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Renormalization-group evolution of chiral gauge theories

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We calculate the ultraviolet to infrared evolution and analyze possible types of infrared behavior for several asymptotically free chiral gauge theories with gauge group $\mathrm{SU}(N)$ and massless chiral fermions transforming according to a symmetric rank-2 tensor representation S and $N+4$ copies (flavors) of a conjugate fundamental representation $\bar{F}$, together with a vectorlike subsector with chiral fermions in higher-dimensional representation(s). We construct and study three such chiral gauge theories. These have respective vectorlike subsectors comprised of (a) p copies of fermions in the adjoint representation, (b) $N=2k$ even and p copies of fermions in the antisymmetric rank-k tensor representation, and (c) p copies of $\{S+\bar{S}\}$ fermions. Results are presented for beta functions, their infrared zeros, and predictions from the most-attractive-channel approach for the formation of bilinear fermion condensates. Importantly, we show that for these theories, the expected ultraviolet to infrared evolution obeys a conjectured inequality concerning the field degrees of freedom for all values of the parameters N and p characterizing each theory.

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I. INTRODUCTION

The question of how the properties of an asymptotically free chiral gauge theory change as a function of the Euclidean momentum scale μ at which one measures these properties is of fundamental physical interest. For sufficiently large μ in the deep ultraviolet (UV), a theory of this type is weakly coupled and can be described by perturbative methods. As μ decreases, the gauge coupling increases, as described by the renormalization group (RG) and associated beta function. To understand the infrared (IR) properties of a strongly coupled chiral gauge theory has long been, and continues to be, an outstanding goal in quantum field theory. If the theory satisfies the 't Hooft global anomaly-matching conditions, then it might confine and produce massless gauge-singlet composite spin-1/2 fermions [1–10]. Alternatively, the strong gauge interaction could produce bilinear fermion condensates. A chiral gauge theory that does not contain any vectorlike fermion subsector is defined as being irreducibly chiral. If a chiral gauge theory has an irreducibly chiral fermion content, then these fermion condensates necessarily break the chiral gauge symmetry [8,10–14], whereas if it contains a vectorlike fermion subsector, then condensates of fermions in this vectorlike subsector may preserve the gauge symmetry. In both cases, the fermion condensates break global chiral flavor symmetries. In general, there can be several stages of condensate formation at different momentum scales, with a resultant sequence of gauge and/or global symmetry breaking. Here and below, we restrict our consideration to asymptotically free chiral gauge theories that have no anomalies in gauged currents, as is required for renormalizability. Thus, in the models that we construct, the numbers of chiral fermions in various representations of the gauge group are chosen to satisfy this requirement. Further, we restrict this paper to theories with only gauge and fermion fields but without any scalar fields.

There are several methods that one can use to investigate the ultraviolet to infrared evolution of a chiral gauge theory. These include (i) (perturbative) calculation of the beta function and analysis of possible IR zeros of this beta function; (ii) use of the most-attractive-channel (MAC) approach, which can suggest in which channel(s) bilinear fermion condensates are most likely to form [12] if the coupling gets sufficiently strong in the infrared; and (iii) a conjectured inequality involving the perturbative degrees of freedom in the massless fields [8,15]. We will denote this as the conjectured DFI, where DFI stands for degree of freedom inequality. As was shown in [8] and discussed further in [9,10], if the types of UV to IR evolution involving either formation of fermion condensates with associated spontaneous chiral gauge and global symmetry breaking or confinement with production of massless composite fermions were to occur over a sufficiently large range of fermion contents (specifically, a sufficiently large range of values of p in the Sp model reviewed in Sec. III), these would violate the conjectured degree-of-freedom inequality. Hence, assuming the validity of the conjectured degree-of-freedom inequality imposes significant restrictions on the behaviors of these theories. Moreover, as noted in [10], the type of UV to IR evolution that would obey the degree-of-freedom inequality over the greatest range of p values is not the one favored by the MAC approach. These results lead one to inquire whether it is possible to achieve the goal of constructing chiral gauge theories where the expected type(s) of UV to IR evolution obey the conjectured degree-of-freedom inequality throughout the full range of parameters specifying the fermion contents of these theories.

In this paper we report a successful achievement of this goal and give several examples of such theories. Our theories have the gauge group SU(N) and massless chiral fermions transforming according to a symmetric rank-2 tensor representation of SU(N), denoted S, and $N+4$ copies (i.e., flavors) of a conjugate fundamental representation, denoted $\bar{F}$, together with a vectorlike subsector consisting of p copies of massless chiral fermions in higher-dimensional representation(s). Because SU(2) has only (pseudo)real representations, it does not yield a chiral gauge theory, so we restrict our considerations to chiral gauge theories having a gauge group SU(N) with $N \geq 3$. We construct and analyze three theories of this type. In the first two, the higher-dimensional representation R of the fermions in the vectorlike subsector is self-conjugate, i.e., $R = \bar{R}$. These theories have p copies of chiral fermions in (a) the adjoint representation, Adj, and (b) for $N = 2k$ even, p copies of chiral fermions in the k-fold antisymmetric tensor representation, denoted $[N/2]_N = [k]_{2k}$. For properties that are common to both of these two theories, we will use the generic symbol R_{sc} to refer to the respective self-conjugate (sc) representations. In the third type of theory, (c), the vectorlike subsector is comprised of p copies of pairs of fermions of the form $\{R + \bar{R}\}$ with $R = S$. Each of these three types of chiral gauge theories thus consists of an irreducibly chiral subsector, namely the S and $N+4$ copies of $\bar{F}$ fermions, together with a vectorlike subsector. Although we shall refer to these as three theories, each one is really a two-parameter class of theories depending on N and p.

We have chosen the representation R of the fermions in the vectorlike subsector of the theories studied in this paper so that for values of N and p that lead to a sufficiently strong gauge coupling in the infrared and associated formation of bilinear fermion condensates, the most attractive channel for condensation involves the fermions in the vectorlike subsector and is of the form $R \times \bar{R} \to 1$, where here, the symbol 1 denotes a singlet under SU(N). This contrasts with the theory studied in [8–10], which has a vectorlike subsector consisting of p copies of massless fermions transforming as $\{F + \bar{F}\}$. In that theory, the most attractive channel is $S \times \bar{F} \to F$ rather than $F \times \bar{F} \to 1$. For each of our new chiral gauge theories, we present results on beta functions, IR zeros of the respective beta functions, and predictions from the most attractive channel approach. We then demonstrate that in each theory, for each type of expected UV to IR evolution, the conjectured degree-of-freedom inequality is obeyed throughout the full parameter range.

If the gauge theory is irreducibly chiral, then the gauge invariance forbids any fermion masses in the Lagrangian. For our purposes we will assume that the masses of the fermions in vectorlike subsector are also zero. This assumption does not entail a significant loss of generality, because, generically, if a fermion in the vectorlike subsector had a nonzero mass m, then as the reference scale μ decreases below m, one would integrate this vectorlike fermion out of the low-energy effective theory applicable below that scale, and the result for the infrared behavior would be equivalent to a theory without this fermion.

This paper is organized as follows. In Sec. II we discuss our general theoretical framework and methods of analysis. Section III is devoted to a brief review of a theory studied previously in [8–10]. In Sec. IV we explain the basic strategy that we use to construct our chiral gauge theories. In Secs. V and VI we present and analyze two new chiral gauge theories with vectorlike subsectors having fermions transforming according to self-conjugate representations of the gauge group. In Sec. VII we discuss the global flavor symmetry group for these two types of theories. For the values of N and p that lead to strong coupling in the infrared and fermion condensation, we then analyze, in Sec. VIII, the further evolution into the infrared of the low-energy effective field theory that is applicable below the scale of this initial condensation. In Sec. IX we demonstrate that for both of these new chiral gauge theories with a given SU(N) gauge group, the expected UV to IR evolution obeys the conjectured degree-of-freedom inequality for the full range of values of p. Section X is devoted to the analysis of the third type of chiral gauge theory, with the type-(c) vectorlike subsector. Again, we show that the conjectured degree-of-freedom inequality is obeyed for this theory. Our conclusions are given in Sec. XI, and some relevant formulas are included in the Appendix.

II. THEORETICAL FRAMEWORK AND METHODS OF ANALYSIS

In this section we discuss the theoretical framework and methods of analysis that we use. As noted above, we consider asymptotically free chiral gauge theories with gauge group $G = \mathrm{SU}(N)$ and denote the gauge coupling measured at a Euclidean momentum scale as $g(\mu)$. It is also convenient to use the quantities $\alpha(\mu) = g(\mu)^2/(4\pi)$ and

$$a(\mu) \equiv \frac{g(\mu)^2}{16\pi^2} = \frac{\alpha(\mu)}{4\pi}. \tag{2.1}$$

(The argument μ in these couplings will often be suppressed in the notation.) Without loss of generality, we write all fermion fields in terms of left-handed chiral components.

A. Beta function

The ultraviolet to infrared evolution of the gauge coupling is described by the beta function, $\beta_g = dg/dt$, or equivalently,

$$\beta_\alpha = \frac{d\alpha}{dt} = \frac{g}{2\pi}\beta_g \tag{2.2}$$

where $dt = d\ln\mu$. This has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell, \qquad (2.3)$$

where we have extracted an overall minus sign, b_ℓ is the ℓ-loop coefficient, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. The n-loop beta function, denoted $\beta_{\alpha,n\ell}$, is given by Eq. (2.3) with the upper limit on the ℓ-loop summation equal to n instead of ∞. The property of asymptotic freedom means that $\beta_\alpha < 0$ for small α. With the minus sign extracted in the perturbative expansion (2.3), this is satisfied if $b_1 > 0$. The one-loop and two-loop coefficients b_1 [16] and b_2 [17] are independent of the scheme used for regularization and renormalization, while the b_ℓ with $\ell \geq 3$ are scheme dependent.

If $b_2 < 0$, then the two-loop beta function, $\beta_{\alpha,2\ell}$, has an IR zero at

$$\alpha_{IR,2\ell} = 4\pi a_{IR,2\ell} = -\frac{4\pi b_1}{b_2}. \qquad (2.4)$$

For sufficiently small fermion content, b_2 is positive, but as one enlarges the fermion content in the theory, the sign of b_2 can become negative while the theory is still asymptotically free, yielding an infrared zero in $\beta_{\alpha,2\ell}$ at the above value. If a theory has such an infrared zero in the beta function, then, as the reference scale μ decreases from large values in the ultraviolet, $\alpha(\mu)$ increases toward this infrared zero. If this IR zero occurs at sufficiently weak coupling, one expects that the theory evolves from the UV to the IR without confinement or spontaneous chiral symmetry breaking (SχSB), to a non-Abelian Coulomb phase. In this case, the infrared zero of beta is an exact IR fixed point (IRFP) of the renormalization group, and as $\mu \to 0$ and the beta function vanishes, and the theory exhibits scaling behavior with nonzero anomalous dimensions. This phenomena was discussed for vectorial gauge theories in [17,18].

B. Most-attractive-channel approach

In a theory whose UV to IR evolution leads to a gauge coupling that is strong enough to produce fermion condensates, one method that has been widely used to predict which type of condensate is most likely to form is the most-attractive-channel (MAC) approach [12]. Let us consider a condensation channel in which fermions in the representations R_1 and R_2 of a given gauge group form a condensate that transforms according to the representation R_{cond} of this group, denoted

$$R_1 \times R_2 \to R_{cond}. \qquad (2.5)$$

An approximate measure, based on one-gluon exchange, of the attractiveness of this condensation channel, is

$$\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{Ch}), \qquad (2.6)$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation R [19], and $R_{Ch} \equiv R_{cond}$. At this level of one-gluon exchange, if ΔC_2 is positive (negative), then the channel is attractive (repulsive). The most attractive channel is the one that yields the maximum (positive) value of ΔC_2. The MAC approach predicts that if, *a priori*, several condensation channels could occur, then the one that actually occurs is the channel that has the largest (positive) value of ΔC_2. The MAC method was applied, for example, in efforts to build reasonably UV-complete models with dynamical electroweak symmetry breaking [14]. These models made use of asymptotically free chiral gauge interactions that became strongly coupled, naturally leading to the formation of certain condensates (of fermions subject to the chiral gauge interaction) in a hierarchy of scales corresponding, via inverse powers, to the observed generational hierarchy of Standard Model fermion mass scales. In these previous applications of the MAC approach, and also in our present application, one bears in mind that the MAC method is based on the one-gluon exchange and hence is only a rough guide to the nonperturbative phenomenon of fermion condensation.

An analysis of the Schwinger-Dyson equation for the propagator of a massless fermion transforming according to the representation R of a gauge group G shows that, in the ladder (i.e., iterated one-gluon exchange) approximation the minimum value of α for which fermion condensation occurs in a vectorial gauge theory is given by the condition that $3\alpha_{cr}C_2(R)/\pi = 1$, or equivalently, $3\alpha_{cr}\Delta C_2/(2\pi) = 1$, since $\Delta C_2 = 2C_2(R)$ in this case [20]. Therefore, an estimate is that as μ decreases and $\alpha(\mu)$ increases, condensation will first occur in a given channel Ch when $\alpha(\mu)$ increases through a critical value

$$\alpha_{cr,Ch} \sim \frac{2\pi}{3\Delta C_2(R)_{Ch}}, \qquad (2.7)$$

where we have labelled $C_2(R)$ with a subscript for the channel Ch. This estimate will be of particular interest for the most attractive channel. Clearly, because of the strong-coupling nature of the fermion condensation process, Eq. (2.7) is only a rough estimate. A measure of the likelihood that the coupling grows large enough in the infrared to produce fermion condensation in a given channel Ch is the ratio

$$\rho_{IR,Ch} \equiv \frac{\alpha_{IR,2\ell}}{\alpha_{cr,Ch}}. \qquad (2.8)$$

If this ratio is significantly larger (smaller) than unity, one may infer that condensation in the channel Ch is likely (unlikely). As with the caveats given above concerning the MAC, in using this ratio $\rho_{IR,Ch}$, one is cognizant of the theoretical uncertainties due to the strong-coupling nature of the physics.

C. Degree-of-freedom inequality

A quantity that can give interesting predictions for renormalization-group evolution involves the relevant perturbative field degrees of freedom in the effective field theory that is applicable at a given reference scale, μ. From the study of second-order phase transitions and critical phenomena in statistical mechanics and condensed matter physics, one is familiar with the Wilsonian thinning of degrees of freedom as one changes the scale on which one measures physical quantities from short distances (UV) to large distances (IR). Given the correspondence between the inverse distance and the reference momentum scale μ, one may naturally expect a similar decrease (or nonincrease) of dynamical degrees of freedom in a quantum field theory as μ decreases from large values in the ultraviolet to small values in the infrared. In conformal field theory in $d = 2$ dimensions, it has been proved that a certain quantity that can be interpreted as a measure of the degrees of freedom (the central charge of the associated Virasoro algebra) decreases as a function of the renormalization-group flow [21].

Given that a theory is asymptotically free, the gauge coupling approaches zero in the deep ultraviolet as $\mu \to \infty$, so that one can identify and enumerate the perturbative degrees of freedom in the fields. Depending on the theory, it may also be true that in the deep infrared, as $\mu \to 0$, the residual (massless) particles are weakly interacting, so that again one can describe them perturbatively and enumerate their degrees of freedom. Although one is describing the UV to IR evolution of a zero-temperature quantum field theory, a natural approach to the enumeration of the perturbative degrees of freedom in the fields is provided by envisioning a finite-temperature field theory, where the temperature T corresponds to the Euclidean scale, μ, and using the count embodied in the free energy density, $F(T)$. This is given by

$$F(T) = f(T)\frac{\pi^2}{90}T^4 \tag{2.9}$$

with

$$f = 2N_v + \frac{7}{4}N_f + \frac{7}{8}N_{f,Maj} + N_s, \tag{2.10}$$

where N_v and N_s are the number of vector and (real) scalar fields, and N_f and $N_{f,Maj}$ are the number of chiral components of Dirac and Majorana fermions in the theory, respectively [22,23]. Assuming that the relevant fields become free in the respective UV and IR limits, we define

$$f_{UV} = f(\infty), \qquad f_{IR} = f(0). \tag{2.11}$$

Since the theories that we consider are required to be asymptotically free, we can always identify the Lagrangian fields in the deep UV and hence calculate f_{UV}.

