

Thermodynamics of higher dimensional topological dilaton black holes with a power-law Maxwell field

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(Received 16 November 2014; published 19 February 2015)

In this paper, we extend the study on the nonlinear power-law Maxwell field to dilaton gravity. We introduce the $(n + 1)$ -dimensional action in which gravity is coupled to a dilaton and power-law nonlinear Maxwell field, and we obtain the field equations by varying the action. We construct a new class of higher dimensional topological black hole solutions of Einstein-dilaton theory coupled to a power-law nonlinear Maxwell field and investigate the effects of the nonlinearity of the Maxwell source as well as the dilaton field on the properties of the spacetime. Interestingly enough, we find that the solutions exist provided one assumes three Liouville-type potentials for the dilaton field, and in the case of the Maxwell field, one of the Liouville potentials vanishes. After studying the physical properties of the solutions, we compute the mass, charge, electric potential and temperature of the topological dilaton black holes. We also study the thermodynamics and thermal stability of the solutions and disclose the effects of the dilaton field and the power-law Maxwell field on the thermodynamics of these black holes. Finally, we comment on the dynamical stability of the obtained solutions in four dimensions.

DOI: [10.1103/PhysRevD.91.044035](https://doi.org/10.1103/PhysRevD.91.044035)

PACS numbers: 04.70.Bw, 04.30.-w, 04.70.Dy

I. INTRODUCTION

At the present epoch, the Universe expands with acceleration instead of deceleration along the scheme of the standard Friedmann model [1]. This fact created much more interest in the alternative theories of gravity in recent years. One of the alternative theories of gravity is dilaton gravity, which can be thought of as the low energy limit of string theory. Indeed, in the low energy limit of string theory, one recovers Einstein gravity along with a scalar dilaton field which is nonminimally coupled to gravity and other fields such as gauge fields [2]. The action of dilaton gravity also contains one or more Liouville-type potentials, which can result from the breaking of spacetime supersymmetry in ten dimensions.

Many attempts have been made to construct exact solutions of Einstein-Maxwell-dilaton (EMd) gravity in the literature. For instance, exact asymptotically flat solutions of EMd gravity with no dilaton potential have been constructed in Refs. [3–6]. But, the asymptotic behavior of the solutions of dilaton gravity with potential may be neither flat nor (anti) de Sitter [(A)dS]. These kinds of solutions which are neither asymptotically flat nor (A)dS are interesting from different points of view. First, it is speculated that the linear dilaton spacetimes which appear as near-horizon limits of the dilatonic black holes may show holography that can be considered as an indication of

the possible extensions of AdS/CFT correspondence [7]. Second, the range of validity of methods and tools originally developed for, and tested in the case of, asymptotically flat or asymptotically AdS black holes may be extended using such solutions. Third, in addition to black holes with spherical horizon, there exist black hole solutions with toroidal or hyperbolic event horizons, as in the case of asymptotically AdS solutions. Having different topologies for the horizon gives rise to different properties for the black holes with drastically different black hole thermodynamics. For instance, it was argued that Schwarzschild-AdS black holes with toroidal or hyperbolic horizons are thermally stable and the Hawking-Page phase transition [8] does not occur [9], while Schwarzschild black holes with spherical horizon are not stable. The topological black holes are studied extensively in many aspects [10–19]. Many authors have explored asymptotically non-flat or non-(A)dS black hole solutions [5,6,10,20–32]. Static charged black hole solutions in the presence of Liouville-type potential, with positive [20], zero or negative constant curvature horizons, [21] have been discovered, and properties of these solutions which are not asymptotically (A)dS have been studied [22]. Also, thermodynamics of $(n + 1)$ -dimensional dilaton black holes with unusual asymptotics have been studied [33,34].

Here, we turn our investigation to dilaton gravity, including the power-law Maxwell field term in the action. This is motivated by the fact that, as in the case of the scalar field which has shown that the particular power of the massless Klein-Gordon Lagrangian shows conformal invariance in arbitrary dimensions [35], one can have a

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conformally electrodynamic Lagrangian in higher dimensions. Although the Maxwell Lagrangian loses its conformally invariant property in higher dimensions, the Lagrangian $[-\exp(-4\alpha\Phi/(n-1))F_{\mu\nu}F^{\mu\nu}]^{(n+1)/4}$ is conformally invariant in $(n+1)$ dimensions. That is, this Lagrangian is invariant under the conformal transformation $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$ and $A_\mu \rightarrow A_\mu$. The studies on the black object solutions coupled to a conformally invariant Maxwell field were carried out in [36,37].

The motivation for studying solutions of Einstein gravity with arbitrary dimensions is based on string theory which predicts spacetime to have more than four dimensions. Although it was thought for a while that the extra spatial dimensions are of the order of the Planck scale, recent theories suggest that if we live on a three-dimensional brane in a higher dimensional bulk, it is possible to have the extra dimensions relatively large and still unobservable [38,39]. All gravitational objects including black holes are higher dimensional in such a scenario. Higher dimensional Ricci flat rotating black branes with a conformally invariant power-Maxwell source in the absence of a dilaton field have been investigated in [40]. Thermodynamics of higher dimensional topological dilaton black holes with a linear Maxwell source have been explored in [10].

