Origin of symmetric PMNS and CKM matrices

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The Pontecorvo–Maki–Nakagawa–Sakata and Cabibbo–Kobayashi–Maskawa matrices are phenomenologically close to symmetric, and a symmetric form could be used as zeroth-order approximation for both matrices. We study the possible theoretical origin of this feature in flavor symmetry models. We identify necessary geometric properties of discrete flavor symmetry groups that can lead to symmetric mixing matrices. Those properties are actually very common in discrete groups such as A_4, S_4, or $\Delta(96)$. As an application of our theorem, we generate a symmetric lepton mixing scheme with $\theta_{12} = \theta_{23} = 36.21°$; $\theta_{13} = 12.20°$, and $\delta = 0$, realized with the group $\Delta(96)$.

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I. INTRODUCTION

The properties of the fermion mixing matrices are expected to give important hints on the underlying flavor physics. Flavor symmetries [1] are an attractive and most often studied approach to explain the rather different structure of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) and Cabibbo–Kobayashi–Maskawa (CKM) mixing matrices. Literally hundreds of models have been proposed in the literature, applying many possible discrete groups in order to explain lepton and quark mixing. Instead of adding simply another model to that list, we study in this paper an interesting possible property of both the CKM and PMNS matrices. Namely, despite the fact that the CKM mixing is a small while the PMNS mixing is large, both can to reasonable precision be estimated to be symmetric. The symmetric form of the CKM matrix was noticed early and has been studied in many references [2–10]. After neutrino oscillation was well established, the possible symmetric PMNS matrix also attracted some attention [11–17]. The symmetric form discussed in these references includes the manifestly symmetric case ($U = U^T$) and the Hermitian case ($U = U^\dagger$). It is easy to get the relation

$$(U = U^T) \Rightarrow (|U| = |U|^T) \Leftarrow (U = U^\dagger) \tag{1}$$

by taking absolute values, which implies any physical prediction from $|U| = |U|^T$ can also be used in the other two cases $U = U^T$ or $U = U^\dagger$. Both of them are special cases of $|U| = |U|^T$, which is what we mean by symmetric mixing matrix from now on.

Using the global fits of the CKM [18] and PMNS [19] matrices, one finds

$$|U_{\rm CKM}| = \begin{pmatrix} \begin{pmatrix}0.97441\\0.97413\end{pmatrix} & \begin{pmatrix}0.22597\\0.22475\end{pmatrix} & \begin{pmatrix}0.00370\\0.00340\end{pmatrix} \\ \begin{pmatrix}0.22583\\0.22461\end{pmatrix} & \begin{pmatrix}0.97358\\0.97328\end{pmatrix} & \begin{pmatrix}0.0426\\0.0402\end{pmatrix} \\ \begin{pmatrix}0.00919\\0.00854\end{pmatrix} & \begin{pmatrix}0.0416\\0.0393\end{pmatrix} & \begin{pmatrix}0.99919\\0.99909\end{pmatrix} \end{pmatrix} \tag{2}$$

$$|U_{\rm PMNS}| = \begin{pmatrix} \begin{pmatrix}0.845\\0.791\end{pmatrix} & \begin{pmatrix}0.592\\0.512\end{pmatrix} & \begin{pmatrix}0.172\\0.133\end{pmatrix} \\ \begin{pmatrix}0.521\\0.254\end{pmatrix} & \begin{pmatrix}0.698\\0.455\end{pmatrix} & \begin{pmatrix}0.782\\0.604\end{pmatrix} \\ \begin{pmatrix}0.521\\0.254\end{pmatrix} & \begin{pmatrix}0.698\\0.455\end{pmatrix} & \begin{pmatrix}0.782\\0.604\end{pmatrix} \end{pmatrix}. \tag{3}$$

Here, the upper (lower) values in each entry are upper (lower) bounds of the matrix elements. The CKM matrix has been measured to a high precision (here, we show the 1σ range), and the relations $|U_{12}| = |U_{21}|$, $|U_{23}| = |U_{32}|$ are still quite compatible with data. The relation $|U_{13}| = |U_{31}|$ is, however, not fulfilled by data. As a symmetric mixing matrix requires that [2,11]

$$|U_{31}|^2 - |U_{13}|^2 = |U_{12}|^2 - |U_{21}|^2 = |U_{23}|^2 - |U_{32}|^2 = 0, \tag{4}$$

we have an interesting option, namely, that some flavor symmetry or other mechanism generates $|U_{12}| = |U_{21}|$ and $|U_{23}| = |U_{32}|$, but $U_{13} = U_{31} = 0$. Higher-order corrections, which are frequently responsible for the smallest mixing angles, are then the source of nonzero $|U_{13}| \neq |U_{31}|$ as well as of CP violation. Rather trivially, matrices with

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only one mixing angle are symmetric, and the same holds for the unit matrix.

The symmetry conjecture for the PMNS mixing is less compatible with data, as shown by the 3σ bounds in Eq. (3) [20]. Similar to the quark sector, the 13 and 31 elements are incompatible with symmetry (the other two relations between the elements are also not favored by data), and a similar situation as mentioned above for the CKM matrix might be realized. Of course, one could also imagine that an originally symmetric mixing matrix is modified by higher-order corrections, vacuum expectation value (VEV) misalignment, renormalization group effects, or other mechanisms that have been studied in the literature.

