Three loop analysis of the critical $O(N)$ models in $6 - \epsilon$ dimensions

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We continue the study, initiated in [L. Fei, S. Giombi, and I. R. Klebanov, Phys. Rev. D 90, 025018 (2014)], of the $O(N)$ symmetric theory of $N + 1$ massless scalar fields in $6 - \epsilon$ dimensions. This theory has cubic interaction terms $\frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3$. We calculate the three loop beta functions for the two couplings and use them to determine certain operator scaling dimensions at the IR stable fixed point up to order ϵ^3. We also use the beta functions to determine the corrections to the critical value of N below which there is no fixed point at real couplings. The result suggests a significant reduction in the critical value as the dimension is decreased to 5. We also study the theory with $N = 1$, which has a Z_2 symmetry under $\phi \to -\phi$. We show that it possesses an IR stable fixed point at imaginary couplings which can be reached by flow from a nearby fixed point describing a pair of $N = 0$ theories. We calculate certain operator scaling dimensions at the IR fixed point of the $N = 1$ theory and suggest that, upon continuation to two dimensions, it describes a nonunitary conformal minimal model.

I. INTRODUCTION AND SUMMARY

This paper is a sequel to [1] where a one loop analysis was carried out for the cubic $O(N)$ symmetric theory of $N + 1$ scalar fields σ and ϕ^i in $6 - \epsilon$ dimensions. The Lagrangian of this theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3, \quad (1.1)$$

and the one loop beta functions showed that for $N > N_{\text{crit}}$ there exists an IR stable fixed point with real values of the two couplings. It was argued that this IR fixed point of the cubic $O(N)$ theory is equivalent to the perturbatively unitary UV fixed point of the $O(N)$ model with interaction $(\phi^i \phi^i)^2$, which exists for large N in $4 < d < 6$ [2–5]. The $1/N$ expansions of various operator scaling dimensions were found in [1] to agree with the corresponding results [6–14] in the quartic $O(N)$ model continued to $6 - \epsilon$ dimensions.

A surprising result of [1] was that the one loop value of N_{crit} is very large: if N_{crit} is treated as a continuous real parameter, then it is ≈ 1038.266. Our main interest is in continuing the $d = 6 - \epsilon$ fixed point to $\epsilon = 1$ in the hope of finding a five-dimensional $O(N)$ symmetric unitary conformal field theory (CFT). In order to study the ϵ expansion of N_{crit}, in Sec. II we calculate the three loop β functions, following the earlier work of [15–17].[1] In Sec. IV we find the following expansion for the critical value of N:

$$N_{\text{crit}} = 1038.266 - 609.840\epsilon - 364.173\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (1.2)$$

Neglecting further corrections, this gives $N_{\text{crit}}(\epsilon = 1) \approx 64$, but higher orders in ϵ can obviously change this value significantly. It is our hope that a conformal bootstrap approach [19–22], perhaps along the lines of [23], can help determine N_{crit} more precisely in $d = 5$. The bootstrap approach may also be applied in noninteger dimensions close to 6, but one should keep in mind that such theories are not strictly unitary [24].[2]

The major reduction of N_{crit} as ϵ is increased from 0 to 1 is analogous to what is known about the Abelian Higgs model in $4 - \epsilon$ dimensions.[3] For the model containing N_f complex scalars, the one loop critical value of N_f is found to be large, $N_{f,\text{crit}} \approx 183$ [26]. However, the $\mathcal{O}(\epsilon)$ correction found from two loop beta functions has a negative coefficient and almost exactly cancels the leading term when $\epsilon = 1$, suggesting that the $N_{f,\text{crit}}$ is small in the physically interesting three-dimensional theory [27].

Another interesting property of the theories (1.1) is the existence of the lower critical value N'_{crit} such that for $N < N'_{\text{crit}}$ there is an IR stable fixed point at *imaginary* values of g_1 and g_2. The simplest example of such a nonunitary theory is $N = 0$, containing only the field σ. Its $6 - \epsilon$ expansion was originally studied by Michael Fisher [28] and the continuation to $\epsilon = 4$ provides an approach to the Yang-Lee edge singularity in the two-dimensional Ising model [this is the (2,5) minimal model [29,30] with central

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[1]These papers considered cubic field theories of $q - 1$ scalar fields that were shown in [18] to describe the q-state Potts model. These theories possess only discrete symmetries and generally differ from the $O(N)$ symmetric theories that we study.

[2]Another possible nonperturbative approach to the theory in $4 < d < 6$ is the exact renormalization group (RG) [25]. This approach does not seem to indicate the presence of a UV fixed point in the theory of N scalar fields, but a search for an IR fixed point in the theory of $N + 1$ scalar fields has not been carried out yet.

[3]We are grateful to Igor Herbut for pointing this out to us.

charge $-22/5$]. From the three loop β functions we find the ϵ expansion

$$N'_{\text{crit}} = 1.02145 + 0.03253\epsilon - 0.00163\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (1.3)$$

The smallness of the coefficients suggests that $N'_{\text{crit}} > 1$ for a range of dimensions below 6. In Sec. V we discuss some properties of the $N = 1$ theory. We show that it possesses an unstable fixed point with $g_1^* = g_2^*$ where the Lagrangian splits into that of two decoupled $N = 0$ theories. There is also an IR stable fixed point where $g_2^* = 6g_1^*/5 + O(\epsilon)$. A distinguishing feature of this nonunitary CFT is that it has a discrete Z_2 symmetry, and it would be interesting to search for it using the conformal bootstrap methods developed in [31]. We suggest that, when continued to two dimensions, it describes the (3,8) nonunitary conformal minimal model.

In Sec. IV A we also discuss unstable unitary fixed points that are present in $6 - \epsilon$ dimensions for all N. For $N = 1$ the fixed point has $g_1^* = -g_2^*$; it is Z_3 symmetric and describes the critical point of the 3-state Potts model in $6 - \epsilon$ dimensions [32].[4]

II. THREE LOOP β-FUNCTIONS IN $d = 6 - \epsilon$

The action of the cubic theory is

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi_0^i)^2 + \frac{1}{2}(\partial_\mu \sigma_0)^2 + \frac{1}{2}g_{1,0}\sigma_0\phi_0^i\phi_0^i + \frac{1}{6}g_{2,0}\sigma_0^3\right), \quad (2.1)$$

where ϕ_0^i and σ_0 are bare fields and $g_{1,0}$ and $g_{2,0}$ are bare coupling constants.[5] As usual, we introduce renormalized fields and coupling constants by

$$\sigma_0 = Z_\sigma^{1/2}\sigma, \qquad \phi_0^i = Z_\phi^{1/2}\phi^i,$$

$$g_{1,0} = \mu^{\frac{\epsilon}{2}}Z_{g_1}Z_\sigma^{-1/2}Z_\phi^{-1}g_1, \qquad g_{2,0} = \mu^{\frac{\epsilon}{2}}Z_{g_2}Z_\sigma^{-3/2}g_2. \quad (2.2)$$

Here g_1, g_2 are the dimensionless renormalized couplings, and μ is the renormalization scale. We may write

$$Z_\sigma = 1 + \delta_\sigma, \qquad Z_\phi = 1 + \delta_\phi,$$

$$Z_{g_1} = 1 + \delta g_1, \qquad Z_{g_2} = 1 + \delta g_2 \quad (2.3)$$

so that, in terms of renormalized quantities, the action reads

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[4]We are grateful to Yu Nakayama for valuable discussions on this issue.

[5]We do not include mass terms as we are ultimately interested in the conformal theory. In the dimensional regularization that we will be using, mass terms are not generated if we set them to zero from the start.

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3\right.$$

$$\left. + \frac{\delta_\phi}{2}(\partial_\mu \phi^i)^2 + \frac{\delta_\sigma}{2}(\partial_\mu \sigma)^2 + \frac{\delta g_1}{2}\sigma\phi^i\phi^i + \frac{\delta g_2}{6}\sigma^3\right). \quad (2.4)$$

To carry out the renormalization procedure, we will use dimensional regularization [33] in $d = 6 - \epsilon$ and employ the minimal subtraction scheme [34]. In this scheme, the counterterms are fixed by requiring cancellation of poles in the dimensional regulator, and have the structure

$$\delta g_1 = \sum_{n=1}^{\infty} \frac{a_n(g_1, g_2)}{\epsilon^n}, \qquad \delta g_2 = \sum_{n=1}^{\infty} \frac{b_n(g_1, g_2)}{\epsilon^n},$$

$$\delta_\phi = \sum_{n=1}^{\infty} \frac{z_n^\phi(g_1, g_2)}{\epsilon^n}, \qquad \delta_\sigma = \sum_{n=1}^{\infty} \frac{z_n^\sigma(g_1, g_2)}{\epsilon^n}. \quad (2.5)$$

The anomalous dimensions and β-functions are determined by the coefficients of the simple $1/\epsilon$ poles in the counterterms [34]. Specifically, in our case we have that the anomalous dimensions are given by

$$\gamma_\phi = -\frac{1}{4}\left(g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2}\right)z_1^\phi, \quad (2.6)$$

$$\gamma_\sigma = -\frac{1}{4}\left(g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2}\right)z_1^\sigma \quad (2.7)$$

and the β-functions are

$$\beta_1(g_1, g_2) = -\frac{\epsilon}{2}g_1 + \frac{1}{2}\left(g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2} - 1\right)$$
$$\times \left(a_1 - \frac{1}{2}g_1(2z_1^\phi + z_1^\sigma)\right),$$

$$\beta_2(g_1, g_2) = -\frac{\epsilon}{2}g_2 + \frac{1}{2}\left(g_1\frac{\partial}{\partial g_1} + g_2\frac{\partial}{\partial g_2} - 1\right)$$
$$\times \left(b_1 - \frac{3}{2}g_2 z_1^\sigma\right). \quad (2.8)$$

In other words, in order to determine the anomalous dimensions and β-functions, we have to calculate the coefficients of the $1/\epsilon$-divergencies in the loop diagrams, from which we can read off the residues $a_1(g_1, g_2), b_1(g_2, g_2), z_1^\phi(g_1, g_2), z_1^\sigma(g_1, g_2)$.