In this paper, we would like to construct a new class of $(n+1)$ -dimensional topological black holes of dilaton gravity in the presence of a power-law Maxwell field $[-\exp(-4\alpha\Phi/(n-1))F_{\mu\nu}F^{\mu\nu}]^p$, where we relax the conformally invariant issue for generality. Of course, the solution exists for the case of a conformally invariant source $p = (n+1)/4$. We find that the solution exists provided one assumes three Liouville-type potentials. The interesting point is that one of the Liouville potentials vanishes for the case of the Maxwell field ($p = 1$). We shall investigate the thermal stability of the black holes and explore the effects of nonlinearity of the Maxwell field on the thermodynamics of these black holes.

This paper is structured as follows. In Sec. II, we introduce the action of Einstein-dilaton gravity coupled to a power-law Maxwell field, and by varying the action we obtain the field equations. Then, we construct the exact topological black hole solutions of this theory and investigate their properties. In Sec. III, we obtain the conserved and thermodynamic quantities of the solutions and verify the validity of the first law of black hole thermodynamics. In Sec. IV, we study thermal stability of the solutions in both canonical and grand canonical ensembles. The last section is devoted to conclusions and discussions.

II. FIELD EQUATIONS AND SOLUTIONS

The action of $(n+1)$ -dimensional ($n \geq 3$) Einstein-power Maxwell-dilaton gravity can be written as

$$S = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left\{ \mathcal{R} - \frac{4}{n-1} (\nabla\Phi)^2 - V(\Phi) + (-e^{-4\alpha\Phi/(n-1)} F)^p \right\}, \quad (1)$$

where \mathcal{R} is the Ricci scalar, Φ is the dilaton field, $V(\Phi)$ is a potential for Φ , and p and α are two constants determining the nonlinearity of the electromagnetic field and the strength of coupling of the scalar and electromagnetic fields, respectively. $F = F_{\lambda\eta} F^{\lambda\eta}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and A_μ is the electromagnetic potential. The equations of motion can be obtained by varying the action (1) with respect to the gravitational field $g_{\mu\nu}$, the dilaton field Φ , and the gauge field A_μ , which yields the following field equations:

$$\mathcal{R}_{\mu\nu} = g_{\mu\nu} \left\{ \frac{1}{n-1} V(\Phi) + \frac{(2p-1)}{n-1} (-F e^{-4\alpha\Phi/(n-1)})^p \right\} + \frac{4}{n-1} \partial_\mu \Phi \partial_\nu \Phi + 2p e^{-4\alpha p \Phi/(n-1)} (-F)^{p-1} F_{\mu\lambda} F_\nu{}^\lambda, \quad (2)$$

$$\nabla^2 \Phi - \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{p\alpha}{2} e^{-4\alpha p \Phi/(n-1)} (-F)^p = 0, \quad (3)$$

$$\partial_\mu (\sqrt{-g} e^{-4\alpha p \Phi/(n-1)} (-F)^{p-1} F^{\mu\nu}) = 0. \quad (4)$$

We would like to find the static topological solutions of the above field equations. The most general form of such a metric can be written as

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) h_{ij} dx^i dx^j, \quad (5)$$

where $f(r)$ and $R(r)$ are functions of r which should be determined, and h_{ij} is a function of coordinates x^i which span an $(n-1)$ -dimensional hypersurface with constant scalar curvature $(n-1)(n-2)k$. Here k is a constant, and it characterizes the hypersurface. Without loss of generality, one can take $k = 0, 1, -1$, such that the black hole horizon in (5) can be a zero (flat), positive (spherical), or negative (hyperbolic) constant curvature hypersurface. The Maxwell equation (4) can be integrated immediately to give

$$F_{tr} = \frac{q e^{\frac{4\alpha p \Phi(r)}{(n-1)(2p-1)}}}{(rR)^{\frac{n-1}{2p-1}}}, \quad (6)$$

where q is an integration constant related to the electric charge of the black hole. Substituting (5) and (6) in the field equations (2) and (3), we arrive at

$$f'' + \frac{(n-1)f'}{r} + \frac{(n-1)f'R'}{R} + \frac{2V}{n-1} - \frac{2[1+(n-3)]p}{n-1} \left(2q^2(rR)^{-\frac{2(n-1)}{2p-1}} e^{\frac{4\alpha\Phi}{(n-1)(2p-1)}} \right)^p = 0, \quad (7)$$

$$f'' + \frac{(n-1)f'}{r} + \frac{(n-1)f'R'}{R} + \frac{2V}{n-1} + \frac{4(n-1)fR'}{rR} + \frac{2(n-1)fR''}{R} + \frac{8f\Phi^2}{n-1} - \frac{2[1+(n-3)]p}{n-1} \left(2q^2(rR)^{-\frac{2(n-1)}{2p-1}} e^{\frac{4\alpha\Phi}{(n-1)(2p-1)}} \right)^p = 0, \quad (8)$$

$$\frac{f'}{r} + \frac{f'R'}{R} + \frac{(n-2)f}{r^2} + \frac{2(n-1)fR'}{rR} + \frac{(n-2)R^2f}{R^2} + \frac{fR''}{R} - \frac{k(n-2)}{(rR)^2} + \frac{V}{n-1} + \frac{2p-1}{n-1} \left(2q^2(rR)^{-\frac{2(n-1)}{2p-1}} e^{\frac{4\alpha\Phi}{(n-1)(2p-1)}} \right)^p = 0, \quad (9)$$

