PHYSICAL REVIEW D **91,** 045013 (2015)

Chiral fermions as classical massless spinning particles

C. Duval[1,*] and P. A. Horváthy[2,3,†]

[1] *Aix-Marseille Université, CNRS CPT UMR 7332, 13288 Marseille, France Université de Toulon, CNRS CPT UMR 7332, 83957 La Garde, France*

[2] *Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, 37200 Tours, France*

[3] *Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, People's Republic of China*

(Received 20 June 2014; published 12 February 2015)

Semiclassical chiral fermion models with Berry term are studied in a symplectic framework. In the free case, the system can be obtained from Souriau's model for a relativistic massless spinning particle by "enslaving" the spin. The Berry term is identified with the classical spin two-form of the latter model. The Souriau model carries a natural Poincaré symmetry that we highlight, but spin enslavement breaks the boost symmetry. However the relation between the models allows us to derive a Poincaré symmetry of unconventional form for chiral fermions. Then we couple our system to an external electromagnetic field. For gyromagnetic ratio $g = 0$ we get curious superluminal Hall-type motions; for $g = 2$ and in a pure constant magnetic field in particular, we find instead spiraling motions.

DOI: 10.1103/PhysRevD.91.045013 PACS numbers: 11.15.Kc, 03.65.Vf, 11.30.-j, 11.30.Cp

I. INTRODUCTION

Massless chiral fermions have attracted considerable recent interest [1–12]. Sophisticated quantum calculations are greatly simplified by using (semi)classical models which can be derived from the Dirac equation [2]. The model proposed in [1,2,5], for example, describes a spin-$1/2$ system with positive helicity and energy, by the phase-space action

$$S = \int \left((\boldsymbol{p} + e\boldsymbol{A}) \cdot \frac{d\boldsymbol{x}}{dt} - (|\boldsymbol{p}| + e\phi) - \boldsymbol{a} \cdot \frac{d\boldsymbol{p}}{dt} \right) dt, \quad (1.1)$$

which also involves the additional momentum-dependent vector potential $\boldsymbol{a}(\boldsymbol{p})$ for the Berry monopole in $\boldsymbol{p}$-space [13],

$$\nabla_{\boldsymbol{p}} \times \boldsymbol{a} = \Theta \equiv \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}, \quad (1.2)$$

where $\hat{\boldsymbol{p}}$ is the unit vector $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$. Here $\boldsymbol{A}(\boldsymbol{x}, t)$ and $\phi(\boldsymbol{x}, t)$ are ordinary vector and scalar potentials and e is the electric charge.

A remarkable feature of the system (1.1) is its *lack of manifest Lorentz symmetry* even in the absence of an external gauge field [14].

In this paper we first show that a *free* chiral fermion model can be related to Souriau's *relativistic model of a massless spinning particle* [15] by "enslaving" the spin,

$$\boldsymbol{s} = s\hat{\boldsymbol{p}}, \quad (1.3)$$

cf. (3.11), viewed as a sort of gauge fixing condition. The massless spinning model carries a natural Poincaré symmetry, which we also generalize to finite transformations. This natural symmetry *is not inherited by the chiral model*, though spin enslaving breaks the Poincaré symmetry to the so-called Aristotle group [15] spanned by rotations and by space- and time- translations: *the chiral system carries no natural boost symmetry.*

The subtle relationship between the two models allows, nevertheless, for a *different*, *twisted Poincaré symmetry* for the chiral fermion (1.1), obtained by exporting the one carried by the massless spinning model. We stress that this twisted Poincaré symmetry should be considered rather as a *dynamical symmetry* in that its action on space-variables also involves the momentum.

Then we study the coupling to an external electromagnetic field. Applying first Souriau's version of minimal coupling [15,16] to the massless spinning model, we obtain a rather peculiar system, described in Sec. V A, which exhibits superluminal velocities with a Hall-type behavior both for 3-space and spin motion.

We consider next a more general, nonminimal coupling scheme, which accommodates any gyromagnetic ratio, g, by allowing the mass-square to depend on the coupling between the spin and the electromagnetic field [16,17]. The resulting, rather complicated system, presented in Sec. V B, combines the equations of motion of the previously studied minimal model ($g = 0$), with new, Stern-Gerlach-type terms, which involve derivatives of the field, reminiscent of recent propositions [9,10].

For the normal value $g = 2$, which is consistent with the Dirac equation [17], the anomalous terms are switched off, leading to considerable simplification. In a uniform, purely magnetic field we find, for example, spiraling motions. Spin enslavement, although not mandatory, is possible in this case.

*duval@cpt.univ-mrs.fr
†horvathy@lmpt.univ-tours.fr

The same procedure applied to chiral fermions allows to recover the results in [2,5]. But the chiral and the massless spinning systems behave differently because of the extra structure of the latter, which remains hidden in the free case, but which comes to light under coupling to an external field: the chiral model has a 6-dimensional phase space, while the massless spinning particle model has an 8-dimensional one.

We present our results in a symplectic framework. (The reader is advised to consult any of the standard textbooks as [18–22], for example.) The particular version we follow throughout this paper, outlined in Appendix A, is taken from Souriau's book [15], chapter III, pp. 123–227. The key point is that the classical motions correspond to curves (or surfaces) in an evolution space, V, where the dynamics takes place and is determined by a two-form σ; this can be thought of as a common generalization of both the Hamiltonian and Lagrangian formalisms.

II. SYMPLECTIC DESCRIPTION OF THE CHIRAL MODEL

Let us assume, for simplicity that we work in a Lorentz frame where the external field is stationary. Variation of the chiral action (1.1) yields the equations of motion for position $\boldsymbol{x}$ and momentum $\boldsymbol{p}\neq 0$ in three-space,

$$\begin{cases} \mathfrak{m}\frac{d\boldsymbol{x}}{dt} = \hat{\boldsymbol{p}} + e\boldsymbol{E}\times\Theta + (\Theta\cdot\hat{\boldsymbol{p}})e\boldsymbol{B}, \\ \mathfrak{m}\frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E} + e\hat{\boldsymbol{p}}\times\boldsymbol{B} + e^2(\boldsymbol{E}\cdot\boldsymbol{B})\Theta, \end{cases} \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field, respectively, and [23]

$$\mathfrak{m} = 1 + e\Theta\cdot\boldsymbol{B}. \tag{2.2}$$

Alternatively and equivalently, the chiral model (1.1) can be described within a symplectic framework [15,18–22], outlined in Appendix A. For chiral fermions, the evolution space is

$$V^7 = T(\mathbb{R}^3\setminus\{0\})\times\mathbb{R} \tag{2.3}$$

described by triples $(\boldsymbol{x},\boldsymbol{p},t)$, and is endowed with the two-form σ in (A.2), i.e.,

$$\sigma = \omega - dh\wedge dt, \tag{2.4}$$

specialized to the case where

$$\omega = \omega_0 + \frac{e}{2}\epsilon_{ijk}B^i dx^j\wedge dx^k, \tag{2.5}$$

$$\omega_0 = dp_i\wedge dx^i - \frac{s}{2|\boldsymbol{p}|^3}\epsilon^{ijk}p_i dp_j\wedge dp_k, \tag{2.6}$$

$$h = |\boldsymbol{p}| + e\phi, \tag{2.7}$$

where $s = 1/2$ [28]. The two-forms ω and thus σ are closed since $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{B} = 0$ and $\nabla_{\boldsymbol{p}}\cdot\Theta = 0$, see (1.2). (Remember that the points such that $\boldsymbol{p} = 0$, where the divergence of $\Theta(\boldsymbol{p})$ would be a Dirac delta-function, do not belong to our manifold, V^7.) Wherever

$$\det(\omega_{\alpha\beta}) = \mathfrak{m}^2 \equiv (1 + e\Theta\cdot\boldsymbol{B})^2 \neq 0, \tag{2.8}$$

the kernel of σ is 1-dimensional, and a curve $(\boldsymbol{x}(\tau),\boldsymbol{p}(\tau),t(\tau))$ is tangent to it iff the equations of motion (2.1) are satisfied [29]. At points where $\det(\omega_{\alpha\beta}) = 0$ the system is degenerate, necessitating symplectic alias Faddeev-Jackiw [30] reduction.

A constant Θ aligned in the z-direction would correspond to the planar case studied in [24,25,27,31]. Then, the vanishing of the analogous determinant, interpreted as the vanishing of an effective mass, merely requires fine-tuning of the magnetic field; the dynamical degrees of freedom drop from 4 to 2, and the only allowed motions are those which follow the Hall law [24,25,27,31]. In the chiral case here, instead, Θ is parallel to the momentum, $\boldsymbol{p}$. The determinant (2.8) can only vanish at particular singular points of phase space, since $\Theta = \Theta(\boldsymbol{p})$ and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x})$. The vanishing of $\mathfrak{m}$ is therefore rather spurious even at such exceptional points, since it requires the magnetic field to be of the order of the squared momentum, which appears inconsistent with the assumed adiabaticity.

Returning to the general case $\mathfrak{m}\neq 0$, Eqs. (2.1) exhibit the so-called *anomalous velocity* terms which have been recognized as the main reason behind *transverse shifts* or *side jumps* in spin-Hall-type effects [32–34]. Let us underline the strong similarities of the chiral system with massive semiclassical models [13,26,35] as well as with their planar counterparts [24,25,27,36]. A recent study indicates that chiral fermions follow a similar pattern and exhibit, in particular, an *Anomalous Hall effect* [37].

III. MASSLESS SPINNING PARTICLES

Now we consider instead a *free relativistic massless spinning particle* that we describe, following [15], by a 9-dimensional evolution space V^9 as follows. (See Appendix B for an overview of the model.) We start with three four-vectors R, I, J in Minkowski space-time $\mathbb{R}^{3,1}$ with signature $(-,-,-,+)$. Then we put

$$V^9 = \{R, I, J\in\mathbb{R}^{3,1}|I_\mu I^\mu = J_\mu J^\mu = 0, I_\mu J^\mu = -1\} \tag{3.1}$$

with I future-directed. Thus I and J are lightlike (nonzero) vectors generating a null 2-plane while R represents a space-time event. This particular evolution space is obtained from the Poincaré group by factoring out a suitable internal $SO(2)$ subgroup (cf. Appendix B), and carries therefore a natural action of the Poincaré group.