Working in perturbation theory, we will denote by a_{1i} the term of ith order in the coupling constants, and similarly for the other residue functions. Then, using the results for the Feynman diagrams collected in the Appendix, we find the anomalous dimensions:

$$\gamma_\phi = -\frac{1}{2}z_{12}^\phi - z_{14}^\phi - \frac{3}{2}z_{16}^\phi$$

$$= \frac{g_1^2}{6(4\pi)^3} - \frac{g_1^2}{432(4\pi)^6}(g_1^2(11N - 26) - 48g_1g_2 + 11g_2^2)$$

$$- \frac{g_1^2}{31104(4\pi)^9}(g_1^4(N(13N - 232) + 5184\zeta(3) - 9064) + g_1^3g_2 6(441N - 544)$$

$$- 2g_1^2g_2^2(193N - 2592\zeta(3) + 5881) + 942g_1g_2^3 + 327g_2^4), \tag{2.9}$$

$$\gamma_\sigma = -\frac{1}{2}z_{12}^\sigma - z_{14}^\sigma - \frac{3}{2}z_{16}^\sigma$$

$$= \frac{Ng_1^2 + g_2^2}{12(4\pi)^3} + \frac{1}{432(4\pi)^6}(2Ng_1^4 + 48Ng_1^3g_2 - 11Ng_1^2g_2^2 + 13g_2^4)$$

$$+ \frac{1}{62208(4\pi)^9}(96N(12N + 11)g_1^5g_2 - 1560Ng_1^3g_2^3 + 952Ng_1^2g_2^4 - 2Ng_1^6(1381N - 2592\zeta(3) + 4280)$$

$$+ g_2^6(2592\zeta(3) - 5195) + 3Ng_1^4g_2^2(N + 4320\zeta(3) - 8882)) \tag{2.10}$$

and the β-functions

$$\beta_1 = -\frac{\epsilon}{2}g_1 + \left(a_{13} - \frac{1}{2}g_1(2z_{12}^\phi + z_{12}^\sigma)\right) + 2\left(a_{15} - \frac{1}{2}g_1(2z_{14}^\phi + z_{14}^\sigma)\right) + 3\left(a_{17} - \frac{1}{2}g_1(2z_{16}^\phi + z_{16}^\sigma)\right)$$

$$= -\frac{\epsilon}{2}g_1 + \frac{1}{12(4\pi)^3}g_1((N - 8)g_1^2 - 12g_1g_2 + g_2^2)$$

$$- \frac{1}{432(4\pi)^6}g_1((536 + 86N)g_1^4 + 12(30 - 11N)g_1^3g_2 + (628 + 11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4)$$

$$+ \frac{1}{62208(4\pi)^9}g_1(g_2^6(5195 - 2592\zeta(3)) + 12g_1g_2^5(-2801 + 2592\zeta(3))$$

$$- 8g_1^2g_2^4(1245 + 119N + 7776\zeta(3)) + g_1^4g_2^2(-358480 + 53990N - 3N^2 - 2592(-16 + 5N)\zeta(3))$$

$$+ 36g_1^5g_2(-500 - 3464N + N^2 + 864(5N - 6)\zeta(3))$$

$$- 2g_1^6(125680 - 20344N + 1831N^2 + 2592(25N + 4)\zeta(3)) + 48g_1^3g_2^3(95N - 3(679 + 864\zeta(3)))), \tag{2.11}$$

$$\beta_2 = -\frac{\epsilon}{2}g_2 + \left(b_{13} - \frac{3}{2}g_2 z_{12}^\sigma\right) + 2\left(b_{15} - \frac{3}{2}g_2 z_{14}^\sigma\right) + 3\left(b_{17} - \frac{3}{2}g_2 z_{16}^\sigma\right)$$

$$= -\frac{\epsilon}{2}g_2 + \frac{1}{4(4\pi)^3}(-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3)$$

$$+ \frac{1}{144(4\pi)^6}(-24Ng_1^5 - 322Ng_1^4g_2 - 60Ng_1^3g_2^2 + 31Ng_1^2g_2^3 - 125g_2^5)$$

$$+ \frac{1}{20736(4\pi)^9}(-48N(713 + 577N)g_1^7 + 6272Ng_1^2g_2^5 + 48Ng_1^3g_2^4(181 + 432\zeta(3))$$

$$- 5g_2^7(6617 + 2592\zeta(3)) - 24Ng_1^5g_2^2(1054 + 471N + 2592\zeta(3))$$

$$+ 2Ng_1^6g_2(19237N - 8(3713 + 324\zeta(3))) + 3Ng_1^4g_2^3(263N - 6(7105 + 2448\zeta(3)))). \tag{2.12}$$

In the case $N = 0$ (the single scalar cubic theory), our results are in agreement with the three loop calculation of [16].

III. THE IR FIXED POINT

Let us introduce the notation

$$g_1 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}}x, \qquad g_2 \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}}y. \tag{3.1}$$

In terms of the new variables x and y, the condition that both β-functions be zero reads

$$
\begin{aligned}
0 = &\frac{1}{2}x(-8x^2 + N(x^2 - 1) - 12xy + y^2) \\
&- \frac{1}{12N}x((536 + 86N)x^4 + 12(30 - 11N)x^3y + (628 + 11N)x^2y^2 + 24xy^3 - 13y^4)\epsilon \\
&- \frac{1}{288N^2}x(12xy^5(2801 - 2592\zeta(3)) + y^6(-5195 + 2592\zeta(3)) + 48x^3y^3(2037 - 95N + 2592\zeta(3)) \\
&+ 8x^2y^4(1245 + 119N + 7776\zeta(3)) + x^4y^2(358480 - 53990N + 3N^2 + 2592(5N - 16)\zeta(3)) \\
&- 36x^5y(-500 - 3464N + N^2 + 864(5N - 6)\zeta(3)) \\
&+ 2x^6(125680 - 20344N + 1831N^2 + 2592(25N + 4)\zeta(3)))\epsilon^2
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
0 = &-\frac{1}{2}(9y^3 + N(12x^3 + y - 3x^2y)) \\
&- \frac{1}{4N}(125y^5 + Nx^2(24x^3 + 322x^2y + 60xy^2 - 31y^3))\epsilon \\
&- \frac{1}{96N^2}(N^2x^4(27696x^3 - 38474x^2y + 11304xy^2 - 789y^3) + 5y^7(6617 + 2592\zeta(3)) \\
&+ 34224Nx^7 - 6272Nx^2y^5 + 16Nx^6y(3713 + 324\zeta(3)) - 48Nx^3y^4(181 + 432\zeta(3)) \\
&+ 48Nx^5y^2(527 + 1296\zeta(3)) + 18Nx^4y^3(7105 + 2448\zeta(3)))\epsilon^2.
\end{aligned}
\tag{3.3}
$$

These equations can be solved order by order in the ϵ expansion. Using also the $1/N$ expansion, we find the fixed point values

$$
\begin{aligned}
x_* = &1 + \frac{22}{N} + \frac{726}{N^2} - \frac{326180}{N^3} - \frac{349658330}{N^4} + \cdots \\
&+ \left(-\frac{155}{6N} - \frac{1705}{N^2} + \frac{912545}{N^3} + \frac{3590574890}{3N^4} + \cdots\right)\epsilon \\
&+ \left(\frac{1777}{144N} + \frac{29093/36 - 1170\zeta(3)}{N^2} + \cdots\right)\epsilon^2,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
y_* = &6\left(1 + \frac{162}{N} + \frac{68766}{N^2} + \frac{41224420}{N^3} + \frac{28762554870}{N^4} + \cdots\right) \\
&+ \left(-\frac{215}{2N} - \frac{86335}{N^2} - \frac{75722265}{N^3} - \frac{69633402510}{N^4} + \cdots\right)\epsilon \\
&+ \left(\frac{2781}{48N} + \frac{270911 - 157140\zeta(3)}{6N^2} + \cdots\right)\epsilon^2.
\end{aligned}
\tag{3.5}
$$

This large N solution corresponds to an IR stable fixed point and generalizes the one loop result of [1]. This fixed point exists and is stable to all orders in the $1/N$ expansion.

If results beyond the $1/N$ expansion are desired, one can determine the ϵ expansions of x_*, y_* for finite N as follows. Plugging the expansions

$$x_* = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + \cdots,$$
$$y_* = y_0(N) + y_1(N)\epsilon + y_2(N)\epsilon^2 + \cdots \qquad (3.6)$$

into (3.2) and (3.3), the leading order terms are found to be [1,35]

$$x_0(N) = \sqrt{\frac{N}{(N-44)z(N)^2 + 1}} z(N),$$

$$y_0(N) = \sqrt{\frac{N}{(N-44)z(N)^2 + 1}}(1 + 6z(N)), \qquad (3.7)$$

where $z(N)$ is the solution to the cubic equation

$$840z^3 - (N-464)z^2 + 84z + 5 = 0 \qquad (3.8)$$

with large N behavior $z(N) = 840N + O(N^0)$.[6] This solution is real only if $N > 1038.27$, as can be seen from the discriminant of the above cubic equation. Once the $x_0(N), y_0(N)$ are known, one can then determine the higher order terms in (3.6) by solving the equations (3.2) and (3.3) order by order in ϵ.

For $N \gg 1038$ the finite N exact results are close to (3.4) and (3.5), but for $N \sim 1038$ they deviate somewhat, indicating that, close to the critical N, the large N expansion is not a good approximation (see also Fig. 1 below).

A. Dimensions of ϕ and σ

In terms of the rescaled couplings x, y defined in (3.1), the anomalous dimensions read

$$\gamma_\phi = \frac{x^2}{N}\epsilon - \frac{x^2}{12N^2}((26 - 11N)x^2 + 48xy - 11y^2)\epsilon^2$$
$$+ \frac{x^2}{144N^3}(6(544 - 441N)x^3y - 942xy^3 - 327y^4 + x^4(9064 + (232 - 13N)N - 5184\zeta(3))$$
$$+ 2x^2y^2(5881 + 193N - 2592\zeta(3))\})\epsilon^3, \qquad (3.9)$$

$$\gamma_\sigma = \frac{Nx^2 + y^2}{2N}\epsilon - \frac{1}{12N^2}(13y^4 + Nx^2(2x^2 + 48xy - 11y^2))\epsilon^2$$
$$+ \frac{1}{288N^3}(N^2x^4(2762x^2 - 1152xy - 3y^2)$$
$$+ 2Nx^2(-528x^3y + 780xy^3 - 476y^4 + 3x^2y^2(4441 - 2160\zeta(3)) + 8x^4(535 - 324\zeta(3)))$$
$$+ y^6(5195 - 2592\zeta(3)))\epsilon^3. \qquad (3.10)$$

Plugging the fixed point values (3.4) and (3.5) into these expressions, we get the conformal dimensions of σ and ϕ at the fixed point:

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi \qquad (3.11)$$

$$= 2 - \frac{\epsilon}{2} + \left(\frac{1}{N} + \frac{44}{N^2} + \frac{1936}{N^3} + \cdots\right)\epsilon + \left(-\frac{11}{12N} - \frac{835}{6N^2} - \frac{16352}{N^3} + \cdots\right)\epsilon^2$$
$$+ \left(-\frac{13}{144N} + \frac{6865}{72N^2} + \frac{54367/2 - 3672\zeta(3)}{N^3} + \cdots\right)\epsilon^3, \qquad (3.12)$$

$$\Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma \qquad (3.13)$$

$$= 2 + \left(\frac{40}{N} + \frac{6800}{N^2} + \cdots\right)\epsilon + \left(-\frac{104}{3N} - \frac{34190}{3N^2} + \cdots\right)\epsilon^2$$
$$+ \left(-\frac{22}{9N} + \frac{47695/18 - 2808\zeta(3)}{N^2} + \cdots\right)\epsilon^3. \qquad (3.14)$$

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[6]The other two roots have large N behavior $z(N) \sim \pm\sqrt{5N}$ and they are unstable IR fixed points [1].