In accord with experience in statistical mechanics, Ref. [15] conjectured the degree-of-freedom inequality

$$\Delta f \equiv f_{UV} - f_{IR} \geq 0 \tag{2.12}$$

for vectorial gauge theories, and Ref. [8] extended this conjecture to chiral gauge theories. In [8] this conjecture was applied to analyze several asymptotically free chiral gauge theories. Subsequent studies have investigated the possible types of IR behavior involving strong coupling and condensate formation; Refs. [9,10] are particularly relevant for our current work.

As noted above, since we restrict our study to asymptotically free theories, the condition that the theory becomes free as $\mu \to \infty$ is always satisfied. There are three types of situations where the condition that the fields are also weakly coupled in the IR is satisfied. In all of these we can calculate f_{IR}. In the first of these, the theory evolves to an exact, weakly coupled IR fixed point, so that the field degrees of freedom in the massless fields are the same as they were in the UV, up to small, calculable perturbative corrections, which obey the inequality (2.12) [8,15]. In the second type of situation, there is global and/or gauge symmetry breaking at one or more scales, so that as μ decreases below these scales toward the infrared, in the applicable low-energy effective field theory, the remaining massless particles are Nambu-Goldstone bosons (NGBs) resulting from the spontaneous chiral symmetry breaking. Since the NGBs have only derivative interactions among themselves, which vanish as $\sqrt{s}/\Lambda \to 0$, where $\sqrt{s}$ is the center-of-mass energy and Λ denotes the scale of chiral symmetry breaking, it follows that these NGBs become free in the infrared limit. A third type of possible situation is one in which the chiral gauge interaction confines and produces massless gauge-singlet composite fermions. The interactions between these gauge-singlet fermions involve higher-dimension operators and hence are also weak in the infrared. In some models, the second and third types of behavior can occur together [10].

A direct test of the conjectured degree-of-freedom inequality (2.12) for asymptotically free chiral gauge theories would probably require lattice simulations. However, because of fermion doubling on the lattice (in which a single continuum fermion produces 2^d fermion modes on a d-dimensional Euclidean lattice, with half corresponding to one sign of γ_5 and the other half corresponding to the opposite sign of γ_5), it has been challenging to simulate chiral gauge theories via lattice methods. A different approach to testing the validity of the conjecture is to study its application to vectorial gauge theories. These have the advantage that they can be simulated on the lattice, and there are well-understood ways of dealing with fermion doubling so that in the continuum limit one should be able to determine the actual number, N_f, of active fermions. Ongoing lattice studies of the infrared behavior of various vectorial gauge theories, such as a gauge theory with $G = \mathrm{SU}(2)$ and $N_f = 6$ Dirac fermions in the fundamental representation [24], are making progress in testing the conjectured degree-of-freedom inequality.

III. THE Sp THEORY

In this section we review the properties of a chiral gauge theory that has been studied before [4,5,8–10] and provides motivation for our present work. The reader who is familiar with this material could skip this section and proceed to Sec. IV. This theory, which we denote the Sp model, has the gauge group $\mathrm{SU}(N)$ and massless chiral fermions transforming according to

(1) a symmetric rank-2 tensor representation, S, with corresponding field $\psi_L^{ab} = \psi_L^{ba}$,
(2) $N+4$ copies of chiral fermions in the conjugate fundamental representation, $\bar{F}$, with fields $\chi_{a,i,L}$, $i = 1, \ldots, N+4$, and
(3) a vectorlike subsector consisting of p copies of pairs of chiral fermions transforming as $\{F + \bar{F}\}$, with fields $\chi^a_{j,L}$ and $\chi_{a,j,L}$, $j = 1, \ldots, p$.

Here and below, $a, b, c\ldots$ are gauge indices and $i, j, \ldots$ are copy (i.e., flavor) indices.

The one- and two-loop coefficients in the beta function of this theory are

$$(b_1)_{Sp} = 3N - 2 - \frac{2p}{3} \tag{3.1}$$

and

$$(b_2)_{Sp} = \frac{13}{2}N^2 - 15N + \frac{1}{2} + 6N^{-1} + p\left(-\frac{13N}{3} + N^{-1}\right). \tag{3.2}$$

The coefficient $(b_1)_{Sp}$ decreases with p and vanishes at $p = p_{b1z,Sp} = (9/2)N - 3$, where the subscript bnz stands for "b_n equals zero." Asymptotic freedom requires $(b_1)_{Sp} > 0$, i.e.,

$$p < \frac{9}{2}N - 3. \tag{3.3}$$

The two-loop coefficient is positive for small p and decreases through zero to negative values as p increases through the value

$$p_{b2z,Sp} = \frac{3(13N^3 - 30N^2 + N + 12)}{2(13N^2 - 3)}. \tag{3.4}$$

In the interval

$$(I_p)_{Sp}\colon\ p_{b2z,Sp} < p < p_{b1z,Sp} \tag{3.5}$$

the two-loop beta function has an infrared zero, which occurs at the value

$$\alpha_{IR,2\ell,Sp} = \frac{8\pi N(9N - 6 - 2p)}{p(26N^2 - 6) - 39N^3 + 90N^2 - 3N - 36}. \tag{3.6}$$

Clearly, the two-loop perturbative calculation that yields this result (3.6) is most accurate if p is near the upper end of the interval $(I_p)_{Sp}$, where $\alpha_{IR,2\ell,Sp}$ is small, and becomes less reliable as p approaches the lower end of the interval $(I_p)_{Sp}$.

For this theory, the most attractive channel for fermion condensation is

$$S \times \bar{F} \to F, \tag{3.7}$$

with attractiveness measure

$$\Delta C_2 = C_2(S) = \frac{(N+2)(N-1)}{N} \quad \text{for } S \times \bar{F} \to F. \tag{3.8}$$

Hence, for this channel,

$$\rho_{IR,S\times\bar{F}} \equiv \frac{\alpha_{IR,2\ell,Sp}}{\alpha_{cr,S\times\bar{F}}} = \frac{12(9N - 6 - 2p)(N+2)(N-1)}{p(26N^2 - 6) - 39N^3 + 90N^2 - 3N - 36}. \tag{3.9}$$

This ratio exceeds unity for

$$p < p_{cr,Sp} = \frac{3(49N^3 - 18N^2 - 95N + 60)}{2(25N^2 + 12N - 27)}. \tag{3.10}$$

If p is only slightly less than p_{b1z}, then $\rho_{IR,S\times\bar{F}} \ll 1$, so the UV to IR evolution is expected to be a deconfined, weakly coupled non-Abelian Coulomb phase. Here, also taking into account perturbative corrections to the free-field count of field degrees of freedom, the DFI is obeyed [8].

If p is sufficiently small [with either $p \in (I_p)_{Sp}$ or $1 \le p < p_{b2z,Sp}$], then the theory becomes strongly coupled in the infrared. For these values of p, one possible type of UV to IR evolution could produce confinement with massless, gauge-singlet composite fermions [4] and no spontaneous chiral symmetry breaking. Alternately, there could be fermion condensation in the most attractive channel (3.7), breaking the gauge group $\mathrm{SU}(N)$ to $\mathrm{SU}(N-1)$ and also breaking global flavor symmetries. The associated fermion condensate has the form

$$\langle \psi_L^{abT} C \chi_{b,i,L} \rangle. \tag{3.11}$$

Without loss of generality, one may pick $a = N$ and $i = 1$. The fermions involved in this condensate gain dynamical masses, and one then constructs the low-energy effective field theory applicable at lower scales. The coupling in this low-energy theory continues to grow and is again expected to produce a condensate in the most attractive channel, $S \times \bar{F}$, where now S and $\bar{F}$ refer to representations of SU$(N-1)$. This process continues sequentially until the original SU(N) gauge symmetry in the UV is completely broken.

The degree-of-freedom measure in the UV is

$$f_{UV,Sp} = 2(N^2-1) + \frac{7}{4}\left[\frac{N(N+1)}{2} + (N+4+2p)N\right]. \tag{3.12}$$

For the possible type of UV to IR evolution that leads to confinement and massless composite fermions (labeled with the subscript *sym*), one finds [8]

$$f_{IR,Sp;sym} = \frac{7}{4}\left[\frac{1}{2}(N+4+p)(N+3+p) + p(N+4+p) + \frac{1}{2}p(p+1)\right]. \tag{3.13}$$

Here and below, the subscripts after IR in a quantity such as $f_{IR,Sp;sym}$ refer to the theory (here, the Sp theory) and then, after the semicolon, the type of UV to IR evolution. Thus, for this type of UV to IR flow,

$$(\Delta f)_{Sp;sym} \equiv f_{UV,Sp} - f_{IR,Sp;sym} = \frac{1}{4}[15N^2 + 7N - 50 - 14p(4+p)]. \tag{3.14}$$

[Here, in the symbol $(\Delta f)_{Sp;sym}$, the first subscript identifies the theory and the subscripts after the semicolon identify the type of UV to IR evolution; the same notation is used for the other theories to be discussed.] The difference $(\Delta f)_{Sp;sym}$ is positive if and only if

$$p < -2 + \sqrt{\frac{15N^2+7N+6}{14}}. \tag{3.15}$$

For the type of UV to IR flow involving sequential fermion condensation in the $S \times \bar{F} \to F$ channels

$$f_{IR,Sp;S\times\bar{F}} = 2N(4+p) + 1 + \frac{7}{4}\left[\frac{N(N-1)}{2} + 4N + 2pN\right]. \tag{3.16}$$

Consequently, for this type of UV to IR flow,

$$(\Delta f)_{Sp;S\times\bar{F}} \equiv f_{UV,Sp} - f_{IR,Sp;S\times\bar{F}} = \frac{1}{4}[15N^2 - 25N - 12 - 8pN]. \tag{3.17}$$

This is positive if and only if

$$p < \frac{15N^2 - 25N - 12}{8N}. \tag{3.18}$$

If, for a given N, the upper bounds (3.15) and (3.18) were substantially greater than the value of $p_{cr,Sp}$ in Eq. (3.10), then they would not be important, since in this region, toward the upper end of the interval $(I_p)_{Sp}$, one would expect that the UV to IR evolution would be to a deconfined non-Abelian Couolmb phase, for which the conjectured DFI is obeyed. However, these upper bounds (3.15) and (3.18) are less then $p_{cr,Sp}$. For example, for $N = 3$, we have $p_{b2z,Sp} = 24/19 = 1.263$ (to the given floating point accuracy) and $p_{b1z,Sp} = 21/2 = 10.5$, so the interval $(I_p)_{Sp}$ consists of the values $2 \le p \le 10$. Furthermore, for this $N = 3$ value, $p_{cr,Sp} = 6$, so that for $p \lesssim 6$, one may anticipate that the UV to IR evolution would plausibly involve strong coupling, as embodied in the two types of evolution discussed above, namely confinement with massless composite fermions and no spontaneous chiral symmetry breaking or production of fermion condensates and associated gauge and global symmetry breaking. Now

$$N = 3 \Rightarrow (\Delta f)_{Sp;sym} > 0 \quad \text{if } p < \frac{-14 + 9\sqrt{7}}{7} = 1.402, \tag{3.19}$$

so that if this UV to IR evolution leading to massless composite fermions without any spontaneous chiral symmetry breaking were to occur for values in the strongly coupled range of p, $2 \le p \lesssim 6$, then it would violate the conjectured degree-of-freedom inequality (2.12). Furthermore,

$$N = 3 \Rightarrow (\Delta f)_{Sp;S\times\bar{F}} > 0 \quad \text{if } p < 2. \tag{3.20}$$

Hence, if the UV to IR evolution were to lead to condensate formation in the successive $S \times \bar{F}$ channels of the SU(N) theory, the SU$(N-1)$ theory, etc., then it would violate the conjectured DFI for much of the strongly coupled range of values of p, including $2 \le p \lesssim 6$.

In general, the Sp model is a two-parameter theory, depending on both N and p. An interesting limit is

$$LNP\colon\ N \to \infty, \qquad p \to \infty \quad \text{with} \quad r \equiv \frac{p}{N} \text{ fixed} \quad \text{and} \quad \alpha(\mu)N \text{ finite}. \tag{3.21}$$

We denote this as the LNP (large N and p) limit [25]. In this LNP limit, the resultant theory evidently depends only on the single parameter r. We define

$$r_{bnz} \equiv \lim_{LNP} \frac{p_{bnz}}{N}, \qquad n = 1, 2. \tag{3.22}$$

One has

$$r_{b1z} = \frac{9}{2} \tag{3.23}$$

and

$$r_{b2z} = \frac{3}{2} \tag{3.24}$$

so that the analogue of $(I_p)_{Sp}$ for this LNP limit is

$$(I_r)_{Sp}\colon\ 1.5 < r < 4.5. \tag{3.25}$$

Further,

$$r_{cr,S\times\bar{F}} \equiv \lim_{LNP} \frac{p_{cr,S\times\bar{F}}}{N} = \frac{147}{50} = 2.94. \tag{3.26}$$

We define a rescaled degree-of-freedom measure that is finite in the LNP limit, namely

$$\bar{f} \equiv \lim_{LNP} \frac{f}{N^2}. \tag{3.27}$$

One has

$$\bar{f}_{UV,Sp} = \frac{37}{8} + \frac{7}{2} r, \tag{3.28}$$

$$\bar{f}_{IR,Sp;sym} = \frac{7}{8} + \frac{7}{2} r(1+r), \tag{3.29}$$

and

$$\bar{f}_{IR,Sp;S\times\bar{F}} = \frac{7}{8} + \frac{11}{2} r. \tag{3.30}$$

Consequently, for the type of UV to IR evolution that leads to confinement and massless composite fermions, which might occur in the strongly coupled IR regime where $r \lesssim 3$,

$$(\Delta\bar{f})_{Sp;sym} \equiv \bar{f}_{UV,Sp} - \bar{f}_{IR,Sp;sym} = \frac{15 - 14r^2}{4}. \tag{3.31}$$

This would obey the conjectured DFI only if [8,10,26]

$$r < \sqrt{\frac{15}{14}} = 1.035. \tag{3.32}$$

For the possible type of UV to IR evolution that leads to sequential fermion condensation in the $S \times \bar{F} \to F$ channels,

$$(\Delta\bar{f})_{Sp;S\times\bar{F}} \equiv f_{UV,Sp} - \bar{f}_{IR,Sp;S\times\bar{F}} = \frac{15 - 8r}{4}. \tag{3.33}$$

This would obey the conjectured DFI only if

$$r < \frac{15}{8} = 1.875. \tag{3.34}$$

Both of the upper limits (3.32) and (3.34) are well below the upper bound from asymptotic freedom, $r < 4.5$. Importantly, they are also below the value of $r \sim 3$ where the estimate Eq. (3.26) suggests that strong-coupling behavior occurs. Hence, in this Sp model, there is considerable uncertainty in the overall prediction for the UV to IR evolution in the case where this involves strong coupling. Assuming the validity of the conjectured degree-of-freedom inequality, this DFI would forbid two types of strongly coupled UV to IR evolution that would otherwise be inferred to be likely, namely confinement without any spontaneous chiral symmetry breaking in the interval $\sqrt{15/14} < r \lesssim 3$ and condensate formation in the MAC with attendant gauge and chiral symmetry breaking in the interval $15/8 < r \lesssim 3$.

This property of the Sp model, noted in [8] and further discussed in [9,10], provides a motivation for the goal of constructing asymptotically free chiral gauge theories where the likely type(s) of UV to IR evolution is (are) in agreement with the conjectured degree-of-freedom inequality for the full range of fermion contents (as parametrized here by the value of p). We have achieved this goal, as we report in the present work.