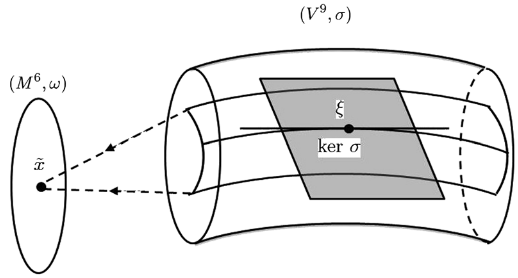

FIG. 1. A free massless spinning particle has a 9-dimensional evolution space V^9; its dynamics is defined by a two-form σ [see (3.5)], whose kernel, $\ker\sigma$, is 3-dimensional. A motion is a leaf tangent to the latter. All points of a leaf can be reached by a Z-shift (also called a Wigner-Souriau translation). The set of leaves forms the space of motions, M^6, whose points are $\tilde{x} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$.

An equivalent, but for our purposes more convenient, description of V^9 uses the spin tensor. Renaming $P = I$ (which will be later interpreted as the linear momentum) the latter is defined as,

$$S_{\mu\nu} = -s\epsilon_{\mu\nu\rho\sigma}P^\rho J^\sigma. \tag{3.2}$$

The spin tensor satisfies $\frac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2$, where $s \neq 0$ is the scalar spin (whose sign is called helicity). The condition

$$S_{\mu\nu}P^\nu = 0 \tag{3.3}$$

is plainly satisfied. Identifying the tensor $S = (S_{\mu\nu})$ with an element of the Lorentz Lie algebra $\mathfrak{o}(3,1)$, the evolution space (3.1) can also be presented as

$$V^9 = \Big\{R, P \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) | P_\mu P^\mu = 0, \\ S_{\mu\nu}P^\nu = 0, \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2\Big\}, \tag{3.4}$$

with, again, P future-pointing. The evolution space V^9 depicted on Fig. 1 is endowed with the closed two-form borrowed from [15], namely [38],

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu. \tag{3.5}$$

The dynamics is given by the foliation whose leaves are tangent to the kernel of σ in V^9; a world sheet [or world line] of the system is obtained by projecting a leaf of the latter to Minkowski space-time, yielding its corresponding space-time track. Calculating the kernel of (3.5) using also the constraints which define the evolution space, readily shows that a curve $(R(\tau), P(\tau), S(\tau))$ in V^9 (where τ is a real parameter) is tangent to $\ker\sigma$ iff

$$\begin{cases} P_\mu \dot{R}^\mu = 0, \\ \dot{P}^\mu = 0, \\ \dot{S}^{\mu\nu} = P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu, \end{cases} \tag{3.6}$$

where the "dot" stands for $d/d\tau$. [The parameter τ is arbitrary since any change of parameter would leave the system (3.6) invariant.] The space-time "velocity," $\dot{R}$, associated to any such curve is hence orthogonal to the momentum P. Indeed, the distribution defined by Eqs. (3.6) can be integrated using space-time vectors Z orthogonal to P, $P_\mu Z^\mu = 0$, namely as,

$$R^\mu \to R^\mu + Z^\mu, \qquad P^\mu \to P^\mu, \\ S^{\mu\nu} \to S^{\mu\nu} + (P^\mu Z^\nu - P^\nu Z^\mu). \tag{3.7}$$

Any point in a leaf of $\ker\sigma$ can be reached by choosing a suitable vector Z. Therefore at each point of V^9 the kernel of the two-form σ is 3-*dimensional* and projects to space-time, according to (3.6), as an affine subspace of $\mathbb{R}^{3,1}$, spanned by all vectors at R orthogonal to the linear momentum P. Thus the motions of a free massless spinning particle take place on a 3-dimensional wave-plane, tangent to the light-cone at each space-time event R: the particle is *not localized* in space-time [15,39].

We insist that all curves which lie in a leaf should be considered to be the same motion, left invariant by a Z-shift in (3.7). The space of motions is the collection $M^6 = V^9/\ker\sigma$ of those leaves and inherits the structure of a 6-*dimensional* symplectic manifold (see below). As we explain it below, spin is responsible for the space-time delocalization of massless particles.

To obtain down-to-earth expressions, we put $R = (\boldsymbol{r}, t)$ where $\boldsymbol{r}$ and t are the position and time coordinates in a chosen Lorentz frame. The two null-vectors are in turn $P = (\boldsymbol{p}, |\boldsymbol{p}|)$ and $J = (\boldsymbol{q}, -|\boldsymbol{q}|)$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are two (necessarily nonzero) 3-vectors which satisfy $\boldsymbol{p}\cdot\boldsymbol{q} + |\boldsymbol{p}||\boldsymbol{q}| = 1$ by (3.1). In these terms we have

$$S_{ij} = \epsilon_{ijk}s^k, \qquad \boldsymbol{s} = s(\boldsymbol{p}|\boldsymbol{q}| + \boldsymbol{q}|\boldsymbol{p}|), \\ S_{j4} = s(\boldsymbol{p}\times\boldsymbol{q})_j = (\hat{\boldsymbol{p}}\times\boldsymbol{s})_j. \tag{3.8}$$

We label each leaf of $\ker\sigma$ by picking a representative point in each of them in a way which is convenient for our purposes. To this end, we first observe that $\tau \to (R + \tau P, P, S)$ is an integral curve of $\ker\sigma$ for any given (R, P, S), i.e., a particular "motion." Next, shifting this curve by

$$Z = \left(\frac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|}\times\boldsymbol{s}, 0\right) \tag{3.9}$$

yields another integral curve lying in the same leaf. Finally, taking $\tau = -t/|\boldsymbol{p}|$ yields the point which has zero time coordinate; this is the point that we choose. See Fig. 2 to illustrate our strategy. The corresponding point on the shifted curve has position

$$R = (\tilde{\boldsymbol{x}}, 0). \tag{3.10}$$

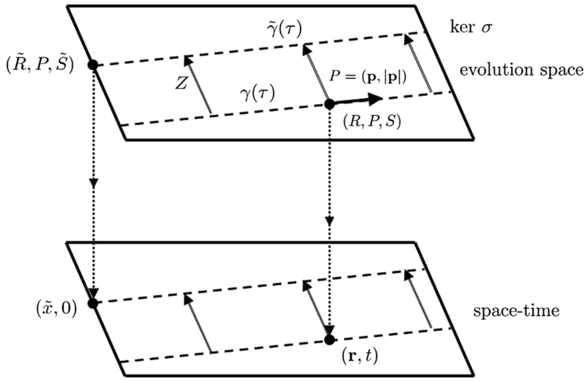

FIG. 2. The spin of a motion tangent to the (3-dimensional) kernel of σ can be enslaved by a suitable Z-shift. Choosing the point $(\tilde{R}, P, \tilde{S})$ on the shifted curve with vanishing time coordinate, $\tilde{R} = (\tilde{\boldsymbol{x}}, 0)$, provides us with a coordinate $\tilde{\boldsymbol{x}}$ of the motion-sheet. The characteristic leaves in V^9 project to Minkowski space-time as 3-planes orthogonal to the momentum: a massless spinning particle cannot be localized.

Spin becomes enslaved to the linear 3-momentum,

$$S_{j4} = 0, \qquad \boldsymbol{s} = s\hat{\boldsymbol{p}}. \tag{3.11}$$

An important observation which follows from (3.8) is that

$$\hat{\boldsymbol{p}} \cdot \boldsymbol{s} = s \tag{3.12}$$

in general, and not only in the enslaved case (3.11). It is thus *not* the length of the 3 vector $\boldsymbol{s}$ but its *projection* along $\hat{\boldsymbol{p}}$ which is a constant.

In terms of $(3+1)$-variables, $Z = (\boldsymbol{Z}, \hat{\boldsymbol{p}} \cdot \boldsymbol{Z})$ the "Z-shift" (3.7) acts as

$$\begin{aligned} &\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{Z}, \qquad t \to t + \hat{\boldsymbol{p}} \cdot \boldsymbol{Z}, \qquad \boldsymbol{p} \to \boldsymbol{p}, \\ &\boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{p} \times \boldsymbol{Z}. \end{aligned} \tag{3.13}$$

Thus, in the free case, the freedom of Z-shifting allows us to *eliminate the spin as an independent degree of freedom* altogether and the entire leaf can be labeled by $\tilde{\boldsymbol{x}}$ [see Definition (3.10)] and $\tilde{\boldsymbol{p}} = \boldsymbol{p} \neq 0$ alone. The latter parametrize the space of motions $M^6 = V^9 / \ker \sigma$, which has therefore the topology of $T(\mathbb{R}^3 \setminus \{0\})$. Finally, the two-form σ in (3.5) descends to the space of motions M^6 as the symplectic two-form (2.6), namely

$$\omega = d\tilde{p}_i \wedge d\tilde{x}^i - \frac{s}{2|\tilde{\boldsymbol{p}}|^3} \epsilon^{ijk} \tilde{p}_i d\tilde{p}_j \wedge d\tilde{p}_k. \tag{3.14}$$

Now we establish the Poincaré symmetry of the model. The Poincaré Lie algebra $\mathfrak{e}(3,1)$, spanned by the pairs (Λ, Γ) where $\Lambda = (\Lambda_{\mu\nu})$ belongs to the Lorentz Lie algebra $\mathfrak{so}(3,1)$, and $\Gamma = (\Gamma^\mu)$ is a translation in Minkowski space-time, $\mathbb{R}^{3,1}$, acts on V^9 by the lift of its action on Minkowski-space-time. This action on V^9 reads as follows

$$\begin{aligned} &\delta R^\mu = \Lambda^\mu_\nu R^\nu + \Gamma^\mu, \qquad \delta P^\mu = \Lambda^\mu_\nu P^\nu, \\ &\delta S^{\mu\nu} = \Lambda^\mu_\rho S^{\rho\nu} - \Lambda^\nu_\rho S^{\rho\mu}, \end{aligned} \tag{3.15}$$

and clearly leaves the two-form (3.5) invariant. It is therefore a symmetry of the system, which descends to the space of motions (M^6, ω). The associated Noetherian conserved quantities are

$$P^\mu = I^\mu, \qquad M^{\mu\nu} = R^\mu P^\nu - R^\nu P^\mu + S^{\mu\nu}, \tag{3.16}$$

which identifies the vector P and the bi-vector M as the *conserved linear and angular momentum*, respectively.