For completeness, we give the phenomenological prediction of a symmetric mixing matrix, using the standard parametrization of the CKM and PMNS mixing matrices [11]:

$$|U_{13}| = \frac{\sin\theta_{12}\sin\theta_{23}}{\sqrt{1-\sin^2\delta\cos^2\theta_{12}\cos^2\theta_{23}}+\cos\delta\cos\theta_{12}\cos\theta_{23}}. \tag{5}$$

This is the unique physical prediction of both $|U| = |U|^T$ and $U = U^T$. Note that $|U| = |U|^T$ has only one prediction as the unitarity requires that the relation in Eq. (4) is fulfilled, so once we set $|U_{13}|^2 = |U_{31}|^2$, we immediately get $|U| = |U|^T$. It is also the unique prediction of $U = U^T$ because any 3-by-3 unitary U with $|U| = |U|^T$ can be transformed to a new U' satisfying $U' = U'^T$ simply by rephasing [2,11]. Note that this does not hold for more than three generations. The Hermitian case $U = U^\dagger$ has not only the prediction of Eq. (5) but also CP conservation [13]. Thus, it would predict $\sin\theta_{13} = \tan\theta_{12}\tan\theta_{23}$.

Despite the apparent deviation from their symmetric forms, one can still use it as an attractive zeroth-order ansatz and attempt to study its theoretical origin. One option, put forward in Ref. [15], is that a single unitary matrix V diagonalizes all mass matrices of quarks and leptons at leading order, in addition to the $SU(5)$ relation $m_d = m_\ell^T$ between the down quark and charged lepton mass matrices. While it is difficult to embed this in realistic mass spectra in grand unified theories, the predictions of this scenario are that $U_{\mathrm{CKM}} = V^\dagger V = 1$ while $U_{\mathrm{PMNS}} = V^T V$ is symmetric.

In this paper, we study the origin of symmetric mixing matrices from an underlying flavor symmetry. We prove a theorem that links geometric properties of discrete symmetry groups to the symmetric form of the mixing matrices. This theorem, explained in detail in Sec. II, holds only for subgroups of $SO(3)$ with real representations and can interestingly be realized in the most often studied groups A_4 and S_4. We find a modification that holds in the complex case in Sec. III, which could be applied to subgroups of $SU(3)$, for instance, to $\Delta(96)$. We use this to reproduce a previously studied, and actually symmetric, mixing scenario for the PMNS matrix in Sec. IV.

Since our analysis links the properties of the symmetry group with the mixing matrix, we end our Introduction with a summary on how the generators of the group can be related to the matrices diagonalizing the mass matrices, following the strategy developed in Refs. [21–23]. In general, if a flavor symmetry group G is applied to, for instance, the lepton sector, then it must be broken to two residual symmetries G_ℓ and G_ν acting on the charged lepton sector and neutrino sector:

$$G \to \begin{cases} G_\nu: & S^T M_\nu S = M_\nu \\ G_\ell: & T^\dagger M_\ell T = M_\ell. \end{cases} \tag{6}$$

Here, the left-handed neutrino (assumed to be Majorana) mass matrix M_ν is invariant under the transformation $S^T M_\nu S$ for $S \in G_\nu$, and M_ℓ (defined by $m_\ell m_\ell^\dagger$, where m_ℓ is the charged lepton mass matrix) is the effective mass matrix of left-handed charged leptons, invariant under $T^\dagger M_\ell T$. Then, the diagonalizing matrices U_ν and U_ℓ defined by

$$M_\nu = U_\nu D_\nu U_\nu^T, \tag{7}$$

$$M_\ell = U_\ell D_\ell U_\ell^\dagger \tag{8}$$

can be directly determined by S and T according to [21–23]

$$U_\nu^\dagger S U_\nu = D_S, \tag{9}$$

$$U_\ell^\dagger T U_\ell = D_T, \tag{10}$$

where all D are diagonal matrices. Note that U_ν obtained from Eq. (9) does not include Majorana phases that rephase each column of U_ν. Equation (9) is independent of such rephasing, which means the Majorana phases are not determined by flavor symmetries in this approach. For Majorana neutrinos, G_ν has to be a direct product of two $\mathbb{Z}_2$, i.e., $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, to fully determine the mixing in neutrino sector [24]. For quarks, we have the same framework but since they are Dirac fermions, we do not have to be limited to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. In this case, note that S^T and U_ν^T in Eqs. (6) and (7) should be replaced with $S^\dagger$ and $U_\nu^\dagger$.