$$f\Phi'' + \Phi'f' + \frac{(n-1)f\Phi'}{r} + \frac{(n-1)f\Phi'R'}{R} - \frac{n-1}{8} \frac{dV}{d\Phi} - \frac{p\alpha}{2} \left(2q^2(rR)^{-\frac{2(n-1)}{2p-1}} e^{\frac{4\alpha\Phi}{(n-1)(2p-1)}} \right)^p = 0, \quad (10)$$

where the prime denotes a derivative with respect to r . Our aim here is to construct exact, $(n+1)$ -dimensional topological solutions to the above field equations with an arbitrary dilaton coupling parameter α . Calculations show that there exist exact topological solutions of physical interest, provided we take the dilaton potential with three Liouville-type potentials as

$$V(\Phi) = 2\Lambda_1 e^{2\zeta_1\Phi} + 2\Lambda_2 e^{2\zeta_2\Phi} + 2\Lambda e^{2\zeta_3\Phi}, \quad (11)$$

where $\Lambda_1, \Lambda_2, \Lambda, \zeta_1, \zeta_2$ and ζ_3 are constants. It is important to note that in the case of topological black holes of Emd theory, one only needs to take two terms in the Liouville potential [10], while here we find that for a power-law Maxwell source in dilaton gravity, we need to add an additional term to the potential and consider Liouville-type dilaton potentials with three terms.

In order to solve the system of equations (7)–(10) for three unknown functions $f(r), R(r)$ and $\Phi(r)$, we make the ansatz

$$R(r) = e^{2\alpha\Phi(r)/(n-1)}. \quad (12)$$

Subtracting (7) from (8), after using (12), we find

$$\Phi'' + \frac{2(\alpha^2 + 1)\Phi^2}{\alpha(n-1)} + \frac{2\Phi'}{r} = 0, \quad (13)$$

which has the following solution:

$$\Phi(r) = \frac{(n-1)\alpha}{2(\alpha^2 + 1)} \ln\left(\frac{b}{r}\right). \quad (14)$$

Substituting (12) and (14) into Eqs. (8)–(10), one can easily show that these equations have a unique consistent solution of the form

$$f(r) = \frac{k(n-2)(1+\alpha^2)^2 r^{2\gamma}}{(1-\alpha^2)(\alpha^2+n-2)b^{2\gamma}} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2^p p(1+\alpha^2)^2 (2p-1)^2 b^{-\frac{2(n-2)p\gamma}{(2p-1)}} q^{2p}}{\Pi(n+\alpha^2-2p)r^{-\frac{2[(n-3)p+1]-2p(n-2)\gamma}{2p-1}}} - \frac{2\Lambda b^{2\gamma}(1+\alpha^2)^2 r^{2(1-\gamma)}}{(n-1)(n-\alpha^2)}, \quad (15)$$

where b is an arbitrary nonzero positive constant, $\gamma = \alpha^2/(\alpha^2 + 1)$, $\Pi = \alpha^2 + (n-1-\alpha^2)p$, and the constants should be fixed as

$$\begin{aligned} \zeta_1 &= \frac{2}{(n-1)\alpha}, \\ \zeta_2 &= \frac{2p(n-1+\alpha^2)}{(n-1)(2p-1)\alpha}, \\ \zeta_3 &= \frac{2\alpha}{n-1}, \\ \Lambda_1 &= \frac{k(n-1)(n-2)\alpha^2}{2b^2(\alpha^2-1)}, \\ \Lambda_2 &= \frac{2^{p-1}(2p-1)(p-1)\alpha^2 q^{2p}}{\Pi b^{\frac{2(n-1)p}{2p-1}}}. \end{aligned} \quad (16)$$

It is worth noting that in the linear Maxwell case where $p = 1$, we have $\Lambda_2 = 0$, and hence the potential has two terms. Indeed, the term $2\Lambda_2 e^{2\zeta_2\Phi}$ in the Liouville potential is necessary in order to have solution (15) for the field equations of the power-law Maxwell field in dilaton gravity. Note that Λ remains as a free parameter which plays the role of the cosmological constant, and we assume it to be negative and take it in the standard form $\Lambda = -n(n-1)/2l^2$. The parameter m in Eq. (15) is the integration constant which is known as the geometrical mass and can be written in terms of the horizon radius as

$$\begin{aligned}
m(r_+) &= \frac{k(n-2)b^{-2\gamma}r_+^{\frac{\alpha^2+n-2}{\alpha^2+1}}}{(2\gamma-1)(\gamma-1)(\alpha^2+n-2)} \\
&+ \frac{2^p p(2p-1)^2 b^{\frac{-2(n-2)\gamma p}{(2p-1)}} q^{2p} r_+^{\frac{\alpha^2-2p+n}{(2p-1)(\alpha^2+1)}}}{(\gamma-1)^2(\alpha^2-2p+n)\Pi} \\
&+ \frac{b^{2\gamma} n r_+^{\frac{\alpha^2-n}{\alpha^2+1}}}{l^2(\gamma-1)^2(n-\alpha^2)}, \tag{17}
\end{aligned}$$