To get explicit formulas in a $(3+1)$-decomposition, we parametrize the Poincaré Lie algebra by $\Lambda_{ij} = \epsilon_{ijk}\omega^k$, $\Lambda_{i4} = \beta^i$, and $\Gamma = (\boldsymbol{\gamma}, \varepsilon)$, where $\omega, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3, \varepsilon \in \mathbb{R}$ are infinitesimal rotations, boosts and space- and time-translations, respectively. In terms of this decomposition, the infinitesimal Poincaré-action on V^9 is given, see (3.15) and (3.8), by

$$\begin{cases} \delta \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{\beta} t + \boldsymbol{\gamma}, \\ \delta t = \boldsymbol{\beta} \cdot \boldsymbol{r} + \varepsilon, \\ \delta \boldsymbol{p} = \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{\beta}|\boldsymbol{p}|, \\ \delta \boldsymbol{s} = \boldsymbol{\omega} \times \boldsymbol{s} - \boldsymbol{\beta} \times (\hat{\boldsymbol{p}} \times \boldsymbol{s}), \end{cases} \tag{3.17}$$

and duly projects to Minkowski space-time as the natural one.

To write down the explicit form of the Poincaré momenta (3.16) we present the matrix $M = (M_{\mu\nu})$ (which belongs to the dual of the Lorentz algebra) as $M_{ij} = \epsilon_{ijk}\ell^k$ and $M_{j4} = g^j$ with ℓ and $\boldsymbol{g}$ two 3-vectors. In terms of the above $(3+1)$-parametrization we find

$$\begin{cases} \ell = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{s}, \\ \boldsymbol{g} = |\boldsymbol{p}|\boldsymbol{r} - \boldsymbol{p}t + \hat{\boldsymbol{p}} \times \boldsymbol{s}. \end{cases} \tag{3.18}$$

Then the quantity

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{g}}{|\boldsymbol{p}|} = \boldsymbol{r} - \hat{\boldsymbol{p}}t + \frac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|} \times \boldsymbol{s} \tag{3.19}$$

is itself conserved. Working out the action of the full Poincaré Lie algebra (3.15) on the space of motions (M^6, ω) provides us with [40]

$$\begin{cases} \delta \tilde{\boldsymbol{p}} = \boldsymbol{\omega} \times \tilde{\boldsymbol{p}} + |\tilde{\boldsymbol{p}}|\boldsymbol{\beta}, \\ \delta \tilde{\boldsymbol{x}} = \boldsymbol{\omega} \times \tilde{\boldsymbol{x}} + s\boldsymbol{\beta} \times \frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|^2} - \boldsymbol{\beta} \cdot \tilde{\boldsymbol{x}} \frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|} + \boldsymbol{\gamma} - \varepsilon \frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|}. \end{cases} \tag{3.20}$$

The 10-parameter vector field (3.20) leaves the free symplectic structure (3.14) invariant, i.e., it generates a family of symmetries, to which the symplectic Noether theorem [15] associates 10 constants of the motion, namely

$$\begin{cases} \boldsymbol{\ell} = \tilde{\boldsymbol{x}} \times \tilde{\boldsymbol{p}} + s\hat{\boldsymbol{p}} & \text{angular momentum} \\ \boldsymbol{g} = |\tilde{\boldsymbol{p}}|\tilde{\boldsymbol{x}} & \text{boost momentum} \\ \boldsymbol{p} = \tilde{\boldsymbol{p}} & \text{linear momentum} \\ \mathcal{E} = |\tilde{\boldsymbol{p}}| & \text{energy} \end{cases} \tag{3.21}$$

whose conservation follows also directly from the free equations of motions. Note that the two terms in the free angular momentum $\boldsymbol{\ell}$ are separately conserved [41].

The Poisson brackets of the quantities in (3.21) calculated using (3.14),

$$\begin{aligned} &\{\ell_i, \ell_j\} = -\epsilon_{ij}^k \ell_k, \qquad \{\ell_i, g_j\} = -\epsilon_{ij}^k g_k, \\ &\{\ell_i, p_j\} = -\epsilon_{ij}^k p_k, \qquad \{\ell_i, \mathcal{E}\} = 0, \\ &\{g_i, g_j\} = \epsilon_{ij}^k \ell_k, \qquad \{g_i, p_j\} = -\mathcal{E}\delta_{ij}, \\ &\{g_i, \mathcal{E}\} = -p_i, \qquad \{p_i, p_j\} = 0, \qquad \{p_i, \mathcal{E}\} = 0, \end{aligned} \tag{3.22}$$

are those of the *Poincaré Lie algebra* $\mathfrak{e}(3,1)$, as expected. Calculating the Casimir invariants

$$m^2 = -\boldsymbol{p}^2 + \mathcal{E}^2 = 0, \qquad \boldsymbol{\ell} \cdot \hat{\boldsymbol{p}} = s, \tag{3.23}$$

shows that the infinitesimal Poincaré symmetry we have just found is realized in the *zero-mass and spin-s representation*.

The reason hidden behind all this is that the (connected) Poincaré group acts on the space of motions symplectically and transitively. Therefore (M^6, ω) is a *coadjoint orbit of the Poincaré group* [15]. The symplectic form (3.14) is, in particular, Souriau's #(17.145) in [15]. The Z-translations in Eq. (3.7), also identified as Wigner translations [42], belong to the stability subgroup, $\mathrm{SO}(2) \times \mathbb{R}^3$, of the Poincaré-action of a basepoint in the orbit.

So far, we have considered the infinitesimal action of the Poincaré *Lie algebra*. The construction allows us to work out the *finite action* of the *connected* (also called neutral) Poincaré *Lie group* on the space of motions (M^6, ω). To that end it is enough to spell out its natural action

$$(R, I, J) \to (R' = \mathcal{L}R + \mathcal{C}, I' = \mathcal{L}I, J' = \mathcal{L}J) \tag{3.24}$$

with $\mathcal{L} \in \mathrm{SO}_+(3,1)$ and $\mathcal{C} \in \mathbb{R}^{3,1}$, integrating the infinitesimal action (3.15) on the evolution space V^9 introduced in (3.1). Parametrizing the connected Poincaré group by A (rotation), $\boldsymbol{b} = b\boldsymbol{u}$ (boost in the direction $\boldsymbol{u}$), $\mathbf{c}$ (space-translation), and e (time-translation), a tedious calculation summarized in Appendix C) yields the action $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}) \to (\tilde{\boldsymbol{p}}', \tilde{\boldsymbol{x}}')$, where

$$\begin{cases} \tilde{\boldsymbol{p}}' = A\tilde{\boldsymbol{p}} + (\gamma - 1)(\boldsymbol{u} \cdot A\tilde{\boldsymbol{p}})\boldsymbol{u} + \gamma|\tilde{\boldsymbol{p}}|\boldsymbol{b}, \\ \tilde{\boldsymbol{x}}' = \frac{1}{|\tilde{\boldsymbol{p}}| + \boldsymbol{b}\cdot A\tilde{\boldsymbol{p}}} \Big[\boldsymbol{b} \times A\Big(\tilde{\boldsymbol{x}} \times \tilde{\boldsymbol{p}} + s\frac{\tilde{\boldsymbol{p}}}{|\tilde{\boldsymbol{p}}|}\Big) \\ + |\tilde{\boldsymbol{p}}|A\tilde{\boldsymbol{x}} + (\gamma - 1)|\tilde{\boldsymbol{p}}|(\boldsymbol{u} \cdot A\tilde{\boldsymbol{x}})\boldsymbol{u} - \gamma|\tilde{\boldsymbol{p}}|(\boldsymbol{b} \cdot A\tilde{\boldsymbol{x}})\boldsymbol{b} + (\boldsymbol{b} \cdot A\tilde{\boldsymbol{p}})\mathbf{c} - \frac{e}{\gamma}(A\tilde{\boldsymbol{p}} + (\gamma - 1)(\boldsymbol{u} \cdot A\tilde{\boldsymbol{p}})\boldsymbol{u}) + |\tilde{\boldsymbol{p}}|(\mathbf{c} - \boldsymbol{b}e)\Big], \end{cases} \tag{3.25}$$

with $\gamma = (1 - |\boldsymbol{b}|^2)^{-1/2}$ as usual; by keeping "tildes" we insist that our variables live on the space of motions (remember that $\tilde{\boldsymbol{p}} = \boldsymbol{p}$ but $\tilde{\boldsymbol{x}} \neq \boldsymbol{r}$). This extends the infinitesimal action (3.20) to finite transformations.

In a Lorentz frame the trajectory labeled with $\tilde{\boldsymbol{x}}$ is given by (3.19), i.e.,

$$\tilde{\boldsymbol{x}} = \boldsymbol{r} - \hat{\boldsymbol{p}}t + \frac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|} \times \boldsymbol{s}, \tag{3.26}$$

which describes motion with the velocity of light, directed along $\hat{\boldsymbol{p}}$. A Z-shift displaces the trajectory; starting, in particular, with "enslaved" spin, the latter is "unchained" and the trajectories one obtains fill a three-plane in 4-space. However, it is easy to see using (3.17) that the right-hand side of (3.26) remains invariant: the motion is not affected.

Intuitively, the freedom of Z-shifting is reminiscent of *gauge freedom*: it can always be performed at will; enslaving spin is in turn a sort of *gauge fixing*, allowing to interpret the result in terms of physical degrees of freedom alone.

We mention for completeness that the Poincaré symmetry of the massless spinning particle actually extends to an $\mathfrak{so}(4,2)$ conformal symmetry. See, e.g., [43].

IV. POINCARÉ SYMMETRY OF THE FREE CHIRAL MODEL

Now we return to chiral fermions. Does the free system (1.1) admit a Poincaré symmetry? For $\boldsymbol{E} = \boldsymbol{B} = 0$ the motions can be determined explicitly: the Θ-term drops out from (2.1), yielding

$$\boldsymbol{x}(t) = \tilde{\boldsymbol{x}} + \hat{\boldsymbol{p}}t, \qquad \boldsymbol{p}(t) = \tilde{\boldsymbol{p}}, \tag{4.1}$$

with $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{p}}$ constant vectors (and $\hat{\boldsymbol{p}} = \tilde{\boldsymbol{p}}/|\tilde{\boldsymbol{p}}|$). As explained in Sec. II, the chiral space of motions $M^6 = V^7/\ker\sigma$ can, therefore, be labeled by the constants of the motion

$$\tilde{\boldsymbol{x}} = \boldsymbol{x}(t) - \hat{\boldsymbol{p}}t \quad \text{and} \quad \tilde{\boldsymbol{p}}. \tag{4.2}$$

With the fields switched off, the *two-form ω in (2.5) becomes precisely (3.14): the free chiral model has the* *same space of motions as that of the massless spinning particle with $s = 1/2$, studied in Sec. III.*

Then, our strategy is to "import" the natural Poincaré symmetry of the massless spinning model to the chiral system through their common space of motions. From the identity of the space-of-motions coordinates $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ we conclude that, in terms of the coordinates $(\boldsymbol{x}, \boldsymbol{p}, t)$ on the chiral evolution space V^7, the strange-looking Poincaré action (3.20) (with $s = 1/2$) becomes,

$$\begin{cases} \delta\boldsymbol{x} = \boldsymbol{\omega}\times\boldsymbol{x} + \boldsymbol{\beta}\times\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} + \boldsymbol{\beta}t + \boldsymbol{\gamma}, \\ \delta\boldsymbol{p} = \boldsymbol{\omega}\times\boldsymbol{p} + |\boldsymbol{p}|\boldsymbol{\beta}, \\ \delta t = \boldsymbol{\beta}\cdot\boldsymbol{x} + \varepsilon. \end{cases} \tag{4.3}$$

By construction, these vector fields generate the same Lie algebra as those in (3.20), namely the Poincaré algebra $\mathfrak{e}(3,1)$.

