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Baby Skyrme model, near-BPS approximations, and supersymmetric extensions

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We study the baby Skyrme model as a theory that interpolates between two distinct BPS systems. For this, a near-BPS approximation can be used when there is a small deviation from each of the two BPS limits. We provide analytical explanation and numerical support for the validity of this approximation. We then study the set of all possible supersymmetric extensions of the baby Skyrme model with $\mathcal{N} = 1$ and the particular ones with extended $\mathcal{N} = 2$ supersymmetries and relate this to the above mentioned almost-BPS approximation.

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I. INTRODUCTION

The baby Skyrme model in $(2+1)$ dimensions [1,2] has been widely investigated, both for its own sake and for being a toy model of more sophisticated theories in higher dimensions. In this paper we focus our attention on its features as a theory that interpolates between two distinct BPS systems [3,4]. We note that, after a convenient rescaling, the model depends on only one parameter ζ that can be set to take values in the interval $0 \leq \zeta \leq 1$. At the edges of this interval there are two distinct BPS models: the $O(3)$ sigma model at $\zeta = 0$ and the restricted baby Skyrme model at $\zeta = 1$. Near both edges of this interval, an almost-BPS approximation can be used to obtain an analytic approximation of the soliton solution. The exact solution, which we obtain numerically for the first topological sector, flows to this approximation as the parameter ζ goes to 0 or to 1.

The near-BPS approximation has been recently used in two different physical contexts. The first one is that of the holographic QCD [5–9] and the second one is the so-called "generalized Skyrme model" [10–13]. The first case is very similar to the $\zeta \to 0$ limit in our toy model, while the second one is very similar to its $\zeta \to 1$ limits (is essentially the same in one higher dimension). In both cases, one of the main physical reasons for studying the near-BPS systems is to have a model that reproduces the small binding energies observed in nuclear physics. So the near-BPS approximation has both mathematical and phenomenological interest. Thus, looking at the baby Skyrme model, we can study this near-BPS limit in a simplified context and, in particular, provide a concrete explanation for its validity.

The baby Skyrme model possesses various supersymmetric extensions which all have in common the same bosonic sector. These supersymmetric extensions of the baby Skyrme model were first discussed in [14,15], following earlier attempts to supersymmetrize Skyrme-like theories in $3+1$ dimensions [16,17] (see also more recently [18]). These supersymmetric theories are, in general, $\mathcal{N} = 1$ supersymmetric, thus with two real supercharges, and $\mathcal{N} = 2$ at the two ends of the interval. The almost-BPS properties of the almost-BPS theory can then be understood in terms of the quantum supersymmetry algebra, at least near the $\zeta = 0$ edge.

The paper is organized as follows: In Sec. II we study the bosonic baby Skyrme model and its near-BPS limits for various choices of potentials. In Sec. III we consider the $\mathcal{N} = 1$ supersymmetric extensions of these theories. In Sec. IV we study the $\mathcal{N} = 2$ extensions and their BPS properties. We conclude in Sec. V with some open questions.

II. THE BOSONIC BABY SKYRME MODEL

The action for the $O(3) = S^2$ baby Skyrme model is

$$S = \int d^3x \left(\frac{\theta_2}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{\theta_4}{2} (\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}) \cdot (\partial^\mu \vec{\phi} \times \partial^\nu \vec{\phi}) - \theta_0 V(\vec{\phi}) \right), \tag{2.1}$$

with the target space subject to the constraint $\vec{\phi} \cdot \vec{\phi} = 1$. We consider this model for a class of potentials of the following form,

$$V(\vec{\phi}) = \left(\frac{1 - \hat{n} \cdot \vec{\phi}}{2} \right)^k, \tag{2.2}$$

where $\hat{n}$ is a unit vector and k an integer. This family of potentials contains, for example, the old baby Skyrme model for $k = 1$ and the so-called holomorphic model for $k = 4$ [1]. In addition to the arbitrariness of the functional

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form of the potential, we have, in general, three parameters $\theta_{0,2,4}$ in the model. Rescaling the action by an overall constant, and rescaling the length scale, we can effectively reduce this arbitrariness to having only a one-parameter family of Lagrangians.

Shortly we choose a parametrization which is the most convenient for us, namely, to describe the flow between two BPS systems that we want to study in this paper.

The first BPS system is the pure sigma model whose Lagrangian is given by

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}, \tag{2.3}$$

which, in the $CP(1)$ formulation, takes the form

$$\mathcal{L}_2 = \frac{1}{(1+|w|^2)^2}\partial_\mu w\partial^\mu\bar{w}. \tag{2.4}$$

This model has a BPS bound which is saturated by the holomorphic and antiholomorphic solutions,

$$E_{BPS} = 4\pi|Q|, \tag{2.5}$$

where Q is the topological charge.

The second BPS system is the so-called restricted baby Skyrme model and its Lagrangian consists of only two terms, the term with quartic derivatives and the potential term:

$$\mathcal{L}_{4,0} = -\frac{1}{2}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})\cdot(\partial^\mu\vec{\phi}\times\partial^\nu\vec{\phi}) - V(\vec{\phi}). \tag{2.6}$$

For the potential of the form (2.2), and using the $CP(1)$ formulation, the Lagrangian of the restricted baby Skyrme model is described by

$$\mathcal{L}_{4,0} = \frac{1}{(1+|w|^2)^4}\partial_\mu w\partial^\nu\bar{w}(\partial_\mu w\partial^\nu\bar{w} - \partial_\nu w\partial^\mu\bar{w}) - \frac{|w|^{2k}}{(1+|w|^2)^k}. \tag{2.7}$$

This model also has a BPS bound and its solutions satisfy

$$E_{BPS} = \frac{8\pi}{k+2}|Q|. \tag{2.8}$$

The full baby Skyrme model can be thought of as an interpolation between these two BPS systems. By rescaling the action and the length scale, we can choose the parameters in (2.1) to be of the form

$$\theta_2 = 1-\zeta, \qquad \theta_0 = \theta_4 = \frac{\zeta(k+2)}{2}, \tag{2.9}$$

and so the full Lagrangian can be written as

$$\mathcal{L} = (1-\zeta)\mathcal{L}_2 + \frac{\zeta(k+2)}{2}\mathcal{L}_{4,0}, \tag{2.10}$$

where $\mathcal{L}_2$ and $\mathcal{L}_{4,0}$ are given in (2.4) and (2.7). The parameter ζ takes the value in an interval [0, 1] and the boundaries of the interval represent the two BPS systems. Note that for this choice of parameters the total bound, which is the sum of the two BPS bounds (2.5) and (2.8), is fixed to be $4\pi|Q|$ for every value of ζ. The existence of this bound for the full system follows directly from the existence of the two bounds of the two BPS systems taken in isolation [3]. In general, the bound can be saturated only at the edges of the interval as we shall demonstrate below.

To find a one-soliton solution, we consider the radial ansatz.

$$w(r,\theta) = e^{i\theta}f(r), \tag{2.11}$$

for which the profile function $f(r)$ has to satisfy the boundary conditions $f(r\to 0) = \infty$ and $f(r\to\infty) = 0$. The energy functional in terms of $f(r)$ is now given by

$$\frac{E}{4\pi} = \int dr\left\{\frac{r(1-\zeta)}{(1+f^2)^2}\left(f'^2 + \frac{f^2}{r^2}\right) + \frac{\zeta(k+2)}{4}\left(\frac{4f'^2f^2}{r(1+f^2)^4} + \frac{rf^{2k}}{(1+f^2)^k}\right)\right\}. \tag{2.12}$$

The exact forms of this profile function can be obtained by minimizing this functional for various values of ζ. The profile functions, for all values of ζ, always diverge like $f(r)\simeq\lambda/r$ as $r\to 0$. To find the profile function numerically, we can use the "shooting method," i.e., varying the parameter λ until we find that the other boundary condition (at infinity) is also satisfied.