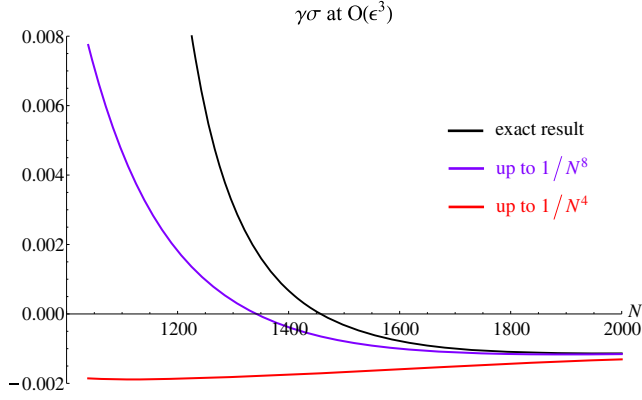FIG. 1 (color online). The $O(\epsilon^3)$ in Δ_σ as a function of N for $N \geq 1039$. The $1/N$ expansion approaches the exact result as we include more terms.

One can verify that these results are in precise agreement with the large N calculation of [6–8] for the critical $O(N)$ model in general d, analytically continued to $d = 6 - \epsilon$. This provides a strong check on our calculations and on our interpretation of the IR fixed point of the cubic $O(N)$ scalar theory.

The $1/N$ expansions are expected to work well for $N \gg 1038$. For any N larger than the critical value, the ϵ expansions of the scaling dimensions may be determined using (3.10) and the exact analytic solutions for the fixed point location (x_*, y_*). For example, in Fig. 1 we plot the coefficient of the $O(\epsilon^3)$ term in Δ_σ as a function of N and compare it with the corresponding $1/N$ expansion.

B. Dimensions of quadratic and cubic operators

In [1] the mixed anomalous dimensions of quadratic operators σ^2 and $\phi^i \phi^i$ were calculated at one loop order. These results were checked against the $O(1/N)$ term in the corresponding operator dimensions for the $O(N)$ ϕ^4 theory [12]. In this paper, we carry out an additional check, comparing with the $O(1/N^2)$ correction found in [36], but still working to the one loop order in ϵ (it should be straightforward to generalize the mixing calculation to higher loops, but we will not do it here).

In the quartic $O(N)$ theory with interaction $\frac{\lambda}{4}(\phi^i \phi^i)^2$, the derivative of the beta function at the fixed point coupling

$$\omega = \beta'(\lambda_*) = 4 - d + \frac{\omega_1}{N} + \frac{\omega_2}{N^2} + \cdots \quad (3.15)$$

is related to the dimension of the operator $(\phi^i \phi^i)^2$ by

$$\Delta_{\phi^4} = d + \omega. \quad (3.16)$$

In [12,36] the coefficient ω_1 was computed as a function of dimension d:

$$\omega_1 = \frac{2(d-4)(d-2)(d-1)\Gamma(d)}{d\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2})^3}. \quad (3.17)$$

The coefficient ω_2 has a more complicated structure for general d which was first found in [36]. Using this result, we get that in $d = 5$,

$$\Delta_{\phi^4} = 4 - \frac{2048}{15\pi^2 N} - \frac{8192(67125\pi^2 - 589472)}{3375\pi^4 N^2}$$
$$+ \cdots \approx 4 - \frac{13.8337}{N} - \frac{1819.66}{N^2} + \cdots. \quad (3.18)$$

Let us also quote the expansion of ω_2 in $d = 4 - \epsilon$ and $d = 6 - \epsilon$:

$$\omega_2 = 102\epsilon^2 + \left(-\frac{259}{2} + 120\zeta(3)\right)\epsilon^3 + \cdots,$$
$$d = 4 - \epsilon, \quad (3.19)$$

$$\omega_2 = -49760\epsilon + \frac{237476}{3}\epsilon^2$$
$$+ \left(-\frac{92480}{9} + 32616\zeta(3)\right)\epsilon^3 + \cdots,$$
$$d = 6 - \epsilon. \quad (3.20)$$

In $d = 4 - \epsilon$, one can check that the above results correctly reproduce the derivative of the β-function [37]

$$\beta = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 - \frac{3(3N+14)}{64\pi^4}\lambda^3$$
$$+ \frac{33N^2 + 480N\zeta(3) + 922N + 2112\zeta(3) + 2960}{4096\pi^6}\lambda^4$$
$$+ O(\lambda^5) \quad (3.21)$$

at the IR fixed point

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon + \frac{24\pi^2(3N+14)}{(N+8)^3}\epsilon^2$$
$$- \frac{\pi^2(33N^3 - 110N^2 + 96(N+8)(5N+22)\zeta(3) - 1760N - 4544)}{(N+8)^5}\epsilon^3 + O(\epsilon^4). \quad (3.22)$$

In $d = 6 - \epsilon$, the dimension of the $(\phi^i \phi^i)^2$ operator in the quartic theory should be matched to the primary operator arising from the mixing of the σ^2 and $\phi^i \phi^i$ operators in our cubic theory. In [1], the mixing matrix of σ^2 and $\phi^i \phi^i$ to one loop order was found to be

$$\gamma^{ij} = \frac{-1}{6(4\pi)^3} \begin{pmatrix} 4g_1^2 - N g_1^2 & 6\sqrt{N} g_1^2 - \sqrt{N} g_1 g_2 \\ 6\sqrt{N} g_1^2 - \sqrt{N} g_1 g_2 & 4g_2^2 - N g_1^2 \end{pmatrix}. \tag{3.23}$$

Computing the eigenvalues $\gamma_\pm$ of this matrix, and inserting the values of one loop fixed point couplings

$$g_{1*} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left(1 + \frac{22}{N} + \frac{726}{N^2} - \frac{326180}{N^3} + \cdots \right), \tag{3.24}$$

$$g_{2*} = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}} \left(1 + \frac{162}{N} + \frac{68766}{N^2} + \frac{41224420}{N^3} + \cdots \right) \tag{3.25}$$

we find the scaling dimensions of the quadratic operators to be

$$\Delta_- = d - 2 + \gamma_-$$
$$= 4 + \left(-\frac{100}{N} - \frac{49760}{N^2} - \frac{27470080}{N^3} + \cdots \right) \epsilon + \mathcal{O}(\epsilon^2), \tag{3.26}$$

$$\Delta_+ = d - 2 + \gamma_+$$
$$= 4 + \left(\frac{40}{N} + \frac{6800}{N^2} + \frac{2637760}{N^3} + \cdots \right) \epsilon + \mathcal{O}(\epsilon^2). \tag{3.27}$$

The operator with dimension Δ_+ is a descendant of σ. The operator with dimension Δ_- is a primary, and comparing with (3.17), (3.20), we see that its dimension precisely agrees with the results of [36] to order $1/N^2$. The higher order terms in ϵ can be determined from mixed anomalous dimension calculations beyond one loop, and we leave this to future work.

We now calculate the mixed anomalous dimensions of the nearly marginal operators $\mathbb{O}^1 = \sigma \phi \phi$ and $\mathbb{O}^2 = \sigma^3$. Using the beta functions written in Eqs. (2.11)and (2.12), we can determine the anomalous dimensions of the nearly marginal operators by computing the eigenvalues of the matrix

$$M_{ij} = \frac{\partial \beta_i}{\partial g_j}. \tag{3.28}$$

Strictly speaking, this matrix is not exactly equal to the anomalous dimension mixing matrix, because it is not symmetric. However, we could make it symmetric by dividing and multiplying the off-diagonal elements by a factor $\sqrt{3N} + O(\epsilon)$, which corresponds to an appropriate rescaling of the couplings. This clearly does not change the eigenvalues of the matrix, and hence we can directly compute the eigenvalues $\lambda_\pm$ of (3.28), and obtain the dimensions of the eigenstate operators as

$$\Delta_\pm = d + \lambda_\pm. \tag{3.29}$$

Plugging in the fixed point values x_* and y_* from Eqs. (3.4) and (3.5), we find that

$$\Delta_+ = 6 + \left(\frac{155}{3} \epsilon^2 - \frac{1777}{36} \epsilon^3 + \cdots \right) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2} \right),$$
$$\Delta_- = 6 + \left(-420\epsilon + 499\epsilon^2 - \frac{1051}{12} \epsilon^3 + \cdots \right) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2} \right). \tag{3.30}$$

The dimension of the σ^k operator in the quartic $O(N)$ model is known to order $1/N$ as a function of d [12], and may be written as

$$\Delta(\sigma^k) = 2k + \frac{k(d-2)((k-1)d^2 - d(3k-1) + 4)\Gamma(d)}{Nd\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2})^3}$$
$$+ \mathcal{O}\left(\frac{1}{N^2} \right). \tag{3.31}$$

Our result for Δ_- agrees with the ϵ expansion of this formula for $k = 3$ in $d = 6 - \epsilon$.

IV. ANALYSIS OF CRITICAL N AS A FUNCTION OF ϵ

We now investigate the behavior of N_{crit} above which the fixed point exists at real values of the couplings. This can be defined as the value of N (formally viewed as a continuous parameter) at which two real solutions of the β-function equations merge, and subsequently go off to the complex plane. Geometrically, this means that the curves on the (g_1, g_2) plane defined by the zeros of β_1 and β_2 are barely touching, i.e. they are tangent to each other. Therefore the critical N, as well as the corresponding critical value of the couplings, can be determined by solving the system of equations

$$\beta_1 = 0, \qquad \beta_2 = 0, \qquad \frac{\partial \beta_1/\partial g_1}{\partial \beta_1/\partial g_2} = \frac{\partial \beta_2/\partial g_1}{\partial \beta_2/\partial g_2}. \tag{4.1}$$

Note that the condition in the second line is equivalent to requiring that the determinant of the anomalous dimension mixing matrix of nearly marginal operators, $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$, vanishes. This means that one of the two eigenstates becomes marginal.