Equation (4.3) confirms the recently proposed action of the Lorentz subalgebra on chiral fermions [9]. The conserved quantities associated with the generators of the latter are, in particular,

$$\begin{cases} \boldsymbol{\ell} = \boldsymbol{x}\times\boldsymbol{p} + \frac{1}{2}\hat{\boldsymbol{p}}, \\ \boldsymbol{g} = |\boldsymbol{p}|\boldsymbol{x} - \boldsymbol{p}t, \end{cases} \tag{4.4}$$

as it can be checked directly by showing that the infinitesimal rotations and boost generators in Eq. (4.3) Lie-transport the two-form σ in (2.4)–(2.6), and then by calculating the associated Noetherian quantities.

We have thus established a twisted Poincaré symmetry of the free chiral system. We insist, however, that the action (4.3) is *not* the usual, natural one on Minkowski space-time. In fact, it is *not* an action on space-time at all, since it also involves the momentum variable $\boldsymbol{p}$; it is rather a sort of dynamical symmetry—but one for the free dynamics.

In conclusion, the chiral model *admits a Poincaré symmetry, but, unlike for the massless spinning model, this symmetry does not act in the usual, natural way*. It follows that $\boldsymbol{x}$ should *not* be considered as a *bona fide* position variable, because it *does not transform* under a boost as positions should: it labels a *motion* and is not a space coordinate. We contend that the *well-founded* position of our particle should rather be regarded as given by the three spatial coordinates, $\boldsymbol{r}$, relatively to a chosen Lorentz frame, of intrinsic space-time translations within the Poincaré group. From the identity of the space-of-motions coordinates $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ we conclude in fact that the coordinates, $\boldsymbol{x}$, of the chiral particle are related to those, $\boldsymbol{r}$, of the massless Poincaré model according to

$$\boldsymbol{x} = \boldsymbol{r} + \frac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|}\times\boldsymbol{s} \quad \text{with} \quad \boldsymbol{s}\cdot\hat{\boldsymbol{p}} = \frac{1}{2}. \tag{4.5}$$

The coordinates coincide, $\boldsymbol{x} = \boldsymbol{r}$, only when spin is enslaved.

We mention that an alternative derivation, based on embedding the chiral system into the one of Souriau was found recently [44].

V. COUPLING TO AN EXTERNAL ELECTROMAGNETIC FIELD

Let us now cope with a number of procedures enabling us to couple our relativistic massless and spinning particle to an external electromagnetic field.

The conventional minimal coupling rule says that the 4-momentum should be shifted by the 4-potential as follows,

$$p_\mu \to p_\mu - eA_\mu. \tag{5.1}$$

This is *not exactly* what is proposed in (1.1), though: while the rule (5.1) is used for the 4-momentum $(\boldsymbol{p}, h)$, the 3-vector $\boldsymbol{p}$ in the Berry term Θ is *not* shifted. Remarkably, this "half-way-rule" is instead consistent with working with the same evolution space as for a free particle, but adding the electromagnetic field strength F to the free two-form (3.5) [15],

$$\sigma \to \sigma + eF, \tag{5.2}$$

where e is the electric charge of the system. This two-form is still closed, $d\sigma = 0$, because F is a closed two-form of Minkowski space-time.

The rules (5.1) and (5.2) are equivalent only in the spinless case. Then why should (5.2) be chosen? An argument in its favor comes from the fact that it is manifestly gauge-invariant [15,21] and leads, for instance, to the Bargmann-Michel-Telegdi equations [45] in a straightforward manner [15,16]. It also happened to yield an insight into Hall-type phenomena in planar physics [24,25,27,31], and in noncommutative mechanics in 3 space dimensions [26,33]. In noncommutative mechanics in the plane, modifications of the principle (5.2) lead to unsatisfactory models, see [27,46]. The merits of (5.2) have been praised, for example, by Sternberg [22]. It is hence this scheme we will be using throughout this paper.

A. Minimal coupling of the massless spinning model

Applying the prescription (5.2) to the massless spinning model of Sec. III yields, on the evolution space V^9 in (3.4), the closed two-form

$$\sigma = -dP_\mu\wedge dR^\mu - \frac{1}{2s^2}dS^\mu_\lambda\wedge S^\lambda_\rho dS^\rho_\mu + \frac{e}{2}F_{\mu\nu}dR^\mu\wedge dR^\nu. \tag{5.3}$$

Then a lengthy calculation using the constraints in the definition (3.4) of V^9 shows that the equations of free motions (3.6) change to [47]

$$\begin{cases} \dot{R}^\mu = P^\mu + \frac{S^{\mu\nu}F_{\nu\rho}P^\rho}{\frac{1}{2}S\cdot F}, \\ \dot{P}^\mu = -eF^\mu_\nu \dot{R}^\nu, \\ \dot{S}^{\mu\nu} = P^\mu\dot{R}^\nu - P^\nu\dot{R}^\mu. \end{cases} \tag{5.4}$$

assuming that $S\cdot F \equiv S_{\alpha\beta}F^{\alpha\beta} \neq 0$. The dimension of $\ker\sigma$ drops from 3 to 1: the spin-field coupling term in the velocity relation breaks the Z-shift-invariance. It follows that the spin degree cannot now be eliminated and we are left with a 8-*dimensional space of motions* (phase space, locally).

Let us now express the equations of motion (5.4) in terms of the $(3+1)$-decomposition we introduced in the previous section. Assuming that

$$(a)\ \frac{1}{2}S\cdot F \equiv \frac{1}{2}S_{\alpha\beta}F^{\alpha\beta} = \boldsymbol{s}\cdot(\boldsymbol{B} - \hat{\boldsymbol{p}}\times\boldsymbol{E}) \neq 0,$$
$$(b)\ \hat{\boldsymbol{p}}\cdot\boldsymbol{B} \neq 0, \tag{5.5}$$

a strange cancellation takes place in the velocity relation in (5.4), which becomes

$$\dot{\boldsymbol{r}} = s|\boldsymbol{p}|\frac{\boldsymbol{B} - \hat{\boldsymbol{p}}\times\boldsymbol{E}}{\boldsymbol{s}\cdot(\boldsymbol{B} - \hat{\boldsymbol{p}}\times\boldsymbol{E})}, \qquad \dot{t} = s|\boldsymbol{p}|\frac{(\hat{\boldsymbol{p}}\cdot\boldsymbol{B})}{\boldsymbol{s}\cdot(\boldsymbol{B} - \hat{\boldsymbol{p}}\times\boldsymbol{E})}. \tag{5.6}$$

Condition (a), the analog of the nonvanishing of the effective mass (2.8), will henceforth be assumed to hold.

Condition (b) in (5.5) requires that the momentum should not be perpendicular to the magnetic field. When it is not satisfied, then $\dot{t} = 0$, so that, while the motion still takes place along a curve, it becomes *instantaneous* [48].

Let us assume that the regularity conditions (5.5) hold; then merging the two equations in (5.6) provides us with

$$\begin{cases} \frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{B} - \hat{\boldsymbol{p}}\times\boldsymbol{E}}{\hat{\boldsymbol{p}}\cdot\boldsymbol{B}}, \\ \frac{d\boldsymbol{p}}{dt} = e\Big(\boldsymbol{E} + \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{B}\Big) = e\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{\hat{\boldsymbol{p}}\cdot\boldsymbol{B}}\hat{\boldsymbol{p}}, \\ \frac{d\boldsymbol{s}}{dt} = \boldsymbol{p}\times\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{p}\times\boldsymbol{B}}{\hat{\boldsymbol{p}}\cdot\boldsymbol{B}} - \frac{\boldsymbol{p}\times(\hat{\boldsymbol{p}}\times\boldsymbol{E})}{\hat{\boldsymbol{p}}\cdot\boldsymbol{B}}. \end{cases} \tag{5.7}$$

We insist on the rather unusual form of these equations. First, the $\hat{\boldsymbol{p}}$ one would have expected on the r.h.s. of the velocity relation cancels out; the electric charge drops out also. The dynamics of the momentum decouples from the spin as long as the latter does not vanish; also the scalar spin $s \neq 0$ disappears from all equations. Equations (5.7) imply that $d\hat{\boldsymbol{p}}/dt = 0$ so that the direction of $\boldsymbol{p}$ is unchanged during the motion. Spin is in fact not an independent variable, its (for space-time dynamics irrelevant) motion is entirely determined by the other dynamical data [50].

Let us put, for example, our massless but charged particle into crossed constant electric and magnetic fields like in the Hall effect, $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, (say). Then $\boldsymbol{p}$ is itself a constant of the motion, and so is the angle ϑ between $\boldsymbol{B}$ and $\boldsymbol{p}$ (which cannot be $\pi/2$ for $\boldsymbol{p}\cdot\boldsymbol{B} \neq 0$). Let us assume for simplicity that the initial momentum lies in the x-z plane.