II. THEOREM FOR $|U| = |U|^T$

In this paper, as mentioned above, we define the symmetric mixing matrix as

$$|U| = |U|^T \tag{11}$$

rather than the original definition of $U = U^T$ or the Hermitian case $U = U^\dagger$. The phenomenology is the same

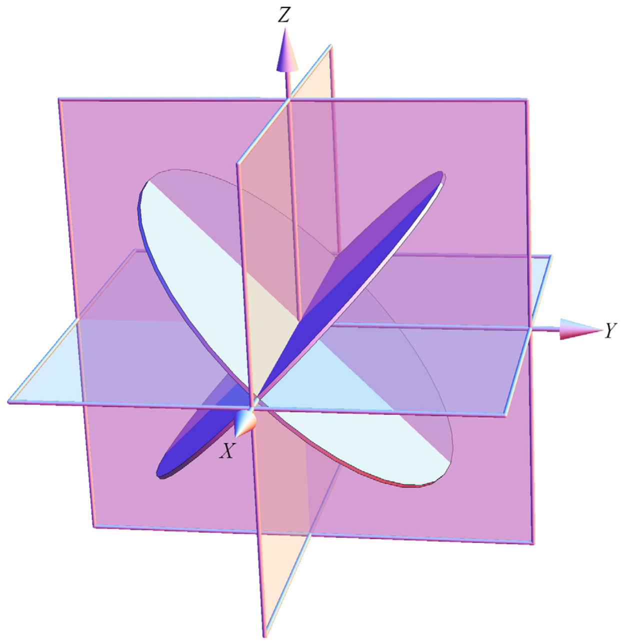

FIG. 1 (color online). The geometrical relation of the mirror planes and the bisecting planes. The mirrors are placed on the $y-z$, $z-x$, or $x-y$ plane. The two round disks are called bisecting planes because they bisect all the square mirror planes. The bisecting planes are boundaries of octants.

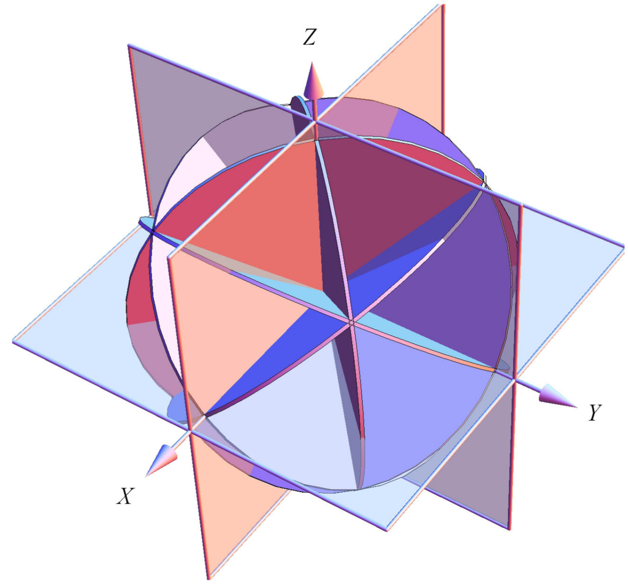

FIG. 2 (color online). The complete collection of all six possible bisecting planes and their geometrical relation with the mirror planes. A rotational symmetry with its axis on one of these bisecting planes can give, according to Theorem A, a symmetric mixing $|U| = |U|^T$.

for $|U| = |U|^T$ and $U = U^T$ but more general than the Hermitian case.

Before we formulate our theorem that links geometrical properties of the flavor symmetry group to a symmetric mixing matrix, we will first define the geometric concepts that will be used.

The $\mathbb{Z}_2$ symmetries used in neutrino sector are actually just reflections or 180° rotations (the difference between them is trivial, and any 180° rotation in three-dimensional space can be changed to a reflection if we add an overall minus sign and vice versa). In three-dimensional flavor space, going without loss of generality in the diagonal neutrino basis, the $\mathbb{Z}_2$ transformations correspond to putting minus signs to neutrino mass eigenstates: $\nu_i \to -\nu_i$. The combined $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ in three-dimensional flavor space corresponds to two reflections with respect to the direction of neutrino mass eigenstates. Or, in a picture with which we are more familiar, if there are planes of which the normal vectors are neutrino mass eigenstates, the $\mathbb{Z}_2$ symmetries are just the mirror symmetries of those planes. Since $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ contains commutative mirror transformations or since the mass eigenstates are orthogonal, the mirrors should be perpendicular to each other, as shown in Fig. 1 by translucent squares. If the transformation from G_ℓ is real (we will discuss the complex case later) in flavor space, it is an $SO(3)$ transformation that can always be represented by a rotation. If the rotation axis is on the *bisecting* plane, which is defined as the plane that bisects the two mirror squares, or as the boundaries of octants in the diagonal neutrino basis, then we define that G_ℓ bisects the two $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. We show two bisecting planes in Fig. 1, while all bisecting planes are shown in Fig. 2. This gives now all definitions necessary for our theorem. *Theorem A.—* If an $SO(3)$ subgroup G contains two noncommutative Abelian subgroups G_ν and G_ℓ, and if G_ν is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ while G_ℓ *bisects* the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, then G as a flavor symmetry can produce a symmetric mixing matrix.

The definition of *bisection* and symmetric mixing were given previously. The subgroups G_ν and G_ℓ are required to be Abelian because the residual flavor symmetries are always Abelian [21–23] and noncommutative so that the mixing is nontrivial.

The proof of this theorem will be obvious after we introduce the general $SO(3)$ rotation and the diagonalization below [see Eqs. (20) and (24)]. The $\mathbb{Z}_2$ symmetries are applied to Majorana neutrinos and the bisecting rotation to charged leptons. However, one can also apply the theorem to quarks and obtain a symmetric CKM mixing. In the case of Dirac fermions, $\mathbb{Z}_2$ is not necessary but sufficient. We will comment further on CKM mixing later. Because the axis of a bisecting rotation can be rotated on its bisecting planes, there are infinite bisecting rotations. Hence, Theorem A can produce infinite symmetric mixing matrices with 1 degree of freedom. Note that the unitary matrix with the constraint $|U| = |U|^T$ has only one prediction; see Eq. (5).