Next we consider a near-BPS approximation to describe the soliton solutions near the two edges of the interval. We first describe our approach in detail for the first edge, $\zeta\to 0$, i.e., the one close to the pure sigma model. This method is very similar to the one discussed in [8] for a holographic model in which the role of the potential was played by the space-time curvature. Earlier uses of this method for different theories can be found in [5–8,10,11,13]. A similar, but not equivalent, approach for the study of the baby Skyrme model can also be found in [19].

A solution of the one-soliton profile of the pure sigma model $\mathcal{L}_2$ can be taken in the form of the holomorphic function,

$$f(r) = \frac{\lambda}{r}, \tag{2.13}$$

where λ describes the scale of the lump and is a free parameter. We put this ansatz into (2.12) and determine the value of λ that minimizes the total energy. The result of the minimization gives us

$$\lambda_* = \frac{2^{1/2}(k-1)^{1/4}}{3^{1/4}}. \tag{2.14}$$

Note that this approach can be used only if $\mathcal{L}_{4,0}$, evaluated on the holomorphic ansatz, is convergent. This is true for $k>1$ and thus excludes the old baby Skyrme model which we will discuss separately. The total energy for the holomorphic ansatz, evaluated for the minimum (2.14), is then

$$E = 4\pi + 4\pi\zeta\left(\frac{k+2}{(k-1)^{1/2}2\sqrt{3}} - 1\right). \quad (2.15)$$

This result can now be used in two different ways. First of all, it provides an upper bound to the exact soliton energy, which is valid for any value of ζ. Second, in the limit $\zeta \to 0$, the exact solutions become well approximated by the holomorphic ansatz (2.13) at the scale (2.14), and (2.15) gives the correct first-order expansion of the soliton mass near $\zeta = 0$. Later we will present an analytic argument explaining why this approximation can be trusted, and we will present numerical evidence for this claim.

The other BPS approximation is very analogous in spirit. To discuss it, we first consider any solution of the restricted baby Skyrme model $\mathcal{L}_{4,0}$. Such a solution would strongly depend on the value of the parameter k, so for the moment we consider $k = 2$. The restricted baby Skyrme model has an infinitely large space of solutions, due to its area-preserving diffeomorphism invariance. The solution with radial symmetry is of the form

$$f(r) = \frac{1}{\sqrt{e^{r^2/2} - 1}}. \quad (2.16)$$

Minimizing $\mathcal{L}_2$ on the space of solutions is quite simple for the one-soliton case; we just pick the radially symmetric solution. The energy of this solution is then given by

$$E = 4\pi + 4\pi(1-\zeta)\left(\frac{\pi^2}{12} + \frac{\log 2}{2} - 1\right). \quad (2.17)$$

Again this result has a double interpretation. It is either an exact upper bound, which can be taken together with (2.15), or it is an approximate solution valid near $\zeta = 1$.

Another case we will consider explicitly is the $k = 6$ case for which the solution with radial symmetry of $\mathcal{L}_{4,0}$ is given by

$$f(r) = \sqrt{\frac{1}{\sqrt{1+r^2} - 1}}, \quad (2.18)$$

and its energy is

$$E = 4\pi + 4\pi(1-\zeta)\left(\log 2 - \frac{5}{8}\right). \quad (2.19)$$

We can determine the profile functions of the one-soliton fields numerically for $k = 2, 6$ and for various values of ζ. Our results are presented in Figs. 1 and 2. The first plot in

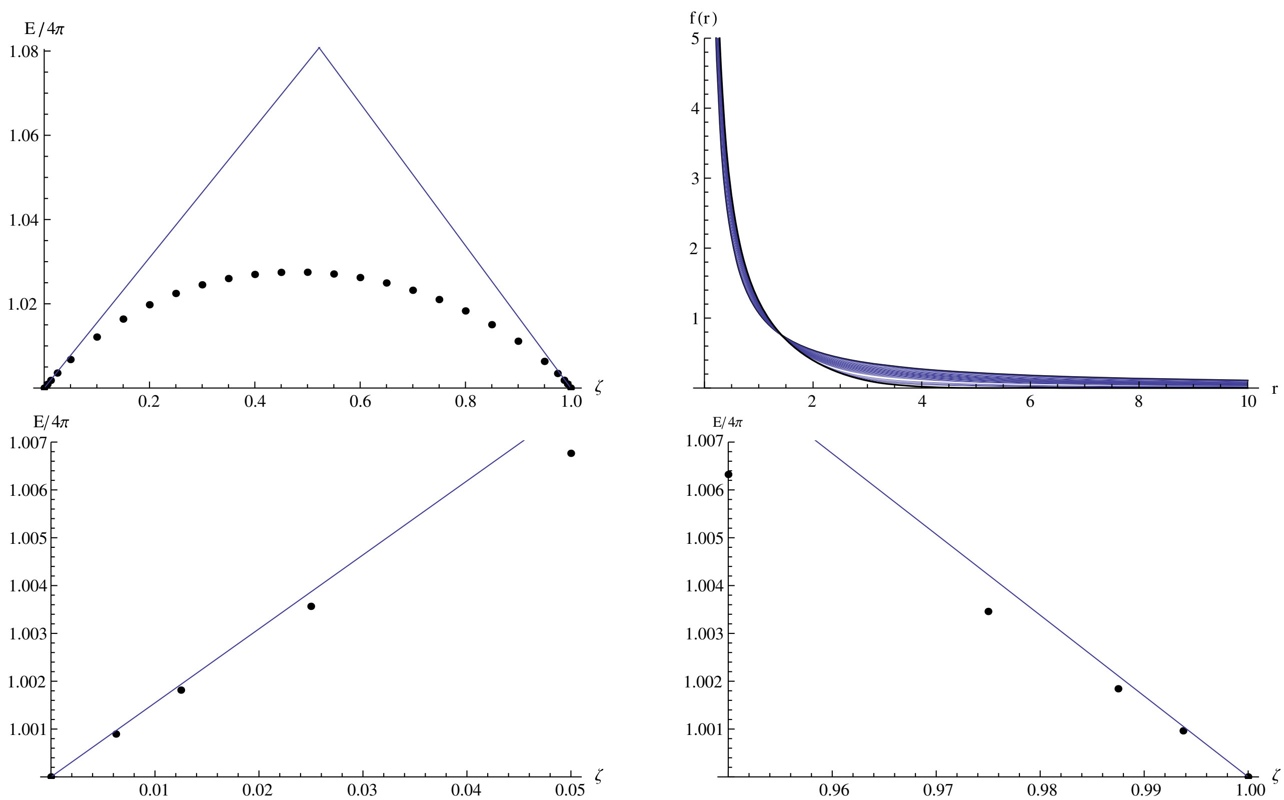

FIG. 1 (color online). In the first plot, first row, the mass of the soliton normalized to the BPS lower bound 4π for $k = 2$ is plotted as a function of ζ. The upper bounds are the two near-BPS approximations (blown up in the plots below). The second plot, first row, presents the corresponding radial profiles for $f(r)$ for various values of ζ. Thus, it shows the flow between the two almost-BPS solutions as ζ varies over the interval [0, 1]. The plots in the second row are the mass plot zoomed near the two edges of the interval.