where r_+ is the positive real root of $f(r_+) = 0$. In the limiting case where $p = 1$, solution (15) reduces to the topological dilaton black holes of EMd gravity presented in Ref. [10]. One may note that in the absence of a nontrivial dilaton ($\alpha = \gamma = 0$) for a linear Maxwell theory ($p = 1$), solution (15) reduces to

$$f(r) = k - \frac{m}{r^{n-2}} + \frac{2q^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2\Lambda}{n(n-1)}r^2, \tag{18}$$

which describes an $(n+1)$ -dimensional asymptotically AdS topological black hole with a positive, zero or negative constant curvature hypersurface (see, for example, Refs. [13,14]). One can easily show that the gauge potential A_t corresponding to the electromagnetic field (6) can be written as

$$A_t = \frac{qb^{\frac{(2p+1-n)\gamma}{(2p-1)}}}{\Upsilon r^\Upsilon}, \tag{19}$$

where $\Upsilon = (n-2p+\alpha^2)/[(2p-1)(1+\alpha^2)]$. Let us discuss the range of parameters p and α for which our obtained solutions have reasonable behavior and are physically more interesting. There are two restrictions on p and α . The first one is due to the fact that the electric potential A_t should have a finite value at infinity. This leads to $\Upsilon > 0$:

$$\frac{n-2p+\alpha^2}{(2p-1)(1+\alpha^2)} > 0. \tag{20}$$

The above equation leads to the following restriction on the range of p :

$$\frac{1}{2} < p < \frac{n+\alpha^2}{2}. \tag{21}$$

The second restriction comes from the fact that the term including m in spatial infinity should vanish. This fact leads to the following restriction on α :

$$\alpha^2 < n-2. \tag{22}$$

Thus, one can summarize (21) and (22) as follows:

$$\text{for } \frac{1}{2} < p < \frac{n}{2}, \quad 0 \leq \alpha^2 < n-2, \tag{23}$$

$$\text{for } \frac{n}{2} < p < n-1, \quad 2p-n < \alpha^2 < n-2. \tag{24}$$

It is worth mentioning that in the above ranges the dilaton potential $V(\Phi)$ has a lower finite limit in the range $1/2 < p < 1$, where $\Lambda_2 < 0$, and therefore the system is stable. One can also easily see that in the above ranges, as in the special cases of $p = 1$ or $\alpha = 0$, the term including q in $f(r)$ vanishes at spacial infinity, as one expects. In addition, one may note that in the allowed ranges of p and α , Π is always positive, and therefore the q term in $f(r)$ is always positive.

Next, we study the physical properties of the solutions. First, we investigate the asymptotic behavior of the solutions. For $\alpha < 1$, the last term in $f(r)$ is dominant at infinity. Thus, in order to have a positive value for $f(r)$ at infinity, $\Lambda < 0$. On the other hand, for $\alpha > 1$, the first term in the metric function is dominant at large r , and therefore k should be equal to -1 or zero. It is notable to mention that these solutions do not exist for the string case where $\alpha = 1$ in the $k = \pm 1$ cases. Also, it is worth mentioning that the solution is well defined in the allowed ranges of α and p . Next, we look for the curvature singularities. The Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r = 0$, it is finite for $r \neq 0$, and it goes to zero as $r \rightarrow \infty$. Thus, there is an essential singularity located at $r = 0$.

As we mentioned, the charge term is positive everywhere, and since the dominant term is the charge term as r goes to zero, the singularity is timelike as in the case of Reissner-Nordstrom black holes. Thus, one cannot have a Schwarzschild-type black hole solution with one event horizon. In order to consider the type of singularity whether it is naked or not, we calculate the Hawking temperature of the topological black holes. The Hawking temperature can be written as

$$\begin{aligned}
T_+ &= \frac{f'(r_+)}{4\pi} = \frac{(1+\alpha^2)}{4\pi} \left\{ \frac{k(n-2)}{b^{2\gamma}(1-\alpha^2)r_+^{1-2\gamma}} \right. \\
&\quad \left. - \frac{\Lambda b^{2\gamma}r_+^{1-2\gamma}}{n-1} - \frac{2^p p(2p-1)b^{\frac{-2(n-2)\gamma p}{(2p-1)}} q^{2p}}{\Pi r_+^{\frac{2p(n-2)(1-\gamma)+1}{2p-1}}} \right\}.
\end{aligned}$$

Extreme black holes occur when r_+ or q are chosen such that $T_+ = 0$. Using (25), one can find that

$$\begin{aligned}
q_{\text{ext}}^{2p} &= \frac{b^{\frac{2p(n-2)\gamma}{(2p-1)}} \Pi}{p(2p-1)2^{2p}} r_{\text{ext}}^{\frac{2p(n(1-\gamma)-1)+2\gamma}{2p-1}} \\
&\quad \times \left[\frac{nb^{2\gamma}}{l^2} + \frac{k(n-2)}{(1-\alpha^2)b^{2\gamma}} r_{\text{ext}}^{4\gamma-2} \right], \tag{25}
\end{aligned}$$

which, for the case of $\alpha = 0$, reduces to

$$(q_{\text{ext}}^{2p})_{\alpha=0} = \frac{p(n-1)r_{\text{ext}}^{\frac{2(n-1)p}{2p-1}}}{2^p p(2p-1)} \left(\frac{n}{l^2} + \frac{k(n-2)}{r_{\text{ext}}^2} \right).$$

Thus, our solutions present black holes with inner and outer horizons located at r_- and r_+ provided $q < q_{\text{ext}}$, an extreme black hole if $q = q_{\text{ext}}$, and a naked singularity provided $q > q_{\text{ext}}$ (see Figs. 1–3).