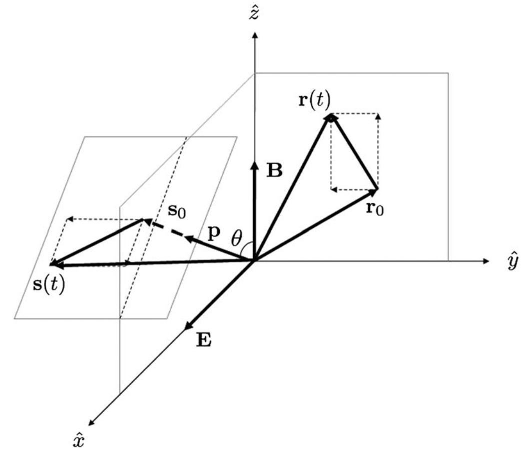

FIG. 3. Motion in Hall-type electric and magnetic fields described by Eq. (5.8). The initial position, $\boldsymbol{r}_0$, is chosen in the y-z plane, and the initial momentum and spin are chosen to be parallel and in the x-z plane. Then, the spatial motion, $\boldsymbol{r}(t)$, is a combination of constant-velocity Hall drift perpendicularly to $\boldsymbol{E}$ and $\boldsymbol{B}$, combined with constant-velocity vertical drift. The momentum, $\boldsymbol{p}$, is conserved, but the spin, $\boldsymbol{s}(t)$, moves in a plane perpendicular to the momentum.

Then the equations of motion are solved by

$$\begin{cases} \boldsymbol{r}(t) = ((\cos\vartheta)^{-1}\hat{\boldsymbol{z}} - \frac{E}{B}\hat{\boldsymbol{y}})t + \boldsymbol{r}_0, \\ \boldsymbol{p}(t) = \boldsymbol{p}_0, \\ \boldsymbol{s}(t) = |\boldsymbol{p}|(-\tan\vartheta\hat{\boldsymbol{y}} + \frac{E}{B}(\cos\vartheta\hat{\boldsymbol{x}} - \sin\vartheta\hat{\boldsymbol{z}}))t + \boldsymbol{s}_0. \end{cases} \tag{5.8}$$

Thus, in addition to a constant-speed vertical motion, the "particle" also drifts perpendicularly to the electric field with *Hall velocity* E/B. The spin vector follows an even more curious motion perpendicularly to $\hat{\boldsymbol{p}}$ so that its projection on $\hat{\boldsymbol{p}}$ remains a constant, $\boldsymbol{s}(t)\cdot\hat{\boldsymbol{p}} = \boldsymbol{s}_0\cdot\hat{\boldsymbol{p}}$. Thus while spin is decoupled, it cannot consistently be enslaved as in (3.11) since $\boldsymbol{s}$ and $\hat{\boldsymbol{p}}$ do not remain parallel even if we start with a such initial condition, see Fig. 3. Note that $S\cdot F = 2sB\cos\vartheta$ for such a motion and the system is regular therefore when $\vartheta \neq \pi/2$.

The velocity is *superluminal* ($|d\boldsymbol{r}/dt| > 1$) and diverges as $\vartheta \to \pi/2$; for $\hat{\boldsymbol{p}}\cdot\boldsymbol{B} = 0$ we get instantaneous motions, i.e. with infinite velocity, parallel to the z axis. This is in fact a general property, as seen from (5.4), because $\dot{R}_\mu\dot{R}^\mu < 0$, the 4-vector $(S^{\mu\nu}F_{\nu\rho}P^\rho)$ being space-like.

B. Anomalous coupling

The model of Sec. VA is not completely satisfactory, and now we generalize our minimal scheme. Our clue is to allow the "mass-square" $P_\mu P^\mu$ to depend on the coupling of spin to the electromagnetic field as suggested in [16,17], i.e.,

$$P_\mu P^\mu = -\frac{eg}{2} S \cdot F, \tag{5.9}$$

where we used once again the shorthand $S \cdot F \equiv S_{\alpha\beta} F^{\alpha\beta}$, cf. (5.5). The real constant g will be interpreted as the *gyromagnetic ratio* [51]. Generalizing the previous relation $P = I$ as

$$P^\mu = I^\mu + \frac{eg}{4} (S \cdot F) J^\mu, \tag{5.10}$$

where I and J are still as in (3.1), helps us to implement the equation of state (5.9). The condition $S_{\mu\nu} P^\nu = 0$ is also automatically satisfied.

Hence we introduce the novel evolution space

$$\tilde{V}^9 = \Big\{ P, R \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) | P_\mu P^\mu = -\frac{eg}{2} S \cdot F, \\ S_{\mu\nu} P^\nu = 0, \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = s^2 \Big\}, \tag{5.11}$$

endowed with the closed two-form,

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu + \frac{1}{2} e F_{\mu\nu} dR^\mu \wedge dR^\nu. \tag{5.12}$$

Note that (5.12) is formally the same as (5.3) up to the mass-shell constraint (5.9).

Some more effort is needed to work our the new equations of motion from the kernel of σ using the constraints which define $\tilde{V}^9$. We find that a curve $(R(\tau), P(\tau), S(\tau))$ is tangent to $\ker \sigma$ in (5.12) iff

$$\begin{cases} \dot{R}^\mu = P^\mu - \frac{1}{(g+1)} \frac{1}{S_{\alpha\beta} F^{\alpha\beta}} [(g-2) S^{\mu\nu} F_{\nu\rho} P^\rho - g S^{\mu\nu} \partial_\nu F_{\rho\sigma} S^{\rho\sigma}], \\ \dot{P}^\mu = -e F^\mu_\nu \dot{R}^\nu - \frac{eg}{4} \partial^\mu F_{\rho\sigma} S^{\rho\sigma}, \\ \dot{S}^{\mu\nu} = P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu + \frac{eg}{2} [S^\mu_\rho F^{\rho\nu} - S^\nu_\rho F^{\rho\mu}]. \end{cases} \tag{5.13}$$

These equations, which reduce to (5.3) for $g = 0$, constitute the zero-rest-mass counterparts of the celebrated Bargmann-Michel-Telegdi equations for massive relativistic particles [45], as well as 4 dimensional analogs of "exotic" anyons in the plane [25]. In the normal case, $g = 2$, resulting from the Dirac equation [17], the previously considered anomalous velocity is canceled but there arises a new, Stern–Gerlach-type contribution involving the derivative of the external electromagnetic field. Thus, an anomalous velocity term shows up for any value of the gyromagnetic ratio g.

Now we turn to a $(3+1)$-decomposition. Things behave as before, up to some subtle differences. First, we have

$$R = (\boldsymbol{r}, t), \qquad P = (\boldsymbol{p}, \mathcal{E}), \qquad S_{j4} = \left(\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s} \right)_j, \tag{5.14}$$

where the spin tensor is still defined as in (3.2), but the new dispersion relation generalizes (3.21) [52], namely,

$$\mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - \frac{eg}{2} S \cdot F}. \tag{5.15}$$

Decomposing the electromagnetic field into its electric and magnetic components, the quantity (5.5) (a) is generalized to

$$\frac{1}{2} S \cdot F = \boldsymbol{s} \cdot \left(\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{E} \right). \tag{5.16}$$

Then a rather tedious calculation yields the following $(3+1)$-form of the equations of motion (5.13), namely

$$\begin{cases} \dot{\boldsymbol{r}} = \frac{3g}{2(g+1)} \boldsymbol{p} - \left(\frac{g-2}{g+1} \right) \left[\frac{\boldsymbol{s} \cdot \boldsymbol{p}}{S \cdot F} \left(\boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{E} \right) - \frac{eg}{2} \boldsymbol{E} \times \frac{\boldsymbol{s}}{\mathcal{E}} \right] - \frac{g}{2(g+1) S \cdot F} \left(\boldsymbol{s} \times (S \cdot \boldsymbol{D} F) - \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s} (S \cdot D_t F) \right), \\ \dot{t} = \frac{g}{2(g+1)\mathcal{E}} (3|\boldsymbol{p}|^2 - (g+1) e S \cdot F) - \left(\frac{g-2}{g+1} \right) \frac{1}{\mathcal{E} S \cdot F} (\boldsymbol{p} \cdot \boldsymbol{B})(\boldsymbol{s} \cdot \boldsymbol{p}) + \frac{eg(g-2)}{2(g+1)\mathcal{E}^2} \boldsymbol{s} \cdot (\boldsymbol{p} \times \boldsymbol{E}) - \frac{g}{(g+1)} \frac{1}{\mathcal{E} S \cdot F} (\boldsymbol{p} \times \boldsymbol{s}) \cdot (S \cdot \boldsymbol{D} F), \\ \dot{\boldsymbol{p}} = e(\boldsymbol{E} \dot{t} + \dot{\boldsymbol{r}} \times \boldsymbol{B}) + \frac{eg}{4} S \cdot \boldsymbol{D} F, \\ \dot{\boldsymbol{s}} = \boldsymbol{p} \times \dot{\boldsymbol{r}} + \frac{eg}{2} ((\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s}) \times \boldsymbol{E} + \boldsymbol{s} \times \boldsymbol{B}), \end{cases} \tag{5.17}$$

where we introduced the new shorthands

$$S \cdot D_j F = 2\boldsymbol{s} \cdot \left(\partial_j \boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \partial_j \boldsymbol{E} \right), \qquad S \cdot D_t F = 2\boldsymbol{s} \cdot \left(\partial_t \boldsymbol{B} - \frac{\boldsymbol{p}}{\mathcal{E}} \times \partial_t \boldsymbol{E} \right). \tag{5.18}$$

When $g = 0$ we recover (5.7).

To gain more insight, we consider the case $g = 2$ and assume that the external fields are constant. Equation (5.13) imply that when the electromagnetic field is constant, $S \cdot F$ in the denominator is a constant of the motion. The system is therefore regular whenever the initial conditions are regular. Then the field-derivative terms drop out as does also the anomalous velocity term [53], and the complicated system (5.17) simplifies to one reminiscent of a massive relativistic particle,

$$(g=2)\begin{cases}\mathcal{E}\frac{d\boldsymbol{r}}{dt} = \boldsymbol{p},\\ \frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} + \frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{B}),\\ \frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}}((\frac{\boldsymbol{p}}{\mathcal{E}} \times \boldsymbol{s}) \times \boldsymbol{E} + \boldsymbol{s} \times \boldsymbol{B}),\end{cases} \tag{5.19}$$

assuming that $\mathcal{E} \neq 0$, which acts as a sort of effective mass, is real. (Recall that $\boldsymbol{p} \neq 0$ implies that $\mathcal{E}$ cannot vanish).

