

Numerical determination of the operator-product-expansion coefficients in the 3D Ising model from off-critical correlators

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We propose a general method for the numerical evaluation of operator product expansion coefficients in three dimensional conformal field theories based on the study of the conformal perturbation of two point functions in the vicinity of the critical point. We test our proposal in the three dimensional Ising model, looking at the magnetic perturbation of the $\langle\sigma(\mathbf{r})\sigma(0)\rangle$, $\langle\sigma(\mathbf{r})\epsilon(0)\rangle$ and $\langle\epsilon(\mathbf{r})\epsilon(0)\rangle$ correlators from which we extract the values of $C_{\sigma\epsilon}^\sigma = 1.07(3)$ and $C_{\epsilon\epsilon}^\epsilon = 1.45(30)$. Our estimate for $C_{\sigma\epsilon}^\sigma$ agrees with those recently obtained using conformal bootstrap methods, while $C_{\epsilon\epsilon}^\epsilon$, as far as we know, is new and could be used to further constrain conformal bootstrap analyses of the 3d Ising universality class.

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I. INTRODUCTION

In the past few years a renewed interest in the conformal bootstrap approach [1] to three dimensional conformal field theories (CFT) led to a set of remarkable results [2–19] on the universality classes of several 3d statistical models. Among these models a particular attention has been devoted to the 3d Ising case, both for its physical relevance and for the fact that in this case very precise numerical estimates exist for the scaling dimensions [20], which allowed highly nontrivial tests of the conformal bootstrap results. It would be interesting to perform a similar comparison also for the operator product expansion (OPE) coefficients [21], but for these constants Monte Carlo estimates with an accuracy comparable with that of the scaling dimensions are still lacking. As an attempt to fill this gap we propose in this paper a general strategy for the numerical estimate of the structure constants and, as a proof of concept, we evaluate $C_{\sigma\epsilon}^\sigma$ and $C_{\epsilon\epsilon}^\epsilon$ in the Ising case. Our estimates should be considered mainly as an exploratory study on the effectiveness of the procedure with a limited use of computer power. We plan in a forthcoming paper to obtain more precise estimates for these and other OPE coefficients.

It is well known that estimating structure constants is much more difficult than estimating scaling dimensions. The most naive approach would be to extract them directly from three point functions (i.e. the connected correlators of three operators) at criticality. However it is easy to see that the subtraction in the connected correlator of the two point components leads to a mix of contributions with similar exponents which makes it very difficult to extract the sought for OPE coefficients. In this paper we propose an alternative method based on conformal perturbation which allows to overcome this problem. The main idea is to study the perturbed two point functions using the OPE to rewrite them as sums of perturbed one point functions and then use the known scaling behavior of one point functions and structure constants to write the perturbed two point function

as an expansion in powers of the scaling variable. Following [21] it is always possible to write the short distance behavior of two-point correlators in the vicinity of a critical point as a short distance expansion:

$$\langle O_i(\mathbf{r})O_j(\mathbf{0})\rangle_h = \sum_k C_{ij}^k(r; h) \langle O_k(\mathbf{0})\rangle_h \quad (1)$$

where $\langle \dots \rangle_h$ denotes the expectation values with respect to the perturbed action, the O_i represent a complete set of operators of the conformal theory and the $C_{ij}^k(r, h)$ are the Wilson coefficients. While the one point functions $\langle O_k(\mathbf{0})\rangle_h$ cannot be determined from the knowledge of the critical correlators and take care of the nonperturbative physics of the model, the Wilson coefficients can be obtained from integrals of critical correlators. Moreover it was shown in [22] that the n th order derivatives of the Wilson coefficients with respect to the perturbing parameter of the theory are always infrared finite. The structure constants can then be easily extracted from the coefficients of the short distance expansion (see [22–24] for further details). In practice one is limited for numerical reasons to the first few coefficients (typically the first two or three) which however (combining all the possible two points functions and relevant perturbations) are enough to obtain all the structure constants among relevant operators.

As it is easy to see the method is based on very general features of CFTs and OPEs. It can be applied to any d -dimensional CFT and in fact it was applied a few years ago to the 2d Ising model perturbed by a magnetic field [23,24], to various perturbations of the Tricritical Ising model [25] and then used to identify the signatures of subleading irrelevant operators in the perturbed two point functions [26]. In all these cases the structure constants were already known from the exact solution of Belavin Polyakov and Zamolodchikov [27] and the main emphasis was on comparing the perturbative results with the numerical simulations.