Actually a lot of discrete flavor symmetries satisfy the conditions required by Theorem A, for example, the tetrahedral group T and octahedral group O, which are

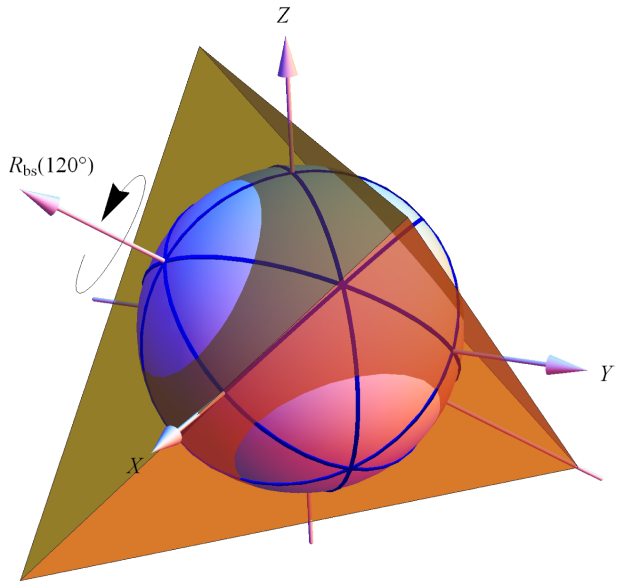

FIG. 3 (color online). Tetrahedron symmetry. The dark blue circles show the bisecting planes. The axes of the 120° rotational symmetries of the tetrahedron are on those planes; therefore, according to Theorem A, the tetrahedral group as a flavor symmetry can produce a symmetric mixing matrix $|U| = |U|^T$.

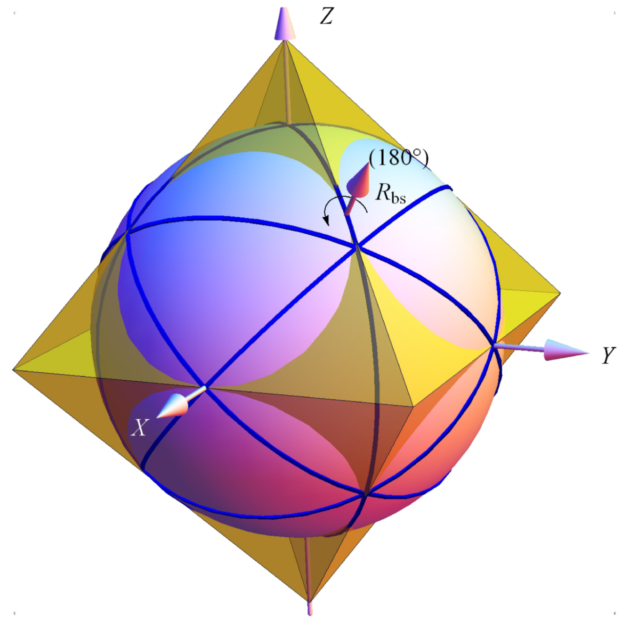

FIG. 4 (color online). Octahedron symmetry. Similar to Fig. 3, according to Theorem A, the octahedral group can also be used to produce a symmetric mixing matrix $|U| = |U|^T$.

just the widely used A_4 and S_4 flavor symmetry groups, respectively (for geometrical interpretations on A_4 and S_4, see, e.g., Ref. [25]). We can see in Fig. 3 that if we choose the three 180° rotational axes as x, y, and z axes, which penetrate the tetrahedron through the two central points of two edges, then the bisecting planes are determined, as shown by dark blues circles. The tetrahedron is also invariant under the 120° rotation marked in Fig. 3, which is a bisecting rotation since the axis is on three bisecting planes. In explicit formulas, we say the tetrahedron is invariant under the rotations

$$R_{\rm bs} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{12}$$

$$S_1 = \mathrm{diag}(1, -1, -1); \quad S_2 = \mathrm{diag}(-1, 1, -1). \tag{13}$$

Here, $R_{\rm bs}$ is the 120° bisecting rotation, and $S_i (i = 1, 2)$ are the 180° rotations around the x and y axes. Equation (12) can be obtained by requiring that $R_{\rm bs}(1, 0, 0)^T = (0, 0, 1)^T$, which means $R_{\rm bs}$ rotates the x axis to the z axis, as well as the other two relations $R_{\rm bs}(0, 1, 0)^T = (-1, 0, 0)^T$ and $R_{\rm bs}(0, 0, 1)^T = (0, -1, 0)^T$.

If $R_{\rm bs}$ and $S_i (i = 1, 2)$ are the residual symmetries of the charged lepton sector and neutrino sector, respectively, i.e.,

$$R_{\rm bs}^\dagger M_\ell R_{\rm bs} = M_\ell; \; S_i^T M_\nu S_i = M_\nu, \tag{14}$$

then according to Eqs. (9) and (10), we can compute U_ℓ and U_ν from $R_{\rm bs}$ and S_i. The result is

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ -1 & -\omega^2 & -\omega \\ 1 & 1 & 1 \end{pmatrix}; \; U_\nu = 1. \tag{15}$$

We see that U_ℓ is the Wolfenstein matrix, up to trivial signs. Therefore, in this case, the PMNS matrix $U = U_\ell^\dagger U_\nu$ is symmetric, i.e, $|U| = |U|^T$.