In a *pure magnetic field* momentum and spin satisfy equations of identical form,

$$\frac{d\boldsymbol{p}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{p} \times \boldsymbol{B}, \qquad \frac{d\boldsymbol{s}}{dt} = \frac{e}{\mathcal{E}}\boldsymbol{s} \times \boldsymbol{B}. \tag{5.20}$$

Thus

$$\begin{cases}|\boldsymbol{p}| = \text{const} \neq 0, & \boldsymbol{p} \cdot \boldsymbol{B} = \text{const},\\ |\boldsymbol{s}| = \text{const} \neq 0, & \boldsymbol{s} \cdot \boldsymbol{B} = \text{const},\end{cases}$$
$$\Rightarrow \begin{cases} p_z = \text{const}, & s_z = \text{const},\\ \mathcal{E} = \sqrt{|\boldsymbol{p}|^2 - e\boldsymbol{s} \cdot \boldsymbol{B}} = \text{const}.\end{cases} \tag{5.21}$$

Choosing the z axis in the direction of the magnetic field, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, for example, the momentum and spin vectors precess and the position spirals around the z axis with common angular velocity $\omega = -eB/\mathcal{E}$,

$$\boldsymbol{p}(t) = (p_0 e^{-i(eB/\mathcal{E})t}, p_z), \qquad \boldsymbol{s}(t) = (s_0 e^{-i(eB/\mathcal{E})t}, s_z),$$
$$\boldsymbol{r}(t) = \left(\frac{ip_0}{eB} e^{-i(eB/\mathcal{E})t}, \frac{p_z}{\mathcal{E}} t\right) + \boldsymbol{r}_0, \tag{5.22}$$

where $p_0 = p_x + ip_y$, $s_0 = s_x + is_y$, cf. Fig. 4. It is worth noting that for weak fields and pure magnetic field, and $\boldsymbol{s} = \frac{1}{2}\hat{\boldsymbol{p}}$,

$$\mathcal{E} \approx |\boldsymbol{p}| - \frac{eg}{4} S \cdot F = |\boldsymbol{p}| - e\frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}, \tag{5.23}$$

which is the modified dispersion relation proposed in [6,9,10].

By Eqs. (5.22) the direction of the rotation is reversed if the sign of the electric charge e is reversed, whereas the direction of the vertical propagation is unchanged. As to chirality, when the sign of spin is reversed, Eqs. (5.22) yield similar spiraling motions, rotating and drifting in the same directions but with different angular velocities, namely with

$$\omega_\pm = -\frac{eB}{\mathcal{E}_\pm} = -\frac{eB}{\sqrt{|\boldsymbol{p}|^2 \mp e\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}} \approx -\frac{eB}{|\boldsymbol{p}|}\left(1 \pm \frac{e\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}\right), \tag{5.24}$$

assuming that the magnetic field is weak.

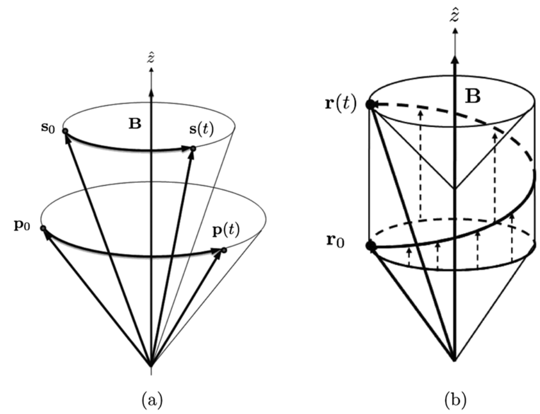

FIG. 4. (a) Motion in pure constant magnetic field $\boldsymbol{B}$. Both momentum $\boldsymbol{p}(t)$ and spin $\boldsymbol{s}(t)$ precess, with identical angular velocity, around the $\boldsymbol{B}$ direction, cf. Eq. (5.22). (b) The position $\boldsymbol{r}(t)$ spirals on a cylinder around the $\boldsymbol{B}$-axis, obtained by combining precession with the vertical drift of the supporting cone itself.

In the purely magnetic case, enslavement (3.11) *can* be consistently required. However, this is *manifestly not so* in the presence of an electric field [54]: *the independent spin degree of freedom can not be switched off* if $\boldsymbol{E} \neq 0$.

VI. CONCLUSION

In this paper we have shown that the semiclassical chiral fermion model, much discussed in recent times in connection with the chiral magnetic and chiral vortical effects [1–9], can, in the free case, be related to the zero mass and spin-$1/2$ particle model of [15]. The latter carries a natural Poincaré symmetry that can be exported to chiral fermions using the above relation.

We obtain, in particular, Lorentz boosts as proposed recently [9]. One could argue that this is what one would expect for a relativistic theory; we would like to stress, however, that this action is *not* the usual, natural one on ordinary space-time—on the contrary, it resembles a dynamical symmetry in that it also involves the momentum. We contend that the variable $\boldsymbol{x}$, viewed commonly as position, does not transform correctly under a boost; it is rather our $\boldsymbol{r}$, which is the *bona fide* position coordinate studied in Secs. III and IV. The situation is reminiscent of that of Newton-Wigner coordinates, familiar for the Dirac equation.

Our model is *similar to but different from* those proposed in [1–10]: while the usual chiral model (1.1) has no independent spin variable, ours has additional degrees of freedom associated with unchained spin and instrumental for having a natural Poincaré action. These additional degrees of freedom do not influence the free dynamics, though, as they can be eliminated by enslaving the spin to the momentum, using the additional symmetry referred to as Wigner-Souriau translations [11,15,39,42,44].

The models become even more different when coupled to an external field: the standard chiral models have a 6-dimensional phase space, whereas ours has, in the coupled case, 8 dimensions. Also the motions appear rather different in the two frameworks. The difference comes from choosing the physically relevant position coordinate: $\boldsymbol{x}$ in the chiral model and $\boldsymbol{r}$ in the one we propose here. The question is not purely academic, since the coupling to a field is expressed precisely in terms of the position.

The difference between the theories originates in that in the usual approach [1–11] they are derived from some widely accepted and physically trusted theory like the Dirac equation, transport theory, fluid dynamics, etc., while we build ours from the principles of Souriau's mechanics, based on group theory, cf. [15,16].

We present our investigations in symplectic, instead of usual variational terms. Although the two frameworks are essentially equivalent [15,55], using the symplectic one is better adapted to study degenerate systems as in the free case.

Non-Abelian generalization could also be considered along the lines discussed in [56].

ACKNOWLEDGMENTS

We would like to thank Mike Stone for calling our attention to this problem, for enlightening correspondence and for sending us his paper [11]. "Information tunneling" may have mutually influenced our work. Pengming Zhang helped us to produce our figures. We are moreover indebted to François Ziegler, Gary Gibbons, Kostya Bliokh and László Fehér for stimulating discussions and correspondence.

APPENDIX A: SOURIAU'S MECHANICS AS GENERALIZED VARIATIONAL CALCULUS

In the framework of [15] the dynamics is determined by a closed two-form σ of constant rank defined on some evolution space V; the motions, described by curves or by surfaces of V, are the so-called "characteristic leaves," tangent to the kernel, $\ker\sigma$, of the two-form σ.

To explain how this comes about, we consider a particle described by a Lagrangian on phase space of which (1.1) is an example that will serve as an illustration. Denoting the phase space variables $\boldsymbol{x}$ and $\boldsymbol{p}$ collectively by $\xi = (\xi^\alpha)$, the Lagrangian in (1.1) is of the form $u_\alpha\dot{\xi}^\alpha - h(\xi)$ and the associated variational equations are

$$\omega_{\alpha\beta}\dot{\xi}^\beta = \partial_\alpha h, \quad \text{where} \quad \omega_{\alpha\beta} = \partial_\alpha u_\beta - \partial_\beta u_\alpha. \tag{A1}$$

We note *en passant* that if the matrix $(\omega_{\alpha\beta})$ is regular, then multiplying (A1) with the inverse matrix would yield Hamilton's equations.

A next step is to extend the 6-dimensional phase space into the 7-dimensional *evolution space* V^7 described by triples $y = (\boldsymbol{x}, \boldsymbol{p}, t)$ and unify the two-form $\omega = \frac{1}{2}\omega_{\alpha\beta}d\xi^\alpha\wedge d\xi^\beta$ with the Hamiltonian into the two-form

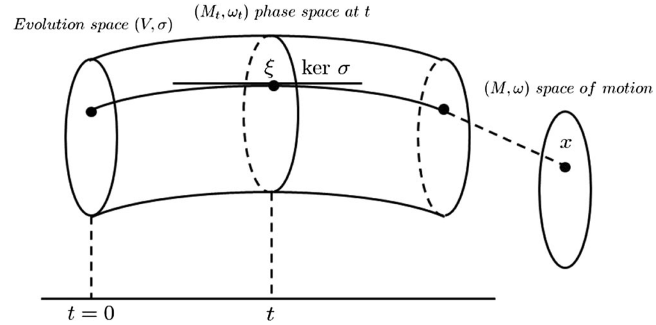

FIG. 5. A motion is described, in the evolution space V, by a submanifold, which is tangent to the kernel of a closed two-form, σ, of constant rank. The space of motions is the symplectic manifold, (M,ω), obtained from (V,σ) by factoring out these characteristic submanifolds. A phase space at time t is a section of the evolution space for a fixed value of t, and provides a local chart of the space of motions.

$$\sigma = \omega - dh\wedge dt. \tag{A2}$$

Then the equations of motion (A1) become finally

$$\sigma(\dot{y}, \cdot) = 0, \tag{A3}$$

expressing that the velocity, $\dot{y}$, of the motion unfolded into the evolution space belongs to $\ker\sigma$, see Fig. 5.

More generally, we can consider an evolution space V of dimension d, endowed with a closed two-form σ of constant rank; let the kernel of σ be an r-dimensional vector space. Then general theorems guarantee that $\ker\sigma$ is tangent, at each point, to an r-dimensional submanifold called a *characteristic leaf* of σ; the latter can be viewed as *solution of a generalized variational problem.*

Factoring out the characteristic leaves provides us with a symplectic manifold (M,ω) of dimension $2n = d - r$, called the *space of motions*, which can be regarded as an abstract substitute for the phase space. For further details the reader is invited to consult, e.g., [15,55].

We just mention that a *symmetry* is a transformation of the evolution space V which preserves its two-form σ. The relation between symmetries and conservation laws is established by the symplectic form of Noether's theorem. Conversely, the space of motions of classical systems with a given symmetry can be constructed, under suitable conditions, by group theoretical considerations [15].

APPENDIX B: MASSLESS, SPINNING RELATIVISTIC PARTICLE MODELS

We recall, here, the model of massless, spinning particle dwelling in Minkowski space-time, as spelled out in [15], Section (14.29). The space of motions (or of classical states) of a free relativistic particle is a homogeneous symplectic manifold of the Poincaré group, E(3,1). These coadjoint orbits are known and classified; those with zero mass, and nonzero spin are constructed as follows.