The aim of this paper is to apply this strategy to the three dimensional case. In particular we shall address, as an example, the magnetic perturbation of the 3d Ising model.

II. ISING MODEL PERTURBED BY A MAGNETIC FIELD h

The action of the continuum Ising model in 3D perturbed by a magnetic field is

$$S = S_{\text{cft}} + h \int \sigma(\mathbf{r}) d\mathbf{r}, \quad (2)$$

where S_{cft} is the action of the conformal field theory which describes the model at the critical point.

At the conformal point there are two quasi-primary relevant fields σ and ϵ , whose dimensions are given by $\Delta_\sigma = 0.51814(5)$, $\Delta_\epsilon = 1.41275(25)$ [20].

The VEVs acquire a dependence on h , that can be fixed using renormalization group arguments:

$$\langle \sigma \rangle = A_\sigma h^{\frac{\Delta_\sigma}{\Delta_h}} \quad (3)$$

$$\langle \epsilon \rangle = A_\epsilon h^{\frac{\Delta_\epsilon}{\Delta_h}}. \quad (4)$$

In order to simplify notations let us introduce the scaling variable $t = |h|r^{\Delta_h}$, and an adimensional version of the structure constants and their derivatives:

$$C_{ij}^k \equiv \lim_{r \rightarrow \infty} C_{ij}^k(r; 0) r^{\dim(C_{ij}^k(r; 0))}$$

$$\partial_h C_{ij}^k \equiv \lim_{r \rightarrow \infty} \partial_h C_{ij}^k(r; 0) r^{\dim(\partial_h C_{ij}^k(r; 0))}.$$

With these definitions the short distance expansion of σ and ϵ correlators becomes:

$$\begin{aligned} r^{2\Delta_\sigma} \langle \sigma(\mathbf{r}) \sigma(0) \rangle &= C_{\sigma\sigma}^1 + A_\epsilon C_{\sigma\sigma}^\epsilon t^{\frac{\Delta_\epsilon}{\Delta_h}} \\ &+ A_\sigma \partial_h C_{\sigma\sigma}^\sigma t^{\frac{\Delta_\sigma}{\Delta_h} + 1} + O(t^2) \end{aligned} \quad (5)$$

$$\begin{aligned} r^{\Delta_\sigma + \Delta_\epsilon} \langle \sigma(\mathbf{r}) \epsilon(0) \rangle &= A_\sigma C_{\sigma\epsilon}^\sigma t^{\frac{\Delta_\sigma}{\Delta_h}} + \partial_h C_{\sigma\epsilon}^1 t \\ &+ A_\epsilon \partial_h C_{\sigma\epsilon}^\epsilon t^{\frac{\Delta_\epsilon}{\Delta_h} + 1} + O\left(t^{\frac{\Delta_\sigma}{\Delta_h} + 2}\right) \end{aligned} \quad (6)$$

$$\begin{aligned} r^{2\Delta_\epsilon} \langle \epsilon(\mathbf{r}) \epsilon(0) \rangle &= C_{\epsilon\epsilon}^1 + A_\epsilon C_{\epsilon\epsilon}^\epsilon t^{\frac{\Delta_\epsilon}{\Delta_h}} \\ &+ A_\sigma \partial_h C_{\epsilon\epsilon}^\sigma t^{\frac{\Delta_\sigma}{\Delta_h} + 1} + O(t^2) \end{aligned} \quad (7)$$

where $\Delta_h = 3 - \Delta_\sigma = 2.48186(5)$ is the dimension of the magnetic field and with the standard CFT normalizations we may set $C_{\epsilon\epsilon}^1 = C_{\sigma\sigma}^1 = 1$.

