**, 1997 (2008).
[22] D. O. Devecioglu and O. Sarioglu, Phys. Rev. D **83**, 021503 (2011); **83**, 124041 (2011).
[23] R. M. Wald and A. Zoupas, Phys. Rev. D **61**, 084027 (2000); G. Barnich and F. Brandt, Nucl. Phys. **B633**, 3 (2002); G. Barnich, Classical Quantum Gravity **20**, 3685 (2003); G. Barnich and G. Compere, Phys. Rev. D **71**, 044016 (2005); **73**, 029904(E) (2006); J. Math. Phys. **49**, 042901 (2008).
[24] S. Hyun, J. Jeong, S. A. Park, and S. H. Yi, arXiv:1410.1312.
[25] R. M. Wald, Phys. Rev. D **48**, R3427 (1993); V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[26] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973); **30**, 521(E) (1973).
[27] O. Hohm and E. Tonni, J. High Energy Phys. 04 (2010) 093.
[28] Y. S. Myung, Y. W. Kim, and Y. J. Park, Eur. Phys. J. C **70**, 335 (2010).
[29] M. Banados and S. Theisen, Phys. Rev. D **72**, 064019 (2005).
[30] Z. Y. Fan and H. Lu, arXiv:1501.00006.