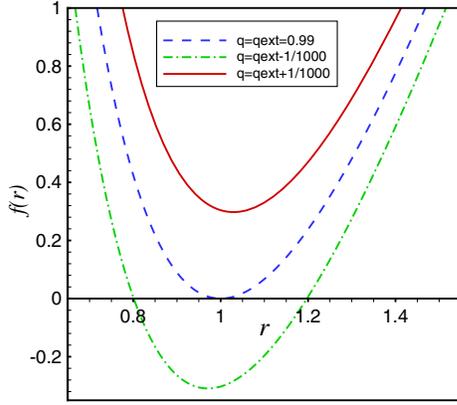


FIG. 1 (color online). The function $f(r)$ versus r for $n = 4$, $\alpha = 0.5$, $p = 2$, $l = b = 1$, $k = 0$ and $r_{\text{ext}} = 1$.

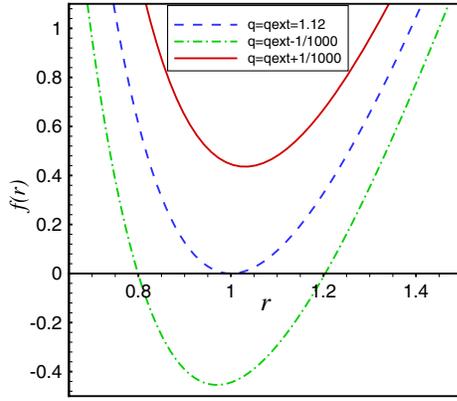


FIG. 2 (color online). The function $f(r)$ versus r for $n = 4$, $\alpha = 0.5$, $p = 2$, $l = b = 1$, $k = 1$ and $r_{\text{ext}} = 1$.

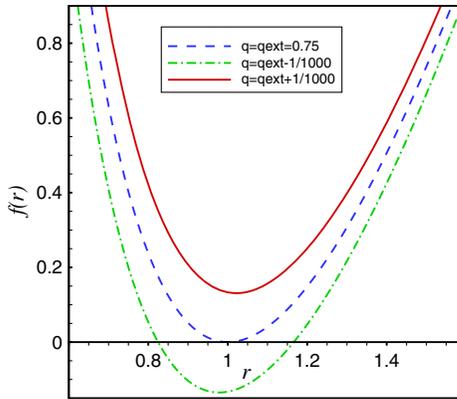


FIG. 3 (color online). The function $f(r)$ versus r for $n = 4$, $\alpha = 0.5$, $p = 2$, $l = b = 1$, $k = -1$ and $r_{\text{ext}} = 1$.

III. THERMODYNAMICS OF TOPOLOGICAL BLACK HOLES

Since discussions on the black hole thermodynamics depend on the mass of the solutions, we first calculate the mass of the dilaton black holes using the modified subtraction method of Brown and York (BY) [41]. In order to use the modified BY method [42], the metric should be written in the form

$$ds^2 = -X(\mathcal{R})dt^2 + \frac{d\mathcal{R}^2}{Y(\mathcal{R})} + \mathcal{R}^2 d\Omega^2. \quad (26)$$

To do this, we perform the following transformation [43]:

$$\mathcal{R} = rR(r).$$

It is a matter of calculations to show that the metric (5) may be written as (26) with the following X and Y :

$$X(\mathcal{R}) = f(r(\mathcal{R})),$$

$$Y(\mathcal{R}) = f(r(\mathcal{R})) \left(\frac{d\mathcal{R}}{dr} \right)^2 = \left(\frac{b}{r} \right)^{2\gamma} (1-\gamma)^2 f(r(\mathcal{R})).$$

The background metric is chosen to be the metric (26) with

$$\begin{aligned} X_0(\mathcal{R}) &= f_0(r(\mathcal{R})) \\ &= \frac{k(n-2)r^{2\gamma}}{(1-2\gamma)(1-\gamma)(\alpha^2+n-2)b^{2\gamma}} \\ &\quad + \frac{2\Lambda b^{2\gamma} r^{2(1-\gamma)}}{(n-1)(1-\gamma)^2(\alpha^2-n)}, \end{aligned} \quad (27)$$

$$Y_0(\mathcal{R}) = \frac{k(1-\gamma)(n-2)}{(1-2\gamma)(\alpha^2+n-2)} + \frac{2\Lambda b^{4\gamma} r^{2(1-2\gamma)}}{(n-1)(\alpha^2-n)}. \quad (28)$$