We include a parenthetical remark here. In the Sp model, although not favored by the MAC criterion, if there were condensation of the fermions in the vectorlike subsector in the $F \times \bar{F} \to 1$ channel, followed at lower scales by either confinement with massless composite fermions or sequential condensate formation in the $S \times \bar{F} \to F$ channels, then

$$\begin{aligned}\bar{f}_{IR,Sp;F\times\bar{F},sym} &= \bar{f}_{IR,Sp;F\times\bar{F},S\times\bar{F}} \\ &= \frac{7}{8} + r(2+r).\end{aligned} \tag{3.35}$$

Hence, for this type of UV to IR evolution,

$$\begin{aligned}(\Delta\bar{f})_{Sp;F\times\bar{F},S\times\bar{F}} &\equiv \bar{f}_{UV,Sp} - \bar{f}_{IR,Sp;F\times\bar{F},S\times\bar{F}} \\ &= \frac{1}{4}(15 + 6r - 4r^2).\end{aligned} \tag{3.36}$$

This is positive for

$$r < \frac{3+\sqrt{69}}{4} = 2.83. \tag{3.37}$$

Thus, of the various possible types of UV to IR evolution in the Sp theory, the type that obeys the DFI conjecture over the largest range of r is condensation in the $F \times \bar{F} \to 1$ channel, followed at lower scales by either confinement with massless composite fermions or sequential condensate formation in the $S \times \bar{F} \to F$ channels. But the initial condensation in this type of evolution is not the one favored by the MAC criterion, which, instead favors initial and then sequential condensation in the $S \times \bar{F} \to F$ channels of the $\mathrm{SU}(N)$ then $S \times \bar{F} \to F$ condensation in the $\mathrm{SU}(N-1)$ theory, and so forth, until the $\mathrm{SU}(N)$ gauge symmetry is completely broken.

IV. STRATEGY FOR CONSTRUCTION OF NEW CHIRAL GAUGE THEORIES

Our general method for constructing the chiral gauge theories presented here is as follows. We take the gauge group to be $G = \mathrm{SU}(N)$ and include, as the irreducibly chiral sector of the theory, fermions transforming as the S and $(N+4)$ copies of $\bar{F}$. We choose the vectorlike subsector to consist of p copies of fermions that transform according to representation(s) R of G such that the channel

$$R \times \bar{R} \to 1 \tag{4.1}$$

is more attractive than other channels. (For some of our theories, $R = \bar{R}$.) In the theories that we consider, the next-most-attractive channel is

$$S \times \bar{F} \to F. \tag{4.2}$$

The ΔC_2 attractiveness measures for these channels are

$$\Delta C_2 = 2C_2(R) \quad \text{for } R \times \bar{R} \to 1 \tag{4.3}$$

and

$$\Delta C_2 = C_2(S) = \frac{(N+2)(N-1)}{N} \quad \text{for } S \times \bar{F} \to \bar{F}, \tag{4.4}$$

so the condition that the $R \times \bar{R} \to 1$ channel is more attractive than the $S \times \bar{F} \to \bar{F}$ channel is that

$$\Delta C_2(R) = 2C_2(R) > \frac{(N+2)(N-1)}{N}. \tag{4.5}$$

In all the cases that we consider, this guarantees that the $R \times \bar{R} \to 1$ channel is the most attractive channel in which condensation thus occurs first as the theory evolves from the UV to the IR. Consequently, if the fermion content is such that the running coupling $\alpha(\mu)$ becomes sufficiently large in the infrared, then, because the MAC is (4.1), the fermion condensation at the highest energy scale occurs among the fermions in the vectorlike subsector of the model, via the channel $R \times \bar{R} \to 1$. The resultant low-energy effective field theory applicable below this scale is thus comprised of the irreducible chiral sector of the theory, equivalent to the $p = 0$ special case of the full theory, with just the S fermion and the $N+4$ copies of the $\bar{F}$ fermion. As reviewed in Sec. III, the various possible types of UV to IR evolution of this $p = 0$ theory obey the conjectured degree-of-freedom inequality [8–10].

V. THEORY WITH $R = Adj$

A. Particle content

In this section we construct and study a chiral gauge theory with gauge group $\mathrm{SU}(N)$ and fermion content consisting of chiral fermions transforming according to

(1) a symmetric rank-2 tensor representation, S, with corresponding field $\psi_L^{ab} = \psi_L^{ba}$,

(2) $N+4$ copies (also called "flavors") of chiral fermions in the conjugate fundamental representation, $\bar{F}$, with fields $\chi_{a,i,L}$, $i = 1, \ldots, N+4$, and

(3) p copies of chiral fermions in the adjoint representation, denoted Adj, with fields $\xi^a_{b,j,L}$, $j = 1, \ldots, p$.

Here and below, $a, b, c\ldots$ are gauge indices and i, j are copy indices. We call this the Adj theory by reference to the choice of the representation $R = R_{sc}$ for the fermions in the vectorlike subsector. This fermion content is summarized in Table I. As noted above, we restrict our discussion to $N \geq 3$ because SU(2) has only (pseudo)real representations and hence a gauge theory based on the gauge group SU(2) is not chiral. This theory thus depends on the two integer parameters, $N \geq 3$ and $p \geq 0$, with an upper limit on p given by Eq. (5.5) below. We will sometimes use the Young tableaux $\Box\Box$ and $\bar{\Box}$ for S and $\bar{F}$. The irreducibly chiral sector of this theory is comprised of the S and the $N+4$ copies of $\bar{F}$ fermions, and the vectorlike subsector is comprised of the Adj fermions. Because of this self-conjugate nature of R_{sc}, the Adj fermions may be considered to be Majorana. Thus, if one were to remove the irreducibly chiral part of this theory and consider the part containing the gauge fields and the Adj fermions alone, the dynamical particle content in the Lagrangian would be analogous to the gluons and gluinos of an $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N)$ gauge theory.

We recall that since the contribution to the triangle anomaly from S satisfies [27]

$$\mathrm{Anom}(S) = (N+4)\mathrm{Anom}(F), \tag{5.1}$$

and since

$$\mathrm{Anom}(R) = -\mathrm{Anom}(\bar{R}), \tag{5.2}$$

TABLE I. Properties of fermions in the chiral gauge theories with vectorlike subsector consisting of p copies of fermions in the self-conjugate representation $R = R_{sc}$. The entries in the columns are (i) fermion, (ii) representation of the SU(N) gauge group, (iii) number of copies, and representations (charges for Abelian factors) of the respective factor groups in the global flavor symmetry group: (iv) $SU(N+4)_{\bar{F}}$, (v) $SU(p)_{R_{sc}}$, (vi) $U(1)_1$, (vii) $U(1)_2$. The notation for the fermion ξ in the R_{sc} is generic; specifically, this is $\xi^a_{b,i,L}$ for the Adj model and $\xi^{a_1,\ldots,a_k}_{i,L}$ for the AT model (with $N = 2k$). See text for further discussion.

Fermion	SU(N)	No. copies	$SU(N+4)_{\bar{F}}$	$SU(p)_{R_{sc}}$	$U(1)_1$	$U(1)_2$
S: ψ^{ab}_L	$\square\square$	1	1	1	$N+4$	$2pT_{R_{sc}}$
$\bar{F}$: $\chi_{a,i,L}$	$\bar{\square}$	$N+4$	$\square$	1	$-(N+2)$	0
R_{sc}: ξ_L	R_{sc}	p	1	$\square$	0	$-(N+2)$

it follows that the set of chiral fermions S plus $(N+4)$ copies of $\bar{F}$ yields a theory that is free of anomalies in gauged currents. Furthermore, from Eq. (5.2), it follows that for any self-conjugate representation R_{sc}, $\text{Anom}(R_{sc}) = 0$. Hence, we are free to add fermions transforming according to a self-conjugate representation to a chiral gauge theory that is free of anomalies in gauged currents and it will retain this anomaly-free property. We use this fact here with $R_{sc} = Adj$.

B. Beta function

The beta function for this Adj theory is given by Eq. (2.3) with the one-loop coefficient

$$(b_1)_{Adj} = \frac{1}{3}[(9-2p)N - 6] \tag{5.3}$$

and the two-loop coefficient

$$(b_2)_{Adj} = \frac{1}{6}[(39-32p)N^2 - 90N + 3 + 36N^{-1}]. \tag{5.4}$$

(See the Appendix for general formulas for b_1 and b_2.) These coefficients contain the maximal scheme-independent information about the dependence of the gauge coupling on the reference scale, μ. This information will suffice for our present purposes. Higher-loop effects for vectorial theories and effects of scheme transformations on higher-loop terms in the beta function for gauge theories have been studied in [28–35].

We denote the values of p for which $(b_1)_{Adj} = 0$ as $p_{b1z,Adj}$ (where the subscript stands for b_1 zero). This value is [36]

$$p_{b1z,Adj} = \frac{3(3N-2)}{2N}. \tag{5.5}$$

Our requirement that the model should be asymptotically free means that $\beta_\alpha < 0$ for small α. This is equivalent to the condition that $b_1 > 0$ or, if b_1 vanishes, then the further requirement that $b_2 > 0$. Now $(b_1)_{Adj} > 0$ if and only if $p < p_{b1z,Adj}$, i.e.,

$$p < \frac{3(3N-2)}{2N}. \tag{5.6}$$

This means that the set of physical, integral values of p allowed by our requirement of asymptotic freedom are $0 \le p \le 3$ for $N = 3,4,5,6$ and $0 \le p \le 4$ for $N \ge 7$. Note that if $N = 6$ and $p = 4$, then $b_1 = 0$, so one must examine the sign of b_2 to determine if the theory is asymptotically free or not, and for this case $(b_2)_{Adj}$ is negative, hence excluding it from consideration. Here and below, for a given theory and value of N, we will denote the maximum allowed value of p as p_{max}.

As a consequence of the asymptotic freedom of the theory, the beta function always has a zero at $\alpha = 0$, which is a UV fixed point (UVFP) of the renormalization group. In general, the two-loop beta function, $\beta_{\alpha,2\ell}$, has an IR zero if b_2 has a sign opposite to that of b_1, i.e., if b_2 is negative. For $p = 0$, $(b_2)_{Adj} > 0$, so $\beta_{\alpha,2\ell}$ has no IR zero. As p increases, $(b_2)_{Adj}$ decreases and eventually passes through zero to negative values, giving rise to an IR zero of $\beta_{\alpha,2\ell,Adj}$. Let us denote the value of p where b_2 vanishes as $p_{b2z,Adj}$. This is

$$p_{b2z,Adj} = \frac{3(13N^3 - 30N^2 + N + 12)}{32N^3}. \tag{5.7}$$

In Table II we list values of $p_{b1z,Adj}$ and $p_{b2z,Adj}$ for this theory. The value $p_{b2z,Adj}$ is less than the upper bound on p, $p_{b1z,Adj}$, i.e.,

$$p_{b2z,Adj} < p_{b1z,Adj}. \tag{5.8}$$

This inequality is proved by analyzing the difference,

$$p_{b1z,Adj} - p_{b2z,Adj} = \frac{3(35N^3 - 2N^2 - N - 12)}{32N^3}. \tag{5.9}$$

This difference is positive for all physical N. Hence, for p in the interval [36]

$$(I_p)_{Adj}\colon\ p_{b2z,Adj} < p < p_{b1z,Adj}, \tag{5.10}$$

TABLE II. Values of $p_{b1z,Adj}$ and $p_{b2z,Adj}$ in the *Adj* theory as functions of N.

N	$p_{b2z,Adj}$	$p_{b1z,Adj}$
3	0.3333	3.5000
4	0.5391	3.7500
5	0.6690	3.9000
6	0.7578	4.0000
7	0.8222	4.0714
8	0.8708	4.1250
9	0.90895	4.1667
10	0.9396	4.2000
11	0.9647	4.2773
12	0.9857	4.2500
13	1.0035	4.2692
14	1.0187	4.2857
15	1.0320	4.3000
10^2	1.1906	4.4700
10^3	1.2159	4.4970
∞	1.21875	4.5000

this theory is asymptotically free, and $\beta_{\alpha,2\ell,Adj}$ has an IR zero. The actual physical, integral values of p in the interval $(I_p)_{Adj}$ depend on the value of N. There are several different sets of N and p values where this IR zero is physical:

$$(I_p)_{Adj}\colon\ 1 \le p \le 3 \quad \text{if } 3 \le N \le 6,\\ 1 \le p \le 4 \quad \text{if } 7 \le N \le 12,\\ 2 \le p \le 4 \quad \text{if } N \ge 13. \tag{5.11}$$

These different cases follow from two properties. First, $p_{b1z,Adj}$ (continued to real numbers) is a monotonically increasing function of N for physical N and ascends through the value 4 as N increases through the value $N = 6$. Second, for $N > (1 + \sqrt{1081})/30 = 1.129$ and hence for the range $N \ge 3$ relevant here, $p_{b2z,Adj}$ is a monotonically increasing function and increases through 1 at $N = 12.7922$ (the largest root of $7N^3 - 90N^2 + 3N + 36$). Hence, if $N \ge 13$, the lowest value of $p \in (I_p)_{Adj}$ is $p = 2$, as indicated in (5.11).

For values of N and p where $\beta_{\alpha,2\ell,Adj}$ has a physical IR zero, it occurs at

$$\alpha_{IR,2\ell,Adj} \equiv 4\pi a_{IR,2\ell,Adj} = -4\pi \frac{(b_1)_{Adj}}{(b_2)_{Adj}} = \frac{8\pi N[(9-2p)N-6]}{(32p-39)N^3 + 90N^2 - 3N - 36}. \tag{5.12}$$

In using this result, it should be recalled that, in general, an IR zero of a beta function at $\alpha_{IR,2\ell} = -4\pi b_1/b_2$ can be reliable if $|b_2|$ is not too small, i.e., when $\alpha_{IR,2\ell}$ is not too large for the perturbative calculation to be applicable. In Table III we list values of $\alpha_{IR,2\ell,Adj}$.

TABLE III. Values of $\alpha_{IR,2\ell,Adj}$ and $\rho_{IR,Adj\times Adj}$ in the *Adj* theory for an illustrative range of values of N and, for each N, the values of p in the respective interval $(I_p)_{Adj}$.

N	p	$\alpha_{IR,2\ell,Adj}$	$\rho_{IR,Adj\times Adj}$
3	1	1.96	5.63
3	2	0.471	1.35
3	3	0.0982	0.281
4	1	2.34	8.95
4	2	0.470	1.80
4	3	0.120	0.457
5	1	2.75	13.1
5	2	0.448	2.14
5	3	0.121	0.579
6	1	3.24	18.6
6	2	0.4215	2.415
6	3	0.117	0.669
7	1	3.88	25.9
7	2	0.395	2.64
7	3	0.110	0.738
7	4	0.00504	0.0337
8	1	4.75	36.3
8	2	0.370	2.82
8	3	0.104	0.793
8	4	0.00784	0.0599
13	2	0.275	3.42
13	3	0.0768	0.954
13	4	0.0109	0.135
14	2	0.261	3.49
14	3	0.0728	0.973
14	4	0.01075	0.144
15	2	0.249	3.56
15	3	0.0692	0.991
15	4	0.0106	0.152

It is of interest to consider the limit [25]

$$N \to \infty \quad \text{with} \quad \zeta(\mu) \equiv \alpha(\mu)N \ \text{finite} \quad \text{and} \quad p \ \text{fixed}. \tag{5.13}$$

In this limit,

$$\lim_{N\to\infty} p_{b1z,Adj} = \frac{9}{2} \tag{5.14}$$

and

$$\lim_{N\to\infty} p_{b2z,Adj} = \frac{39}{32} = 1.21875, \tag{5.15}$$

so that the interval $(I_p)_{Adj}$ becomes

$$\lim_{N\to\infty}(I_p)_{Adj}\colon\ \frac{39}{32} < p < \frac{9}{2}, \tag{5.16}$$

containing the physical, integral values $p = 2, 3, 4$. In the large-N limit (5.13), the combination of α, or equivalently, a, and N that remains finite is

$$\zeta \equiv \lim_{N\to\infty} \alpha N. \tag{5.17}$$

Correspondingly, the rescaled beta function that is finite has the form

$$\beta_\zeta \equiv \frac{d\zeta}{dt}, \tag{5.18}$$

where, as in Eq. (2.2), $t = \ln\mu$. In this limit, for physical $p \in (I_p)_{Adj}$, the (rescaled, finite) $\beta_{\zeta,2\ell}$, has an IR zero at

$$\zeta_{IR,2\ell,Adj} = \frac{8\pi(9-2p)}{32p-39}. \tag{5.19}$$

The approach to this limit of $N \to \infty$ involves correction terms that are powers in $1/N$:

$$N\alpha_{IR,2\ell,Adj} = \frac{8\pi(9-2p)}{32p-39} - \frac{96\pi(p+48)}{(32p-39)^2 N} + O\left(\frac{1}{N^2}\right). \tag{5.20}$$

One may compare the approach to the $N \to \infty$ limit here with that in a (vectorial) SU(N) gauge theory with N_f fermions in the fundamental representation in the limit $N \to \infty$, $N_f \to \infty$ with the ratio N_f/N fixed and finite [and $\alpha(\mu)N$ a finite function of μ], denoted by the LNN limit in [30]. In that case [29,30] the leading correction term to the limit was suppressed like $1/N^2$ instead of $1/N$, and the correction terms formed a series in powers of $1/N^2$ instead of powers in $1/N$. Hence, the approach to the $N \to \infty$ limit here is not as rapid as in the LNN limit.