As another example, we show in Fig. 4 that the octahedral symmetry, which is isomorphic to the widely used S_4 symmetry, also has the required properties for Theorem A. Figures 3 and 4 show that the properties required by Theorem A are quite common in discrete groups with three-dimensional irreducible real representations.

Now we present the theorem in explicit formulas. For simplicity, we choose a basis under which the mirror has a normal vector $(1, 0, 0)^T$, $(0, 1, 0)^T$, or $(0, 0, 1)^T$, so the mirror symmetry is just a reflection with respect to the $y - z$, $z - x$, or $x - y$ plane. Then, the mirror transformations through the $y - z$ and $z - x$ planes are

$$S_1 = \mathrm{diag}(-1, 1, 1) \quad \text{and} \quad S_2 = \mathrm{diag}(1, -1, 1),$$

respectively. Under this basis, the normal vectors $\boldsymbol{n} = (n_1, n_2, n_3)^T$ of the six bisecting planes satisfy one of the six conditions,

$$|n_i| = |n_j|, \qquad (i, j = 1, 2, 3; i \neq j), \tag{16}$$

i.e., $n_1 = \pm n_2$, $n_2 = \pm n_3$, or $n_1 = \pm n_3$. The bisecting rotations $R_{\rm bs}$ with such an axis have special forms that we will show below. The neutrino and charged lepton mass matrices are now invariant under transformations of (S_1, S_2) and $R_{\rm bs}$, respectively:

$$S_i^T M_\nu S_i = M_\nu;\ (i = 1, 2) \tag{17}$$

$$R_{\rm bs}^\dagger M_\ell R_{\rm bs} = M_\ell. \tag{18}$$

Diagonalizing these matrices with the transformation

$$M_\nu = U_\nu D_\nu U_\nu^T, \qquad M_\ell = U_\ell D_\ell U_\ell^\dagger \tag{19}$$

gives the PMNS mixing matrix $U = U_\ell^\dagger U_\nu$. According to Theorem A, U will be symmetric with proper ordering of the eigenvectors.

Since we choose the basis in which S_1 and S_2 are diagonal, M_ν is constrained to be diagonal by Eq. (17), and hence U_ν is diagonal. As discussed at the end of the Introduction, diagonalization of (S_1, S_2) and $R_{\rm bs}$ will give U_ν and U_ℓ. So actually the key point of Theorem A can be stated as follows: For any $SO(3)$ matrix R, if $R = R_{\rm bs}$, there must be a unitary matrix U that is symmetric ($|U| = |U|^T$) and can diagonalize R. The converse is also true, which means $R = R_{\rm bs}$ is the necessary and sufficient condition for $|U| = |U|^T$.

Thus, in the basis we choose, we have a bisecting rotation to generate $|U_\ell| = |U_\ell|^T$ and the mirror symmetries to make U_ν diagonal, and therefore we get a symmetric PMNS matrix. In the above discussion, we have explained the theorem in a specific basis; however, the physical result is independent of any basis. One can choose another basis in which the mirrors are not on the $x-y$, $y-z$, and $z-x$ planes, in which case the neutrino sector is not diagonal and in general $|U_\ell| \neq |U_\ell|^T$. However, the geometrical relation of the bisecting planes and the mirror planes makes sure that the product $U_\ell^\dagger U_\nu$ is symmetric.

As for the explicit form of the bisecting rotation $R_{\rm bs}$, we should first introduce the general rotation. The most general rotation in Euclidean space that rotates the whole space around an axis $\boldsymbol{n} = (n_1, n_2, n_3)^T$ ($\boldsymbol{n}.\boldsymbol{n} = 1$) by an angle ϕ is

$$R(\boldsymbol{n}, \phi) = \begin{pmatrix} n_1^2 + c(n_2^2 + n_3^2) & (1-c)n_1 n_2 + s n_3 & -s n_2 + (1-c) n_1 n_3 \\ (1-c) n_1 n_2 - s n_3 & c + n_2^2 - c n_2^2 & s n_1 + (1-c) n_2 n_3 \\ s n_2 + (1-c) n_1 n_3 & -s n_1 + (1-c) n_2 n_3 & c + n_3^2 - c n_3^2 \end{pmatrix}, \tag{20}$$

where $c = \cos\phi$ and $s = \sin\phi$.

One can check that Eq. (20) does rotate the whole space around $\boldsymbol{n}$ by an angle ϕ while keeping $\boldsymbol{n}$ invariant. For example, when $\boldsymbol{n} = \boldsymbol{n}_z \equiv (0, 0, 1)^T$, we have

$$R(\boldsymbol{n}_z, \phi) = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

which is the familiar form of a rotation in the $x-y$ plane around the z axis.