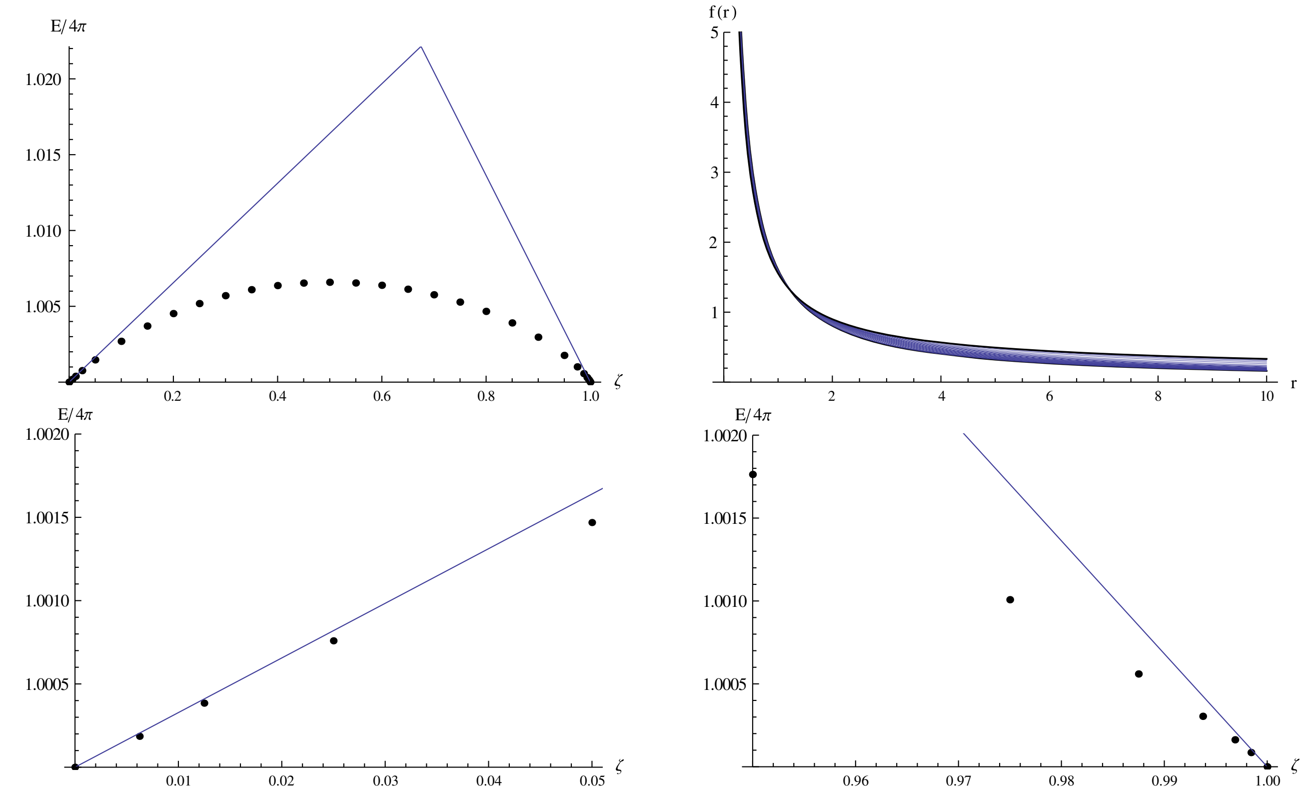

FIG. 2 (color online). As Fig. 1 but for $k = 6$.

the first row in both sets of figures is of the soliton mass, normalized to 4π, and compared with the various bounds. The plots show that the lower BPS bound and the two upper bounds (2.15) and (2.17) form a triangle. The energy near the two edges is well approximated by the upper bounds. The second plot in the first row in the figures shows the corresponding profile functions $f(r)$ for different values of ζ. It is quite clear that the full functions converge to the BPS solutions near the two edges. In the second row of the figures, we zoom near the two edges of the mass plot to show that the linear expansion is well captured by the near-BPS ansatz.

The case $k = 4$ is special. This is the case of the holomorphic potential for which a holomorphic solution for the charge one sector exists for all values of ζ. This is due to the fact that the moduli space for $\mathcal{L}_2$ and the moduli space for $\mathcal{L}_{4,0}$ intersect at one point. In this case, there is no flow and the total BPS bound is always saturated.

So we see that for the cases $k > 1$, everything works very well. The only exception is, as stated before, the case of $k = 1$ near the first edge of the interval $\eta \to 0$. Clearly, when $k = 1$ we cannot use Eq. (2.15). The reason for this is that the holomorphic solution (2.13) diverges when evaluated on $\mathcal{L}_{4,0}$. The only information we can extract from this analysis is that the solution converges to a singular holomorphic function as $\lambda_* \to 0$, and the derivative of the energy with respect to ζ is infinite at $\zeta = 0$. For the other edge, we can still use the near-BPS approximation. The solution at $\zeta = 1$ is given by the following function (with compact support):

$$f(r) = \begin{cases} \frac{4-r^2}{r\sqrt{8-r^2}} & r \leq 2 \\ 0 & r \geq 2 \end{cases}. \tag{2.20}$$

The energy evaluated for this function is given by

$$E = 4\pi + 4\pi(1-\zeta)\left(2\log 2 - \frac{17}{24}\right). \tag{2.21}$$

All this is confirmed by the numerical calculations, the results of which are presented in Fig. 3.

Now we present an explanation of why the near-BPS approximation works in a general case, using a finite-dimensional toy model. In general, the energy is of the following form,

$$E = E_{BPS}(\Phi) + \zeta V(\Phi), \tag{2.22}$$

with the property that $E_{BPS}(\Phi)$ has a flat direction in a subspace $\Phi_{BPS}(\lambda)$, while $V(\Phi)$ is a generic potential which lifts this degeneracy, and ζ is a parameter which we want to send to zero. The BPS property implies that, as we perform any expansion around a BPS solution

$$\Phi = \Phi_{BPS}(\lambda) + \Delta\Phi_{\perp BPS} + \Delta\Phi_{\parallel BPS}, \tag{2.23}$$

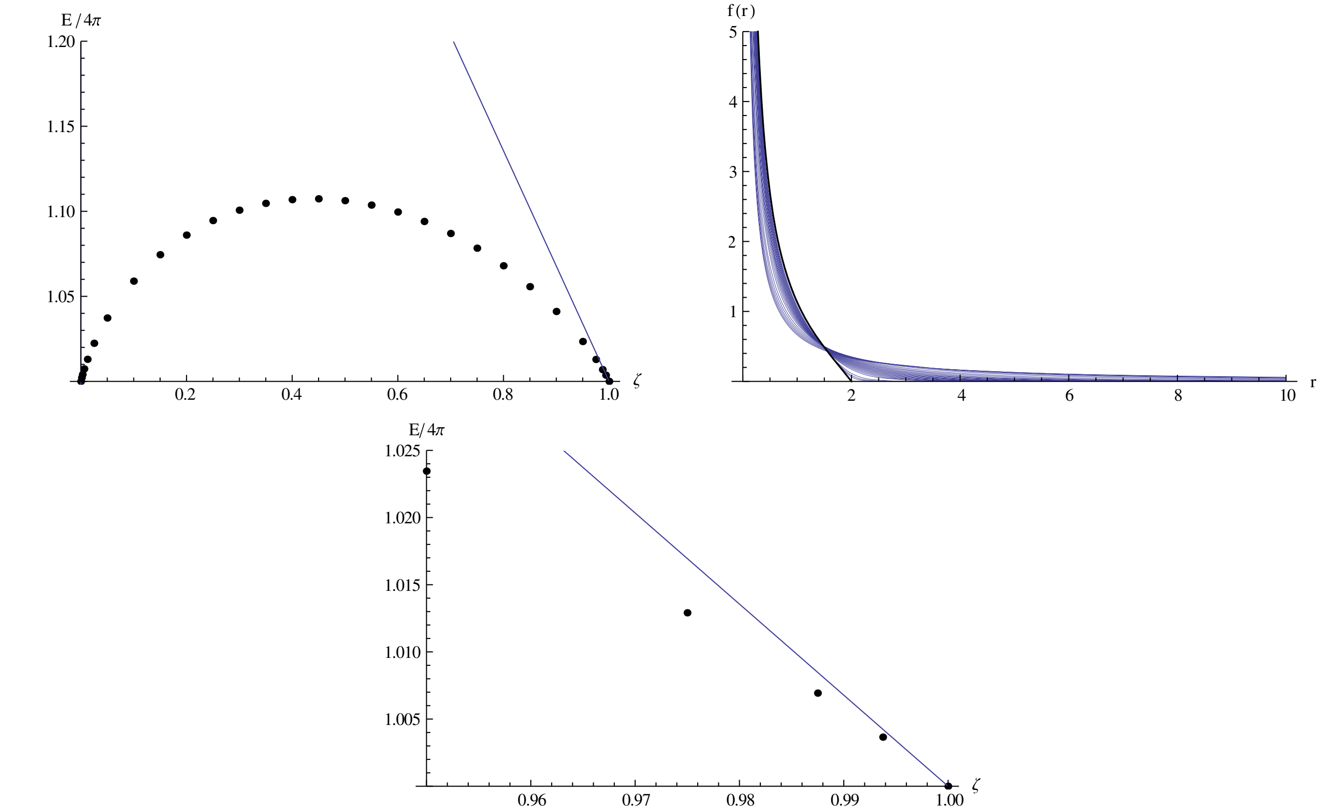

FIG. 3 (color online). As Figs. 1 and 2 but this time for $k = 1$.

the energy is sensitive only to the fluctuations in the space perpendicular to the moduli space $\delta\Phi_{\perp BPS}(x)$:

$$E_{BPS}(\Phi) = E_{BPS\,\mathrm{bound}} + \frac{1}{2}\frac{\partial^2 E_{BPS}(\Phi)}{\partial \Phi_\perp{}^2}\Delta\Phi^2_{\perp BPS} + \cdots \tag{2.24}$$

Note that the fluctuations $\Delta_\perp$ and $\Delta_\parallel$ live in a vector space, but for simplicity we avoid writing explicitly the vectorial indices.