C. Analysis of UV to IR Flows

Because of the asymptotic freedom of the theory, i.e., the fact that the beta function is negative for small α, it follows that, as the Euclidean reference momentum scale μ decreases from the ultraviolet toward the infrared, $\alpha(\mu)$ increases. There are several possibilities for the behavior that can occur:

(1) First, if the beta function has an IR zero at a sufficiently small value of $\alpha = \alpha_{IR}$, then one expects that the theory will evolve into the infrared without any spontaneous chiral symmetry breaking. In this case, the IR zero of β_α is an exact IRFP of the renormalization group, so that as $\mu \to 0$, the theory exhibits scale invariance with nonzero anomalous dimensions. In the IR limit $\mu \to 0$, one anticipates that the theory is in a deconfined, massless non-Abelian Coulomb phase.

(2) For smaller values of p, the IR zero of the beta function is larger, and correspondingly, $\alpha(\mu)$ becomes larger as μ decreases from the UV to the IR. Then the strongly coupled gauge interaction can produce fermion condensates that break global and possibly also local gauge symmetries. This behavior also applies if p is sufficiently small that the beta function has no IR zero, so that $\alpha(\mu)$ keeps increasing with decreasing μ until it exceeds the interval where the perturbative beta function describes its evolution. In this general category of UV to IR evolution, there can be a sequence of condensate formations at various energy scales.

(3) In the strongly coupled case (including both the subcases where the beta function has an IR zero at sufficiently large coupling and where the beta function has no IR zero), an alternate possibility is, if the fermion content satisfies the 't Hooft anomaly-matching conditions [1], then the gauge interaction might confine and produce massless gauge-singlet composite fermions.

The beta function describes the growth of $\alpha(\mu)$ as the reference momentum scale μ decreases from the UV to the IR. If the fermion content is such that the beta function has no IR zero, then the interaction definitely becomes strongly coupled in the infrared. If, on the other hand, the beta function does have an IR zero, then one must investigate how large the value of the coupling is at this zero. In conjunction with knowledge of the probable channel in which fermions may condense and the corresponding estimate of the minimum critical coupling, α_{cr} that triggers this condensation, one can then draw a plausible inference as to whether the condensation takes place or whether, in contrast, the theory evolves into the infrared without any fermion condensation or associated spontaneous chiral symmetry breaking.

The only composite fermions that one can form are those of the $p = 0$ theory, and we find that these do not match the global anomalies of $G_{fl,Rsc}$ [given below in Eq. (7.1) for $R_{sc} = Adj$]. This rules out the possibility that the original theory can form massless composite fermions involving the full set of massless fermions in the theory with $p > 0$. As we will discuss below, however, if the UV to IR evolution leads to sufficiently strong coupling so that there is condensation in the $R_{sc} \times R_{sc} \to 1$ channel, giving the R_{sc} fermions dynamical masses, then in the low-energy effective field theory below the condensation scale, with these fermions removed, the descendant theory is equivalent to the original theory with $p = 0$. In this descendant theory (called the $S\bar{F}$ theory below), further evolution into the infrared might produce massless gauge-singlet composite fermions.

To obtain information concerning the likely type of UV to IR evolution among types 1 and 2 in the list above, as a function of p, we first identify the most attractive channel, which is

$$Adj \times Adj \to 1. \tag{5.21}$$

This clearly preserves the SU(N) gauge symmetry, and has attractiveness measure

$$\Delta C_2 = 2N \quad \text{for } Adj \times Adj \to 1. \tag{5.22}$$

In particular, this channel is more attractive than the $S \times \bar{F} \to F$ channel, in accordance with the inequality (4.5). Quantitatively, the difference in ΔC_2 values for these two channels is

$$\Delta C_2(Adj \times Adj \to 1) - \Delta C_2(S \times \bar{F} \to F) = \frac{N^2 - N + 2}{N}, \tag{5.23}$$

which is positive for all physical N. The condensates for the $Adj \times Adj \to 1$ channel are

$$\langle \xi^{aT}_{b,i,L} C \xi^{b}_{a,j,L} \rangle, \qquad i,j = 1, \ldots, p. \tag{5.24}$$

From Eq. (5.22), we obtain the rough estimate of the minimal critical coupling for condensation in the $Adj \times Adj \to 1$ channel:

$$\alpha_{cr,Adj\times Adj} \simeq \frac{\pi}{3N}. \tag{5.25}$$

Thus, an approximate indication of the size of the IR fixed point relative to the size that would lead to the formation of fermion condensates in the channel (5.21) is the ratio

$$\rho_{IR,Adj\times Adj} \equiv \frac{\alpha_{IR,2\ell,Adj}}{\alpha_{cr,Adj\times Adj}} = \frac{24N^2[(9-2p)N - 6]}{(32p-39)N^3 + 90N^2 - 3N - 36}. \tag{5.26}$$

As p decreases, $\alpha_{IR,2\ell}$ increases. Therefore, considering N and p as being extended from the non-negative integers to the non-negative real numbers, one can calculate a rough estimate of the critical value of p, denoted $p_{cr,Adj\times Adj}$, such that, as p decreases through this value, $\alpha_{IR,2\ell}$ increases through the value $\alpha_{cr,Adj\times Adj}$. This critical value of $p_{cr,Adj\times Adj}$ is thus obtained by setting $\rho_{IR,Adj\times Adj} = 1$ and solving for p, yielding

$$p_{cr,Adj\times Adj} \simeq \frac{3(85N^3 - 78N^2 + N + 12)}{80N^3}. \tag{5.27}$$

This critical value $p_{cr,Adj}$ is a monotonically increasing function of N for physical N, increasing from $67/30 = 2.23$ for $N = 3$ and, as $N \to \infty$,

$$\lim_{N\to\infty} p_{cr,Adj\times Adj} = \frac{51}{16} = 3.1875, \tag{5.28}$$

where the limit is approached from below as N increases.

We list values of the ratio $\rho_{IR,Adj\times Adj}$ in Table III for several illustrative values of N and p. For all of the values of N presented in this table, the respective values of the ratio $\rho_{IR,Adj\times Adj}$ for $p = 4$ are much smaller than 1, so that one can conclude that for $p = 4$, the theory evolves from the UV to a scale-invariant, non-Abelian Coulomb phase in the IR. As is evident from Table III, for a given N, as p decreases, $\alpha_{IR,2\ell,Adj}$ increases. As this IR coupling becomes of O(1), the uncertainties in the use of perturbation theory increase. For most of the $p = 3$ cases shown with various N, the ratio $\rho_{IR,Adj\times Adj}$ is sufficiently close to 1 that, taking account of these uncertainties, one cannot draw a definite conclusion as to whether fermion condensate does or does not take place. For the cases shown in Table III with $p = 1$ [where this is in $(I_p)_{Adj}$] and $p = 2$, the ratio $\rho_{IR,Adj\times Adj}$ is substantially larger than 1, so that in these cases, one expects that the gauge interaction becomes strong enough to produce fermion condensation in the channel (5.21).

In the large-N limit defined above,

$$\lim_{N\to\infty} \rho_{IR,Adj\times Adj} = \frac{24(9-2p)}{32p-39}. \tag{5.29}$$

In particular,

$$\lim_{N\to\infty} \rho_{IR,Adj\times Adj} = \frac{24}{89} = 0.270 \quad \text{for } p = 4, \tag{5.30}$$

$$\lim_{N\to\infty} \rho_{IR,Adj\times Adj} = \frac{72}{57} = 1.26 \quad \text{for } p = 3, \tag{5.31}$$

$$\lim_{N\to\infty} \rho_{IR,Adj\times Adj} = \frac{24}{5} = 4.80 \quad \text{for } p = 2 \tag{5.32}$$

(where the floating-point results are given to the indicated accuracy). Hence, in this large-N limit, since the limit of the ratio $\rho_{IR,Adj\times Adj}$ for $p = 4$ is sufficiently small compared to 1 that it is plausible that in the IR the theory is in a deconfined Coulombic phase, while if $p = 3$, $\rho_{IR,Adj\times Adj}$ is too close to unity for one to be able to draw a definite conclusion. Finally, if $p = 2$, then $\rho_{IR,Adj\times Adj}$ is sufficiently large compared with 1 that one expects that the theory can produce bilinear condensates in the most attractive channel, as discussed above.

We continue with the analysis of the UV to IR evolution for the smaller values of p that produce a strongly coupled gauge interaction. As the momentum scale μ decreases through a scale denoted Λ_{Adj}, $\alpha(\mu)$ exceeds $\alpha_{cr,Adj}$, and, from our discussion above, we infer that the gauge interaction produces the bilinear fermion condensates (5.24) in the MAC, $Adj \times Adj \to 1$. These condensates preserve the SU(N) gauge symmetry and the U$(1)_1$ global symmetry, while breaking the U$(1)_2$ and SU(p) global symmetries (these global symmetries are defined in Sect. VII). By the use of a vacuum alignment argument [37], one can plausibly infer that the condensates (5.24) have $i = j$, with $i = 1, \ldots, p$ and hence preserve an SO(p) global isospin symmetry defined by the transformation

$$\xi^a_{b,i,L} \to \sum_{j=1}^{p} \mathcal{O}_{ij} \xi^a_{b,j,L}, \qquad \mathcal{O} \in \mathrm{SO}(p). \tag{5.33}$$

Just as light quarks gain dynamical, constituent quark masses of order Λ_{QCD} due to the formation of $\langle \bar{q} q \rangle$ condensates in quantum chromodynamics (QCD), so also, the $p(N^2-1)$ components, $\xi^a_{b,i,L}$, of the Adj fermions involved in these condensates pick up a common dynamical mass of order Λ_{Adj}.

At scales $\mu < \Lambda_{Adj}$, the analysis proceeds by integrating out the massive $\xi^a_{b,j,L}$ fermions, constructing the low-energy effective field theory applicable for these lower scales, and then exploring the further evolution of this descendant theory into the infrared. Since the condensation (5.24) gives dynamical masses to all of the Adj fermions $\xi^a_{b,j,L}$, $j = 1, \ldots, p$, the low-energy effective theory below this condensation scale Λ_{Adj} is just the $p = 0$ theory. Since the evolution of this theory is the same as for our second type of chiral gauge theory, we first study this second theory, and then discuss the further IR evolution.

VI. THEORY WITH $N = 2k$ AND $R = [N/2]_N$

A. Particle content

In this section we construct and study a chiral gauge theory with gauge group $G = \mathrm{SU}(N)$ with even $N = 2k$, and fermions transforming according to

(1) a symmetric rank-2 tensor representation, S, with corresponding field $\psi^{ab}_L = \psi^{ba}_L$,

(2) $N + 4$ copies chiral fermions in the conjugate fundamental representation, $\bar{\square}$, with fields $\chi_{a,i,L}$, $i = 1, \ldots, N + 4$, and

(3) p copies of chiral fermions in the totally antisymmetric k-fold tensor representation $[N/2]_N = [k]_{2k}$, with fields $\xi^{a_1 \ldots a_k}_{j,L}$, $j = 1, \ldots, p$.

We again label this theory by the representation of the fermions in the vectorlike subsector, namely AT, for antisymmetric k-fold tensor. This fermion content is summarized in Table I.

The representation $[k]_N$ has the dimension (for general N)

$$\dim([k]_N) = \binom{N}{k} \tag{6.1}$$

and satisfies the equivalence property

$$[N-k]_N = \overline{[k]}_N. \tag{6.2}$$

Here we have used the standard notation for the binomial coefficient, $\binom{a}{b} \equiv a!/[b!(a-b)!]$. An important property that follows from Eq. (6.2) that we will use here is the fact that for our case of interest, $N = 2k$, the representation $[k]_{2k}$ is self-conjugate:

$$[k]_{2k} = \overline{[k]}_{2k}. \tag{6.3}$$

Combining the self-conjugate property of $[N/2]_N = [k]_{2k}$ with the relation (5.2), it follows that

$$\mathrm{Anom}([k]_{2k}) = 0. \tag{6.4}$$

Thus, this theory has the same irreducibly chiral sector as the theory discussed in the previous section, and a vectorlike subsector that consists of the p copies of the fermions in the $[N/2]_N$ representation.

B. Beta function

We calculate that the one- and two-loop terms in the beta function of this theory are, in terms of $k = N/2$,

$$(b_1)_{AT} = 6k - 2 - \frac{p(2k-2)!}{3[(k-1)!]^2} \tag{6.5}$$

and

$$(b_2)_{AT} = \frac{52k^3 - 60k^2 + k + 6}{2k} - \frac{pk(43+6k)(2k-2)!}{12[(k-1)!]^2}. \tag{6.6}$$

For small p, $(b_1)_{AT}$ is positive, and as p increases, $(b_1)_{AT}$ decreases and passes through zero as p exceeds the value

$$p_{b1z,AT} = \frac{6(3k-1)[(k-1)!]^2}{(2k-2)!}. \tag{6.7}$$

The requirement that the theory should be asymptotically free is thus satisfied if

$$p < \frac{6(3k-1)[(k-1)!]^2}{(2k-2)!}. \tag{6.8}$$

This upper bound decreases rapidly as a function of $k = N/2$, so that as k increases, eventually the requirement of asymptotic freedom precludes any nonzero value of p. Thus, the AT theory has no asymptotically free large-N limit with nonzero p, in contrast to the Adj and $S\bar{S}$ theories constructed and studied here and the Sp theory reviewed in Sec. III.

The beta function of the AT theory has an IR zero if b_2 is negative. For small p, $(b_2)_{AT}$ is positive, and it decreases through zero to negative values as p (continued to the real numbers) increases through the value

$$p_{b2z,AT} = \frac{6(52k^3 - 60k^2 + k + 6)[(k-1)!]^2}{k^2(6k+43)(2k-2)!}. \quad (6.9)$$

We observe that $p_{b1z,AT} > p_{b2z,AT}$. This is proved by considering the difference,

$$p_{b1z,AT} - p_{b2z,AT} = \frac{6(18k^4 + 71k^3 + 17k^2 - k - 6)[(k-1)!]^2}{k^2(43+6k)[(2k-2)!]}. \quad (6.10)$$

This difference is positive for all k values of relevance here (with k extended to the positive reals, it is positive for $k > 0.3724$). By itself, this inequality does not guarantee that there is an integral value of p that lies above $p_{b2z,AT}$ and below $p_{b1z,AT}$, but in fact we find that for each relevant case, there are one or more such integral values. These then define the respective intervals $(I_p)_{AT}$,

$$(I_p)_{AT}: \ p_{b2z,AT} < p < p_{b1z,AT} \quad (6.11)$$

for each k. For the (integral) values of $p \in (I_p)_{AT}$, the beta function of the SU($2k$) AT theory has an IR zero. We list the values of p_{b1z}, p_{b2z}, p_{max}, and $(I_p)_{AT}$ in Table IV. Note that for the cases $G = \mathrm{SU}(N)$ with $k \geq 2$ under consideration here, the requirement of asymptotic freedom allows nonzero values of p only for $k \leq 5$.