A. Conversion from lattice to continuum normalizations

The partition function of the 3d Ising model is given by:

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta \left(\sum_{\langle i, j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i \right)} \quad (8)$$

where $\langle i, j \rangle$ indicates nearest neighbors sites in the lattice which we assume to be a three-dimensional cubic lattice of size L . Fixing β at its critical value $\beta_c = 0.2216544$ [28] and defining $h_l = \beta_c H$, we have:

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta_c \sum_{\langle i, j \rangle} \sigma_i \sigma_j + h_l \sum_i \sigma_i}. \quad (9)$$

It is natural to define the lattice discretization of σ as $\sigma^l \equiv \frac{1}{L^3} \sum_i \sigma_i$, so that its mean value coincides with the magnetization. In a similar way one can define the energy operator ϵ^l as $\epsilon^l \equiv \frac{1}{3L^3} \sum_{\langle i, j \rangle} \sigma_i \sigma_j - \epsilon_{\text{cr}}$, where ϵ_{cr} is the energy at the critical point, so that the mean value of ϵ^l coincides with the singular part of the internal energy.

It is important to notice that measuring the mean values (3), (4) and the two point functions (5), (6), (7) on the lattice we find the lattice versions of the amplitudes A and structure constants C (which we shall denote in the following with the index l : A^l and C^l). To relate these constants with the continuum ones we must first fix the relative normalization of σ^l , ϵ^l and h_l with respect to σ , ϵ and h . The simplest way to do this is by comparing the correlators at the critical point. In fact from

$$\langle \sigma_i \sigma_j \rangle = \frac{R_\sigma^2}{|r_{ij}|^{2\Delta_\sigma}}, \quad (10)$$

we get (assuming the standard normalization of the two point function in the continuum) $\sigma^l = R_\sigma \sigma$. By considering the ϵ^l correlator we get similarly $\epsilon^l = R_\epsilon \epsilon$. Finally, h_l is fixed by matching the lattice and continuum perturbation terms: $h_l = R_\sigma^{-1} h$.

Using R_σ and R_ϵ we can easily obtain continuum amplitudes and OPE constants from the lattice ones. For example from Eq. (4) we get $A_\epsilon^l = R_\sigma^{\frac{\Delta_\epsilon}{\Delta_h}} R_\epsilon A_\epsilon$ and from this and Eqs. (5) and (10) we have the rescaling for the structure constant: $C_{\sigma\sigma}^{\epsilon l} R_\epsilon = C_{\sigma\sigma}^\epsilon$.

III. MONTE CARLO SIMULATIONS

A. Estimating R_σ and R_ϵ

As a first step we extracted the values of R_σ and R_ϵ from a finite size scaling analysis of the two point correlators at the critical point. Table I summarizes our results. We are not aware of independent estimates of these quantities except for ϵ_{cr} which was estimated in [29] as $\epsilon_{\text{cr}} = 0.3302022(5)$ in good agreement with our result. The quoted uncertainties

TABLE I. Rescaling and other useful constants.

R_σ	0.550 (4)
R_ϵ	0.237 (3)
A_σ^l	1.0125 (5)
A_ϵ^l	0.608 (1)
ϵ_{cr}	0.330213 (12)

combine both statistical and systematic errors. It should be noticed, in view of possible future improvements of our analysis, that the main source of error was due to the infinite volume extrapolation of our results (we used lattices in the range $120 \leq L \leq 200$).

B. Perturbed correlators

To extract the structure constants we then estimated the same correlators in presence of a small magnetic field h_l . The choice of h_l is constrained by two main requirements: h_l should be small enough to keep the correlation length ξ as large as possible, at the same time it should not be too small to avoid finite size effects. The optimal choice is thus fixed by the maximal lattice size which we could simulate with the computer resources at our disposal, which was fixed to be $L = 200$. Given this constraint the optimal range turned out to be $2 \times 10^{-5} < h_l < 0.75 \times 10^{-5}$. In this range the correlation length spans from about 20 to 40 lattice spacings, allowing us to sample a sufficient number of different distances of the correlators.

For all values of h we performed our simulations on lattices of size $L = 200$ using a state of art Monte Carlo algorithm with about 10^7 configurations for each simulation. We fitted the correlators with Eqs. (5), (6), (7), setting $C_{\sigma\sigma}^1 = C_{\epsilon\epsilon}^1 = 1$, using $A_\sigma, A_\epsilon, \Delta_\sigma, \Delta_\epsilon$ as fixed inputs and keeping as only free parameters the structure constants. It turned out that in all correlators the last terms quoted in Eqs. (5), (7) and the last two terms in Eq. (6) were negligible within the errors and in all three cases we ended up with a linear fit with only one free parameter. A major source of uncertainty in our estimates is the systematic error due to the uncertainty in the estimates of R_σ and R_ϵ . We quote separately in the following these errors (which we report in square brackets) from the statistical ones. Due to the different scaling behavior, the relative size of statistical and systematic errors is different for the three correlators. For $C_{\sigma\sigma}^\epsilon$ estimated from $\langle \sigma(\mathbf{r})\sigma(0) \rangle$ the two errors are of similar magnitude. $C_{\epsilon\epsilon}^\epsilon$ extracted from $\langle \epsilon(\mathbf{r})\epsilon(0) \rangle$ is dominated by the systematic error while for $C_{\sigma\epsilon}^\sigma$ extracted from $\langle \sigma(\mathbf{r})\epsilon(0) \rangle$ the systematic error is negligible (due to the fact that the critical correlator vanishes in this case).