Working in terms of the rescaled coupling constants defined in (3.1), we can solve the system of equations (4.1) order by order in ϵ. We assume a perturbative expansion

$$x = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3),$$
$$y = y_0 + y_1\epsilon + y_2\epsilon^2 + O(\epsilon^3),$$
$$N = N_0 + N_1\epsilon + N_2\epsilon^2 + O(\epsilon^3) \qquad (4.2)$$

and plugging this into (4.1), we can solve for the undetermined coefficients uniquely. At the zeroth order, we get the equations

$$N_0 + 8x_0^2 - N_0 x_0^2 + 12x_0 y_0 - y_0^2 = 0,$$
$$12N_0 x_0^3 + N_0 y_0 - 3N_0 x_0^2 y_0 + 9y_0^3 = 0,$$
$$6 + \frac{(N_0 - 44)x_0}{6x_0 - y_0} = \frac{6N_0 x_0(y_0 - 6x_0)}{3N_0 x_0^2 - 27y_0^2 - N_0}. \qquad (4.3)$$

The above system of equations can be solved analytically, as was done in [1]. We find that, up to the signs of x_0 and y_0, there are three inequivalent solutions

$$x_0 = 1.01804, \quad y_0 = 8.90305, \quad N_0 = 1038.26605, \qquad (4.4)$$

$$x_0' = 0.23185i, \quad y_0' = 0.25582i, \quad N_0' = 1.02145, \qquad (4.5)$$

$$x_0'' = 0.13175, \quad y_0'' = -0.03277, \quad N_0'' = -0.08750. \qquad (4.6)$$

The first of these solutions, with $N_{\text{crit}} = 1038.26605 + O(\epsilon)$, is of most interest to us because it is related to the large N limit of the theory. For $N > N_{\text{crit}}$, we find a stable IR fixed point at real couplings g_1 and g_2.[7] This fixed point is shown with the red dot in Fig. 2 [there is a second stable IR fixed point obtained by the transformation $(g_1, g_2) \to (-g_1, -g_2)$, which is a symmetry of this theory]. There is also a nearby unstable fixed point, shown with a black dot, which has one stable and one unstable direction. As N approaches N_{crit} from above, the nearby unstable fixed point approaches the IR stable fixed point, and they merge at N_{crit}. At $N < N_{\text{crit}}$, both fixed points disappear into the complex plane. As discussed in [38], this is a rather generic behavior at the lower edge of the conformal window: the conformality is lost through the annihilation of a UV fixed point and an IR fixed point. In [38] this was argued to happen at the lower (strongly coupled) edge of the conformal window for four-dimensional $SU(N_c)$ gauge theory with N_f flavors. It is

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[7]It is stable with respect to flows of the nearly marginal couplings g_1 and g_2. As usual, there are some $O(N)$ invariant relevant operators that render this fixed point not perfectly stable.
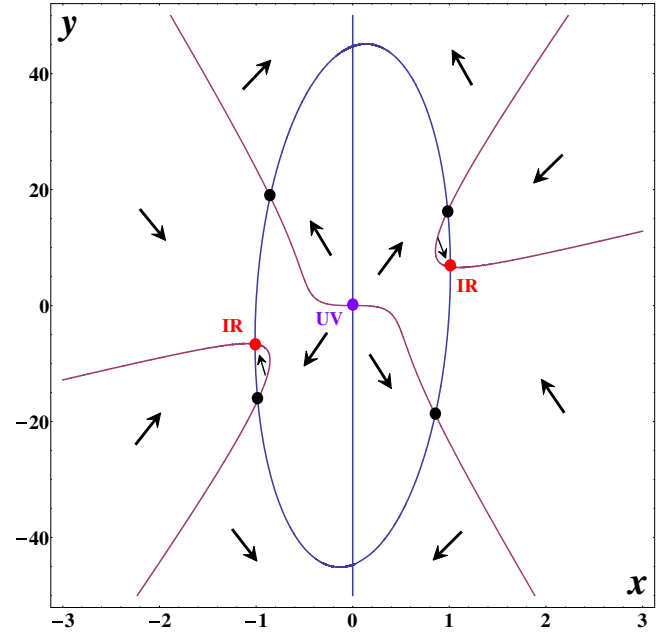

FIG. 2 (color online). The zeros of the one loop β functions and the RG flow directions for $N = 2000$. The red dots correspond to the stable IR fixed points, while the black dots are unstable fixed points. As $N \to N_{\text{crit}}$, the red dot merges with the nearby black dot, and the two fixed points move into the complex plane.

interesting to observe that the same type of behavior occurs at the lower edge of the conformal window of the $O(N)$ model in $d = 6 - \epsilon$, which extends from N_{crit} to infinity.

Let us identify the operator that causes the flow between the unstable fixed point and the IR stable fixed point of our primary interest. It is one of the two nearly marginal operators cubic in the fields that were studied in Sec. III B. By studying the behavior of the dimensions Δ_1 and Δ_2 as $N \to N_{\text{crit}}$ we find that $\Delta_2 \to 6 - \epsilon$. Therefore, it is the operator corresponding to Δ_2 that becomes exactly marginal for $N = N_{\text{crit}}$ and causes the flow between the IR fixed point and the nearby UV fixed point for N slightly above N_{crit}. In bootstrap studies of the quartic $O(N)$ model this operator was denoted by σ^3 [12], i.e. it can be thought of as the "triple-trace operator" $(\phi^i \phi^i)^3$. The theory at the unstable fixed point has an unconventional large N behavior where $x \sim \mathcal{O}(1)$ and $y \sim \mathcal{O}(\sqrt{N})$, so that corrections to scaling dimension proceed in powers of $N^{-1/2}$ [1].

Let us now go back to finding the higher order corrections to N_{crit} given by (4.4) [the higher order corrections to the other critical values (4.5) and (4.6) will be discussed in the next section]. Once we have solved the leading order system (4.3), we can plug the solution into (4.2) and expand (4.1) up to order ϵ^2. From this we obtain simple systems of linear equations from which we can determine x_1, y_1, N_1 and x_2, y_2, N_2. We find

$$x_1 = -0.00940, \quad y_1 = -0.21024, \quad N_1 = -609.93980,$$

$$x_2 = 0.00690, \quad y_2 = 1.01680, \quad N_2 = -364.17333. \tag{4.7}$$

Thus, to three loop order, we conclude that

$$N_{\text{crit}} = 1038.26605 - 609.83980\epsilon - 364.17333\epsilon^2 + O(\epsilon^3). \tag{4.8}$$

We have also checked these expansion coefficients via a direct high-precision numerical calculation of N_{crit} for very small values of ϵ. The large and negative coefficients

indicate that in the physically interesting case of $d = 5$, N_{crit} seems to be much lower than the zeroth order value (this is analogous to the result [27] for the Abelian Higgs model). If we just use the first three terms and plug in $\epsilon = 1$, we get

$$N_{\text{crit}} \approx 64.253. \tag{4.9}$$

For $N < N_{\text{crit}}$ the anomalous dimensions, such as γ_ϕ, are no longer positive (in fact, they become complex). This loss of positivity of γ_ϕ can also be seen as N is reduced in the quartic $O(N)$ model. For example, using the $1/N$ expansion of γ_ϕ in $d = 5$ [8]

$$\gamma_\phi = \frac{32}{15\pi^2 N} - \frac{1427456}{3375\pi^4 N^2}$$
$$+ \left(\frac{275255197696}{759375\pi^6} - \frac{89735168}{2025\pi^4} + \frac{32768\ln 4}{9\pi^4} - \frac{229376\zeta(3)}{3\pi^6} \right) \frac{1}{N^3} + \cdots$$
$$= \frac{3}{2} + \frac{0.216152}{N} - \frac{4.342}{N^2} - \frac{121.673}{N^3} + \cdots \tag{4.10}$$

we find that it stops being positive for $N < 35$. This critical value is not too far from (4.9).

It is also instructive to study the theory using the $4 - \epsilon$ expansion. The anomalous dimension of ϕ^i is [37]

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\epsilon^2 + \frac{N+2}{16(N+8)^4}(-N^2 + 56N + 272)\epsilon^3$$
$$+ \frac{N+2}{64(N+8)^6}(-5N^4 - 230N^3 + 1124N^2 + 17920N + 46144 - 384\zeta(3)(5N+22)(N+8))\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{4.11}$$

For positive ϵ this expansion gives accurate information about the Wilson-Fisher IR fixed points [39]. For nega + ve ϵ there exist formal UV fixed points at negative quartic coupling where we can apply this formula as well. In that case, γ becomes negative for sufficiently large $|\epsilon|$ and $N < N_{\text{crit}}$, indicating that the operator ϕ^i violates the unitarity bound. For example for $d = 5$, corresponding to $\epsilon = -1$, we find $N_{\text{crit}} \approx 8$. Inclusion of the $\mathcal{O}(\epsilon^5)$ term raises this to $N_{\text{crit}} \approx 14$.

We see, therefore, that the estimates of N_{crit} using the quartic $O(N)$ theory in $d = 5$ are even lower than the three loop estimate (4.9). However, our analysis so far has not taken into account the large orders in perturbation theory or non-perturbative effects. To determine N_{crit} in $d = 5$ more precisely, one needs a nonperturbative approach to the $d = 5$ theory, perhaps along the lines of the conformal bootstrap calculation in [23].

A. Unitary fixed points for all positive N

Let us note that not all real fixed points disappear for $N < N_{\text{crit}}$. The unstable real fixed points that are located in the upper left and lower right corners of Fig. 2 exist for all positive N, and we would like to find their interpretation.