To prove our claim we want to determine the minimum of this expression for small ζ. We use the expansion (2.23) around a generic point in the BPS moduli space. Our derivation will provide at the end the correct value $\Phi_{BPS}(\lambda)$ to which the solution is flowing as $\zeta \to 0$.

It is convenient to separate the fluctuations into two parts,

$$\begin{aligned} \Delta\Phi_{\perp BPS} &= \bar{\delta}\Phi_{\perp BPS} + \delta\Phi_{\perp BPS}, \\ \Delta\Phi_{\parallel BPS} &= \bar{\delta}\Phi_{\perp BPS} + \delta\Phi_{\perp BPS}, \end{aligned} \tag{2.25}$$

where $\bar{\delta}$ is the fluctuation of the solution around the $\zeta \to 0$ limit, while δ describes any other fluctuation which we may consider when we try to minimize the energy. The total energy expansion, up to the second order, is then given by

$$\begin{aligned} E = E_{BPS\,\mathrm{bound}} &+ \frac{1}{2}\frac{\partial^2 E_{BPS}(\Phi)}{\partial \Phi_\perp{}^2}\Delta\Phi^2_{\perp BPS} \\ &+ \zeta\Bigg(V(\Phi_{BPS}(\lambda)) + \frac{\partial V(\Phi)}{\partial \Phi_\perp}\Delta\Phi_{\perp BPS} \\ &+ \frac{\partial V(\Phi)}{\partial \Phi_\parallel}\Delta\Phi_{\parallel BPS}\frac{1}{2}\frac{\partial^2 V(\Phi)}{\partial \Phi_\perp{}^2}\Delta\Phi^2_{\perp BPS} \\ &+ \frac{\partial^2 V(\Phi)}{\partial \Phi_\parallel \partial \Phi_\perp}\Delta\Phi_{\perp BPS}\Delta\Phi_{\parallel BPS} + \frac{1}{2}\frac{\partial^2 V(\Phi)}{\partial \Phi_\parallel{}^2}\Delta\Phi^2_{\parallel BPS}\Bigg), \end{aligned} \tag{2.26}$$

where Δ's are given by (2.25).

We first evaluate the perpendicular part of the fluctuation $\bar{\delta}\Phi_{\perp BPS}$. For this we have to set to zero the term in (2.26) proportional to $\delta\Phi_{\perp BPS}$:

$$\left(\frac{\partial^2 E_{BPS}(\Phi)}{\partial \Phi_\perp{}^2}\bar{\delta}\Phi_{\perp BPS} + \zeta\frac{\partial V(\Phi)}{\partial \Phi_\perp}\right)\delta\Phi_{\perp BPS} = 0. \tag{2.27}$$

Thus, we have

$$\bar{\delta}\Phi_{\perp BPS} = -\zeta\left(\frac{\partial^2 E_{BPS}(\Phi)}{\partial \Phi_\perp{}^2}\right)^{-1}\frac{\partial V(\Phi)}{\partial \Phi_\perp}, \tag{2.28}$$

and so we see that $\bar{\delta}\Phi_{\perp BPS}$ goes to zero linearly in ζ.

Then we find the right value of λ to which the solution flows as $\zeta \to 0$ and also the fluctuation $\bar{\delta}\Phi_{\parallel BPS}$. This time the term in (2.26) is proportional to $\delta\Phi_{\parallel BPS}$ which must be set to zero:

$$\zeta\left(\frac{\partial V(\Phi)}{\partial\Phi_\parallel}+\frac{\partial^2 V(\Phi)}{\partial\Phi_\parallel\partial\Phi_\perp}\bar{\delta}\Phi_{\perp BPS}+\frac{\partial^2 V(\Phi)}{\partial{\Phi_\parallel}^2}\bar{\delta}\Phi_{\parallel BPS}\right)\delta\Phi_{\parallel BPS}=0. \tag{2.29}$$

The leading term must be set to zero separately, and this gives

$$\frac{\partial V(\Phi)}{\partial\Phi_\parallel}=0. \tag{2.30}$$

This, as anticipated before, is the condition that determines the correct point of the BPS moduli space. Setting to zero the higher-order terms in (2.29), we get the fluctuation in the parallel direction

$$\bar{\delta}\Phi_{\parallel BPS}=-\left(\frac{\partial^2 V(\Phi)}{\partial{\Phi_\parallel}^2}\right)^{-1}\frac{\partial^2 V(\Phi)}{\partial\Phi_\parallel\partial\Phi_\perp}\bar{\delta}\Phi_{\perp BPS}, \tag{2.31}$$

where $\bar{\delta}\Phi_{\perp BPS}$ is given in (2.28). So $\bar{\delta}\Phi_{\parallel BPS}$ also goes to zero linearly in ζ.

So we note that the energy evaluated on the solution has the following expansion in ζ:

$$E=E_{BPS\,\mathrm{bound}}+\zeta V(\Phi_{BPS}(\lambda))+\mathcal{O}(\zeta^2). \tag{2.32}$$

All the terms in this expression that depend on the fluctuations $\bar{\delta}\Phi_{\perp BPS}$ and $\bar{\delta}\Phi_{\parallel BPS}$ are at least of order ζ^2.

We give an illustrative example which supports these claims. It involves a two-dimensional (x,y) model with

$$E_{BPS}=x^2, \qquad V=y^2+\alpha x+\beta x^2+\gamma xy. \tag{2.33}$$

The moduli space in this case is the line $x=0$, so $\Phi_\perp$ corresponds to x and $\Phi_\parallel$ to y. The minimum of $E_{BPS}(\Phi)+\zeta V(\Phi)$ can be computed exactly in this case, and it corresponds to

$$x=-\frac{\alpha\zeta}{2+2\beta\zeta-\gamma^2\zeta/2}, \qquad y=\frac{\alpha\gamma\zeta}{4+4\beta\zeta+\gamma^2\zeta}. \tag{2.34}$$

As $\zeta\to 0$, this minimum flows to the point $(x,y)=(0,0)$ which is exactly the minimum of V restricted to the line $x=0$. Moreover, the perpendicular and parallel fluctuations as $\zeta\to 0$ are exactly the ones given by (2.28) and (2.31), namely,

$$\bar{\delta}x=-\zeta\left(\frac{\partial^2 E_{BPS}}{\partial x^2}\right)^{-1}\frac{\partial V}{\partial x}=-\frac{\alpha\zeta}{2},$$
$$\bar{\delta}y=-\left(\frac{\partial^2 V}{\partial y^2}\right)^{-1}\frac{\partial^2 V}{\partial x\partial y}\bar{\delta}x=\frac{\alpha\gamma\zeta}{4}. \tag{2.35}$$

III. THE SUPERSYMMETRIC BABY SKYRME MODEL

In this section we consider various types of supersymmetric extensions of the baby Skyrme model. We use the conventions of [20] for $\mathcal{N}=1$ supersymmetry in $(2+1)$ dimensions, apart from the metric signature which we take as $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1)$. We will follow closely the supersymmetric constructions of Refs. [14,15], but with the inclusion of some important extra terms.