To compute the conserved mass of the spacetime, we choose a timelike Killing vector field ξ on the boundary surface \mathcal{B} of the spacetime (26). Then, the quasilocal conserved mass can be written as

$$\begin{aligned} M &= \frac{1}{8\pi} \int_{\mathcal{B}} d^2\varphi \sqrt{\sigma} \{ (K_{ab} - K h_{ab}) \\ &\quad - (K_{ab}^0 - K^0 h_{ab}^0) \} n^a \xi^b, \end{aligned} \quad (29)$$

where σ is the determinant of the metric of the boundary \mathcal{B} , K_{ab}^0 is the extrinsic curvature of the background metric, and n^a is the timelike unit normal vector to the boundary \mathcal{B} . Thus, using the above modified BY formalism and denoting the volume of constant curvature hypersurface $h_{ij} dx^i dx^j$ by ω_{n-1} , one can calculate the mass of the black hole per unit volume ω_{n-1} as

$$M = \frac{b^{(n-1)\gamma}(n-1)}{16\pi(\alpha^2+1)} m. \quad (30)$$

In the following, we are going to explore thermodynamics of the topological dilaton black hole we have just found. The entropy of the topological black hole typically satisfies the so-called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [44]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [45]. It is a matter of calculation to show that the entropy of the topological black hole per unit volume ω_{n-1} is

$$S = \frac{b^{(n-1)\gamma} r_+^{(n-1)(1-\gamma)}}{4}. \quad (31)$$

Using (4), the electric charge can be calculated through the Gauss law

$$Q = \frac{1}{4\pi} \int e^{-\frac{4\alpha p \Phi(r)}{n-1}} (rR)^{n-1} (-F)^{p-1} F_{\mu\nu} n^\mu u^\nu d\Omega, \quad (32)$$

where n^μ and u^ν are the unit spacelike and timelike normals to a sphere of radius r given as

$$n^\mu = \frac{1}{\sqrt{-g_{tt}}} dt = \frac{1}{\sqrt{f(r)}} dt,$$

$$u^\nu = \frac{1}{\sqrt{g_{rr}}} dr = \sqrt{f(r)} dr.$$

Using (32), we obtain

$$Q = \frac{\tilde{q}}{4\pi}, \quad (33)$$

as the charge per unit volume ω_{n-1} , where

$$\tilde{q} = 2^{p-1} q^{2p-1}.$$

One may note that $\tilde{q} = q$ for $p = 1$. The electric potential U , measured at infinity with respect to the horizon, is defined by

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}, \quad (34)$$

where $\chi = C\partial_t$ is the null generator of the horizon. Therefore, using (19) the electric potential may be obtained as

$$U = \frac{Cqb \frac{(2p-n+1)\gamma}{(2p-1)}}{\Upsilon r_+^\Upsilon}. \quad (35)$$

Now, we are in the position to explore the first law of thermodynamics for the topological dilaton black holes. In order to do this, we obtain the mass M as a function of extensive quantities S and Q . Using the expression for the charge, the mass and the entropy given in Eqs. (30), (31),

(33), and the fact that $f(r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S, Q) = (1 + \alpha^2) \left\{ -\frac{b\alpha^2 \Lambda (4S)^{\frac{n-\alpha^2}{n-1}}}{8\pi(n-\alpha^2)} \right. \\ + \frac{k(n-1)(n-2)(4S)^{\frac{\alpha^2+n-2}{n-1}}}{16\pi b\alpha^2(\alpha^2+n-2)(1-\alpha^2)} \\ + \frac{(2p-1)^2 p(n-1)}{2\Pi(\alpha^2-2p+n)} \left(\frac{\pi b\alpha^2}{2^{p-3}} \right)^{\frac{1}{2p-1}} \\ \left. \times Q^{\frac{2p}{2p-1}} (4S)^{-\frac{\alpha^2-2p+n}{(2p-1)(n-1)}} \right\}. \quad (36)$$

One may then regard the parameters S and Q as a complete set of extensive parameters for the mass $M(S, Q)$ and define the intensive parameters conjugate to S and Q . These quantities are the temperature and the electric potential

$$T = \left(\frac{\partial M}{\partial S} \right)_Q, \quad U = \left(\frac{\partial M}{\partial Q} \right)_S, \quad (37)$$

provided C is chosen as $C = (n-1)p^2/\Pi$. It is notable to mention that $C = 1$ in the case of the linear Maxwell field [10]. Calculations show that the intensive quantities calculated by Eq. (37) coincide with Eqs. (25) and (35) as the temperature and electric potential. Thus, these thermodynamics quantities satisfy the first law of thermodynamics

$$dM = TdS + UdQ. \quad (38)$$

IV. STABILITY IN THE CANONICAL AND GRAND-CANONICAL ENSEMBLE

Finally, we study thermal stability of the topological dilaton black holes. The stability of a thermodynamic system with respect to small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy $S(M, Q)$ or its Legendre transformation $M(S, Q)$ around the equilibrium. The local stability in any ensemble requires that the energy $M(S, Q)$ be a convex function of its extensive variable [46,47]. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter, and therefore, the positivity of the heat capacity $C_v = T/(\partial^2 M/\partial S^2)_Q$ is sufficient to ensure the local stability. Hence, in the ranges where T is positive, the positivity of $(\partial^2 M/\partial S^2)_Q$ guarantees the local stability of the solutions. For the spacetime under consideration, we find

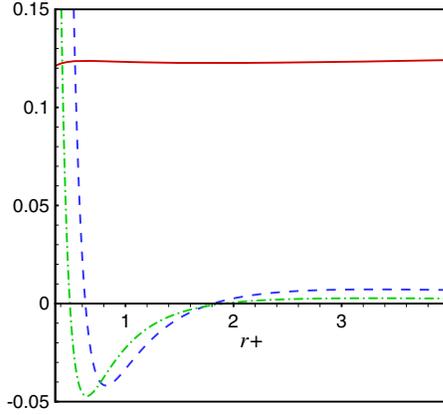


FIG. 7 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M / \partial S^2)_Q$ (dashed curve) and $10^{-2}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus r_+ for $k=1$ with $l=b=1$, $q=0.4$, $\alpha=0.8$, $n=4$ and $p=2$.

shows that there exists a minimum value p_{\min} for which black holes are stable, provided $p > p_{\min}$.