The fixed point with $N = 1$ has a particularly simple property that $g_1^* = -g_2^*$. This property of the solution holds for the three loop β functions, and we believe that it is exact. Using this, we note that the action at the fixed point is proportional to $(\sigma + i\phi)^3 + (\sigma - i\phi)^3$. Therefore, the theory at this fixed point enjoys a Z_3 symmetry acting by the phase rotation on the complex combination $\sigma + i\phi$. This cubic classical action appears in the Ginzburg-Landau theory for the 3-state Potts model (see, for example, [32]).[8] Therefore, we expect the Z_3 symmetric fixed point to describe the 3-state Potts model in $d = 6 - \epsilon$. The dimensions of operators at this fixed point are related by the Z_3 symmetry. For example, we find

$$\Delta_\phi = \Delta_\sigma = 2 - \frac{1}{3}\epsilon + \frac{2}{3}\epsilon^2 + \frac{443}{54}\epsilon^3 + \mathcal{O}(\epsilon^4). \tag{4.12}$$

This is in agreement with the result of [16]. By calculating the eigenvalues $\lambda_\pm$ of the matrix $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$, we also find the dimensions (3.29) of the two cubic operators to order ϵ^3:

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[8]We are grateful to Yu Nakayama for pointing this out to us.

$$\Delta_- = 6 - \frac{14}{3}\epsilon - \frac{158}{9}\epsilon^2 - \left(\frac{17380}{81} + 16\zeta(3)\right)\epsilon^3$$
$$= 6 - 4.66667\epsilon - 17.5556\epsilon^2 - 233.801\epsilon^3,$$

$$\Delta_+ = 6 - \frac{83}{18}\epsilon^2 - \left(\frac{38183}{648} + 4\zeta(3)\right)\epsilon^3$$
$$= 6 - 4.61111\epsilon^2 - 63.7326\epsilon^3. \tag{4.13}$$

The dimension Δ_+ corresponds to the operator $(\sigma + i\phi)^3 + (\sigma - i\phi)^3$ which preserves the Z_3 symmetry and is slightly irrelevant for small ϵ. The dimension Δ_- corresponds to the relevant operator $\sigma(\sigma^2 + \phi^2)$ which breaks the Z_3. Thus, the Z_3 symmetry helps stabilize the fixed point at small ϵ.

Unfortunately, the $6 - \epsilon$ expansions (4.13) have growing coefficients, and it is not clear for what range of ϵ the fixed point exists. Thus, one may not be able to interpolate smoothly from the Z_3 symmetric fixed point in $d = 6 - \epsilon$ to $d = 2$ where the 3-state Potts model is described by the unitary (5,6) minimal model [29].

For all $N \geq 2$ we find unstable fixed points with $O(N)$ symmetry. These fixed points always have a relevant cubic operator, corresponding to a negative eigenvalue of the matrix $\frac{\partial \beta_i}{\partial g_j}$. Also, they exhibit an unconventional large N behavior involving half-integer powers of N, similarly to the unstable fixed points that appear for $N > N_{\text{crit}}$ and are shown by the black dots in the upper right and lower left corners of Fig. 2. We leave a discussion of these fixed points for the future.

V. NONUNITARY THEORIES

In addition to the fixed points studied so far, which are perturbatively unitary and appear for $N > N_{\text{crit}}$, there exist nonunitary fixed points for $N''_{\text{crit}} < N < N'_{\text{crit}}$. The leading values of N'_{crit} and N''_{crit} are given in (4.5) and (4.6), respectively. Using the method developed above for finding the higher order in ϵ corrections to N_{crit} we get

$$N'_{\text{crit}} = 1.02145 + 0.03253\epsilon - 0.00163\epsilon^2,$$
$$x' = i(0.23185 + 0.08887\epsilon - 0.03956\epsilon^2),$$
$$y' = i(0.25582 + 0.11373\epsilon - 0.04276\epsilon^2) \tag{5.1}$$

and

$$N''_{\text{crit}} = -0.08750 + 0.34726\epsilon - 0.88274\epsilon^2,$$
$$x'' = 0.13175 - 0.16716\epsilon + 0.12072\epsilon^2,$$
$$y'' = -0.03277 + 0.13454\epsilon - 0.35980\epsilon^2. \tag{5.2}$$

Unfortunately, the latter expansion has growing coefficients, and we cannot extract any useful information from it. On the other hand, the higher order corrections to N'_{crit} are very small, which suggests that $N'_{\text{crit}} > 1$ for range of dimensions below 6.

The theory with $N = 0$, which contains only the field σ, was originally studied by Michael Fisher as an approach to the Yang-Lee edge singularity in the Ising model [28]. Since the coupling is imaginary, it describes a nonunitary theory where some operator dimensions (e.g. σ) are below the unitarity bounds. In $d = 2$, this CFT corresponds to the (2,5) minimal model [30], which has $c = -22/5$. A conformal bootstrap approach to this model [31] has produced good results for a range of dimensions below 6.

The $N = 1$ theory, which has two fields and two coupling constants, has a more intricate structure. This theory is distinguished from the $N = 0$ case by the presence of a Z_2 symmetry $\phi \to -\phi$. Examining the β functions at $N = 1$ and the eigenvalues of the matrix $\frac{\partial \beta_i}{\partial g_j}$, we observe that there exist a stable fixed point with $g_2^* = 6g_1^*/5 + O(\epsilon)$, and an unstable one with $g_1^* = g_2^*$. Introducing the field combinations

$$\sigma_1 = \sigma + \phi, \qquad \sigma_2 = \sigma - \phi, \tag{5.3}$$

we note that for $g_1^* = g_2^*$ the interactions of the $N = 1$ model decouple as $\sim \sigma_1^3 + \sigma_2^3$, i.e. at this fixed point the theory is a sum of two Fisher's $N = 0$ theories. However, one of the flow directions at this fixed point is unstable, since the corresponding operator has $\Delta_{\mathcal{O}} = 6 - 10\epsilon/9 + O(\epsilon^2)$ and is relevant (this value of the dimension corresponds to the negative eigenvalue of the matrix $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$ at the $g_1^* = g_2^*$ fixed point). This dimension has a simple explanation as follows. The flow away from the decoupled fixed point is generated by the operator $\mathcal{O} = \sigma_1 \sigma_2^2 + \sigma_2 \sigma_1^2$. This is allowed by the original Z_2 symmetry $\phi \to -\phi$, which translates into the interchange of σ_1 and σ_2. Thus,

$$\Delta_{\mathcal{O}} = \Delta_\sigma^{N=0} + \Delta_{\sigma^2}^{N=0} = 2 + 2\Delta_\sigma^{N=0}, \tag{5.4}$$

where we used the fact that in the $N = 0$ theory, $\Delta_{\sigma^2}^{N=0} = 2 + \Delta_\sigma^{N=0}$ because σ^2 is a descendant. Using (3.10) for $N = 0$, we find

$$\Delta_\sigma = 2 - \frac{5}{9}\epsilon - \frac{43}{1458}\epsilon^2 + \left(\frac{8\zeta(3)}{243} - \frac{8375}{472392}\right)\epsilon^3$$
$$= 2 - 0.555556\epsilon - 0.0294925\epsilon^2 + 0.021845\epsilon^3. \tag{5.5}$$

Substituting this into (5.4) we find the dimension of the relevant operator $\mathcal{O}$, which indeed precisely agrees with $\Delta_{\mathcal{O}} = d + \lambda_-$, where λ_- is the negative eigenvalue of $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$ at the $g_1^* = g_2^*$ fixed point. Using the ϵ expansion (5.5), we find that $\mathcal{O}$ continues to be relevant as ϵ is increased. For $\epsilon = 4$, i.e. $d = 2$, we know the exact result in the (2,5) minimal model that $\Delta_\sigma^{N=0} = -2/5$, which implies $\Delta_{\mathcal{O}} = 6/5$. This strongly suggests that $\mathcal{O}$ is relevant, and the decoupled fixed point is unstable, for the entire range

$2 \leq d < 6$. To describe this CFT in $d = 2$ more precisely, we note the existence of the modular invariant minimal model $M(3, 10)$, which is closely related to the product of two Yang-Lee $(2,5)$ minimal models [40,41].

The flow away from the unstable fixed point with $g_1^* = g_2^*$ can lead the $N = 1$ theory to the IR stable fixed point where $g_2^* = 6g_1^*/5 + O(\epsilon)$. Using our results we can deduce the ϵ expansion of various operator dimensions at this fixed point. For example,

$$\Delta_\phi = 2 - 0.5501\epsilon - 0.0234477\epsilon^2 + 0.0200649\epsilon^3 + \cdots,$$
$$\Delta_\sigma = 2 - 0.561122\epsilon - 0.0358843\epsilon^2 + 0.0236057\epsilon^3 + \cdots.$$
$$(5.6)$$

By calculating the eigenvalues $\lambda_\pm$ of the matrix $M_{ij} = \frac{\partial \beta_i}{\partial g_j}$, we find the dimensions of two operators that are slightly irrelevant in $d = 6 - \epsilon$:

$$\Delta_- = d + \lambda_- = 6 - 0.88978\epsilon + 0.0437732\epsilon^2 - 0.039585\epsilon^3,$$
$$\Delta_+ = d + \lambda_+ = 6 - 0.773191\epsilon^2 + 1.59707\epsilon^3. \quad (5.7)$$

As ϵ is increased, these expansions suggest that the two operators become more irrelevant. It would be interesting to study this Z_2 symmetric fixed point using a conformal bootstrap approach along the lines of [31].

Assuming that the $N = 1$ IR fixed point continues to be stable in $d = 5, 4, 3, 2$, it is interesting to look for statistical mechanical interpretations of this nonunitary CFTs. A distinguishing feature of the $N = 1$ CFT is that it has a discrete Z_2 symmetry, while the $N = 0$ theory has no symmetries at all. As we have noted, in $d = 2$ the CFT can be obtained via deforming the $(3, 10)$ minimal model by a Virasoro primary field of dimension $6/5$ (this is the highest dimension relevant operator in that minimal model). After analyzing the spectra of several candidate minimal models, we suggest that the end point of this RG flow is described by the $(3, 8)$ minimal model with $c = -21/4$.[9] Let us note that $M(2, 5)$ and $M(3, 8)$ are members of the series of nonunitary minimal models $M(k, 3k - 1)$.

In addition to the identity operator, the $M(3, 8)$ model has three Virasoro primary fields which are Z_2 odd and three that are Z_2 even. Comparing with the theory in $6 - \epsilon$ dimensions, we can tentatively identify the leading Z_2 odd operator as ϕ and the leading Z_2 even one as σ. Obviously, further work is needed to check if the stable fixed point in $6 - \epsilon$ dimensions with $g_2^* = 6g_1^*/5 + O(\epsilon)$ continued to $\epsilon = 4$ is described by the nonunitary minimal model $M(3, 8)$.