The tables below report our results for the structure constants. r_{\min} and r_{\max} denote the range of distances included in the fit. In all cases we chose $r_{\min} = 5$. We verified that in all three cases this was enough to eliminate lattice artifacts and to neglect short distance subleading contributions in the correlators.

TABLE II. Results for the structure constant $C_{\sigma\sigma}^\epsilon$ obtained from the spin-spin correlator. We report systematic errors in square brackets.

h_l	r_{\min}	r_{\max}	$C_{\sigma\sigma}^\epsilon$
2×10^{-5}	6	20	1.06 (3) [7]
10^{-5}	5	35	1.04 (2) [7]
0.85×10^{-5}	5	35	1.06 (3) [8]
0.75×10^{-5}	5	40	1.07 (3) [8]

TABLE III. Results for the structure constant $C_{\epsilon\epsilon}^\epsilon$ obtained from the energy-energy correlator. We report systematic errors in square brackets.

h_l	r_{\min}	r_{\max}	$C_{\epsilon\epsilon}^\epsilon$
2×10^{-5}	5	16	1.47 (10) [30]
10^{-5}	5	18	1.42 (10) [50]
0.85×10^{-5}	5	18	1.43 (12) [50]

TABLE IV. Results for the structure constant $C_{\sigma\epsilon}^\sigma$ obtained from the spin-energy correlator.

h_l	r_{\min}	r_{\max}	$C_{\sigma\epsilon}^\sigma$
2×10^{-5}	5	16	1.10 (3)
10^{-5}	5	16	1.07 (4)
0.85×10^{-5}	5	14	1.07 (5)

Looking at Tables II, IV we see that, within our statistical errors $C_{\sigma\sigma}^\epsilon = C_{\sigma\epsilon}^\sigma$. Combining the results listed in Tables II, III, IV we quote as our final estimate for the two structure constants $C_{\sigma\epsilon}^\sigma = 1.07(3)$ and $C_{\epsilon\epsilon}^\epsilon = 1.45(30)$. The first value is in good agreement with a recent conformal bootstrap calculation [19]: $(C_{\sigma\epsilon}^\sigma)^2 = 1.10636(9)$ [30]. We are not aware of any estimate for the second constant $C_{\epsilon\epsilon}^\epsilon$. The very fact that it is different from zero is rather nontrivial. In fact in the two dimensional model $C_{\epsilon\epsilon}^\epsilon = 0$ as a consequence of dual symmetry. Our calculation shows that indeed this is not a generic property of the Ising model but is a specific feature of the two dimensional self-dual case.

IV. CONCLUDING REMARKS

In this paper we proposed a general strategy for the Monte Carlo estimate of OPE coefficients in d-dimensional spin models. As a proof of concept of our method we performed an exploratory study in the 3d Ising case and found a value of $C_{\sigma\epsilon}^\sigma$ in agreement with the known conformal bootstrap results and a preliminary estimate for $C_{\epsilon\epsilon}^\epsilon$ which could be used as input for further constrain existing conformal bootstrap calculations. Similarly, the knowledge of these structure constants could be used to

constrain the linear response dynamics at finite temperature of conformal quantum critical systems [31].

We see a few possible improvements of our approach:

- (i) The systematic uncertainty which is the major source of error in C_{ee}^e could be improved, following the approach proposed in [20], looking at different realizations of the 3d universality class with improved scaling behavior.
- (ii) It would be useful to combine the magnetic perturbation that we studied in this paper with other types of perturbation.

- (iii) It would be interesting to extend the method also to boundary CFTs [8] and to CFTs containing conformal defects [32,33]

We plan to address these issues in a forthcoming publication.

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