For convenience we deal with the neutral Poincaré group $G = \mathrm{SE}_+(3,1)$ whose elements are pairs $g = (\mathcal{L}, \mathcal{C})$ with $\mathcal{L} \in \mathrm{SO}_+(3,1)$, and $\mathcal{C} \in \mathbb{R}^{3,1}$, a space-time translation. Every element of the dual $\mathfrak{e}(3,1)^*$ of the Lie algebra, $\mathfrak{e}(3,1)$, of G is a pair $\mu = (M, P)$ with $M \in \mathfrak{so}(3,1)$, the Lorentz momentum, and $P \in \mathbb{R}^{3,1}$, the linear momentum. The pairing between these spaces is given by $\mu \cdot Z = \frac{1}{2} M_{\mu\nu}\Lambda^{\mu\nu} - P_\mu \Gamma^\mu$ with $Z = (\Lambda, \Gamma) \in \mathfrak{e}(3,1)$.

We will deal with oriented and time-oriented Lorentz frames $E = (I, J, K, L)$ of Minkowski space-time such that the only nonzero scalar products are $I_\mu J^\mu = K_\mu K^\mu = L_\mu L^\mu = -1$, with I (null) future-pointing. It is useful to identify those frames with the neutral Lorentz group via $E = \mathcal{L}E_0$, where E_0 is some fixed frame, as well as space-time translations, $\mathcal{C}$, with Minkowskian events, R.

Picking then a fixed Poincaré-momentum $\mu_0 = (M_0, P_0)$ such that $M_0 = sI_0 \times J_0$ (the cross-product of I_0 and J_0, i.e., $(M_0)_{\mu\nu} = -s\epsilon_{\mu\nu\rho\sigma} I_0^\rho J_0^\sigma$) with $s > 0$ interpreted as the classical spin, and $P_0 = I_0$, we may define the one-form

$$\alpha = \mu_0 \cdot g^{-1} dg \tag{B1}$$

on G. Then, as a general result, the two-form

$$\sigma = d\alpha \tag{B2}$$

descends to the coadjoint orbit $M = G/G_{\mu_0}$ as its canonical symplectic form; the leaves generated by the stabilizer G_{μ_0} of μ_0 are interpreted as the motions of our (free) particle and integrate, by construction, the null distribution, $\ker \sigma$, on (G, σ). These leaves project down to space-time as the world sheets of our particle.

In the case under study the one-form (B1) of G reads,

$$\alpha = -I_\mu dR^\mu + sK_\mu dL^\mu, \tag{B3}$$

whereas its derivative (B2) descends to the evolution space $V^9 = G/\mathrm{SO}(2)$ in (3.1), the SO(2)-action on G being $(I,J,K,L,R) \to (I,J,K\cos\theta + L\sin\theta, -K\sin\theta + L\cos\theta, R)$. This two-form, still denoted by σ with a slight abuse of notation, is finally given by (3.5) where we have put $P = I$ for the four-momentum, and $S = sI \times J$ for the spin tensor.

We note for, completeness, that the 1-form α in (B1) can give rise to a Lagrangian in a variational framework, generalizing that outlined in Appendix A.

APPENDIX C: FINITE COADJOINT ACTION OF THE POINCARÉ GROUP

We recall that a Lorentz transformation of $\mathbb{R}^{3,1}$ is of the form

$$\mathcal{L} = \exp\begin{pmatrix} 0 & \alpha\boldsymbol{u} \\ \alpha\boldsymbol{u}^T & 0 \end{pmatrix} . \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}_+(3,1), \tag{C1}$$

where $\alpha \in \mathbb{R}$ is the rapidity and $\boldsymbol{u} \in S^2$ the direction of the boost, $\boldsymbol{b} = \tanh\alpha\boldsymbol{u}$, and $A \in \mathrm{SO}(3)$; we put $\gamma = \cosh\alpha = (1 - |\boldsymbol{b}|^2)^{-\frac{1}{2}}$. Using the shorthand $B = (B^i_j) = (\delta^i_j + (\gamma - 1)u^i u_j)$, an element of the connected (also called neutral) Poincaré group is of the form

$$g = \begin{pmatrix} BA & \gamma\boldsymbol{b} & \mathbf{c} \\ \gamma\boldsymbol{b}^T A & \gamma & e \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SE}_+(3,1), \tag{C2}$$

where $\mathbf{c} \in \mathbb{R}^3$ is a space-translation, and $e \in \mathbb{R}$ a time-translation. The Lie algebra of the Poincaré group is therefore spanned by the matrices

$$Z = \begin{pmatrix} \tilde{\boldsymbol{\omega}} & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ \boldsymbol{\beta}^T & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{e}(3,1) \tag{C3}$$

where $\tilde{\boldsymbol{\omega}} \in \mathfrak{so}(3)$ is identified with $\boldsymbol{\omega} \in \mathbb{R}^3$, also $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}$.

Then the adjoint action $Z \to Z' = (\boldsymbol{\omega}', \boldsymbol{\beta}', \boldsymbol{\gamma}', \varepsilon') = \mathrm{Ad}(g^{-1})Z$ reads

$$\boldsymbol{\omega}' = A^T(\gamma\boldsymbol{\omega} - (\gamma - 1)\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{\omega}) + \gamma\boldsymbol{b} \times B\boldsymbol{\beta}) \tag{C4}$$

$$\boldsymbol{\beta}' = A^T(\gamma A(\boldsymbol{\omega} \times \boldsymbol{b}) - \gamma^2\boldsymbol{b}(\boldsymbol{b} \cdot \boldsymbol{\beta}) + \gamma A\boldsymbol{\beta}) \tag{C5}$$

$$\boldsymbol{\gamma}' = A^T(A(\boldsymbol{\omega} \times \mathbf{c}) - \gamma\boldsymbol{b}(\mathbf{c} \cdot \boldsymbol{\beta}) + A\boldsymbol{\beta}e + A\boldsymbol{\gamma} - \gamma\boldsymbol{b}\varepsilon) \tag{C6}$$

$$\varepsilon' = \gamma((\boldsymbol{b} \times \mathbf{c}) \cdot \boldsymbol{\omega} + \boldsymbol{\beta} \cdot \mathbf{c} - (\boldsymbol{b} \cdot \boldsymbol{\beta})e - \boldsymbol{b} \cdot \boldsymbol{\gamma} + \varepsilon). \tag{C7}$$

Denoting by $\mu = (\boldsymbol{\ell}, \boldsymbol{g}, \boldsymbol{p}, \mathcal{E})$ a "moment" in $\mathfrak{e}(3,1)^*$ where $\mu \cdot Z = \boldsymbol{\ell} \cdot \boldsymbol{\omega} - \boldsymbol{g} \cdot \boldsymbol{\beta} + \boldsymbol{p} \cdot \boldsymbol{\gamma} - \mathcal{E}\varepsilon$, we then find the coadjoint representation $\mu \to \mu' = \mu \circ \mathrm{Ad}(g^{-1})$ where

$$\begin{aligned} \boldsymbol{\ell}' = {} & \gamma A\boldsymbol{\ell} - (\gamma - 1)(\boldsymbol{u} \cdot A\boldsymbol{\ell})\boldsymbol{u} - \gamma\boldsymbol{b} \times A\boldsymbol{g} + \mathbf{c} \times A\boldsymbol{p} \\ & - \gamma\mathcal{E}\boldsymbol{b} \times \mathbf{c} - (\gamma - 1)(\boldsymbol{u} \cdot A\boldsymbol{p})\boldsymbol{u} \times \mathbf{c} \end{aligned} \tag{C8}$$

$$\begin{aligned} \boldsymbol{g}' = {} & \gamma\boldsymbol{b} \times A\boldsymbol{\ell} + \gamma A\boldsymbol{g} - (\gamma - 1)(\boldsymbol{u} \cdot A(\boldsymbol{g} + \boldsymbol{p}e))\boldsymbol{u} \\ & + \gamma(\boldsymbol{b} \cdot A\boldsymbol{p})\mathbf{c} - eA\boldsymbol{p} + \gamma\mathcal{E}(\mathbf{c} - \boldsymbol{b}e) \end{aligned} \tag{C9}$$

$$\boldsymbol{p}' = A\boldsymbol{p} + (\gamma - 1)(\boldsymbol{u} \cdot A\boldsymbol{p})\boldsymbol{u} + \gamma\mathcal{E}\boldsymbol{b} \tag{C10}$$

$$\mathcal{E}' = \gamma(\mathcal{E} + \boldsymbol{b} \cdot A\boldsymbol{p}). \tag{C11}$$

At last, restricting ourselves to positive helicity and energy, the $\mathrm{SE}_+(3,1)$-action is expressed in terms of the quantities $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ describing the space of motions (M^6, ω) given in (3.21); the Poincaré-action (3.25) follows then at once.

[1] D. T. Son and N. Yamamoto, Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids, Phys. Rev. Lett. **109**, 181602 (2012).

[2] M. A. Stephanov and Y. Yin, Chiral Kinetic Theory, Phys. Rev. Lett. **109**, 162001 (2012).

[3] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, The chiral magnetic effect, Phys. Rev. D **78**, 074033 (2008); Real-Time Dynamics of the Chiral Magnetic Effect, Phys. Rev. Lett. **104**, 212001 (2010).

[4] G. Basar, G. V. Dunne, and D. E. Kharzeev, Chiral Magnetic Spiral, Phys. Rev. Lett. **104**, 232301 (2010).

[5] M. Stone and V. Dwivedi, Classical version of the Non-Abelian gauge anomaly, Phys. Rev. D **88**, 045012 (2013); Classical chiral kinetic theory and anomalies in even space-time dimensions, J. Phys. A **47**, 025401 (2013).

[6] D. T. Son and N. Yamamoto, Kinetic theory with berry curvature from quantum field theories, Phys. Rev. D **87**, 085016 (2013).

[7] J.-W. Chen, S. Pu, Q. Wang, and X.-N. Wang, Berry Curvature and Four-Dimensional Monopoles in the Relativistic Chiral Kinetic Equation, Phys. Rev. Lett. **110**, 262301 (2013).

[8] J.-W. Chen, J.-y. Pang, S. Pu, and Q. Wang, Non-Abelian berry phase in a semi-classical description of massive Dirac fermions, Phys. Rev. D **89**, 094003 (2014).

[9] J.-Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, and Y. Yin, Lorentz Invariance in Chiral Kinetic Theory, Phys. Rev. Lett. **113**, 182302 (2014).