For each of the six conditions in Eq. (16) we can get a bisecting rotation matrix from Eq. (20). We use the symbol $R^{(\pm ij)}$ to denote these bisecting rotations:

$$R^{(\pm ij)} \equiv R(\boldsymbol{n}|_{n_i = \pm n_j}, \phi).$$

As an example, for $n_1 = n_3$, we have

$$R^{(13)} = \begin{pmatrix} d & a & p \\ b & h & a \\ q & b & d \end{pmatrix}, \tag{22}$$

where $a = s n_1 + (1-c) n_1 n_2$, $b = (1-c) n_1 n_2 - s n_1$, and $d = c + n_1^2 (1-c)$. The remaining parameters p, q, and h are determined by $RR^T = 1$ if a, b, and d are fixed. In general, they are not equal to each other, but their precise forms are not important here. The point we should stress here is that, if R has $n_1 = n_3$, then the 12 element equals the 23 element, the 21 the 32 element, and the 11 the 33 element. Conversely, if an $SO(3)$ matrix has the form of Eq. (22), then it must be a bisecting rotation with its axis on the $x = z$ plane. This can be seen by solving Eq. (22) as an equation for $(\boldsymbol{n}, \phi)$ [the solution always exists since Eq. (20) contains all possible $SO(3)$ matrices] and finding that the solutions always have $n_1 = n_3$.

R can be diagonalized by

$$U_R^\dagger R U_R = \mathrm{diag}(e^{i\phi}, 1, e^{-i\phi}), \tag{23}$$

where the eigenvalues only depend on ϕ while U_R only depends on $\boldsymbol{n}$. As one can check numerically or by direct analytic calculation, $|U_R|$ has the following form:

$$|U_R|^2 = \frac{1}{2}\begin{pmatrix} 1 - n_1^2 & 2n_1^2 & 1 - n_1^2 \\ 1 - n_2^2 & 2n_2^2 & 1 - n_2^2 \\ 1 - n_3^2 & 2n_3^2 & 1 - n_3^2 \end{pmatrix}. \tag{24}$$

Here, $|U_R|^2$ is not $|U_R||U_R|$, but each element x_{ij} of $|U_R|^2$ is the absolute value squared of the ij element of U_R.

Note that the order of the columns in Eq. (24) can be changed since reordering of the columns of a diagonalization matrix is just a matter of permutation of eigenvectors. For $n_1 = n_3$, we recommend writing it in this order so that once one takes $n_1 = n_3$ one immediately obtains a symmetric matrix. For the other cases such as $n_1 = n_2$, etc., we can always reorder the columns to get a symmetric matrix.

From Eq. (24), the proof of Theorem A is easy. One just sets Eq. (24) equal to its transpose and finds $n_1^2 = n_3^2$. There are two other possible permutations of the columns in which $(n_1^2, n_2^2, n_3^2)^T$ is the first or the last column of U_R, from which we can get $n_2^2 = n_3^2$ or $n_1^2 = n_2^2$.

This completes our proof of Theorem A.

III. GENERALIZATION TO THE COMPLEX CASE

The previous theorem only applies for flavor symmetries with real representations, while some groups used in flavor symmetry model building enjoy complex representations. For the complex case, we cannot find a clear geometrical picture as was possible for real representations in three-dimensional Euclidean space. However, we can somewhat generalize the previous theorem to the complex case by finding some connections between the real and complex cases [26]. In the following discussion, all unitary matrices are elements of $SU(3)$ since the difference between $U(3)$ and $SU(3)$ is a trivial phase. *Theorem B.—* If an $SU(3)$ matrix T can be rephased to a real matrix R as

$$T = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) R \mathrm{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}), \tag{25}$$

if the R is one of the bisecting rotations with $n_i = n_j$ [27] from Theorem A, and if further

$$\alpha_k + \beta_k = 0 (k \neq i, j), \tag{26}$$

then T gives a symmetric mixing matrix [28].

As a note on Theorem B, k in Eq. (26) is the remaining number among $\{1, 2, 3\}$ when $|n_i| = |n_j|$ picks out two numbers for i and j. Since the rephasing matrices $\mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ and $\mathrm{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$ should be in $SU(3)$, it must hold that $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 0$. So actually Eq. (26) is equivalent to $\alpha_i + \alpha_j = -\beta_i - \beta_j$.

As an example, consider that the bisecting rotation is $R^{(13)}$ in Eq. (22); then we have $\alpha_2 + \beta_2 = 0$. In this case, T is

$$T^{(13)} = \begin{pmatrix} f + ig & a\eta_1 & p\eta_5 \\ b\eta_3 & h & a\eta_2 \\ q\eta_6 & b\eta_4 & f - ig \end{pmatrix}, \tag{27}$$

where η_i are some phases, i.e., $|\eta_i| = 1$. The 22 element is still h (it is real) and 11 element is the conjugate of the 33 element, as a result of $\alpha_2 + \beta_2 = 0$. We also have $\eta_1\eta_2\eta_3\eta_4 = 1$ because $\alpha_1 + \alpha_3 + \beta_1 + \beta_3 = 0$. $T^{(13)}$ can be diagonalized by a unitary matrix that we call U_{T13} and one can check that

$$|U_{T13}|^2 = \begin{pmatrix} \frac{t}{2} + \frac{g}{2s_\varphi} & 1 - t & \frac{t}{2} - \frac{g}{2s_\varphi} \\ 1 - t & \frac{h - c_\varphi}{1 - c_\varphi} & 1 - t \\ \frac{t}{2} - \frac{g}{2s_\varphi} & 1 - t & \frac{t}{2} + \frac{g}{2s_\varphi} \end{pmatrix}, \tag{28}$$

where $c_\varphi = \cos\varphi$, $s_\varphi = \sin\varphi$ and

$$c_\varphi = (-1 + 2f + h)/2, \tag{29}$$

$$t = 1 - \frac{1 - h}{2(1 - c_\varphi)}. \tag{30}$$

We can see that indeed $|U_{T13}| = |U_{T13}|^T$.