First of all, let us say a few words about our notation. An $\mathcal{N}=1$ superfield in $(2+1)$ dimensions has the following expansion in Grassmannian coordinates,

$$U=u+\theta^\alpha\psi_\alpha-\theta^2 F, \tag{3.1}$$

where θ^α is a Majorana spinor. The tensors for raising and lowering the spinorial indices are $C_{\alpha\beta}=\sigma_2=-C^{\alpha\beta}$. The covariant derivative which commutes with the supersymmetry generators is given by

$$D_\alpha=\partial_\alpha+i\gamma^{\mu\beta}_\alpha\theta_\beta\partial_\mu, \tag{3.2}$$

and the gamma matrices are of the purely imaginary form:

$$\gamma^0=\sigma_2, \qquad \gamma^1=i\sigma^3, \qquad \gamma^2=i\sigma_1. \tag{3.3}$$

We consider the following terms in the Lagrangian density, of which we will write down explicitly only their bosonic terms in the action. The first term is the quadratic derivative term:

$$\mathcal{L}_2=-\int d^2\theta g(U,\bar{U})D^\alpha\bar{U}D_\alpha U$$
$$=g(u,\bar{u})(|F|^2+\partial_\mu\bar{u}\partial^\mu u)+\text{ferm}. \tag{3.4}$$

Then we have five different higher-derivative terms. The first three of them are generated by considering a superfield of the following form,

$$D_\alpha UD_\beta\bar{U}D_\xi D_\tau UD_\rho D_\sigma\bar{U}, \tag{3.5}$$

with different contractions of the spinorial indices performed with the $C^{\alpha\beta}$ tensor. The three such terms and their bosonic parts in the Lagrangian are given by

$$\mathcal{L}_{4,1}=-\frac{1}{4}\int d^2\theta h_1(U,\bar{U})D_\alpha UD_\alpha\bar{U}D_\beta D_\gamma UD^\beta D^\gamma\bar{U}$$
$$=h_1(u,\bar{u})(|F|^4+2|F|^2\partial_\mu\bar{u}\partial^\mu u+(\partial_\mu\bar{u}\partial^\mu u)^2)+\text{ferm}, \tag{3.6}$$

$$\mathcal{L}_{4,2} = -\frac{1}{2}\int d^2\theta h_2(U,\bar{U}) D_\alpha U D_\beta \bar{U} D^\alpha D^\beta U D^\gamma D^\gamma \bar{U} + \text{H.c.}$$
$$= h_2(u,\bar{u})(4|F|^4 + 8|F|^2 \partial_\mu \bar{u}\partial^\mu u - F^2 \partial_\mu \bar{u}\partial^\mu \bar{u} - \bar{F}^2 \partial_\mu u \partial^\mu u) + \text{ferm}, \quad (3.7)$$

$$\mathcal{L}_{4,3} = -\frac{1}{2}\int d^2\theta h_3(U,\bar{U}) D_\alpha U D_\beta \bar{U} D^\alpha D_\gamma U D^\beta D^\gamma \bar{U}$$
$$= h_3(u,\bar{u})(|F|^4 + 2|F|^2 \partial_\mu \bar{u}\partial^\mu u + |\partial_\mu u \partial^\mu u|^2 - F^2 \partial_\mu \bar{u}\partial^\mu \bar{u} - \bar{F}^2 \partial_\mu u \partial^\mu u) + \text{ferm}. \quad (3.8)$$

Note that the last two of these terms ($\mathcal{L}_{4,2}$ and $\mathcal{L}_{4,3}$) were not included in [14,15], and they will be important in what follows.

The remaining two higher-derivative contributions are constructed from a superfield of the form

$$D_\alpha U D_\beta U D_\xi D_\tau \bar{U} D_\rho D_\sigma \bar{U} + \text{H.c.}, \quad (3.9)$$

with different contractions of its spinorial indices. The two terms that we need are

$$\mathcal{L}_{4,4} = -\frac{1}{8}\int d^2\theta h_4(U,\bar{U})(D_\alpha U D^\alpha U D_\beta D^\beta \bar{U} D_\gamma D^\gamma \bar{U} + \text{H.c.})$$
$$= h_4(u,\bar{u})(2|F|^4 + F^2 \partial_\mu \bar{u}\partial^\mu \bar{u} + \bar{F}^2 \partial_\mu u \partial^\mu u) + \text{ferm}, \quad (3.10)$$

$$\mathcal{L}_{4,5} = -\frac{1}{8}\int d^2\theta h_5(U,\bar{U})(D_\alpha U D^\alpha U D_\beta D_\gamma \bar{U} D^\beta D^\gamma \bar{U} + \text{H.c.})$$
$$= h_5(u,\bar{u})(|F|^4 + |\partial_\mu u \partial^\mu u|^2 + F^2 \partial_\mu \bar{u}\partial^\mu \bar{u} + \bar{F}^2 \partial_\mu u \partial^\mu u) + \text{ferm}. \quad (3.11)$$

There are many other possible scalar superfield combinations which have the same number of superfields U and same number of covariant derivatives D_α. The previous list does not provide a complete classification. But for our purposes, our choice of five terms is the minimal number we have to take into consideration. The reason for this is the following. The bosonic sector of the higher-derivative terms has five possible terms, which can be combined into a five-vector B_i,

$$B_i = (|F|^4, |F|^2 \partial_\mu \bar{u}\partial^\mu u, (\partial_\mu \bar{u}\partial^\mu u)^2, |\partial_\mu u \partial^\mu u|^2, F^2 \partial_\mu \bar{u}\partial^\mu \bar{u} + \text{H.c.}), \quad (3.12)$$

with $i = 1, ..5$. A sum of the previous five terms in the Lagrangian, (3.6), (3.7), (3.8), (3.10), and (3.11), gives a generic linear combination of these terms in the bosonic sector

$$\sum_{i=1}^{5} \mathcal{L}_{4,i} = h_i(u,\bar{u}) M_{ij} B_j, \quad (3.13)$$

with the matrix M_{ij} being

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 4 & 8 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (3.14)$$

Since the determinant of the matrix M is different from zero, the five terms are all linearly independent. These five terms are then sufficient to construct any possible combination of such bosonic terms in the Lagrangian. With the inclusion of more general higher-derivative terms, we could have a different fermionic sector with the same bosonic part. Exploring the full set of possibilities is beyond the scope of this project. The four terms $\mathcal{L}_{4,i}$ with $i = 2, 3, 4, 5$ are the ones that arise in the $\mathcal{N} = 2$ extended model, as we shall derive in Eq. (4.10), so it is natural to choose these. If we want to consider a generic $\mathcal{N} = 1$ bosonic sector, we need to add a fifth linearly independent one which is $\mathcal{L}_{4,1}$.

We also have a potential term with no derivatives:

$$\mathcal{L}_0 = -\int d^2\theta w(U,\bar{U}) = \partial_u w F + \partial_{\bar{u}} w \bar{F} + \text{ferm}. \quad (3.15)$$

So the total Lagrangian is a sum of these various terms:

$$\mathcal{L} = \mathcal{L}_2 + \sum_{i=1}^{5} \mathcal{L}_{4,i} + \mathcal{L}_0. \quad (3.16)$$

Next we look at the ways of recovering the baby Skyrme model in the bosonic sector. For this we want to sum all the bosonic terms, integrate out the auxiliary field F, and then constrain the remaining bosonic Lagrangian to be the one of the baby Skyrme model. The fermionic part of the Lagrangian contains purely fermionic terms and also the mixed ones. In fact, it is not possible to integrate out, in a closed form, the auxiliary field for the full Lagrangian, including the fermionic sector. So we do not have an on-shell form of the supersymmetric baby Skyrme model, with only the fields u and ψ. Only for the solution in which the fermions have been set to zero, which is always possible due to the form of their equation of motion, do we recover the baby Skyrme model after integrating out the auxiliary field.