[10] C. Manuel and J. M. Torres-Rincon, Chiral transport equation from the quantum dirac hamiltonian and the on-shell effective field theory, Phys. Rev. D **90**, 076007 (2014).

[11] M. Stone, V. Dwivedi, and T. Zhou, Berry phase, lorentz covariance, and anomalous velocity for dirac and weyl particles, Phys. Rev. D **91**, 025004 (2015); Wigner translations and the observer-dependence of the position of masslesss spinning particles, arXiv:1501.04586.

[12] D. Karabali and V. P. Nair, Relativistic particle and relativistic fluids: Magnetic moment and spin-orbit interactions, Phys. Rev. D **90**, 105018 (2014).

[13] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. **82**, 1959 (2010).

[14] Alternative equations with Lorentz symmetry were proposed in [7,9], and the model was generalized to the non-Abelian context [5,8].

[15] J.-M. Souriau, *Structure des Systèmes Dynamiques* (Dunod, Paris, 1970); *Structure of Dynamical Systems. A Symplectic View of Physics* (Birkhäuser, Boston, 1997).

[16] J.-M. Souriau, Modèle de particule à spin dans le champs électromagnétique et gravitationnel, Ann. Inst. Henri Poincaré, A, **20**, 315 (1974).

[17] C. Duval, Doctorat de 3ème cycle en physique théorique, Université de Provence, 1972; The general relativistic Dirac-Pauli particle: An underlying classical model, Ann. Inst. Henri Poincaré Phys. Theor. **25**, 345 (1976).

[18] R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1978).

[19] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics (Springer Verlag, Berlin, 1997), Vol. 60.

[20] A. Trautman, *Differential Geometry for Physicists*, Monographs and Textbooks in Physical Science. Stony Brook Lectures (Bibliopolis, Napoli, 1985).

[21] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge University Press, Cambridge, England, 1990).

[22] S. Sternberg, On The Role Of Field Theories In Our Physical Conception Of Geometry, in *Proceedings, Differential Geometrical Methods in Mathematical Physics II, Lecture Notes in Mathematics 676, Bonn, 1977* (Springer, Berlin, 1978), p. 1.

[23] In the case of massive noncommutative mechanics, $1 + e\Theta \cdot \boldsymbol{B}$ is the ratio of the *effective mass* by the rest mass, see [24–27].

[24] C. Duval and P. A. Horváthy, The "Peierls substitution" and the exotic Galilei group, Phys. Lett. B **479**, 284 (2000); Exotic galilean symmetry in the non-commutative plane, and the Hall effect, J. Phys. A **34**, 10097 (2001).

[25] C. Duval and P. A. Horváthy, Anyons with anomalous gyromagnetic ratio, and the Hall effect, Phys. Lett. B **594**, 402 (2004); Noncommuting coordinates, exotic particles, & anomalous anyons in the hall effect, Theor. Math. Phys. **144**, 899 (2005).

[26] C. Duval, Z. Horváth, P. A. Horváthy, L. Martina, and P. C. Stichel, Berry phase correction to electron density in solids and 'exotic' dynamics, Mod. Phys. Lett. B **20**, 373 (2006).

[27] P. A. Horváthy, L. Martina, and P. C. Stichel, Exotic galilean symmetry and non-commutative mechanics, SIGMA **6** (2010) 060.

[28] The symplectic form ω in (2.5) is the sum of the free form ω_0 (2.6) with the magnetic term, and the Hamiltonian, (2.7), is the addition of the scalar potential to the free expression. This form is limited to time-independent fields. However, choosing instead

$$\sigma = \omega_1 - dh_0 \wedge dt,$$
$$\omega_1 = \omega_0 + \frac{e}{2}\epsilon_{ijk}B^k dx^i \wedge dx^j + eE_i dx^i \wedge dt, \qquad h_0 = |\boldsymbol{p}|$$

would accommodate time-dependent fields, though [15]. When the electric field is stationary and curl-free, σ clearly takes the form (2.4). The subtle novelty of Souriau's prescription was emphasised by Sternberg [22].

[29] The equations of motion (2.1) are also Hamiltonian, with Hamiltonian h as in (2.7) and fundamental Poisson brackets given by [26]

$$\{x^i, x^j\} = \frac{-\epsilon^{ijk}\Theta_k}{1 + e\Theta \cdot \boldsymbol{B}}, \qquad \{p_i, p_j\} = \frac{e\epsilon_{ijk}B^k}{1 + e\Theta \cdot \boldsymbol{B}},$$
$$\{p_i, x^j\} = \frac{\delta_i^j + e\Theta_i B^j}{1 + e\Theta \cdot \boldsymbol{B}}.$$

It follows that the coordinates do not commute, let alone in the free case.

[30] L. D. Faddeev, Teor. Mat. Fiz. **1**, 3 (1969) [Feynman integral for singular Lagrangians, Theor. Math. Phys. **1**, 1 (1969)]; L. D. Faddeev and R. Jackiw, Hamiltonian Reduction of Unconstrained and Constrained Systems, Phys. Rev. Lett. **60**, 1692 (1988).

[31] P. M. Zhang and P. A. Horvathy, Chiral decomposition in the non-commutative Landau problem, Ann. Phys. (N.Y.) **327**, 1730 (2012).

[32] R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954); E. N. Adams and E. I. Blount, J. Phys. Chem. Solids **10**, 286 (1959).

[33] P. A. Horváthy, Anomalous hall effect in non-commutative mechanics, Phys. Lett. A **359**, 705 (2006).

[34] K. Y. Bliokh and Y. P. Bliokh, Topological spin transport of photons: The optical magnus effect and berry phase, Phys. Lett. A **333**, 181 (2004); M. Onoda, S. Murakami, and N. Nagaosa, Hall Effect of Light, Phys. Rev. Lett. **93**, 083901 (2004); C. Duval, Z. Horváth, and P. A. Horváthy, Fermat principle for polarized light, Phys. Rev. D **74**, 021701 (2006); Geometrical spinOptics and the optical hall effect, J. Geom. Phys. **57**, 925 (2007).

[35] D. Xiao, J.-r. Shi, and Q. Niu, Berry phase correction to electron density of states in solids, Phys. Rev. Lett. **95**, 137204 (2005).

[36] H. Balasin, D. N. Blaschke, F. Gieres, and M. Schweda, Wong's equations and charged relativistic particles in non-commutative space, SIGMA **10**, 099 (2014).

[37] P.-M. Zhang and P. A. Horváthy, arXiv:1409.4225.

[38] More generally [15] the first term in (3.5) could have a coefficient $\eta = \pm 1$ which represents the *sign of the energy*. In this paper, we consider positive energy only, $\eta = 1$.

[39] R. Penrose, Twistor algebra, J. Math. Phys. (N.Y.) **8**, 345 (1967); R. Penrose and M. A. H. MacCallum, Twistor theory: An Approach to the quantization of fields and space-time, Phys. Rep. **6**, 241 (1973).

[40] One of us (CD) discussed the infinitesimal action (3.20) and the quantities listed in (3.21) with F. Ziegler long ago (unpublished).

[41] This is unlike to the Dirac equation, where internal and external degrees of freedom are intricately mixed, due to the need of using Dirac matrices to represent the Lorentz group. Our classical model might comply with squaring the Dirac equation, allowing indeed to separate the orbital and spin terms.

[42] E. Wigner, On unitary representation of the inhomogeneous Lorentz group, Ann. Math. **40**, 149 (1939).

[43] P. A. Horváthy, B. Cordani, and L. Fehér, Kepler-type dynamical symmetries of long-range monopole interactions, J. Math. Phys. (N.Y.) **31**, 202 (1990).

[44] C. Duval, M. Elbistan, P. A. Horvathy, and P.-M. Zhang, arXiv:1411.6541.

[45] V. Bargmann, L. Michel, and V. L. Telegdi, Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field, Phys. Rev. Lett. **2**, 435 (1959).

[46] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, Non-commutative planar particle dynamics with gauge interactions, Ann. Phys. (N.Y.) **306**, 78 (2003).

[47] The system would become singular at those points of V^9 where $S_{\alpha\beta}F^{\alpha\beta} = 0$; this would change locally the dimension of $\ker\sigma$, destroying *a priori* the smooth manifold structure of the space of motions.

[48] Instantaneous motions with infinite velocity are familiar in nonrelativistic optics [34]. Intriguingly, motion with superluminal velocity also appears in certain higher-order massless relativistic models [49].

[49] M. S. Plyushchay, Massless point particle with rigidity, Mod. Phys. Lett. A **04**, 837 (1989); Massless particle with rigidity as a model for description of bosons and fermions, Phys. Lett. B **243**, 383 (1990).

[50] Somewhat surprisingly, switching off the external fields in (5.7) does not yield the free system. Remember, though, that the Eqs. (5.7) are derived under the assumption that the regularity conditions (5.5) be satisfied—which is clearly not the case when the electromagnetic field vanishes. Accordingly, the transition from the interacting to the free case is not smooth, as highlighted by the dimension of the space of motions dropping from 8 to 6. The correct way of tackling the problem would be, once again, Faddeev-Jackiw reduction [30] we do not consider here.

[51] Equation (5.9) can be generalized by putting $P_\mu P^\mu = f(eS.F)$ where f is an otherwise arbitrary function such that $f(0) = 0$. We refer to [16,17] for the case of massive spinning particles nonminimally coupled to an external electromagnetic field.

[52] As the term $S \cdot F$ itself involves $\mathcal{E}$, Eq. (5.15) is a third-order algebraic equation for $\mathcal{E}$.

[53] This is also what happens in the plane for $g = 2$, just like [25].

[54] This behavior is once again related to *gauge covariance*: a constant electric field could be eliminated by a boost—but boost freedom and enslavement seem to be incompatible.

[55] P. A. Horváthy and L. Úry, Analogy between statics and dynamics, related to variational mechanics, Acta Phys. Hung. **42**, 251 (1977); P. A. Horváthy, Variational formalism for spinning particles., J. Math. Phys. (N.Y.) **20**, 49 (1979).

[56] C. Duval, On the prequantum description of spinning particles in an external gauge field, *Proc. Aix Conference on Diff. Geom. Math. in Math. Phys., Lecture Notes in Mathematics 836*, (Springer, Berlin, 1980), 49; C. Duval and P. A. Horváthy, Particles with internal structure: The geometry of classical motions and conservation laws, Report No. CPT-81-P-1304; Ann. Phys. (N.Y.) **142**, 10 (1982).