- (ii) $k=1$: In this case, from Fig. 7 we see that the Hawking-Page phase transition occurs between small and large black holes. Choosing $q < (q_{\text{ext}})_{\alpha=0}$, one ensures that $T > 0$ for the allowed region $\alpha < 1$. Again, there is a maximum value for α such that black holes are stable for $\alpha < \alpha_{\text{max}}$ (see Fig. 8). The effects of p in the range $1/2 < p < n/2$ on the stability in both canonical and grand canonical ensembles are shown in Fig. 9. One can see that for $p > p_{\min}$, these solutions represent stable black holes. Numerical calculations show that for $n/2 < p < n-1$, where α has a p -dependent lower limit, there is also a minimum value p_{\min} that, for values greater than p_{\min} , the black holes are stable.
- (iii) $k=-1$: As in the case of $k=0$, the stability of black holes in both canonical and grand canonical

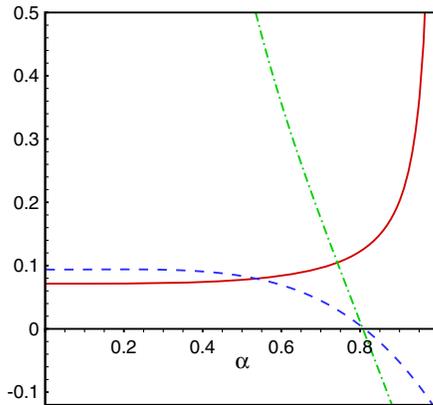


FIG. 8 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M / \partial S^2)_Q$ (dashed curve) and $10^{-1}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus α for $k=1$ with $l=b=1$, $q=0.45$, $r_+=2$, $n=4$ and $p=2$.

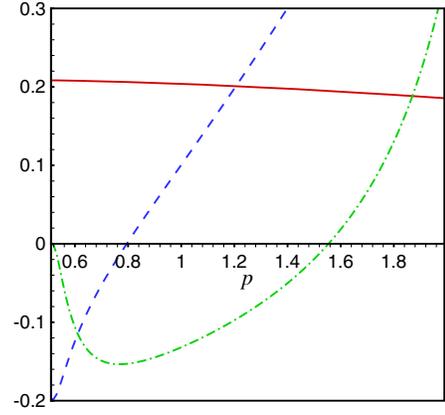


FIG. 9 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M / \partial S^2)_Q$ (dashed curve) and $10^{-1}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus p for $k=1$ with $l=b=1$, $q=0.8$, $r_+=1.1$, $\alpha=0.9$ and $n=4$.

ensembles should be investigated separately for the $\alpha < 1$ and $\alpha > 1$ cases. As one can see from Eq. (39), $(\partial^2 M / \partial S^2)_Q$ is positive for $\alpha < 1$. Therefore, for $q < q_{\text{ext}}$, where black holes exist, they are stable. However, in the grand canonical ensemble, black holes may be unstable in the range $2p-1 < \alpha^2 \leq 1$. For $\alpha > 1$, as one can see in Fig. 10, the black hole solutions are stable, provided $\alpha < \alpha_{\text{max}}$. Of course, the value of α_{max} depends on the ensemble. Figure 11 shows a Hawking-Page phase transition between small and large black holes. In order to investigate the effect of p on the stability of the solutions, we plot both $(\partial^2 M / \partial S^2)_Q$ and the determinant of the Hessian matrix versus p in the range $1/2 < p < n/2$. This is plotted in Fig. 12, which shows that the black hole solutions are stable, provided $p > p_{\min}$. Numerical calculations also show that for $n/2 < p < n-1$,

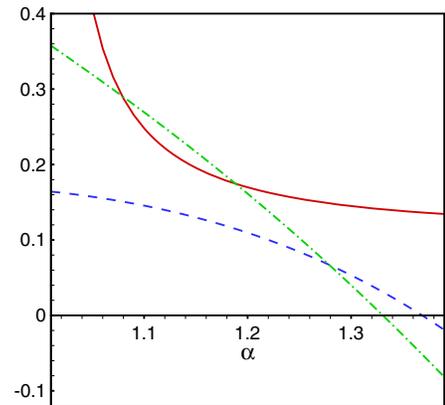


FIG. 10 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M / \partial S^2)_Q$ (dashed curve) and $10^{-1}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus α for $k=-1$ with $l=b=1$, $q=0.45$, $r_+=2$, $n=4$ and $p=2$.