IV. APPLICATION

In this section, we will apply our theorems to an actual mixing scheme. After the T2K neutrino experiment measured a large nonzero θ_{13} in 2011 [29], many models have been proposed to explain the result. References [30,31] scanned a series of discrete groups [$\Delta(6n^2)$ and Γ_N], and one of the found schemes was quite close to the T2K result at the time. In the standard parametrization, the angles are

$$\theta_{23} = \theta_{12} = \tan^{-1}\frac{2}{\sqrt{3} + 1} = 36.21° \tag{31}$$

and

$$\theta_{13} = \sin^{-1}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) = 12.20°. \tag{32}$$

In total, the PMNS matrix is

$$U = \begin{pmatrix} \frac{1}{6}(3 + \sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(3 - \sqrt{3}) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}(3 - \sqrt{3}) & -\frac{1}{\sqrt{3}} & \frac{1}{6}(3 + \sqrt{3}) \end{pmatrix}. \tag{33}$$

While this mixing scheme is ruled out by current data, it fulfills our criterion of a symmetric mixing matrix and could serve as a starting point or zeroth-order approximation.

The mixing scheme can be produced in the $\Delta(96)$ group, which can be defined by three generators a, b, and c with the following properties [32]:

$$\begin{aligned} a^3 = b^2 = (ab)^2 = c^4 = 1, \\ cac a^{-1} = a^{-1}c^{-1}a = bcb^{-1} = b^{-1}cb, \\ cbc^{-1}b^{-1} = bc^{-1}b^{-1}c. \end{aligned} \tag{34}$$

In a three-dimensional faithful representation, a, b, and c can be represented by [32]

$$a^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad b^{(3)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$c^{(3)} = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

The mixing scheme is produced if a $\mathbb{Z}_3$ subgroup generated by $T = a^2(cb)^2c$ and a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ subgroup generated by $S_1 = b, S_2 = a^2c^2ac^2$ is applied to charged leptons and neutrinos, respectively. To be precise,

$$\mathbb{Z}_3 : T = \begin{pmatrix} 0 & 0 & i \\ -1 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \tag{36}$$

and the two $\mathbb{Z}_2$ are generated by

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 : S_1 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{37}$$

From our theorem, it is easy to see that this can produce symmetric mixing. In the diagonal neutrino basis, we transform T to T_d,

$$T_d = \begin{pmatrix} \frac{i}{2} & \frac{i}{\sqrt{2}} & \frac{i}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{2} & \frac{i}{\sqrt{2}} & -\frac{i}{2} \end{pmatrix}, \tag{38}$$

and (S_1, S_2) to (S_{1d}, S_{2d}),

$$S_{1d} = \mathrm{diag}(-1,-1,1); \; S_{2d} = \mathrm{diag}(-1,1,-1). \tag{39}$$

Then, according to our theorem, we see that T_d can be rephased via a transformation defined in Eq. (25) to a bisecting rotation R with $\alpha_2 = \beta_2 = 0$, $(\alpha_1, \alpha_3) = (\pi/2, -\pi/2)$, and $(\beta_1, \beta_3) = (0, 0)$. Or, in a simpler way, T_d has the form of Eq. (27). So the mixing matrix should be symmetric.

A dynamical realization of the mixing scheme in $\Delta(96)$ has been studied in Ref. [33]. That model is rather complicated using both three- and six-dimensional representations. Here, we present a simpler model that only uses two additional sets of scalar fields ϕ^ν, ϕ^ℓ, and features all particles in the same three-dimensional representation of $\Delta(96)$ of Eqs. (36) and (37),

$$\ell, \ell^c, \nu, \phi^\nu, \phi^\ell \sim 3. \tag{40}$$

We use the representation in Eqs. (36) and (37) rather than Eqs. (38) and (39) because the Clebsch—Gordan (CG) coefficients are simpler. The result does not depend on the basis. The CG coefficients we will use in this representation are

$$3 \otimes \bar{3} \to 1 : \delta_{ij} \tag{41}$$

$$3 \otimes 3 \otimes 3 \to 1 : \epsilon_{ijk} \tag{42}$$

$$3 \otimes 3 \otimes 3 \otimes 3 \to 1 : \delta_{ijmn} \tag{43}$$

$$3 \otimes 3 \otimes \bar{3} \otimes \bar{3} \to 1 : \delta_{im}\delta_{jn}\delta_{in}\delta_{jm}. \tag{44}$$

Here, ϵ_{ijk} is the Levi-Civitá tensor (or order 3 antisymmetric tensor), and δ_{ijmn} is defined as

$$\delta_{ijmn} = \begin{cases} 1 & (i = j = m = n), \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