The most general bosonic baby Skyrme model is parametrized by three real and positive functions $V(u,\bar{u})$, $K(u,\bar{u})$, $S(u,\bar{u})$ and is of the form

$$\mathcal{L} = K(u,\bar{u})\partial_\mu\bar{u}\partial^\mu u + S(u,\bar{u})(|\partial_\mu u\partial^\mu u|^2 - \partial_\mu\bar{u}\partial^\mu u) - V(u,\bar{u}). \tag{3.17}$$

The specific cases considered in Sec. II are

$$K = \frac{1}{(1+|u|^2)^2}, \qquad S = \frac{1}{(1+|u|^2)^4}, \qquad V = \frac{|u|^{2k}}{(1+|u|^2)^k}. \tag{3.18}$$

To proceed further, we note that there are two different strategies to obtain the baby Skyrme Lagrangian; both strategies have been considered in Refs. [14] and [15]. We will adopt the same strategies, but taking into consideration also the extra terms (3.7) and (3.8).

The first strategy is the one discussed in [14]. Here we try to combine the higher-derivative terms in order to reproduce the baby Skyrme term with no additional auxiliary field terms. The baby Skyrme higher-derivative term, in the notation of (3.12), correspond to the vector $(0,0,-1,1,0)$. We then have the equation

$$S(u,\bar{u})(0,0,-1,1,0) = h_i(u,\bar{u})M_{ij}, \tag{3.19}$$

which is solved by

$$h_i(u,\bar{u}) = S(u,\bar{u})\frac{1}{5}(-5,1,1,-2,4). \tag{3.20}$$

The Lagrangian in this case becomes

$$\mathcal{L} = g|F|^2 + g\partial_\mu\bar{u}\partial^\mu u + S(u,\bar{u})(|\partial_\mu u\partial^\mu u|^2 - \partial_\mu\bar{u}\partial^\mu u) + \partial_u wF + \partial_{\bar{u}} w\bar{F} + \text{ferm.} \tag{3.21}$$

After setting $\psi = 0$, the auxiliary field can be solved by

$$\bar{F} = -\frac{\partial_u w}{g}, \tag{3.22}$$

and so the Lagrangian becomes

$$\mathcal{L} = g\partial_\mu\bar{u}\partial^\mu u - \frac{|\partial_u w|^2}{g} + S(u,\bar{u})(|\partial_\mu u\partial^\mu u|^2 - \partial_\mu\bar{u}\partial^\mu u). \tag{3.23}$$

In this case we can then recover the baby Skyrmion theory by making the following choice:

$$g(u,\bar{u}) = K(u,\bar{u}), \qquad |\partial_u w|^2 = V(u,\bar{u})K(u,\bar{u}). \tag{3.24}$$

For example, for the specific choice (3.18), the solution for w is given by the real integral

$$w(u,\bar{u}) = \int^{|u|} dx \frac{x^k}{(1+x^2)^{1+k/2}}. \tag{3.25}$$

A solution of this form can be easily found whenever V and K are simply functions of $|u|$.

In the second approach (see [15]), we do not use any superpotential term, so we set $\mathcal{L}_0 = 0$. We then arrange the coefficients of the higher-derivative terms $\mathcal{L}_{4,i}$ so that only the terms proportional to $|F|^4$, $|F|^2$, or $|F|^0$ appear in the bosonic sector; i.e., we set to zero coefficients of the terms F^2 and $\bar{F}^2$. In this case, we can integrate out explicitly the auxiliary field after we have also set $\psi = 0$. Finally, we arrange the coefficient of the term with four time derivatives to vanish.

For the terms proportional to F^2 and $\bar{F}^2$ to vanish, we have

$$(-h_2 - h_3 + h_4 + h_5)(F^2\partial_\mu\bar{u}\partial^\mu\bar{u} + \bar{F}^2\partial_\mu u\partial^\mu u) = 0, \tag{3.26}$$

and, thus, we require that

$$h_5 = h_2 + h_3 - h_4. \tag{3.27}$$

Then the total Lagrangian becomes

$$\mathcal{L} = g|F|^2 + g\partial_\mu\bar{u}\partial^\mu u + (h_1 + 5h_2 + 2h_3 + h_4)|F|^4 + (2h_1 + 8h_2 + 2h_3)|F|^2\partial_\mu\bar{u}\partial^\mu u + h_1(\partial_\mu\bar{u}\partial^\mu u)^2 + (h_2 + 2h_3 - h_4)|\partial_\mu u\partial^\mu u|^2 + \text{ferm.} \tag{3.28}$$

Next we set $\psi = 0$ and find that the auxiliary field can be solved by

$$|F|^2 = -\frac{g + (2h_1 + 8h_2 + 2h_3)\partial_\mu\bar{u}\partial^\mu u}{2(h_1 + 5h_2 + 2h_3 + h_4)}. \tag{3.29}$$

Thus, the bosonic Lagrangian, at this stage, becomes

$$\mathcal{L} = \frac{g(h_2 + h_3 + h_4)}{h_1 + 5h_2 + 2h_3 + h_4}\partial_\mu\bar{u}\partial^\mu u + \frac{h_1(h_4 - 3h_2) - (4h_2 + h_3)^2}{h_1 + 5h_2 + 2h_3 + h_4}(\partial_\mu\bar{u}\partial^\mu u)^2 + (h_2 + 2h_3 - h_4)|\partial_\mu u\partial^\mu u|^2 - \frac{g^2}{4(h_1 + 5h_2 + 2h_3 + h_4)}. \tag{3.30}$$

Finally, we have to impose the vanishing of the coefficient of the terms with four time derivatives, and this gives us the following constraint:

$$-11h_2^2 - 2h_1(h_2 - h_3) + 3h_3^2 + 4h_2(h_3 - h_4) - h_4^2 = 0. \tag{3.31}$$

The final bosonic Lagrangian of the baby Skyrme type is, thus,

$$\mathcal{L} = \frac{g(h_2 + h_3 + h_4)}{h_1 + 5h_2 + 2h_3 + h_4}\partial_\mu \bar{u}\partial^\mu u + (h_2 + 2h_3 - h_4)(|\partial_\mu u \partial^\mu u|^2 - \partial_\mu \bar{u}\partial^\mu u) - \frac{g^2}{4(h_1 + 5h_2 + 2h_3 + h_4)}, \tag{3.32}$$

with $h_{1,2,3,4}$ related by the condition (3.31).

If we want a restricted baby Skyrme Lagrangian, which is (3.17) with $K = 0$, we have to also impose the vanishing of the coefficient of the kinetic term in (3.32), and this gives us

$$h_2 = -h_3 - h_4. \tag{3.33}$$

The constraint (3.31) then becomes

$$2(h_1 - 3h_3 - 4h_4)(2h_3 + h_4) = 0. \tag{3.34}$$

We note that we have two branches of solutions of this equation. The first branch, $h_1 - 3h_3 - 4h_4 = 0$, gives an infinite potential, so we exclude it. The second one is

$$h_4 = -2h_3, \tag{3.35}$$

and it gives us the following bosonic Lagrangian:

$$\mathcal{L} = 5h_3(|\partial_\mu u\partial^\mu u|^2 - \partial_\mu \bar{u}\partial^\mu u) - \frac{g^2}{4(h_1 + 5h_3)}. \tag{3.36}$$

To match the restricted baby Skyrme model we can make the following choice:

$$h_3(u,\bar{u}) = \frac{1}{5}S(u,\bar{u}),$$
$$h_1(u,\bar{u}) = \frac{\epsilon}{5}S(u,\bar{u}),$$
$$g(u,\bar{u}) = 2\sqrt{\left(\frac{1}{5} + \epsilon\right)V(u,\bar{u})S(u,\bar{u})}, \tag{3.37}$$

with $\epsilon > 1/5$. This can also be rewritten using (3.27) and (3.35) as

$$h_i(u,\bar{u}) = S(u,\bar{u})\frac{1}{5}(\epsilon, 1, 1, -2, 4). \tag{3.38}$$

At the value $\epsilon = -1/5$, we meet the first branch of solutions of (3.34). Not only can we then recover any restricted baby Skyrme model, but we also have a one-parameter family labeled by ϵ.