For a given $N = 2k$ with a nonvacuous interval $(I_p)_{AT}$, the $\beta_{\alpha,2\ell}$ has an IR zero at

$$\alpha_{IR,2\ell,AT} = -\frac{4\pi(b_1)_{AT}}{(b_2)_{AT}} \quad (6.12)$$

where $(b_1)_{AT}$ and $(b_2)_{AT}$ were given in Eqs. (6.5) and (6.6) above. We list the values of $\alpha_{IR,2\ell,AT}$ in Table V.

C. UV to IR evolution

Here we analyze the UV to IR evolution of this AT chiral gauge theory. By construction, the most attractive channel involves fermion condensation in the channel (4.1), with $R = [N/2]_N = [k]_{2k}$ in this case, i.e.,

TABLE IV. Values of $p_{b1z,AT}$, $p_{b2z,AT}$, p_{max}, and the intervals $(I_p)_{AT}$ as functions of N in the AT model with gauge group SU(N) with $N = 2k$.

N	$p_{b2z,AT}$	$p_{b1z,AT}$	p_{max}	$(I_p)_{AT}$
4	2.509	15	14	$3 \leq p \leq 14$
6	1.590	8	7	$2 \leq p \leq 7$
8	0.665	3.3	3	$1 \leq p \leq 3$
10	0.235	1.2	1	$p = 1$

TABLE V. Values of $\alpha_{IR,2\ell,AT}$ and $\rho_{IR,AT}$ in the AT theory for the relevant values of N and, for each N, the values of p in the respective interval $(I_p)_{AT}$.

N	p	$\alpha_{IR,2\ell,AT}$	$\rho_{IR,AT}$
4	3	11.170	26.67
4	4	3.371	8.05
4	5	1.8345	4.38
4	6	1.178	2.81
4	7	0.814	1.94
4	8	0.583	1.39
4	9	0.422	1.01
4	10	0.305	0.728
4	11	0.215	0.514
4	12	0.144	0.345
4	13	0.0871	0.208
4	14	0.0398	0.095
6	2	4.021	20.16
6	3	0.974	4.88
6	4	0.460	2.29
6	5	0.242	1.21
6	6	0.125	0.625
6	7	0.0508	0.255
8	1	1.290	11.08
8	2	0.183	1.57
8	3	0.0241	0.207
10	1	0.0360	0.473

$$[N/2]_N \times [N/2]_N \to 1. \quad (6.13)$$

This preserves the SU(N) gauge symmetry and has the attractiveness measure

$$\Delta C_2 = 2C_2([N/2]_N) = \frac{k(2k+1)}{2}, \quad (6.14)$$

where we have used the result for $C_2([k]_N)$ given in the Appendix. The condensates are

$$\langle \epsilon_{a_1,\ldots a_{2k}} \xi_{i,L}^{a_1,\ldots,a_k T} C \xi_{j,L}^{a_{k+1},\ldots,a_{2k}} \rangle, \qquad i,j = 1,\ldots,p. \quad (6.15)$$

By a vacuum alignment argument, one may infer that these condensates have $i = j$ [37]. To show that the channel (6.13) is more attractive than the next-most-attractive channel, $S \times \bar{F} \to F$, we examine the difference

$$\Delta C_2([N/2]_N \times [N/2]_N \to 1) - \Delta C_2(S \times \bar{F} \to F) = 2C_2([N/2]_N) - C_2(S) = \frac{2k^3 - 3k^2 - 2k + 2}{2k}. \quad (6.16)$$

This difference is positive for all values of $k \geq 2$ of interest here.

If the beta function has no IR zero, then as the scale μ decreases and $\alpha(\mu)$ increases, it will eventually become large enough to cause condensation, which, according to the MAC criterion, will be in this channel (6.13). If the beta function does have a zero, then the next step in the analysis is to determine how the value of the coupling at this zero compares with α_{cr} for the most attractive channel, (6.13). Substituting (6.14) into the general formula for Eq. (2.7), we calculate

$$\alpha_{cr,AT} = \frac{4\pi}{3k(2k+1)}. \quad (6.17)$$

As discussed above, an approximate measure of how strong the coupling gets in the infrared, compared with the minimum critical value for condensation in the MAC, is then given by the ratio

$$\rho_{IR,AT} \equiv \frac{\alpha_{IR,2\ell,AT}}{\alpha_{cr,AT}}. \quad (6.18)$$

We list values of $\rho_{IR,AT}$ for the relevant N and p in Table V. In cases where condensation occurs in this theory we denote the scale at which it occurs as $\Lambda_{[N/2]_N}$.

1. AT theory with $G = \mathrm{SU}(4)$

In this subsection and the following ones we discuss three illustrative cases with various values of $N = 2k$ and their corresponding intervals $(I_p)_{AT}$. For each value of N, if p is nonzero and $p < p_{b2z}$, i.e., below the lower end of the interval $(I_p)_{AT}$, then the theory has no IR fixed point, even an approximate one, so that the gauge coupling continues to grow in the infrared and will cause condensation in the MAC. Hence, we restrict our consideration here to $p \in (I_p)_{AT}$. The reader is referred to Tables IV and V for numerical values of relevant quantities. As indicated in Table IV, for this SU(4) AT theory the interval $(I_p)_{AT}$ is $3 \le p \le 14$. For p in this interval, $\beta_{\alpha,2\ell,AT}$ has an IR zero at

$$N = 4: \ \alpha_{IR,2\ell,AT} = \frac{8\pi(15-p)}{55p-138}. \quad (6.19)$$

The ratio $\rho_{IR,AT}$ is

$$N = 4: \ \rho_{IR,AT} = \frac{60(15-p)}{55p-138}. \quad (6.20)$$

As listed in Table V, for the range of p from 3 to 7, this ratio takes on values decreasing from 26.7 to 1.94, all well above unity. Thus, one may plausibly expect that for these values of p, in the UV to IR evolution, as the reference scale μ decreases sufficiently and the running coupling approaches $\alpha_{IR,2\ell,AT}$, the gauge interaction will become strong enough to cause fermion condensation in the most attractive channel, $[2]_4 \times [2]_4 \to 1$. For $p = 8, 9, 10, 11$, $\rho_{IR,AT}$ has the respective values 1.39, 1.01, 0.728, 0.514. Given the theoretical uncertainties in these estimates, the IR behavior might or might not involve the formation of the condensates (6.15). For the largest values of p, namely $p = 12, 13, 14$, $\rho_{IR,AT}$ has the respective values 0.345, 0.208, 0.095, so for these cases, it is likely that the theory evolves from the UV to a scale-invariant, deconfined, Coulombic IR phase. This inference is, of course, most reliable for the largest allowed value of p, namely $p = 14$, which leads to the smallest value of $\alpha_{IR,2\ell,AT}$ and $\rho_{IR,AT}$. As discussed above, in the cases where there is condensate formation and chiral symmetry breaking, the IRFP is only approximate, while in the cases where there is no such chiral symmetry breaking the IRFP (calculated to all orders) is exact.

2. AT theory with $G = \mathrm{SU}(6)$

In the SU(6) (i.e., $k = 3$) AT theory, $(I_p)_{AT}$ is the interval $2 \le p \le 7$. For p in this interval, $\beta_{\alpha,2\ell}$ has an IR zero at

$$N = 6: \ \alpha_{IR,2\ell,AT} = \frac{16\pi(8-p)}{3(61p-97)}. \quad (6.21)$$

The ratio $\rho_{IR,AT}$ is

$$N = 6: \ \rho_{IR,AT} = \frac{84(8-p)}{61p-97}. \quad (6.22)$$

As listed in Table V, for $2 \le p \le 7$, this has the respective values 20.16, 4.89, 2.29, 1.21, 0.625, 0.255. Thus, for $p = 2$, $p = 3$, and $p = 4$, it is likely that condensation occurs in the MAC, $[3]_6 \times [3]_6 \to 1$ channel; for $p = 7$, it is likely that there is no condensation; and for the middle two values $p = 5$ and $p = 6$, taking account of the intrinsic theoretical uncertainties, one cannot give a very definite prediction from this analysis.

3. AT theory with $G = \mathrm{SU}(10)$

In the SU(10) ($k = 5$) AT theory, the interval $(I_p)_{AT}$ reduces to just a single nonzero value, $p = 1$, and the resultant $\alpha_{IR,2\ell,AT} = 0.036$, yielding the ratio $\rho_{IR,AT} = 0.473$. It is thus likely that this theory evolves from the UV to the IR to a non-Abelian Coulomb phase, although there are obvious uncertainties in this inference due to the strong-coupling physics involved.

VII. GLOBAL FLAVOR SYMMETRY FOR THEORIES WITH SELF-CONJUGATE R

In analyzing the global flavor symmetry of these chiral gauge theories, it is useful to consider a more general class of theories, in which the vectorlike fermion subsector is comprised of fermions transforming under a general self-conjugate representation, $R = R_{sc}$. The results will then be applied to the two specific theories discussed above, namely those with $G = \mathrm{SU}(N)$, $N \ge 3$, and $R_{sc} = Adj$; and the AT theory with $G = \mathrm{SU}(N)$ with even $N = 2k$, $k \ge 2$, and $R_{sc} = [N/2]_N$.

The classical global chiral flavor symmetry of a theory in this class of theories is

$$G_{fl,cl,R_{sc}} = \mathrm{U}(1)_S \otimes \mathrm{U}(N+4)_{\bar F} \otimes \mathrm{U}(p)_{R_{sc}} = \mathrm{U}(1)_S \otimes \mathrm{SU}(N+4)_{\bar F} \otimes \mathrm{U}(1)_{\bar F} \otimes \mathrm{SU}(p)_{R_{sc}} \otimes \mathrm{U}(1)_{R_{sc}}. \tag{7.1}$$

The representations of the fermions in the two theories with $R = R_{sc}$ under this symmetry are given in Table I. The corresponding global unitary transformations are

$$\psi_L^{ab} \to U_S \psi_L^{ab}, \qquad U_S \in \mathrm{U}(1)_S, \tag{7.2}$$

$$\chi_{a,i,L} \to \sum_{j=1}^{N+4} (U_{\bar F})_{ij} \chi_{a,j,L}, \qquad U_{\bar F} \in \mathrm{U}(N+4)_{\bar F}, \tag{7.3}$$

and

$$\xi_{i,L} \to \sum_{j=1}^{p} (U_{R_{sc}})_{ij} \xi_{j,L}, \qquad U_{R_{sc}} \in \mathrm{U}(p)_{R_{sc}} \tag{7.4}$$

where we have suppressed the $\mathrm{SU}(N)$ gauge indices in Eq. (7.4), which applies to each theory with the corresponding ξ field, i.e., $\xi^a_{b,i,L}$ in the *Adj* theory and $\xi^{a_1,\ldots,a_k}_{i,L}$ in the AT theory.

Each of the three global U(1) symmetries is broken by the instantons of the $\mathrm{SU}(N)$ gauge theory [38]. One may define a three-dimensional vector of anomaly factors,

$$\vec v = (N_S T(S), N_{\bar F} T(\bar F), N_{R_{sc}} T(R_{sc})) = \left(\frac{N+2}{2}, \frac{N+4}{2}, pT_{R_{sc}}\right), \tag{7.5}$$

where the basis is $(S, \bar F, R_{sc})$, and we have inserted the values $N_S = 1$, $N_{\bar F} = N+4$, and $N_{R_{sc}} = p$. One can construct two linear combinations of the three original currents that are conserved in the presence of $\mathrm{SU}(N)$ instantons. The fermions have charges under these global U(1) symmetries given by the vectors

$$\vec Q^{(j)} \equiv (Q_S^{(j)}, Q_{\bar F}^{(j)}, Q_{R_{sc}}^{(j)}), \qquad j = 1, 2, \tag{7.6}$$

where $j = 1$ for $\mathrm{U}(1)_1$ and $j = 2$ for $\mathrm{U}(1)_2$. The condition that the corresponding currents are conserved, i.e., the $\mathrm{U}(1)_j$ global symmetries are exact, in the presence of instantons is that

$$\sum_f N_f T(R_f) Q_f^{(j)} = \vec v \cdot \vec Q^{(j)} = 0 \quad \text{for } j = 1, 2. \tag{7.7}$$

As indicated, this condition is equivalent to the condition that the vectors of charges under the $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ symmetries are orthogonal to the vector $\vec v$. [Note that the condition (7.7) does not uniquely determine the vectors $\vec Q^{(j)}$, $j = 1, 2$.] It will be convenient to choose the first vector, $\vec Q^{(1)}$, so that $Q^{(1)}_{R_{sc}} = 0$. We thus choose

$$\vec Q^{(1)} = (N+4, -(N+2), 0). \tag{7.8}$$

For the vector of charges under $\mathrm{U}(1)_2$, we choose

$$\vec Q^{(2)} = (2pT_{R_{sc}}, 0, -(N+2)). \tag{7.9}$$

[Note that in contrast to Gram-Schmidt orthogonalization of the three vectors $\vec v$, $\vec Q^{(1)}$, and $\vec Q^{(2)}$, here it is not necessary that $\vec Q^{(1)} \cdot \vec Q^{(2)} = 0$.]

The actual nonanomalous global chiral flavor symmetry group of the class of chiral gauge theories with $R = R_{sc}$ is then

$$G_{fl,R_{sc}} = \mathrm{SU}(N+4)_{\bar F} \otimes \mathrm{SU}(p)_{R_{sc}} \otimes \mathrm{U}(1)_1 \otimes \mathrm{U}(1)_2. \tag{7.10}$$

For the two respective theories with (i) $R_{sc} = Adj$ and (ii) $R_{sc} = [N/2]_N$, Eqs. (7.9) and (7.10) apply with (i) $T_{R_{sc}} = T(Adj) = N$ and (ii) $T_{[N/2]_N}$ given by Eq. (A8) in the Appendix. We summarize these properties in Table I.

In general, one must also check to see if either of the chiral gauge theories with $R_{sc} = Adj$ or $R_{sc} = [N/2]_N$ satisfies the 't Hooft anomaly-matching conditions, which are necessary conditions for the possible formation of massless gauge-singlet composite fermions. The possible gauge-singlet fermions can be described by wavefunctions of the form

$$B_{ij} = \bar F_{a,i,L} S_L^{ab} \bar F_{b,j,L}, 1 \le i, j \le N+4. \tag{7.11}$$

Given the minus sign from Fermi statistics and the fact that S^{ab} is a rank-2 symmetric tensor representation (□□) of $\mathrm{SU}(N)$, it follows that $B_{ij} = -B_{ji}$, i.e., B_{jk} is a rank-2 antisymmetric tensor representation ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) of the $\mathrm{SU}(N+4)_{\bar F}$ factor group in the global flavor symmetry group G_{fl}. There are thus $(N+4)(N+3)/2$ components of B_{ij}. The charges of B_{ij} under the two global Abelian factor groups in $G_{fl,R_{sc}}$, $\mathrm{U}(1)_k$, $k = 1, 2$ are determined by the relation

$$Q_B^{(k)} = Q_S^{(j)} + 2Q_{\bar F}^{(k)}, \qquad k = 1, 2. \tag{7.12}$$

Hence,

$$Q_B^{(1)} = -N \tag{7.13}$$

and

$$Q_B^{(2)} = 2pT_{R_{sc}}. \tag{7.14}$$

We find that the global anomalies of a theory with these massless composite fermions do not match those of the original G_{fl} group except in the degenerate case $p = 0$. This $p = 0$ case describes a descendant low-energy effective field theory that occurs if there is condensation in the $R_{sc} \times R_{sc} \to 1$ channel, and will be discussed below.

VIII. ANALYSIS OF LOW-ENERGY EFFECTIVE THEORY FOR $\mu < \Lambda_{R_{sc}}$

In the cases where the values of N and p are such as to lead to the respective bilinear fermion condensates (5.24) or (6.15) at the corresponding scales Λ_{Adj} or $\Lambda_{[N/2]_N}$, we analyze the further UV to IR evolution below these scales. We denote these scales generically as $\Lambda_{R_{sc}}$. Because of this condensation, the p fermions $\xi^a_{b,i,L}$ involved in the condensate (5.24) in the Adj model and the p fermions $\xi^{a_1,\ldots,a_l}_{i,L}$ involved in the condensate (6.15) in the AT theory gain dynamical masses of order Λ_{Adj} and $\Lambda_{[N/2]_N}$, respectively.