The invariant Lagrangian in the lepton sector is

$$\begin{aligned} \mathcal{L} = {}& y_1^\ell \epsilon_{ijk} \phi_i^\ell \ell_j \ell_k^c + y_2^\ell \delta_{ijmn} \phi_i^\ell \phi_j^\ell \ell_m \ell_n^c \\ & + y_3^\ell \delta_{im}\delta_{jn} \bar{\phi}_i^\ell \bar{\phi}_j^\ell \ell_m \ell_n^c + y_1^\nu \delta_{ijmn} \phi_i^\nu \phi_j^\nu \nu_m \nu_n \\ & + y_2^\nu \delta_{im}\delta_{jn} \phi_i^\nu \phi_j^\nu \nu_m \nu_n. \end{aligned} \tag{46}$$

After symmetry breaking, ϕ^ν and ϕ^ℓ obtain the following VEVs:

$$\langle \phi^\ell \rangle = v^\ell (1, -1, i), \qquad \langle \phi^\nu \rangle = v^\nu (1, 0, 1). \tag{47}$$

In the charged lepton sector, $M_\ell = m_\ell m_\ell^\dagger$ is

$$M_\ell = |v^\ell|^2 \begin{pmatrix} u & x+iy & -ix-y \\ x-iy & u & ix-y \\ ix-y & -ix-y & u \end{pmatrix}, \tag{48}$$

while the neutrino mass matrix is

$$M_\nu = |v^\nu|^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}, \tag{49}$$

where $A = y_1^\nu + y_2^\nu; B = y_2^\nu$ with

$$u = 2|y_1^\ell|^2 + 2|y_3^\ell|^2 + |y_2^\ell + y_3^\ell|^2 \tag{50}$$

$$x = |y_1^\ell|^2 - 3|y_3^\ell|^2 - 2\mathrm{Re}[y_2^{\ell *} y_3^\ell], \tag{51}$$

$$y = 2\mathrm{Re}[y_1^{\ell *} y_2^\ell]. \tag{52}$$

M_ℓ and M_ν can be diagonalized by the following unitary matrices:

$$U_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (53)$$

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^2 & 1 & \omega \\ -\omega & -1 & -\omega^2 \\ -i & -i & -i \end{pmatrix}. \quad (54)$$

Thus, the PMNS matrix is

$$U_{\rm PMNS} = \begin{pmatrix} \frac{i+\omega}{\sqrt{6}} & -\frac{\omega^2}{\sqrt{3}} & -\frac{-i+\omega}{\sqrt{6}} \\ \frac{1+i}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{6}} \\ \frac{i+\omega^2}{\sqrt{6}} & -\frac{\omega}{\sqrt{3}} & -\frac{-i+\omega^2}{\sqrt{6}} \end{pmatrix}. \quad (55)$$

It is related to the matrix in Eq. (33) via

$$U_{\rm PMNS} = {\rm diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}) U {\rm diag}(1, e^{i\alpha_1}, e^{i\alpha_2}), \quad (56)$$

where $\beta_1 = 105°$, $\beta_2 = 225°$, $\beta_3 = 165°$, $\alpha_1 = 45°$, and $\alpha_2 = 90°$. Here, α_1 and α_2 would be the Majorana phases if the couplings y_1^ν and y_2^ν in Eq. (46) were real. Therefore, even though the Dirac-type CP is conserved in this model, generally there is still CP violation due to the nonzero Majorana phases, unless the phases of y_1^ν and y_2^ν are tuned to exactly cancel α_1 and α_2.

Our theorem can also be applied to the quark sector. One just assigns the bisecting rotational symmetry to the residual symmetry of up-type (or down type) quarks and the mirror symmetries to that of down-type (or up-type) quarks; then, the CKM mixing will be symmetric. However, building a realistic model for the CKM mixing is a somewhat more difficult task. Compared to the lepton sector in which hundreds of flavor symmetry models have been proposed, for the quark sector, much fewer models exist. This is due to the fact that the small CKM mixing angles do not have straightforward geometric interpretation, which is the basis of discrete flavor symmetry building. Among the existing models for the CKM mixing, we cannot find one that fulfills our criteria (exceptions are of course the trivial cases in which one interprets the CKM matrix as the unit matrix or as a matrix that only consists of the Cabibbo angle), and scanning all discrete groups for the flavor symmetry of quarks is out of the main purpose of this paper. Anyway, when looking for flavor groups to build models for the quark sector, our theorem could be a guidance because when a mixing scheme generated from a flavor symmetry is close to realistic CKM mixing, it must be also close to a symmetric form.

V. CONCLUSION

A possible zeroth order, but surely aesthetically attractive, mixing ansatz for the CKM and PMNS matrices is that they are symmetric. The origin of symmetric PMNS and CKM matrices from the viewpoint of flavor symmetry models has been the focus of our paper.

We have proposed a theorem on the relation between symmetric mixing matrices and geometric properties of discrete flavor symmetry groups. An illustrative connection between the rotation axes of the geometric body associated to the symmetry group exists and shows that popular subgroups of $SO(3)$ such as A_4 and S_4 can lead to symmetric mixing matrices.

Groups with complex irreducible representations do not easily allow for a geometrical interpretation, but a partial generalization of our theorem is possible, which can then apply to $SU(3)$ subgroups such as $\Delta(96)$. A previously studied mixing scheme that turns out to correspond to a symmetric PMNS matrix was used as an explicit example.

The connection of geometric properties of discrete groups and possible features of the mixing matrices may have further applications.

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[27] $n_i = -n_j$ is not necessary consider here because the minus sign can always be absorbed by rephasing in the complex case.
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