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[9]Note that this value is greater than the central charge of the UV theory $M(3, 10)$, which is equal to $-44/5$. For flows between nonunitary theories the Zamolodchikov c-theorem does not hold, and it is possible that $c_{UV} < c_{IR}$.
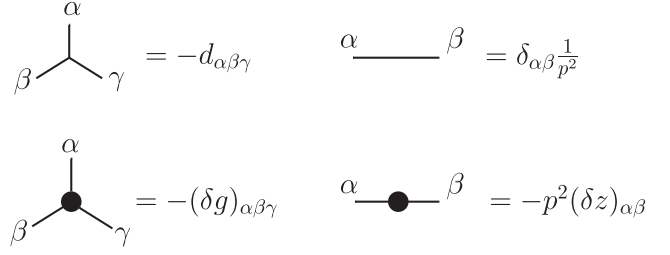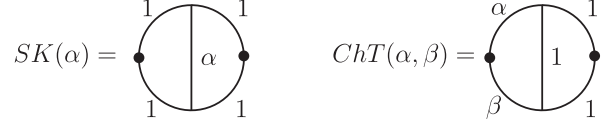

FIG. 3.　Feynman rules.

FIG. 4.　The Special *KITE* and *ChT* diagrams, the numbers labeling each propagator denote its index.

ACKNOWLEDGMENTS

APPENDIX A: SUMMARY OF THREE LOOP RESULTS

The Feynman rules for our theory are depicted in Fig. 3, where we introduced symmetric tensor coupling $d_{\alpha\beta\gamma}$ and counterterms $(\delta g)_{\alpha\beta\gamma}$, $(\delta z)_{\alpha\beta}$ with $\alpha, \beta, \gamma = 0, 1, ..., N$ as

$$d_{000} = g_2, \quad d_{ii0} = d_{i0i} = d_{0ii} = g_1,$$
$$(\delta g)_{000} = \delta g_2, \quad (\delta g)_{ii0} = (\delta g)_{i0i} = (\delta g)_{0ii} = \delta g_1,$$
$$(\delta z)_{00} = \delta_\sigma, \quad (\delta z)_{ii} = \delta_\phi, \quad (A1)$$

where $i = 1, ..., N$. The general form of a Feynman diagram in our theory could be schematically represented as

Feynman diagram = Integral × Tensor structure factor.
$$(A2)$$

The "Tensor structure factors" are products of the tensors $d_{\alpha\beta\gamma}$ and $(\delta g)_{\alpha\beta\gamma}$, $(\delta z)_{\alpha\beta}$, with summation over the dummy indices. Their values for different diagrams are represented in Figs. 7 and 8 after the parentheses.[10] The "Integrals" already include symmetry factors and are the same as in the usual φ^3-theory; their values are listed in Figs. 7 and 8 before the parentheses.

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[10]To find the "Tensor structure factor" we used the fact that it is a polynomial in N, so we calculated sums of products of $d_{\alpha\beta\gamma}$, $(\delta g)_{\alpha\beta\gamma}$, $(\delta z)_{\alpha\beta}$ explicitly for $N = 1, 2, 3, 4, ...$, using Wolfram MATHEMATICA. Having answers for $N = 1, 2, 3, 4, ...$ it is possible to restore the general N form.

1. Counterterms

$$z_{12}^{\phi} = -\frac{g_1^2}{3(4\pi)^3}, \qquad z_{12}^{\sigma} = -\frac{Ng_1^2 + g_2^2}{6(4\pi)^3}, \qquad a_{13} = -\frac{g_1^2(g_1 + g_2)}{(4\pi)^3}, \qquad b_{13} = -\frac{Ng_1^3 + g_2^3}{(4\pi)^3}, \tag{A3}$$

$$z_{14}^{\phi} = \frac{g_1^2}{432(4\pi)^6}(g_1^2(11N - 26) - 48g_1g_2 + 11g_2^2),$$

$$z_{14}^{\sigma} = -\frac{1}{432(4\pi)^6}(2Ng_1^4 + 48Ng_1^3g_2 - 11Ng_1^2g_2^2 + 13g_2^4),$$

$$a_{15} = -\frac{1}{144(4\pi)^6}g_1^2(g_1^3(11N + 98) - 2g_1^2g_2(7N - 38) + 101g_1g_2^2 + 4g_2^3),$$

$$b_{15} = -\frac{1}{48(4\pi)^6}(4Ng_1^5 + 54Ng_1^4g_2 + 18Ng_1^3g_2^2 - 7Ng_1^2g_2^3 + 23g_2^5), \tag{A4}$$

$$z_{16}^{\phi} = \frac{g_1^2}{46656(4\pi)^9}(g_1^4(N(13N - 232) + 5184\zeta(3) - 9064) + g_1^3g_26(441N - 544)$$
$$- 2g_1^2g_2^2(193N - 2592\zeta(3) + 5881) + 942g_1g_2^3 + 327g_2^4),$$

$$z_{16}^{\sigma} = -\frac{1}{93312(4\pi)^9}(2Ng_1^6(1381N - 2592\zeta(3) + 4280) - 96N(12N + 11)g_1^5g_2$$
$$- 3Ng_1^4g_2^2(N + 4320\zeta(3) - 8882) + 1560Ng_1^3g_2^3 - 952Ng_1^2g_2^4 - g_2^6(2592\zeta(3) - 5195)),$$

$$a_{17} = \frac{g_1^2}{15552(4\pi)^9}(-g_1^5(N(531N + 10368\zeta(3) - 2600) + 23968)$$
$$+ g_1^4g_2(99N^2 + 2592(5N - 6)\zeta(3) - 9422N - 2588) + 2g_1^3g_2^2(1075N + 2592\zeta(3) - 16897)$$
$$+ 2g_1^2g_2^3(125N - 5184\zeta(3) - 3917) - g_1g_2^4(5184\zeta(3) + 721) + g_2^5(2592\zeta(3) - 2801)),$$

$$b_{17} = -\frac{1}{2592(4\pi)^9}(2g_1^7N(577N + 713) - 48g_1^6g_2N(31N - 59) + g_1^5g_2^2N(423N + 2592\zeta(3) + 1010)$$
$$- g_1^4g_2^3N(33N - 1296\zeta(3) - 6439) - 27g_1^3g_2^4N(32\zeta(3) + 11) - 301Ng_1^2g_2^5 + g_2^7(432\zeta(3) + 1595)). \tag{A5}$$

APPENDIX B: SAMPLE DIAGRAM CALCULATIONS

1. Some useful integrals

Many of the diagrams listed in Fig. 7 are recursively primitive, so they can be easily evaluated using the integral:

$$I(\alpha, \beta) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^{2\alpha}(p - k)^{2\beta}} = \frac{L_d(\alpha, \beta)}{(k^2)^{\alpha + \beta - d/2}}, \tag{B1}$$
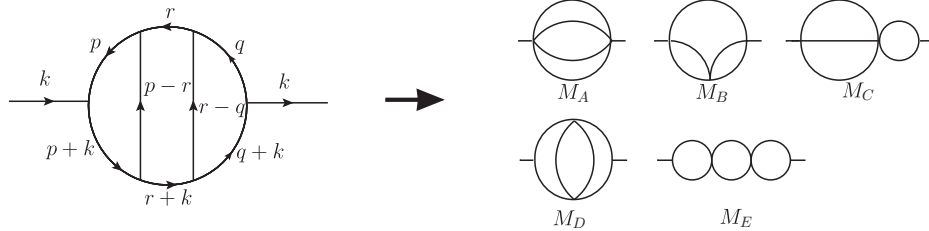
where

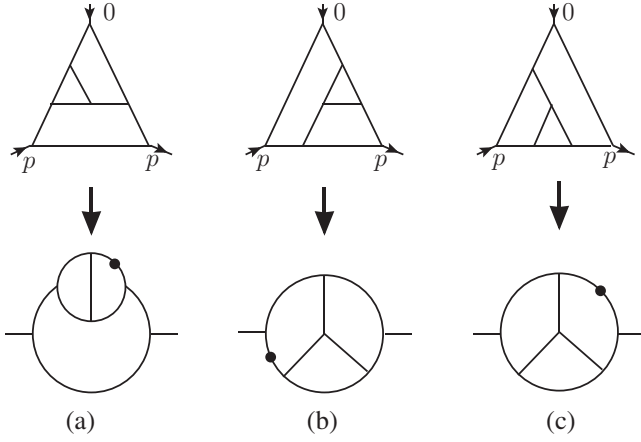FIG. 5. The LADDER diagram can be reduced to five master integrals.

FIG. 6. The three orientations of the same diagram topology correspond to different integrals.

$$L_d(\alpha,\beta) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}. \quad \text{(B2)}$$

For the more complicated integrals, we use the MATHEMATICA program FIRE [42], which uses integration-by-parts relations to turn them into simpler "master integrals," which we then evaluate by hand.

There are two categories of diagrams which show up quite frequently as subdiagrams, the "special *KITE*" diagrams and the "ChT" diagrams shown in Fig. 4.

The special *KITE* diagram is a two loop diagram corresponding to the following integral:

$$SK(\alpha) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^2 (p+k)^2 q^2 (q+k)^2 (p-q)^{2\alpha}}. \quad \text{(B3)}$$

Notice that the power of the middle propagator is arbitrary. Via the Gegenbauer Polynomial technique as described in [43], this integral can be expressed as an infinite sum of gamma functions:

$$SK(\alpha) = -\frac{2}{(4\pi)^d} \frac{1}{(k^2)^{4+\alpha-d}} \frac{\Gamma^2(\lambda)\Gamma(\lambda-\alpha)\Gamma(\alpha+1-2\gamma)}{\Gamma(2\lambda)\Gamma(3\lambda-\alpha-1)}$$
$$\times \left(\frac{\Gamma^2(1/2)\Gamma(3\lambda-\alpha-1)\Gamma(2\lambda-\alpha)\Gamma(\alpha+1-2\lambda)}{\Gamma(\lambda)\Gamma(2\lambda+1/2-\alpha)\Gamma(1/2-2\lambda+\alpha)} \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\alpha+1)} \frac{1}{(n+1-\lambda+\alpha)} \right), \quad \text{(B4)}$$

where $\lambda = d/2 - 1$. In the case of $d = 6 - \epsilon$ we have found that, for example,