To recover the most general baby Skyrme model, we need first to solve explicitly the constraint (3.31). We can express h_1 as a function of the others as follows,

$$h_1 = \frac{-11h_2^2 + 4h_2h_3 + 3h_3^2 - 4h_2h_4 - h_4^2}{2(h_2 - h_3)}, \tag{3.39}$$

and then the bosonic Lagrangian becomes

$$\mathcal{L} = \frac{2g(h_3 - h_2)}{h_2 + h_3 + h_4}\partial_\mu\bar{u}\partial^\mu u + (h_2 + 2h_3 - h_4)(|\partial_\mu u\partial^\mu u|^2 - \partial_\mu\bar{u}\partial^\mu u) - \frac{g^2(h_3 - h_2)}{2(h_2 + h_3 + h_4)^2}. \tag{3.40}$$

We are, thus, left with having to solve the following three equations:

$$K = \frac{g(h_3 - h_2)}{h_2 + h_3 + h_4},$$
$$S = h_2 + 2h_3 - h_4,$$
$$V = \frac{g^2(h_3 - h_2)}{2(h_2 + h_3 + h_4)^2}. \tag{3.41}$$

One possible set of solutions is

$$g = \alpha\sqrt{VS} \tag{3.42}$$

and

$$h_i = \left(\frac{K^2 + SV(\alpha^2 - 4) - 2K\sqrt{SV}\alpha}{4V}, \frac{-3K^2 + 8SV + 2K\sqrt{SV}\alpha}{40V}, \frac{K^2 + 4SV + K\sqrt{SV}\alpha}{20V}, \frac{K^2 - 16SV + 6K\sqrt{SV}\alpha}{40V}, \frac{-K^2 + 16SV - K\sqrt{SV}\alpha}{20V}\right), \tag{3.43}$$

with $\alpha > 0$. So we have a one-parameter family of models for any bosonic baby Skyrme model. When $K = 0$ we recover the solutions (3.37) and (3.38) with $\alpha = 2\sqrt{1/5 + \epsilon}$.

IV. $\mathcal{N} = 2$ SUPERSYMMETRIC EXTENSIONS

In $\mathcal{N} = 2$, the superspace spinor is complex. We can write it as a sum of real and imaginary components as

$$\Theta^\alpha = \theta^\alpha + i\delta^\alpha, \tag{4.1}$$

where θ and δ are two Majorana spinors. In particular, θ is the one that in our conventions corresponds to the $\mathcal{N}=1$ supersymmetry of the previous section.

The $\mathcal{N}=2$ covariant derivatives are

$$\mathcal{D}_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}\partial_\mu, \qquad \bar{\mathcal{D}}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\Theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \quad (4.2)$$

To proceed further, we decompose the $\mathcal{N}=2$ covariant derivatives into sums of the $\mathcal{N}=1$ ones,

$$\mathcal{D}_\alpha = D^{(\theta)}_\alpha + iD^{(\delta)}_\alpha, \qquad \bar{\mathcal{D}}_{\dot{\alpha}} = -D^{(\theta)}_{\dot{\alpha}} + iD^{(\delta)}_{\dot{\alpha}}, \quad (4.3)$$

where $D^{(\theta)}_\alpha$ and $D^{(\delta)}_\alpha$ are the same as (3.2), respectively, for θ and δ. $\mathcal{N}=2$ chiral and antichiral superfields satisfy the constraints $\bar{\mathcal{D}}_{\dot{\alpha}}U_{\mathcal{N}=2}=0$ and $\mathcal{D}_\alpha\bar{U}_{\mathcal{N}=2}=0$, and when expanded into components, they become

$$\begin{aligned} U_{\mathcal{N}=2} &= u + i\Theta\sigma^\mu\bar{\Theta}\partial_\mu u + \frac{1}{4}\Theta\Theta\bar{\Theta}\,\bar{\Theta}\,\Box u + \sqrt{2}\Theta\psi \\ &\quad - \frac{i}{\sqrt{2}}\Theta\Theta\partial_\mu\psi\sigma^\mu\bar{\Theta} + \Theta\Theta F, \\ \bar{U}_{\mathcal{N}=2} &= \bar{u} - i\Theta\sigma^\mu\bar{\Theta}\partial_\mu\bar{u} + \frac{1}{4}\Theta\Theta\bar{\Theta}\,\bar{\Theta}\,\Box\bar{u} + \sqrt{2}\bar{\Theta}\,\bar{\psi} \\ &\quad + \frac{i}{\sqrt{2}}\bar{\Theta}\,\bar{\Theta}\,\Theta\sigma^\mu\partial_\mu\bar{\psi} + \bar{\Theta}\,\bar{\Theta}\,\bar{F}. \end{aligned} \quad (4.4)$$

When the $\mathcal{N}=2$ superfields are chiral or antichiral, the following relations between the $\mathcal{N}=1$ covariant derivatives are satisfied:

$$D^{(\theta)}_\alpha U_{\mathcal{N}=2} = iD^{(\delta)}_\alpha U_{\mathcal{N}=2}, \qquad D^{(\theta)}_\alpha\bar{U}_{\mathcal{N}=2} = -iD^{(\delta)}_\alpha\bar{U}_{\mathcal{N}=2}. \quad (4.5)$$

So all derivatives can be expressed as a function of a unique derivative which we take to be $D^{(\theta)}_\alpha$. From now on we will denote $D^{(\theta)}_\alpha$ simply as D_α.

The $\mathcal{N}=2$ superfields can be expanded in powers of δ as follows,

$$\begin{aligned} U_{\mathcal{N}=2} &= U + i\delta^\alpha D_\alpha U - \frac{1}{2}\delta^\alpha\delta_\alpha D^\beta D_\beta U, \\ \bar{U}_{\mathcal{N}=2} &= \bar{U} - i\delta^\alpha D_\alpha\bar{U} - \frac{1}{2}\delta^\alpha\delta_\alpha D^\beta D_\beta\bar{U}, \end{aligned} \quad (4.6)$$

where U is the $\mathcal{N}=1$ superfield, like the one defined in (3.1), but with a different normalization for the fermionic field,

$$U = u + \sqrt{2}\theta^\alpha\psi_\alpha - \theta^2 F. \quad (4.7)$$

In this formulation the θ dependence is hidden inside the $\mathcal{N}=1$ superfields U and $\bar{U}$.

Returning to our problem, we note that one $\mathcal{N}=2$ model is the pure sigma model arising from the Kahler potential, namely,

$$\begin{aligned} \mathcal{L}_2 &= \int d^2\Theta d^2\bar{\Theta}\mathcal{K}(\bar{U}_{\mathcal{N}=2}, U_{\mathcal{N}=2}) \\ &= -\int d^2\theta\bar{\partial}\partial\mathcal{K}(\bar{U}, U)D^\alpha\bar{U}D_\alpha U. \end{aligned} \quad (4.8)$$

When expressed in the $\mathcal{N}=1$ form, this shows that a $\mathcal{N}=1$ sigma model with a Kahler metric $g(\bar{U}, U) = \bar{\partial}\partial\mathcal{K}(\bar{U}, U)$ has a hidden $\mathcal{N}=2$ supersymmetry [21–24]. This model is one particular case of the theories arising from the first strategy of the previous section. We schematically describe these theories in Fig. 4. The first strategy leads to the general $\mathcal{N}=1$ extension of the baby Skyrme model. Any theory with a parameter ζ defined in Sec. II can be extended to $\mathcal{N}=1$. Among these theories, only one with $\delta=0$ is extendable to $\mathcal{N}=2$, and this is only the case if the metric is Kahler.

Another $\mathcal{N}=2$ extension is provided by the model discussed in [15] which is a particular extension of the restricted baby Skyrme model. So let us consider this model and expand it in the $\mathcal{N}=1$ formalism to see where it lies in the more general $\mathcal{N}=1$ extensions.