For momentum scales μ slightly below the condensation scale $\Lambda_{R_{sc}}$, the resultant global symmetry is

$$G'_{fl} = \mathrm{SU}(N+4)_{\bar F} \otimes \mathrm{SO}(p) \otimes \mathrm{U}(1)_1. \tag{8.1}$$

Here the $\mathrm{SU}(N+4)_{\bar F} \otimes \mathrm{U}(1)_1$ is a global chiral symmetry operating on the massless S and $\bar F$ fermions, leaving their covariant derivatives invariant, while the $\mathrm{SO}(p)$ is a global isospin symmetry of the condensate in each of our two theories with $R = R_{sc}$, or equivalently, the corresponding effective mass term. These mass terms are

$$\Lambda_{Adj} \sum_{i=1}^{p} \xi^{aT}_{b,i,L} C \xi^b_{a,i,L} + \mathrm{H.c.} \tag{8.2}$$

in the Adj theory and

$$\Lambda_{[N/2]_N} \sum_{i=1}^{p} \langle \epsilon_{a_1,\ldots a_{2k}} \xi^{a_1,\ldots,a_k T}_{i,L} C \xi^{a_{k+1},\ldots,a_{2k}}_{i,L} \rangle + \mathrm{H.c.} \tag{8.3}$$

in the AT theory produced by the bilinear fermion condensations in these respective theories. This $\mathrm{SO}(p)$ symmetry also leaves the covariant derivatives of these ξ fields invariant.

The spontaneous symmetry breaking of the initial nonanomalous global symmetry G_{fl} in Eq. (7.10) to the final global symmetry (8.1) produces

$$o(\mathrm{SU}(p)) + 1 - o(\mathrm{SO}(p)) = \frac{p(p+1)}{2} \tag{8.4}$$

massless Nambu-Goldstone bosons, where $o(H)$ denotes the order of a group H.

As the reference scale μ decreases well below $\Lambda_{R_{sc}}$, we integrate these now-massive ξ fermions out of the low-energy (LE) effective field theory (LEEFT) applicable for $\mu \ll \Lambda_{R_{sc}}$. Focusing on the infrared region $\mu \ll \Lambda_{R_{sc}}$, with the ξ fermions integrated out, both the theory with $R_{sc} = Adj$ and the theory with $R_{sc} = [N/2]_N$ reduce to the same low-energy descendant theory, with (massless) S fermion and $N+4$ copies of $\bar F$ fermions. We denote this as the $S\bar F$ theory. This theory has been well studied in the past [2,4,8–10,12]. We recall the results from these earlier studies that we will need for our present analysis.

The value of f_{UV} for the $S\bar F$ model, which we denote as $f_{UV,S\bar F M}$ (M standing for model), is given by the $p = 0$ special case of Eq. (9.1), namely

$$f_{UV,S\bar F M} = 2(N^2-1) + \frac{7}{4}\left[\frac{N(N+1)}{2} + (N+4)N\right]. \tag{8.5}$$

The $S\bar F$ theory is invariant under a nonanomalous global flavor symmetry group

$$G_{fl,S\bar F M} = \mathrm{SU}(N+4)_{\bar F} \otimes \mathrm{U}(1)_{S\bar F}. \tag{8.6}$$

For this theory the three-dimensional vector (7.5) reduces to a two-dimensional vector with the third entry deleted, and the vector of charges that is orthogonal to it and hence defines the charge assignments of the $\mathrm{U}(1)_{S\bar F}$ is given by the first two entries in $Q^{(1)}$, namely

$$\vec{Q}^{(1)} = (N+4, -(N+2)). \tag{8.7}$$

The $S\bar F$ theory is asymptotically free, so the gauge coupling continues to grow as μ decreases. The beta function of this $S\bar F$ theory has one-loop and two-loop coefficients given by Eqs. (5.3) and (5.4) with $p = 0$ or equivalently, the $p = 0$ special case of Eqs. (3.1) and (3.2). In the relevant range $N \ge 3$, b_2 is positive. Since b_1 and b_2 thus have the same sign, the beta function, calculated to the maximal scheme-independent order of two loops, does not have any IR zero. Hence, as μ decreases from the UV to the IR, the running coupling $\alpha(\mu)$ increases, eventually exceeding the region where the perturbatively calculated beta function is applicable.

There are two possible types of UV to IR evolution in the $S\bar F$ theory. First, the strongly coupled gauge interaction may produce bilinear fermion condensates. The most attractive channel is $S \times \bar F \to F$, with condensates

$$\left\langle \sum_{b=1}^{N} \psi^{abT}_L C \chi_{b,i,L} \right\rangle. \tag{8.8}$$

Without loss of generality, one may take $a = N$ and $i = 1$ for the first condensate. This breaks the $\mathrm{SU}(N)$ gauge symmetry down to $\mathrm{SU}(N-1)$, so that the $2N-1$ gauge bosons in the coset $\mathrm{SU}(N)/\mathrm{SU}(N-1)$ gain masses of order this scale of condensation, which we denote Λ_N. The fermions ψ_L^{Nb} and $\chi_{b,1,L}$ with $b = 1, \ldots, N$ involved in this condensate also gain dynamical masses of order Λ_N. In the low-energy theory applicable for scales $\mu < \Lambda_N$, these now massive fermions are integrated out.

The descendant theory is again asymptotically free, so the gauge coupling inherited from the SU(N) theory continues to increase. There is then a second condensation, again in the MAC, $S \times \bar{F} \to F$ channel, breaking the gauge symmetry from SU($N-1$) to SU($N-2$). Without loss of generality, we may take the breaking direction to be $a = N-1$ and the $\bar{F}$ fermion involved in the condensate to be labeled as $\chi_{b,2,L}$, so that this condensate is

$$\left\langle \sum_{b=1}^{N-1} \psi_L^{N-1,bT} C \chi_{b,2,L} \right\rangle. \tag{8.9}$$

We denote the scale at which this occurs as Λ_{N-1}. The $2N-3$ gauge bosons in the coset SU($N-1$)/SU($N-2$) gain masses of order Λ_{N-1} and the fermions $S_L^{N-1,b}$ and $\chi_{b,2,L}$ with $b = 1, \ldots, N-1$ involved in this condensate gain dynamical masses of order Λ_{N-1}. This sequential breaking via condensation in the respective $S \times \bar{F} \to F$ channels continues at the scales Λ_{N-2}, etc. until the gauge symmetry is completely broken. Thus, the sequence of gauge symmetry breaking is

$$\mathrm{SU}(N) \to \mathrm{SU}(N-1) \to \cdots \to SU(2) \to \emptyset. \tag{8.10}$$

The gauge bosons in the respective cosets SU(N)/SU($N-1$), SU($N-1$)/SU($N-2$), etc. gain masses of order Λ_N, Λ_{N-1}, etc, as do the components of the fermions involved in the respective condensates.

Considering the $S\bar{F}$ theory, for this type of UV to IR evolution [8–10],

$$f_{IR,S\bar{F}M;S\times\bar{F}} = 8N + 1 + \frac{7}{4}\left[\frac{N(N-1)}{2} + 4N\right], \tag{8.11}$$

where here the subscript $S\bar{F}M$ means the $S\bar{F}$ model, and the subscript $S\bar{F}$ refers to the condensation channel. For the $S\bar{F}$ model, with this type of UV to IR evolution, one then has

$$(\Delta f)_{S\bar{F}M;S\times\bar{F}} = f_{UV,S\bar{F}M} - f_{IR,S\bar{F}M;S\times\bar{F}} = \frac{15N^2 - 25N - 12}{4}. \tag{8.12}$$

This is positive for all relevant values of N. [For N extended to the positive reals, it is positive for $N > (25 + \sqrt{1345})/30 = 2.056$.]

The low-energy effective $S\bar{F}$ theory applicable below $\Lambda_{R_{sc}}$ could also undergo a different type of flow deeper into the infrared, namely one leading to confinement with massless gauge-singlet composite fermions with wavefunctions (7.11). In this case, for this $S\bar{F}$ theory, considered in isolation,

$$f_{IR,S\bar{F}M;sym} = \frac{7}{4}\left[\frac{(N+4)(N+3)}{2}\right]. \tag{8.13}$$

Hence, for this type of UV to IR evolution,

$$(\Delta f)_{S\bar{F}M;sym} = \frac{15N^2 + 7N - 50}{4}. \tag{8.14}$$

This is positive for all relevant values of N. [For N extended to the positive reals, it is positive for $N > (-7 + \sqrt{3049})/30 = 1.607$.] Thus, for both of these types of UV to IR evolution of the $S\bar{F}$ theory, the conjectured degree-of-freedom inequality (2.12) is obeyed.

IX. COMPARISON WITH DEGREE-OF-FREEDOM INEQUALITY

We now combine the results for the $S\bar{F}$ theory with our calculations of UV and IR degree-of-freedom counts for the different types of UV to IR evolution in the *Adj* and AT chiral gauge theories and compare with the conjectured degree-of-freedom inequality (2.12).

A. UV count

Given that we have required our theories to be asymptotically free, they are weakly coupled in the UV, so we can identify the perturbative degrees of freedom and calculate f_{UV}. From the general formula (2.10), we have

$$f_{UV,R_{sc}} = 2(N^2-1) + \frac{7}{4}\left[\frac{N(N+1)}{2} + (N+4)N\right] + \frac{7}{8}p\,\dim(R_{sc}), \tag{9.1}$$

where the respective terms represent the contributions of the SU(N) gauge fields, the S fermions, the $N+4$ copies of $\bar{F}$ fermions, and the R_{sc} fermions. Explicitly, for the *Adj* theory,

$$f_{UV,Adj} = 2(N^2-1) + \frac{7}{4}\left[\frac{N(N+1)}{2} + (N+4)N\right] + \frac{7}{8}p(N^2-1) \tag{9.2}$$

and for the AT theory, with $N = 2k$,

$$f_{UV,AT} = 2(N^2-1) + \frac{7}{4}\left[\frac{N(N+1)}{2} + (N+4)N\right] + \frac{7}{8}p\binom{N}{N/2}, \quad (9.3)$$

where $\binom{a}{b}$ is the binomial coefficient.

B. f_{IR} calculations

Next, we calculate f_{IR} for the two types of chiral gauge theories discussed above in the cases where the UV to IR evolution involves a high-scale condensation in the respective channels (5.21) or (6.13), followed by sequential condensations in the $S \times \bar{F} \to F$ channel. Taking account of the $p(p+1)/2$ NGBs from the higher-scale symmetry breaking at $\Lambda_{R_{sc}}$, we find, for either of these two types of chiral gauge theories, for this type of infrared evolution below $\Lambda_{R_{sc}}$,

$$\begin{aligned} f_{IR,Adj;Adj\times Adj,S\times\bar{F}} &= f_{IR,AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} \\ &\equiv f_{IR,R_{sc};R_{sc}\times R_{sc},S\times\bar{F}} \\ &= 8N+1+\frac{7}{4}\left[\frac{N(N-1)}{2}+4N\right]+\frac{p(p+1)}{2}, \quad (9.4)\end{aligned}$$

where the subscript R_{sc} identifies the chiral fermion representation in the vectorlike subsector, the next subscript $R_{sc} \times R_{sc}$ is shorthand for the MAC $R_{sc} \times R_{sc} \to 1$ in which the highest-scale condensation takes place, and the last subscript, $S \times \bar{F}$ or sym are shorthand for the two types of IR flow in the low-energy descendant theory, namely sequential $S \times \bar{F} \to F$ condensation formation and gauge and global symmetry breaking in the descendent theory, or confinement with formation of massless composite fermions and retention of exact chiral symmetry (sym) in the infrared. Thus, the subscripts here and below placed after the semicolon in quantities such as $f_{IR,Adj;Adj\times Adj,S\times\bar{F}}$ refer to the sequence of steps in the UV to IR evolution.

For the alternate type of evolution involving high-scale condensation in the respective channels (5.21) or (6.13), followed by confinement leading to massless gauge-singlet composite fermions, we calculate, for either of our two types of chiral gauge theory with $R = R_{sc}$,

$$\begin{aligned} f_{IR,Adj;Adj\times Adj,sym} &= f_{IR,AT;[k]_{2k}\times[k]_{2k},sym} \\ &\equiv f_{IR,R_{sc};R_{sc}\times R_{sc},sym} \\ &= \frac{7}{4}\left[\frac{(N+4)(N+3)}{2}\right]+\frac{p(p+1)}{2}. \quad (9.5)\end{aligned}$$

C. Comparison with DFI for *Adj* theory

Using these inputs, we can now calculate Δf for these chiral gauge theories and compare with the conjectured degree-of-freedom inequality (2.12). For both theories, if the UV to IR evolution is such as to lead to a deconfined non-Abelian Coulomb phase, the perturbative degrees of freedom are the same as in the UV, so the DFI is obeyed. (The perturbative corrections also obey the DFI [8,15].)

We first discuss the possible cases for the theory with $R = Adj$. If N and p are such that the gauge interaction produces the high-scale condensation in the channel (5.24), followed by Eqs. (8.4) with (8.11), we calculate

$$\begin{aligned} (\Delta f)_{Adj;Adj\times Adj,S\times\bar{F}} &\equiv f_{UV,Adj} - f_{IR,Adj;Adj\times Adj,S\times\bar{F}} \\ &= \frac{1}{8}[30N^2-50N-24+7pN^2-11p-4p^2]. \quad (9.6)\end{aligned}$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\left[7N^2-11+\sqrt{49N^4+326N^2-800N-263}\right]. \quad (9.7)$$

The upper bound on the right-hand side of Eq. (9.7) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (5.6). Hence, the conjectured degree-of-freedom inequality (2.12) is obeyed for all N and allowed p with this type of UV to IR evolution.

For the case where the low-energy effective $S\bar{F}$ theory confines without any spontaneous chiral symmetry breaking, producing massless composite fermions, we calculate

$$\begin{aligned} (\Delta f)_{Adj;Adj\times Adj,sym} &\equiv f_{Adj,UV} - f_{IR,Adj;Adj\times Adj,sym} \\ &= \frac{1}{8}[30N^2+14N-100+7pN^2-11p-4p^2]. \quad (9.8)\end{aligned}$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\left[7N^2-11 + \sqrt{49N^4+326N^2+224N-1479}\right]. \quad (9.9)$$

The upper bound on the right-hand side of Eq. (9.9) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (5.6). Hence, the conjectured degree-of-freedom inequality (2.12) is also obeyed for all N and allowed p with this type of UV to IR evolution.

As illustrative numerical examples, we may consider the cases $N=3$ and $N=4$. In these cases, the respective upper bounds on p from Eq. (5.6) are $p \le 3$, while the respective values of the right-hand side of (9.7) are 14.64 and 27.57 and the respective values of the right-hand side of (9.9) are 16.26 and 29.01. Note that if p is close to the upper bound p_{b1z} arising from the requirement of asymptotic freedom, then b_1 is small, so that $\alpha_{IR,2\ell}$ is sufficiently small that the UV to IR evolution is to a non-Abelian Coulomb phase, so that one knows that the DFI is satisfied without going through the present analysis.

These expressions simplify in the limit $N \to \infty$ (with p fixed) in Eq. (5.13). We define rescaled degree-of-freedom measures that are finite in this limit, of the form

$$\bar{f} \equiv \lim_{N\to\infty} \frac{f}{N^2}. \tag{9.10}$$

[We use the same notation, $\bar{f}$ for this $N \to \infty$ limit and for the quantity (3.27) defined in the LNP limit; the context will always make clear which limit is meant.] We calculate

$$\bar{f}_{UV,Adj} = \frac{37+7p}{8}, \tag{9.11}$$

$$\bar{f}_{IR,Adj;Adj\times Adj,S\times\bar{F}} = \bar{f}_{IR,Adj;Adj\times Adj,sym} = \frac{7}{8}, \tag{9.12}$$

and hence

$$(\Delta\bar{f})_{Adj;Adj\times Adj,S\times\bar{F}} = (\Delta\bar{f})_{Adj;Adj\times Adj,sym} = \frac{30+7p}{8}. \tag{9.13}$$

This obviously obeys the conjectured degree-of-freedom inequality (2.12).