		Value of $\frac{\partial}{\partial p^2}$ (sum of graphs)
(a)		$-\frac{1}{(4\pi)^3}\frac{1}{6\epsilon} \times \begin{cases} Ng_1^2 + g_2^2 \\ 2g_1^2 \end{cases}$
(b)		$\frac{1}{(4\pi)^6}\frac{1}{6\epsilon^2}\left(1-\frac{\epsilon}{3}\right) \times \begin{cases} Ng_1^3(g_1+2g_2) + g_2^4 \\ 2g_1^3(g_1+g_2) \end{cases}$
(c)		$-\frac{1}{(4\pi)^6}\frac{1}{36\epsilon^2}\left(1-\frac{11\epsilon}{12}\right) \times \begin{cases} 2Ng_1^4 + Ng_1^2 g_2^2 + g_2^4 \\ g_1^2((N+2)g_1^2 + g_2^2) \end{cases}$
(d)		$-\frac{1}{(4\pi)^9}\frac{1}{9\epsilon^3}\left(1-\frac{5\epsilon}{6}+\frac{7\epsilon^2}{144}\right) \times \begin{cases} Ng_1^3(g_1^3 + 3g_1^2 g_2 + 2g_1 g_2^2 + g_2^3) + g_2^6 \\ g_1^3((N+2)g_1^3 + 3g_1^2 g_2 + g_1 g_2^2 + g_2^3) \end{cases}$
(e)		$-\frac{1}{(4\pi)^9}\frac{1}{9\epsilon^3}\left(1-\frac{7\epsilon}{12}+\frac{71\epsilon^2}{288}\right) \times \begin{cases} Ng_1^3((N+1)g_1^3 + 2g_1^2 g_2 + g_1 g_2^2 + 2g_2^3) + g_2^6 \\ 2g_1^4(g_1+g_2)^2 \end{cases}$
(f)		$\frac{1}{(4\pi)^9}\frac{1}{108\epsilon^3}\left(1-\frac{3\epsilon}{2}+\frac{103\epsilon^2}{144}\right) \times \begin{cases} Ng_1^2(Ng_1^4 + 4g_1^3 g_2 + g_1^2 g_2^2 + g_2^4) + g_2^6 \\ 2g_1^3(2g_1^3 + Ng_1^2 g_2 + g_2^3) \end{cases}$
(g)		$\frac{1}{(4\pi)^9}\frac{1}{18\epsilon^3}\left(1-\frac{11\epsilon}{12}+\frac{2\epsilon^2}{9}\right) \times \begin{cases} Ng_1^2(2g_1^4 + (N+2)g_1^3 g_2 + g_1 g_2^3 + g_2^4) + g_2^6 \\ g_1^3(g_1+g_2)((N+2)g_1^2 + g_2^2) \end{cases}$
(h)		$\frac{1}{(4\pi)^9}\frac{1}{54\epsilon^3}\left(1-\frac{19\epsilon}{12}+\frac{121\epsilon^2}{144}\right) \times \begin{cases} Ng_1^3(2g_1^3 + 2g_1^2 g_2 + g_1 g_2^2 + 2g_2^3) + g_2^6 \\ g_1^2((N+2)g_1^4 + 2(N+1)g_1^3 g_2 + g_2^4) \end{cases}$
(i)		$\frac{1}{(4\pi)^9}\frac{1}{18\epsilon^2}\left(1+\frac{\epsilon\zeta(3)}{2}-\frac{7\epsilon}{8}\right) \times \begin{cases} Ng_1^4(2g_1^2 + 5g_2^2) + g_2^6 \\ 4g_1^4(g_1^2 + g_2^2) \end{cases}$
(j)		$-\frac{1}{(4\pi)^9}\frac{1}{648\epsilon^3}\left(1-\frac{11\epsilon}{12}+\frac{23\epsilon^2}{144}\right) \times \begin{cases} Ng_1^2(4g_1^4 + Ng_1^2 g_2^2 + 2g_2^4) + g_2^6 \\ 4g_1^4(Ng_1^2 + g_2^2) \end{cases}$
(k)		$-\frac{1}{(4\pi)^9}\frac{1}{324\epsilon^3}\left(1-\frac{11\epsilon}{4}+\frac{103\epsilon^2}{36}\right) \times \begin{cases} Ng_1^2((N+2)g_1^4 + 3g_1^2 g_2^2 + g_2^4) + g_2^6 \\ g_1^2((3N+2)g_1^4 + (N+1)g_1^2 g_2^2 + g_2^4) \end{cases}$
(l)		$-\frac{1}{(4\pi)^9}\frac{1}{324\epsilon^3}\left(1-\frac{11\epsilon}{12}-\frac{13\epsilon^2}{144}\right) \times \begin{cases} Ng_1^2(4g_1^4 + Ng_1^2 g_2^2 + 2g_2^4) + g_2^6 \\ g_1^2((N^2+4)g_1^4 + 2Ng_1^2 g_2^2 + g_2^4) \end{cases}$

FIG. 7. Values of derivatives of two-point diagrams. The upper row in parentheses is for $\langle\sigma\sigma\rangle$ and the lower is for $\langle\phi\phi\rangle$.

$$SK\left(2 - \frac{d}{2}\right) = \frac{1}{(4\pi)^d} \frac{1}{(k^2)^{6-3d/2}} \left(\frac{-54}{\epsilon^2} + \frac{-71 + 24\gamma}{1296\epsilon} + \frac{-14641 + 8520\gamma - 1440\gamma^2 + 120\pi^2}{15520} + \cdots\right). \quad \text{(B5)}$$

The ϵ-expansion of the above result can also be verified indirectly with the MATHEMATICA packages MBTOOLS implementing the Mellin-Barnes representation [44].

The *ChT* diagram is another variation of the *KITE* diagram. It corresponds to the integral:

$$ChT(\alpha, \beta) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^{2\alpha}(p + k)^{2\beta} q^2 (q + k)^2 (p - q)^2}. \quad \text{(B6)}$$

In this diagram, which has the same topology as the *KITE* diagram, one triangle has all propagators with index 1, and the other two lines have arbitrary indices α and β. This diagram was evaluated in position space by Vasiliev *et al.* in [7]. Their answer is

$$ChT(\alpha, \beta) = \frac{\pi^d v(d-2)}{\Gamma(\frac{d}{2} - 1)} \frac{1}{(x^2)^{d/2-3+\alpha+\beta}}$$

$$\times \left(\frac{v(\alpha)v(2-\alpha)}{(1-\beta)(\alpha+\beta-2)} + \frac{v(\beta)v(2-\beta)}{(1-\alpha)(\alpha+\beta-2)} + \frac{v(\alpha+\beta-1)v(3-\alpha-\beta)}{(\alpha-1)(\beta-1)}\right), \quad \text{(B7)}$$

where $v(\alpha) = \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}$. For our purpose, we just need to Fourier transform this expression to momentum space.

We also need variations of the *SK* and *ChT* diagrams, with a particular index raised by 1, for example. However, we can use FIRE to relate them to the original version of these diagrams.

2. Example of a two-point function diagram

We will evaluate the three loop ladder diagram which is the first diagram in Fig. 7(e). It corresponds to the integral:

$$\text{LADDER} = \int \frac{d^d p \, d^d q \, d^d r}{(2\pi)^{3d}} \frac{1}{p^2 (p + k)^2 (p - r)^2 r^2 (r + k)^2 (r - q)^2 q^2 (q + k)^2}, \quad \text{(B8)}$$

where the loop momenta are p, q, and r. The external momentum is k. Using FIRE, it can be reduced to a sum of five master integrals, denoted as $M_A, \ldots, M_E$:

$$\text{LADDER} = \frac{4(2d - 5)(3d - 8)(9d^2 - 65d + 118)M_A}{(d - 4)^4 k^8} - \frac{12(d - 3)(3d - 10)(3d - 8)M_C}{(d - 4)^3 k^6}$$

$$+ \frac{32(d - 3)^2 (2d - 7)M_B}{(d - 4)^3 k^6} + \frac{4(d - 3)^2 M_E}{(d - 4)^2 k^4} + \frac{3(d - 3)(3d - 10)M_D}{(d - 4)^2 k^4}. \quad \text{(B9)}$$

The diagrams corresponding to the master integrals are listed in Fig. 5. Among these master integrals, only M_D is nonprimitive, the rest can be calculated easily. However, if we integrate over the middle loop, we see that M_D is in fact related to the special *KITE* diagram $SK(2 - d/2)$. We have

$$M_A = \frac{L_d(1, 1)L_d(1, 2 - d/2)L_d(1, 3 - d)}{(k^2)^{4-3d/2}}, \qquad M_B = \frac{(L_d(1, 1))^2 L_d(1, 4 - d)}{(k^2)^{5-3d/2}},$$

$$M_C = \frac{(L_d(1, 1))^2 L_d(1, 2 - d/2)}{(k^2)^{5-3d/2}}, \qquad M_D = L_d(1, 1)SK(2 - d/2), \qquad M_E = \frac{(L_d(1, 1))^3}{(k^2)^{6-3d/2}}. \quad \text{(B10)}$$

Plugging in $d = 6 - \epsilon$ and expanding in ϵ, we find that

$$\text{LADDER} = \frac{k^2}{(4\pi)^{3d/2}} \left(-\frac{2}{9\epsilon^3} + \frac{-115 + 36\gamma + \log k^2}{108\epsilon^2}\right.$$

$$\left. + \frac{-4043 + 18(115 - 18\gamma)\gamma + 18\pi^2 - 18\log k^2(-115 + 36\gamma + 18\log k^2)}{1296\epsilon} + \cdots\right). \quad \text{(B11)}$$

(a) $-\dfrac{1}{(4\pi)^3}\dfrac{1}{\epsilon} \times \begin{cases} Ng_1^3 + g_2^3 \\ g_1^2(g_1+g_2) \end{cases}$

(b) $\dfrac{1}{(4\pi)^6}\dfrac{1}{2\epsilon^2}\left(1-\dfrac{\epsilon}{4}\right) \times \begin{cases} 3(Ng_1^3(g_1^2+g_1g_2+g_2^2)+g_2^5) \\ g_1^2((N+3)g_1^3+5g_1^2g_2+2g_1g_2^2+g_2^3) \end{cases}$

(c) $-\dfrac{1}{(4\pi)^6}\dfrac{1}{12\epsilon^2}\left(1-\dfrac{7\epsilon}{12}\right) \times \begin{cases} 3(Ng_1^2(2g_1^3+g_2^3)+g_2^5) \\ g_1^2((N+4)g_1^3+2(N+1)g_1^2g_2+g_1g_2^2+2g_2^3) \end{cases}$

(d) $-\dfrac{1}{(4\pi)^6}\dfrac{1}{4\epsilon} \times \begin{cases} g_2(3Ng_1^4+g_2^4) \\ 2g_1^3(g_1^2+g_2^2) \end{cases}$

(e) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{6\epsilon^3}\left(1-\dfrac{3\epsilon}{4}+\dfrac{3\epsilon^2}{4}\right) \times \begin{cases} 3(N(N+1)g_1^7+Ng_1^6g_2+Ng_1^5g_2^2+2Ng_1^4g_2^3+Ng_1^3g_2^4+g_2^7) \\ g_1^2((2N+3)g_1^5+(N+7)g_1^4g_2+(N+6)g_1^3g_2^2+3g_1^2g_2^3+g_2^5) \end{cases}$