This model is defined by

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4, \quad (4.9)$$

where $\mathcal{L}_2$ is (4.8) and $\mathcal{L}_4$ is

$$\begin{aligned} \mathcal{L}_4 &= -\int d^2\Theta d^2\bar{\Theta}\frac{1}{10}S(\bar{U}_{\mathcal{N}=2}, U_{\mathcal{N}=2}) \\ &\quad \times D^\alpha U_{\mathcal{N}=2}D_\alpha U_{\mathcal{N}=2}D^\alpha\bar{U}_{\mathcal{N}=2}D_\alpha\bar{U}_{\mathcal{N}=2} \\ &= \int d^2\theta\frac{1}{10}S(\bar{U}, U)(-D_\alpha UD^\alpha UD_\beta D_\gamma\bar{U}D^\beta D^\gamma\bar{U} \\ &\quad + \text{H.c.} - 2D_\alpha UD_\beta\bar{U}D^\alpha D_\gamma UD^\beta D^\gamma\bar{U} \\ &\quad + \frac{1}{2}D_\alpha UD^\alpha UD_\beta D^\beta\bar{U}D_\gamma D^\gamma\bar{U} + \text{H.c.} \\ &\quad + D_\alpha UD_\beta\bar{U}D^\alpha D^\beta UD^\gamma D^\gamma\bar{U} + \text{H.c.}) + \cdots. \end{aligned} \quad (4.10)$$

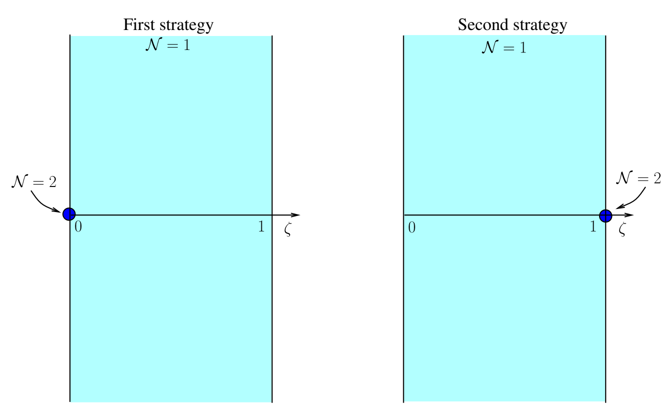

FIG. 4 (color online). Supersymmetric extensions of the baby Skyrme model.

For the last two lines we have performed an integration by parts of D_α, and the … terms depend on derivatives of $S(\bar{U}, U)$, but these do not affect the bosonic part of the Lagrangian. This expansion, taking into account also the coefficients in (3.7), (3.8), (3.10), (3.11), coincides exactly with (3.38) for the choice $\epsilon = 0$.

This other $\mathcal{N} = 2$ extension belongs to the second strategy, as depicted in Fig. 4. This extension is also surrounded by other more general $\mathcal{N} = 1$ extensions of the baby Skyrme model. Note that the two $\mathcal{N} = 2$ extensions cannot be continuously connected by $\mathcal{N} = 1$ extensions since they belong to two disconnected families.

The extended supersymmetry algebra is usually modified by the presence of topological central charges [25]. For the $\mathcal{N} = 2$ $CP(1)$ sigma model in $(2+1)$ dimensions, this charge has been computed explicitly in [26,27]. The algebra is given by

$$\{Q^I_\alpha, Q^J_\beta\} = \delta^{IJ} C_{\beta\rho} \gamma^{\mu}_{\alpha}{}^{\rho} P_\mu + i\epsilon^{IJ} C_{\alpha\beta} T, \tag{4.11}$$

where Q^I_α with $I = 1, 2$ are the two supersymmetry generators and T is the topological charge. Using the linear combinations

$$Q_\alpha = \frac{1}{\sqrt{2}}(1+\gamma^2)^\beta_\alpha (Q^1_\alpha + iQ^2_\alpha),$$
$$\bar{Q}_\alpha = \frac{1}{\sqrt{2}}(1-\gamma^2)^\beta_\alpha (Q^1_\alpha - iQ^2_\alpha), \tag{4.12}$$

and going into the soliton rest frame $P_\mu = (M, 0, 0)$, we have

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\begin{pmatrix} M-T & 0 \\ 0 & M+T \end{pmatrix}. \tag{4.13}$$

Solitons are the half-BPS states that annihilate the two supercharges Q_1 and $\bar{Q}_1$; antisolitons annihilate instead Q_2 and $\bar{Q}_2$. We can rewrite the four supercharges as $\mathcal{Q}_{1,2,3,4}$:

$$\mathcal{Q}_1 = \frac{1}{\sqrt{2}}(Q^1_1 - Q^2_2), \qquad \mathcal{Q}_2 = \frac{1}{\sqrt{2}}(Q^1_2 + Q^2_1),$$
$$\mathcal{Q}_3 = \frac{1}{\sqrt{2}}(Q^1_1 + Q^2_2), \qquad \mathcal{Q}_4 = \frac{1}{\sqrt{2}}(Q^1_2 - Q^2_1). \tag{4.14}$$

The soliton annihilates $\mathcal{Q}_1$ and $\mathcal{Q}_2$, while the antisoliton annihilates $\mathcal{Q}_3$ and $\mathcal{Q}_4$. The supersymmetric multiplet is built around the bosonic soliton state $|s\rangle$ by acting with the broken supercharges: $|s\rangle$, $\mathcal{Q}_3|s\rangle$, $\mathcal{Q}_4|s\rangle$, $\mathcal{Q}_3\mathcal{Q}_4|s\rangle$. Since $\mathcal{Q}_1|s\rangle = \mathcal{Q}_2|s\rangle = 0$, we can equivalently write the multiplet as $|s\rangle$, $Q^1_1|s\rangle$, $Q^1_2|s\rangle$, $Q^1_1 Q^1_2|s\rangle$. When supersymmetry is broken to $\mathcal{N} = 1$, the multiplet is simply lifted in a continuous way from the BPS bound. A short multiplet for $\mathcal{N} = 2$ has, in fact, the same number of states of a long multiplet of $\mathcal{N} = 1$ theory (see Fig. 5).

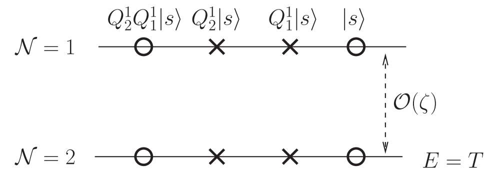

FIG. 5. A "short" $\mathcal{N} = 2$ multiplet is lifted to a "long" $\mathcal{N} = 1$ multiplet when supersymmetry is partially broken.

The discussion of the model near the other $\mathcal{N} = 2$ theory is more subtle. In [15] it was shown that the soliton preserves locally half of the supersymetric generators. In order for the soliton to be really $1/2$ BPS, these local generators must be globally extended. So far only $1/4$ of the global generators have been proven to be left unbroken by the soliton [18]. Furthermore, the quantum algebra has not yet been computed in an explicit form for the $\mathcal{N} = 2$ theory corresponding to the restricted baby Skyrme model. So this problem is left to some further study in the future.

V. CONCLUSIONS

In the first part of this paper, we have laid the groundwork for the near-BPS approximation, both analytically and numerically, using the baby Skyrme as a prototype model. Our analytical arguments also predict the rate of convergence to the BPS moduli space of solutions and, in particular, the rate of the deviation from the BPS moduli space (2.28) and (2.31). To test this rate of convergence, we would need more powerful numerical methods than those we currently have at our disposal. Also, a rigorous analytic proof would require more powerful functional analysis methods. It would also be interesting to extend this analysis to the multisoliton sector and to the bound states of baby Skyrmions.

We have also given a more complete construction of the $\mathcal{N} = 1$ supersymmetric extensions of the baby Skyrme model, generalizing the results of [14,15]. Using two different strategies, we were able to construct two disconnected families of $\mathcal{N} = 1$ theories, each of which possesses an $\mathcal{N} = 2$ extension in which the solitons become BPS saturates. It has not, however, been possible, within the theories we have constructed, to construct a theory with a flow between these two $\mathcal{N} = 2$ models without breaking all the supersymmetries. It is not clear, at present, whether a more general $\mathcal{N} = 1$ framework exists that would allow such a continuous flow to be present.

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