D. Comparison with DFI for AT theory

We next calculate Δf for the AT chiral gauge theory with gauge group $G = \mathrm{SU}(N)$ with even $N = 2k$ and $R_{sc} = [N/2]_N = [k]_{2k}$. As noted above, for values of N and p such that the UV to IR evolution is to a deconfined non-Abelian Coulomb phase in the IR, the perturbative degrees of freedom are the same as in the UV, and the conjectured degree-of-freedom inequality is obeyed.

If N and p are such that the gauge interaction produces high-scale condensation in the channel (5.21) followed at lower scales by condensations in the successive $S\times\bar{F} \to F$ channels in $\mathrm{SU}(N)$, $\mathrm{SU}(N-1)$, etc., then, using Eqs. (8.4) and (8.11), we compute

$$(\Delta f)_{AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} \equiv f_{UV,AT} - f_{IR,AT;[k]_{2k}\times[k]_{2k},S\times\bar{F}} = \frac{1}{8}[30N^2 - 50N - 24 + 7pd_R - 4p(p+1)], \tag{9.14}$$

where here $d_R \equiv \binom{N}{N/2}$. This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\left[7d_R - 4 + \sqrt{480N^2 - 800N - 368 + 49d_R^2 - 56d_R}\right]. \tag{9.15}$$

The upper bound on the right-hand side of Eq. (9.15) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (6.8). Hence, the conjectured degree-of-freedom inequality (2.12) is also obeyed for all N and allowed p with this type of UV to IR evolution in the AT model.

For the alternate type of UV to IR evolution in which the low-energy effective $S\bar{F}$ theory confines without any spontaneous chiral symmetry breaking, producing massless composite fermions, we calculate

$$(\Delta f)_{AT;[k]_{2k}\times[k]_{2k},sym} \equiv f_{AT,UV} - f_{IR,AT;[k]_{2k}\times[k]_{2k},sym} = \frac{1}{8}[30N^2 + 14N - 100 + 7pd_R - 4p(p+1)]. \tag{9.16}$$

This is positive for p satisfying the upper bound

$$p < \frac{1}{8}\left[7d_R - 4 + \sqrt{480N^2 + 224N - 1584 + 49d_R^2 - 56d_R}\right]. \tag{9.17}$$

The upper bound on the right-hand side of Eq. (9.17) is larger than the upper limit on p imposed by the requirement of asymptotic freedom, (6.8). Hence, the conjectured degree-of-freedom inequality (2.12) is also obeyed for all N and allowed p with this type of UV to IR evolution in the AT model.

As numerical examples, for $N = 4$ and $N = 6$, the respective upper bounds on p from Eq. (6.8) are $p \le 14$ and $p \le 7$, while the respective right-hand sides of (9.15) are 14.05 and 38.86 and the respective right-hand sides of (9.17) are 16.22 and 40.56. As before, it should be noted that if p is close to the upper bound from asymptotic freedom, b_1 is small, so that $\alpha_{IR,2\ell}$ is sufficiently small that the UV to IR evolution is to a non-Abelian Coulomb phase, so that one knows that the conjecture degree-of-freedom inequality (2.12) is satisfied.

X. A CHIRAL GAUGE THEORY WITH $S\bar{S}$ VECTORLIKE SUBSECTOR

A. Particle content

In this section we construct and study a chiral gauge theory with gauge group $\mathrm{SU}(N)$ and (massless) chiral fermion content such that the irreducibly chiral part of the theory is the same as in our previous two theories, and the vectorlike subsector consists of p copies of fermions in $\{R + \bar{R}\}$ where R is a non-self-conjugate higher-dimensional representation, namely the symmetric rank-2 tensor, S. Explicitly, the chiral fermions include

(1) a symmetric rank-2 tensor representation, S, with corresponding field $\psi^{ab}_{i,L} = \psi^{ba}_{i,L}$, where $i = p+1$,
(2) $N+4$ copies of chiral fermions in the conjugate fundamental representation, $\bar{F}$, with fields $\chi_{a,j,L}$, where $j = 1,\ldots,N+4$, and
(3) p copies of chiral fermions $\{S + \bar{S}\}$ in the symmetric rank-2 tensor and conjugate tensor representations, with fields $\psi^{ab}_{i,L}$ and $\psi_{ab,i,L}$, $i = 1,\ldots,p$.

TABLE VI. Properties of fermions in the $S\bar{S}$ theory with vectorlike subsector consisting of p copies of fermions in the $\{S+\bar{S}\}$ representations. The entries in the columns are (i) fermion, (ii) representation of the $SU(N)$ gauge group, (iii) number of copies, and representations (charges for Abelian factors) of the respective factor groups in the global flavor symmetry group $G_{fl,S\bar{S}}$: (iv) $SU(1+p)_S$; (v) $SU(N+4)_{\bar{F}}$, (vi) $SU(p)_{\bar{S}}$, (vi) $U(1)_1$, (vii) $U(1)_2$. See text for further discussion.

Fermion	SU(N)	No. copies	$SU(1+p)_S$	$SU(N+4)_{\bar{F}}$	$SU(p)_{\bar{S}}$	$U(1)_1$	$U(1)_2$
S: $\psi_L^{ab,i}$	$\square\square$	$1+p$	$\square$	1	1	$N+4$	0
$\bar{F}$: $\chi_{a,j,L}$	$\bar{\square}$	$N+4$	1	$\square$	1	$-(1+p)(N+2)$	$p(N+2)$
$\bar{S}$: $\psi_{ab,k,L}$	$\overline{\square\square}$	p	1	1	$\square$	0	$-(N+4)$

This fermion content is summarized in Table VI. It is again clear that this theory is free of any anomalies in gauged currents. We will refer to this as the $S\bar{S}$ theory.

B. Beta function

The one- and two-loop terms in the beta function of this theory are

$$(b_1)_{S\bar{S}} = 3N - 2 - \frac{2p(N+2)}{3} \tag{10.1}$$

and

$$(b_2)_{S\bar{S}} = \frac{1}{2}(13N^2 - 30N + 1 + 12N^{-1}) - \frac{2p}{3}(8N^2 + 19N - 12N^{-1}). \tag{10.2}$$

The values of $p_{b1z,S\bar{S}}$ and $p_{b2z,S\bar{S}}$ are listed in Table VII. As p increases, the coefficient $(b_1)_{S\bar{S}}$ decreases and passes through zero as p ascends through the value

$$p_{b1z,S\bar{S}} = \frac{3(3N-2)}{2(N+2)}. \tag{10.3}$$

TABLE VII. Values of $p_{b1z,S\bar{S}}$ and $p_{b2z,S\bar{S}}$ in the $S\bar{S}$ theory, as functions of N.

N	$p_{b2z,S\bar{S}}$	$p_{b1z,S\bar{S}}$
3	0.1920	2.1000
4	0.3433	2.5000
5	0.4573	2.7857
6	0.5456	3.0000
7	0.6159	3.1667
8	0.6730	3.3000
9	0.7203	3.4091
10	0.7602	3.5000
20	0.9642	3.9455
25	1.0106	4.0555
50	1.1098	4.2692
10^2	1.1630	4.3824
10^3	1.2131	4.4880
∞	1.21875	4.5000

The asymptotic freedom requirement requires $b_1 > 0$, i.e.,

$$p < \frac{3(3N-2)}{2(N+2)}. \tag{10.4}$$

There are two marginal cases to consider, consisting of values of N and p for which $(b_1)_{S\bar{S}} = 0$, so that one must determine the sign of $(b_2)_{S\bar{S}}$ to see if the theory is asymptotically free. These are the pairs $(N, p) = (6, 3)$ and (22, 4). However, for both of these cases, $(b_2)_{S\bar{S}}$ is negative, so they are excluded by the condition of asymptotic freedom. The upper bound $p_{b1z,S\bar{S}}$, is a monotonically increasing function of N for all physical N, increasing from 2.1 for $N = 3$ and approaching the limiting value 4.5 from below as $N \to \infty$. The resultant physical, integral values of p that are allowed by the inequality (10.4) are

$$\begin{aligned} &p = 0,\ 1,\ 2 \quad \text{if } 3 \le N \le 6, \\ &p = 0,\ 1,\ 2,\ 3 \quad \text{if } 7 \le N \le 22, \\ &p = 0,\ 1,\ 2,\ 3,\ 4 \quad \text{if } N \ge 23. \end{aligned} \tag{10.5}$$

For small p values, $(b_2)_{S\bar{S}}$ is positive, so the two-loop beta function $\beta_{\alpha,2\ell}$ has no IR zero. The coefficient $(b_2)_{S\bar{S}}$ decreases and passes through zero to negative values as p increases through the value

$$(p_{b2z})_{S\bar{S}} = \frac{3(13N^3 - 30N^2 + N + 12)}{4(N+2)(8N^2 + 3N - 6)}. \tag{10.6}$$

This is a monotonically increasing function of N for all physical N, increasing from the value $24/125 = 0.192$ at $N = 3$ and approaching the limiting value $39/32 = 1.21875$ from below as $N \to \infty$. We list the values of $p_{b1z,S\bar{S}}$ and $p_{b2z,S\bar{S}}$ in Table VII. Since $p_{b1z,S\bar{S}} > p_{b2z,S\bar{S}}$, it follows that there is an interval $(I_p)_{S\bar{S}}$ of values of p for which $\beta_{\alpha,2\ell}$ has an IR zero. This zero occurs at $\alpha_{IR,2\ell,S\bar{S}} = -4\pi(b_1)_{S\bar{S}}/(b_2)_{S\bar{S}}$, where these coefficients were given above. As $N \to \infty$, the product $\alpha_{IR,2\ell,S\bar{S}}N$ approaches the same limit as for the *Adj* model, given above in Eq. (5.19).

C. Global flavor symmetry

The classical global chiral flavor symmetry of this theory is

$$G_{fl,cl,S\bar{S}} = \mathrm{U}(1+p)_S \otimes \mathrm{U}(N+4)_{\bar{F}} \otimes \mathrm{U}(p)_{\bar{S}} = \mathrm{SU}(1+p)_S \otimes \mathrm{U}(1)_S \otimes \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{U}(1)_{\bar{F}} \otimes \mathrm{SU}(p)_{\bar{S}} \otimes \mathrm{U}(1)_{\bar{S}}. \tag{10.7}$$

The representations of the various fermion fields under this symmetry are given in Table VI. The corresponding global unitary transformations are

$$\psi^{ab}_{i,L} \to \sum_{j=1}^{1+p} (U_S)_{ij} \psi^{ab}_{j,L}, \quad U_S \in \mathrm{U}(1+p)_S, \tag{10.8}$$

$$\chi_{a,i,L} \to \sum_{j=1}^{N+4} (U_{\bar{F}})_{ij} \chi_{a,j,L}, \quad U_{\bar{F}} \in \mathrm{U}(N+4)_{\bar{F}}, \tag{10.9}$$

and

$$\psi_{ab,i,L} \to \sum_{i=1}^{p} (U_{\bar{S}})_{ij} \psi_{ab,j,L}, \quad U_{\bar{S}} \in \mathrm{U}(p)_{\bar{S}}. \tag{10.10}$$

Each of the three global U(1) symmetries is broken by SU(N) instantons. As before, we define the vector

$$\vec{v} = (N_S T(S), N_{\bar{F}} T(\bar{F}), N_{\bar{S}} T(\bar{S})) = \left[(1+p)\left(\frac{N+2}{2}\right), \frac{N+4}{2}, p\left(\frac{N+2}{2}\right)\right], \tag{10.11}$$

where the basis is $(S, \bar{F}, \bar{S})$, and we have used the values $N_S = 1+p$, $N_{\bar{F}} = N+4$, and $N_{\bar{S}} = p$. In the same manner as before, we can construct two linear combinations of the three original currents that are conserved in the presence of SU(N) instantons. These have charges given by

$$\vec{Q}^{(j)} \equiv (Q^{(j)}_S, Q^{(j)}_{\bar{F}}, Q^{(j)}_{\bar{S}}), \quad j = 1, 2, \tag{10.12}$$

where $j = 1$ for $\mathrm{U}(1)_1$ and $j = 2$ for $\mathrm{U}(1)_2$. Next, we apply the conditions (7.7) and solve for the vectors of charges $\vec{Q}^{(1)}$ and $\vec{Q}^{(2)}$ under the nonanomalous global symmetries $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$. The condition (7.7) does not uniquely determine the vectors $\vec{Q}^{(j)}$, $j = 1, 2$. We choose

$$\vec{Q}^{(1)} = (N+4, -(1+p)(N+2), 0) \tag{10.13}$$

and

$$\vec{Q}^{(2)} = (0, p(N+2), -(N+4)). \tag{10.14}$$

Then the (nonanomalous) global chiral flavor symmetry group of the theory is

$$G_{fl,S\bar{S}} = \mathrm{SU}(1+p)_S \otimes \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{SU}(p)_{\bar{S}} \otimes \mathrm{U}(1)_1 \otimes \mathrm{U}(1)_2. \tag{10.15}$$

For a given N and p that would lead to strong coupling in the infrared, we check if the infrared theory could consist of confined, gauge-singlet massless composite fermions that satisfy the 't Hooft anomaly-matching conditions. The possible gauge-singlet fermions that could, *a priori*, form are described by the wavefunctions

$$B_{ijk} = \bar{F}_{a,i,L} S^{ab}_{j,L} \bar{F}_{b,k,L}, \quad \text{with} \quad 1 \le i, k \le N+4; \quad 1 \le j \le 1+p \tag{10.16}$$

and

$$B'_{ijk} = (\bar{F}^\dagger)^a_{i,L} \bar{S}_{ab,j,L} (\bar{F}^\dagger)^b_{k,L}, \quad \text{with} \quad 1 \le i, k \le N+4, \quad 1 \le j \le p. \tag{10.17}$$

The composite fermion B_{ijk} transforms as a rank-2 antisymmetric tensor of $\mathrm{SU}(N+4)_{\bar{F}}$ and a fundamental representation of $\mathrm{SU}(1+p)_S$. From the analogue of the relation (7.12), its charges under the two global Abelian factor groups in $G_{fl,S\bar{S}}$, $\mathrm{U}(1)_k$, $k = 1, 2$, are

$$Q^{(1)}_B = -N - 2p(N+2) \tag{10.18}$$

and

$$Q^{(2)}_B = 2p(N+2). \tag{10.19}$$

The composite fermion B'_{ijk} transforms as a rank-2 conjugate antisymmetric tensor of $\mathrm{SU}(N+4)_{\bar{F}}$ and a fundamental representation of $\mathrm{SU}(p)_{\bar{S}}$. Its charges under the two global Abelian factor groups in $G_{fl,S\bar{S}}$, $\mathrm{U}(1)_k$, $k = 1, 2$, are determined by the relation

$$Q^{(k)}_{B'} = Q^{(k)}_{\bar{S}} - 2Q^{(k)}_{\bar{F}}, \quad k = 1, 2. \tag{10.20}$$

Hence,

$$Q^{(1)}_{B'} = 2(1+p)(N+2) \tag{10.21}$$

and

$$Q^{(2)}_{B'} = -(N+4) - 2p(N+2). \tag{10.22}$$

We find that a hypothetical low-energy theory with these two massless composite fermions would not satisfy the 't Hooft anomaly-matching conditions for any nonzero value of p. [In the $p=0$ case, the theory degenerates to the $S\bar{F}$ model, for which there is only the one type of composite fermion (7.11), and the dynamics in the strongly coupled case would allow the formation of this massless composite fermion.] As with the *Adj* and AT theories, in the present $S\bar{S}$ theory, if N and p are such that the theory becomes strongly coupled in the infrared, then the resultant fermion condensation in the $S\times\bar{S}\to 1$ channel leaves, as the descendant low-energy effective field theory below the scale of this condensation, the $S\bar{F}$ theory. This is equivalent to the original $S\bar{S}$ theory with $p=0$.