(f) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{3\epsilon^3}\left(1-\dfrac{3\epsilon}{4}-\dfrac{\epsilon^2}{4}\right) \times \begin{cases} 3(Ng_1^7+2Ng_1^6g_2+2Ng_1^5g_2^2+Ng_1^4g_2^3+Ng_1^3g_2^4+g_2^7) \\ g_1^2((2N+3)g_1^5+(2N+7)g_1^4g_2+(N+5)g_1^3g_2^2+2g_1^2g_2^3+g_1g_2^4+g_2^5) \end{cases}$

(g) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{3\epsilon^3}\left(1-\dfrac{\epsilon}{4}-\dfrac{5\epsilon^2}{16}\right) \times \begin{cases} 3(Ng_1^7+N(N+2)g_1^6g_2+Ng_1^5g_2^2+2Ng_1^3g_2^4+g_2^7) \\ g_1^3((2N+3)g_1^4+(2N+7)g_1^3g_2+5g_1^2g_2^2+3g_1g_2^3+2g_2^4) \end{cases}$

(h) $\dfrac{1}{(4\pi)^9}\dfrac{1}{18\epsilon^3}\left(1-\dfrac{5\epsilon}{12}-\dfrac{47\epsilon^2}{144}\right) \times \begin{cases} 3(2Ng_1^7+2Ng_1^6g_2+N^2g_1^5g_2^2+Ng_1^3g_2^4+Ng_1^2g_2^5+g_2^7) \\ g_1^3((3N+4)g_1^4+(3N+4)g_1^3g_2+(2N+1)g_1^2g_2^2+5g_1g_2^3+2g_2^4) \end{cases}$

(i) $\dfrac{1}{(4\pi)^9}\dfrac{1}{9\epsilon^3}\left(1-\dfrac{5\epsilon}{12}-\dfrac{47\epsilon^2}{144}\right) \times \begin{cases} 3(2Ng_1^7+2Ng_1^6g_2+N^2g_1^5g_2^2+Ng_1^3g_2^4+Ng_1^2g_2^5+g_2^7) \\ g_1^2((N(N+1)+4)g_1^5+(2N+6)g_1^4g_2+(2N+3)g_1^3g_2^2+(N+2)g_1^2g_2^3+g_1g_2^4+g_2^5) \end{cases}$

(j) $\dfrac{1}{(4\pi)^9}\dfrac{1}{18\epsilon^3}\left(1-\dfrac{5\epsilon}{4}+\dfrac{23\epsilon^2}{48}\right) \times \begin{cases} 3(2Ng_1^7+2Ng_1^6g_2+Ng_1^4g_2^3+2Ng_1^3g_2^4+g_2^7) \\ g_1^2((N+4)g_1^5+(4N+6)g_1^4g_2+(4N+2)g_1^3g_2^2+g_1g_2^4+2g_2^5) \end{cases}$

(k) $\dfrac{1}{(4\pi)^9}\dfrac{1}{18\epsilon^3}\left(1-\dfrac{17\epsilon}{12}+\dfrac{10\epsilon^2}{9}\right) \times \begin{cases} 3(2Ng_1^7+N^2g_1^6g_2+2Ng_1^5g_2^2+Ng_1^4g_2^3+Ng_1^2g_2^5+g_2^7) \\ g_1^2((3N+4)g_1^5+(3N+4)g_1^4g_2+(N+3)g_1^3g_2^2+(N+3)g_1^2g_2^3+g_1g_2^4+g_2^5) \end{cases}$

(l) $\dfrac{1}{(4\pi)^9}\dfrac{1}{36\epsilon^3}\left(1-\dfrac{17\epsilon}{12}+\dfrac{11\epsilon^2}{18}\right) \times \begin{cases} 3(N^2g_1^7+2Ng_1^6g_2+3Ng_1^5g_2^2+Ng_1^2g_2^5+g_2^7) \\ g_1^2((3N+4)g_1^5+(2N+6)g_1^4g_2+(2N+1)g_1^3g_2^2+(N+2)g_1^2g_2^3+2g_1g_2^4+g_2^5) \end{cases}$

(m) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{108\epsilon^3}\left(1-\dfrac{29\epsilon}{12}+\dfrac{19\epsilon^2}{9}\right) \times \begin{cases} 3(N(N+2)g_1^7+Ng_1^5g_2^2+2Ng_1^4g_2^3+Ng_1^2g_2^5+g_2^7) \\ g_1^2((4N+4)g_1^5+(5N+2)g_1^4g_2+(N+2)g_1^3g_2^2+(2N+1)g_1^2g_2^3+g_1g_2^4+2g_2^5) \end{cases}$

(n) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{108\epsilon^3}\left(1-\dfrac{7\epsilon}{12}-\dfrac{11\epsilon^2}{48}\right) \times \begin{cases} 3(4Ng_1^7+N^2g_1^4g_2^3+2Ng_1^2g_2^5+g_2^7) \\ g_1^2((N^2+8)g_1^5+(2N^2+4)g_1^4g_2+2Ng_1^3g_2^2+4Ng_1^2g_2^3+g_1g_2^4+2g_2^5) \end{cases}$

(o) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{108\epsilon^3}\left(1-\dfrac{7\epsilon}{12}-\dfrac{11\epsilon^2}{48}\right) \times \begin{cases} 3(4Ng_1^7+N^2g_1^4g_2^3+2Ng_1^2g_2^5+g_2^7) \\ g_1^2((4N+4)g_1^5+N(N+4)g_1^4g_2+4g_1^3g_2^2+(2N+4)g_1^2g_2^3+g_2^5) \end{cases}$

(p) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{36\epsilon^2}\left(1-\dfrac{11\epsilon}{12}\right) \times \begin{cases} 3(N(N+4)g_1^6g_2+Ng_1^4g_2^3+Ng_1^2g_2^5+g_2^7) \\ 2g_1^3((N+4)g_1^4+(2N+3)g_1^2g_2^2+2g_2^4) \end{cases}$

(q) $\dfrac{1}{(4\pi)^9}\dfrac{1}{12\epsilon^2}\left(1-\dfrac{\epsilon}{4}\right) \times \begin{cases} 3(N^2g_1^7+2Ng_1^6g_2+2Ng_1^5g_2^2+Ng_1^4g_2^3+Ng_1^3g_2^4+g_2^7) \\ 2g_1^3(3g_1^4+(N+3)g_1^3g_2+2g_1^2g_2^2+2g_1g_2^3+g_2^4) \end{cases}$

(r) $\dfrac{1}{(4\pi)^9}\dfrac{1}{6\epsilon^2}\left(1-\dfrac{23\epsilon}{12}+2\epsilon\zeta(3)\right) \times \begin{cases} 3(Ng_1^6g_2+3Ng_1^5g_2^2+Ng_1^3g_2^4+g_2^7) \\ g_1^3(2g_1^4+(N+2)g_1^3g_2+g_1^2g_2^2+g_1g_2^3+g_2^4) \end{cases}$

(s) $\dfrac{1}{(4\pi)^9}\dfrac{1}{6\epsilon^2}\left(1-\dfrac{29\epsilon}{24}+\epsilon\zeta(3)\right) \times \begin{cases} 3(2Ng_1^7+2Ng_1^5g_2^2+3Ng_1^4g_2^3+g_2^7) \\ g_1^2(6g_1^5+(3N+4)g_1^4g_2+6g_1^3g_2^2+4g_1^2g_2^3+g_2^5) \end{cases}$

(t) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{3\epsilon} \times \begin{cases} 4Ng_1^6g_2+3Ng_1^4g_2^3+g_2^7 \\ (N+2)g_1^7+4g_1^5g_2^2+g_1^3g_2^4 \end{cases}$

(u) $-\dfrac{1}{(4\pi)^9}\dfrac{1}{3\epsilon}\left(\zeta(3)-\dfrac{1}{3}\right) \times \begin{cases} 3(Ng_1^7+Ng_1^6g_2+3Ng_1^5g_2^2+2Ng_1^4g_2^3+g_2^7) \\ g_1^3((2N+5)g_1^4+7g_1^3g_2+3g_1^2g_2^2+5g_1g_2^3+2g_2^4) \end{cases}$

FIG. 8. Values of three-point diagrams. The upper row in parentheses is for $\langle\sigma\sigma\sigma\rangle$ and the lower is for $\langle\sigma\phi\phi\rangle$.

3. Example of a three-point function diagram

We will evaluate the three loop diagram found in Fig. 8(f). In order to employ the same techniques used for the two-point functions, we impose that the momentum running through the three points are p, $-p$, and 0, as a three-point function with three arbitrary momenta is much more difficult to compute.

However, since the momenta are asymmetric, it is necessary to consider all three "orientations" of each topology of the diagram. Notice that the tensor factors mentioned in the previous section will also be different. As an illustration, let us denote the three orientations of the

diagram we are considering by I_1, I_2, and I_3. After taking into account that one of the external momenta is zero, they are each equivalent to a two-point function as shown in Fig. 6. All lines have indices 1, except those with black dots, which have indices 2.

I_1 contains a subdiagram that is equivalent to $ChT(1, 2)$, which can be evaluated easily using our formula before; after that, the diagram is primitive. The other two diagrams can be reduced via FIRE into the master integrals M_A, M_B, M_C, and M_D as in the LADDER diagram. Again, in $d = 6 - \epsilon$, we find that

$$I_1 = \frac{1}{(4\pi)^{3d/2}} \left(\frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2\log k^2}{8\epsilon^2} + \frac{173 + 18(\gamma - 5)\gamma - \pi^2 + 18\log k^2(-5 + 2\gamma + \log k^2)}{96\epsilon} + \cdots \right), \quad \text{(B12)}$$

$$I_2 = \frac{1}{(4\pi)^{3d/2}} \left(\frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2\log k^2}{8\epsilon^2} + \frac{125 + 18(\gamma - 5)\gamma - \pi^2 + 18\log k^2(-5 + 2\gamma + \log k^2)}{96\epsilon} + \cdots \right), \quad \text{(B13)}$$

$$I_3 = \frac{1}{(4\pi)^{3d/2}} \left(\frac{1}{6\epsilon^3} + \frac{5 - 2\gamma - 2\log k^2}{8\epsilon^2} + \frac{125 + 18(\gamma - 5)\gamma - \pi^2 + 18\log k^2(-5 + 2\gamma + \log k^2)}{96\epsilon} + \cdots \right). \quad \text{(B14)}$$

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