D. UV to IR evolution

In order to investigate the nature of the UV to IR evolution in this $S\bar{S}$ theory, we first note that, again by design, the most attractive channel is

$$S\times\bar{S}\to 1, \tag{10.23}$$

preserving the SU(N) gauge symmetry. This has the attractiveness measure

$$\Delta C_2(S\times\bar{S}\to 1) = 2C_2(S) = \frac{2(N+2)(N-1)}{N}. \tag{10.24}$$

That this is larger than the ΔC_2 for the next-most-attractive channel $S\times\bar{F}\to F$ is clear since

$$\Delta C_2(S\times\bar{S}\to 1) - \Delta C_2(S\times\bar{F}\to F) = C_2(S) > 0. \tag{10.25}$$

Using the rough estimate (2.7), the minimal critical coupling for condensation in the channel (10.23) is

$$\alpha_{cr,S\times\bar{S}} = \frac{\pi N}{3(N+2)(N-1)}. \tag{10.26}$$

In order to get an approximate measure of the size of the coupling at the IR fixed point as compared with the minimum size for condensation, we define the ratio

TABLE VIII. Values of $\alpha_{IR,2\ell,S\bar{S}}$ and $\rho_{IR,S\times\bar{S}}$ for $3\le N\le 8$ in the $S\bar{S}$ theory, and, for each N, the values of $p\in(I_p)_{S\bar{S}}$.

N	p	$\alpha_{IR,2\ell,S\bar{S}}$	$\rho_{IR,S\times\bar{S}}$
3	1	0.684	2.18
3	2	0.0278	0.0885
4	1	0.857	3.68
4	2	0.113	0.486
5	1	0.989	5.29
5	2	0.153	0.819
6	1	1.106	7.04
6	2	0.173	1.10
7	1	1.219	8.98
7	2	0.182	1.34
7	3	0.015	0.111
8	1	1.334	11.15
8	2	0.186	1.55
8	3	0.0245	0.204

$$\rho_{IR,S\times\bar{S}} \equiv \frac{\alpha_{IR,2\ell,S\bar{S}}}{\alpha_{cr,S\times\bar{S}}}, \tag{10.27}$$

depending on N and $p\in(I_p)_{S\bar{S}}$. In Table VIII we list values of this ratio for a range of N and p values.

As $N\to\infty$ (with p fixed [25]), the ratio $\rho_{IR,S\times\bar{S}}$ approaches the same limit as $\rho_{IR,Adj\times Adj}$ in the *Adj* model, namely Eq. (5.29), and the specific values for the allowed range $p=4$, 3, 2 are the same as were given in Eqs. (5.30)–(5.32). Also, the same comments about the likely evolution to various IR phases that were made in the $N\to\infty$ limit there also apply here.

E. Comparison with DFI

Since the theory is asymptotically free and hence weakly coupled in the UV, one can enumerate the perturbative field degrees of freedom, with the result

$$f_{UV,S\bar{S}M} = 2(N^2-1) + \frac{7}{4}\left[(2p+1)\frac{N(N+1)}{2} + (N+4)N\right], \tag{10.28}$$

where here and below, the subscript $S\bar{S}M$ means $S\bar{S}$ model. For values of N and p such that the beta function has an IR zero $\alpha_{IR,2\ell}$ at a value significantly smaller than $\alpha_{cr,S\bar{S}}$, i.e., for which $\rho_{IR,S\bar{S};S\times\bar{S}}$ is well below unity, one expects that the UV to IR evolution of this theory will not involve any spontaneous chiral symmetry breaking but instead will lead to a deconfined non-Abelian Coulomb phase in the infrared. In this case, as for the other two theories discussed above, at the weakly coupled perturbative level, $f_{UV}=f_{IR}$, and the perturbative corrections obey the conjectured degree-of-freedom inequality (2.12).

For values of N and p such that the beta function has no IR zero or an IR zero $\alpha_{IR,2\ell}$ that is moderately large, i.e., for which $\rho_{IR,S\bar{S};S\times\bar{S}} \gtrsim O(1)$, one expects that as the reference scale μ decreases sufficiently, the gauge coupling will become large enough to produce a bilinear fermion condensate, and this condensate is expected to be in the most attractive channel, (10.23). We denote the scale at which this happens as $\Lambda_{S\bar{S}}$. The associated condensate is

$$\langle \psi^{ab\,T}_{i,L} C \psi_{ab,j,L} \rangle. \tag{10.29}$$

A vacuum alignment argument suggests that the dynamics would yield condensates with $i = j$, which would thus take the values $i = j = 1, \ldots, p$, namely

$$\left\langle \sum_{i=1}^{p} \psi^{ab\,T}_{i,L} C \psi_{ab,j,L} \right\rangle. \tag{10.30}$$

The condensate (10.30) preserves an $\mathrm{SO}(p)$ isospin symmetry defined by

$$\psi^{ab}_{i,L} \to \sum_{i=1}^{p} \mathcal{O}_{ij} \psi^{ab}_{j,L},$$
$$\psi_{ab,i,L} \to \sum_{i=1}^{p} \mathcal{O}_{ij} \psi_{ab,j,L}. \tag{10.31}$$

Here the orthogonal transformation $\mathcal{O} \in \mathrm{SO}(p)$ is in 1-1 correspondence with the special case of unitary transformation in $\mathrm{SU}(p+1)_S$ that, furthermore, leaves the $i = p+1$ component of the $(p+1)$-dimensional vector $(\psi^{ab}_{1,L}, \ldots, \psi^{ab}_{p+1,L})^T$ unchanged, and is also a special case of the unitary transformation in $\mathrm{SU}(p)_{\bar{S}}$. Assuming that the condensate takes the form (10.30), this process breaks the initial (nonanomalous) global flavor symmetry group $G_{fl,S\bar{S}}$ to

$$G'_{fl,S\bar{S}} = \mathrm{SO}(p) \otimes \mathrm{SU}(N+4)_{\bar{F}} \otimes \mathrm{U}(1)'. \tag{10.32}$$

Here the $\mathrm{SU}(N+4)$ is (7.3), and the $U(1)'$ is the linear combination of $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ for which the fields $(S, \bar{F}, \bar{S})$ have charges of the form $Q = (a, b, -a)$. The $2p$ chiral fermions involved in the condensate (10.29), namely the S fields $\psi^{ab}_{i,L}$ and the $\bar{S}$ fields $\psi_{ab,i,L}$ with $i = 1, \ldots, p$, gain dynamical masses of order $\Lambda_{S\bar{S}}$. Note that this leaves the $(p+1)$th component $\psi^{ab}_{i,L}$ with $i = p+1$ still massless. It follows that the number of Nambu-Goldboson bosons produced by this spontaneous symmetry breaking of $G_{fl,S\bar{S}}$ to $G'_{fl,S\bar{S}}$ is

$$o(G_{fl,S\bar{S}}) - o(G'_{fl,S\bar{S}}) = \frac{p(3p+5)}{2}. \tag{10.33}$$

In the low-energy effective field theory applicable at scales $\mu \ll \Lambda_{S\bar{S}}$, one integrates out the now-massive S and $\bar{S}$ fermions $\psi^{ab}_{i,L}$ and $\psi_{ab,i,L}$ with $i = 1, \ldots, p$. The resultant global flavor symmetry group describing the massless degrees of freedom in this low-energy effective theory is just that of the $S\bar{F}$ model, $G_{fl,S\bar{F}}$ given in Eq. (8.6).

The further evolution of this $S\bar{F}$ theory into the infrared and the two possibilities of confinement without chiral symmetry breaking or sequential condensate formation in the $S\bar{F} \to F$ channel and associated gauge and global symmetry breaking have been reviewed above. From these we can calculate the resultant IR degrees of freedom and check the degree-of-freedom inequality (2.12).

For the $S \times \bar{S} \to 1$ condensation followed by sequential $S \times \bar{F}$ condensations in the $\mathrm{SU}(N)$ theory, $\mathrm{SU}(N-1)$ theory, etc., we have

$$\begin{aligned} f_{IR,S\bar{S}M;S\times\bar{S},S\times\bar{F}} &= 8N + 1 + \frac{7}{4}\left[\frac{N(N-1)}{2} + 4N\right] \\ &\quad + \frac{p}{2}(3p+5) \\ &= \frac{N(7N+113)}{8} + 1 + \frac{p}{2}(3p+5). \end{aligned} \tag{10.34}$$

Hence,

$$\begin{aligned} &(\Delta f)_{S\bar{S}M;S\times\bar{S},S\times\bar{F}} \equiv f_{UV,S\bar{S}M} - f_{IR,S\bar{S}M;S\times\bar{S},S\times\bar{F}} \\ &= \frac{1}{4}[15N^2 - 25N - 12 + p\{7N(N+1) - 2(3p+5)\}]. \end{aligned} \tag{10.35}$$

This is positive for all N and p values of relevance here. Explicitly, for (non-negative) p, $(\Delta f)_{S\bar{S}M;S\times\bar{S},S\times\bar{F}}$ is positive if

$$\begin{aligned} p < \frac{1}{12}\Big[&7N(N+1) - 10 \\ &+ \sqrt{49N^4 + 98N^3 + 269N^2 - 740N - 188}\Big]. \end{aligned} \tag{10.36}$$

The right-hand side of Eq. (10.36) is greater than the upper bound p_{b1z} allowed by asymptotic freedom. For example, for $N = 3$, the physical, integral values of p are required by asymptotic freedom to be ≤ 2, whereas the right-hand side of (10.36) is 12.95; and for $N = 4$, asymptotic freedom again requires $p \le 2$, whereas the right-hand side of (10.36) is 22.61, and similarly for larger values of N.

For a UV to IR evolution involving $S\bar{S} \to 1$ condensation followed by confinement without spontaneous chiral symmetry breaking, we find

$$f_{IR,S\bar{S}M;S\times\bar{S},sym} = \frac{7}{4}\left[\frac{(N+4)(N+3)}{2}\right] + \frac{p(3p+5)}{2} \tag{10.37}$$

and hence

$$(\Delta f)_{S\bar{S}M;S\times\bar{S},sym} \equiv f_{UV,S\bar{S}M} - f_{IR,S\bar{S}M;S\times\bar{S},sym}$$
$$= \frac{1}{4}[15N^2 + 7N - 50 + p\{7N(N+1) - 2(3p+5)\}]. \tag{10.38}$$

This is positive for all N and p values of relevance here. Explicitly, for (non-negative) p, $(\Delta f)_{S\bar{S}M;S\bar{S},S\bar{F}}$ is positive if

$$p < \frac{1}{12}\Big[7N(N+1) - 12 + \sqrt{49N^4 + 98N^3 + 269N^2 + 28N - 1100}\Big]. \tag{10.39}$$

The right-hand side of Eq. (10.39) is greater than the upper bound p_{b1z} allowed by asymptotic freedom. For example, for $N=3$, $p \le 2$ for asymptotic freedom, while the right-hand side of (10.39) is 13.63; and for $N=4$, again, $p \le 2$ for asymptotic freedom, while the right-hand side of (10.39) is 23.23, and similarly for larger values of N.

In the $N \to \infty$ limit (5.13) (with p fixed), we have

$$\bar{f}_{UV,S\bar{S}} = \frac{37+14p}{8} \tag{10.40}$$

and

$$\bar{f}_{IR,S\bar{S}M;S\times\bar{S},S\times\bar{F}} = \bar{f}_{IR,S\bar{S}M;S\times\bar{S},sym} = \frac{7}{8}, \tag{10.41}$$

and hence

$$(\Delta\bar{f})_{S\bar{S}M;S\times\bar{S},S\times\bar{F}} \equiv \bar{f}_{UV,S\bar{S}M} - \bar{f}_{UV,S\bar{S}M;S\times\bar{S},S\times\bar{F}}$$
$$= (\Delta\bar{f})_{S\bar{S}M;S\times\bar{S},sym} \equiv \bar{f}_{UV,S\bar{S}M} - \bar{f}_{UV,S\bar{S}M;S\times\bar{S},sym}$$
$$= \frac{15+7p}{4}. \tag{10.42}$$

This difference is manifestly positive, in agreement with the conjectured degree-of-freedom inequality (2.12).

XI. CONCLUSIONS

In summary, we have constructed three asymptotically free chiral gauge theories and analyzed their renormalization-group evolution from the ultraviolet to the infrared. These theories have the gauge group SU(N) and massless fermions transforming according to a symmetric rank-2 tensor representation, S, and $N+4$ copies of a conjugate fundamental representation, $\bar{F}$, together with a vectorlike subsector with p copies of fermions in higher-dimensional representation(s). We first studied two theories with the vectorlike fermions in different self-conjugate representations, namely theories with p copies of fermions in (a) the adjoint representation and (b) in the antisymmetric rank-k tensor representation of SU($2k$). We have also studied a third type of theory, with a vectorlike subsector consisting of p pairs of fermions transforming as $\{S+\bar{S}\}$. We have presented results on beta functions, IR zeros of these beta functions, and possible types of UV to IR evolution. In analyzing fermion condensate formation, we have made use of the most-attractive-channel approach. We have shown that for these three types of chiral gauge theories, the various types of likely UV to IR evolution satisfy the conjectured degree-of-freedom inequality (2.12) for all relevant values of N and p. It is hoped that the new chiral gauge theories constructed and analyzed here may serve as useful theoretical laboratories for the study of chiral gauge theories in future work.

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APPENDIX: BETA FUNCTION COEFFICIENTS AND RELEVANT GROUP INVARIANTS

For reference, we list the one-loop and two-loop coefficients [16,17] in the beta function (2.3) for a non-Abelian chiral gauge theory with gauge group G and a set of chiral fermions comprised of N_i fermions transforming according to the representations $\{R_i\}$:

$$b_1 = \frac{1}{3}\left[11C_2(G) - 2\sum_{R_i} N_i T(R_i)\right] \tag{A1}$$

and

$$b_2 = \frac{1}{3}\left[34C_2(G)^2 - 2\sum_{R_i} N_i\{5C_2(G) + 3C_2(R_i)\}T(R_i)\right]. \tag{A2}$$

We list below the group invariants that we use for the relevant case $G = \mathrm{SU}(N)$. We have $C_2(G) = C_2(Adj) = T(Adj) = N$, and, as in the text, we use the symbols F for □ and S for □□. We have

$$C_2(F) = \frac{N^2-1}{2N}, \qquad T(F) = \frac{1}{2}, \tag{A3}$$

$$C_2(S) = \frac{(N+2)(N-1)}{N}, \qquad T(S) = \frac{N+2}{2}, \tag{A4}$$

$$C_2([k]_N) = \frac{k(N+1)(N-k)}{2N}, \tag{A5}$$

and

$$T([k]_N) = \frac{1}{2}\binom{N-2}{k-1}. \tag{A6}$$

Hence, for our case $N = 2k$,

$$C_2([k]_{2k}) = \frac{k(2k+1)}{4} \tag{A7}$$

and

$$T([k]_{2k}) = \frac{(2k-2)!}{2[(k-1)!]^2}. \tag{A8}$$

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[23] The term $(7/4)N_f$ for non-self-conjugate fermions is the sum of a term $(7/8)N_f$ for the fermions and $(7/8)N_f$ for the antifermions; in the case of a self-conjugate, Majorana fermion, these are the same.

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[26] Note that in the line below Eq. (5.3) in [10], there is a misprint; the text should read "satisfying the inequality for $r \le \sqrt{15/14}$," as is obvious from Eq. (4.9) in [10].

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[35] As noted in the text, we confine ourselves here to chiral gauge theories without fundamental scalar fields; there have also been many studies of renormalization-group behavior of chiral gauge theories with both fermion and scalar fields.
[36] Here and elsewhere, when an expression is given for N and/or p that formally evaluates to a nonintegral real value, it is understood implicitly that one infers an appropriate integral value from it, either the greatest integer smaller than the real value, or the least integer greater than the real value, or the closest integer, depending on the context.
[37] We recall that an analogous vacuum alignment argument is used in quantum chromodynamics to infer that among the various $3 \times \bar{3} \to 1$ quark condensates, $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$, and $\langle \bar{s}s \rangle$ form, but others such as $\langle \bar{u}d \rangle$ and $\langle \bar{u}s \rangle$, and $\langle \bar{d}s \rangle$ do not.
[38] Parenthetically, we note that in the *Adj* theory, the anomaly in the global $\mathrm{U}(1)_{R_{sc}}$ symmetry is analogous to the anomaly in the $\mathrm{U}(1)_R$ global symmetry in supersymmetric $\mathrm{SU}(N)$ gauge theory.