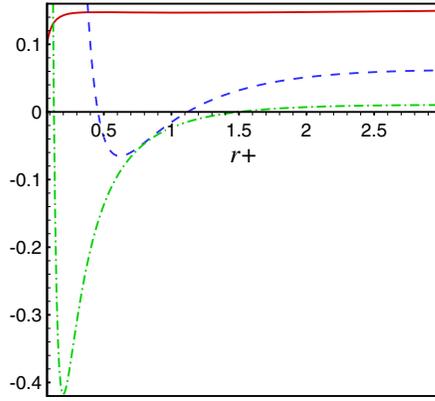


FIG. 11 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M/\partial S^2)_Q$ (dashed curve) and $10^{-2}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus r_+ for $k = -1$ with $l = b = 1$, $q = 0.4$, $\alpha = 1.28$, $n = 4$ and $p = 2$.

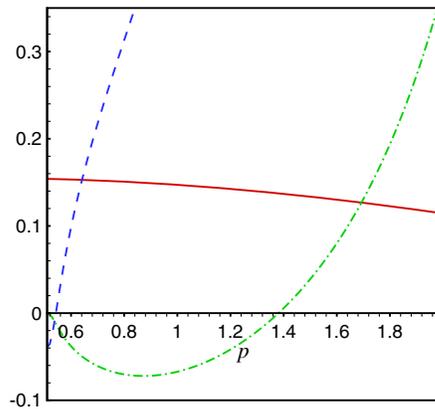


FIG. 12 (color online). The behavior of $10^{-1}T$ (solid curve), $(\partial^2 M/\partial S^2)_Q$ (dashed curve) and $10^{-1}\mathbf{H}_{S,Q}^M$ (dash-dot curve) versus p for $k = -1$ with $l = b = 1$, $q = 0.8$, $r_+ = 1.1$, $\alpha = 1.25$, $n = 4$.

where α has a p -dependent lower limit, black holes are stable.

V. DYNAMICAL STABILITY OF FOUR-DIMENSIONAL BLACK HOLE SOLUTIONS

Besides thermal stability, it is worthwhile to study the dynamical stability of solutions under perturbations. Since the study of dynamical stability for higher-dimensional topological solutions is difficult in general, we study the case of four-dimensional black holes. Regge and Wheeler showed that in four-dimensional static and spherically symmetric background, perturbations can be decomposed into odd- and even-parity sectors according to their transformation properties under a two-dimensional rotation [48]. Perturbations can also be decomposed into a sum of spherical harmonics Y_m^ℓ . In the Regge-Wheeler formalism,

stability is investigated by studying the behavior of perturbation modes.

Using the formalism mentioned, we show that in the framework of scalar-tensor gravity models with a general form of the action [49],

$$S = \int d^4x \sqrt{-g} [G(\Phi, Z)\mathcal{R} + K(\Phi, Z) + G_Z[(\square\Phi)^2 - (\nabla_\mu\nabla_\nu\Phi)^2]], \quad (41)$$

where G and K are arbitrary functions of Φ and $Z = -(\nabla\Phi)^2/2$ and $G_Z = \partial G/\partial Z$, there are dynamically stable solutions under odd-type perturbations, provided

$$\mathcal{F} := 2G > 0, \quad \mathcal{G} := 2G - 4ZG_Z > 0, \quad (42)$$

when the single-mode propagates radially with the squared speed of $c_r^2 = \mathcal{G}/\mathcal{F}$. In the case of uncharged solutions, Eq. (41) matches our action (1), provided $G = 1$ and $K = Z - V(\Phi)$. It is obvious from (42) that in this case our solutions are dynamically stable. Under even-type perturbations, there is again a single mode in our case that propagates with the same radial speed as odd-type perturbations [50]. In this case we have stable solutions, provided $\ell \geq 2$.

Dynamical stability of nonlinear electrodynamics (NED) sources in general relativity is studied in [51]. In the case of a NED Lagrangian $\mathcal{L}(\hat{F})$ where $\hat{F} = 1/4F$, the corresponding Hamiltonian can be defined as $\mathcal{H} \equiv 2\mathcal{L}_{\hat{F}}\hat{F} - \mathcal{L}$. It is also convenient to study the stability using the so-called P frame, where $P = \mathcal{L}_{\hat{F}}^2\hat{F}$. For odd-type perturbations, there are stable solutions, provided \mathcal{H}_P does not vanish anywhere outside the horizon, while for even-type ones, we encounter instability, provided $\mathcal{H}_{xx} > 0$ where $x = \sqrt{-2Q^2P}$. In our case with $\alpha = 0$, one can calculate

$$\mathcal{H}_P = \frac{1}{p} \left(\frac{-4P}{p^2} \right)^{\frac{1-p}{2p-1}}, \quad (43)$$

where $P = -p^2(2F_{rr}^2)^{2p-1}/4$. Obviously, \mathcal{H}_P does not vanish anywhere outside the horizon, and therefore, under odd-type perturbations we have stable solutions. Since

$$\mathcal{H}_{xx} = \frac{x\mathcal{H}_P}{Q^4} \left[1 + \frac{\sqrt{2Q^2}(p-1)}{2p-1} (-P)^{3/2} \right], \quad (44)$$

one encounters dynamically unstable solutions for $p \geq 1$.

VI. SUMMARY AND CONCLUSIONS

To sum up, we generalized the investigations on the power-law Maxwell field to dilaton gravity. We first proposed the suitable Lagrangian in Einstein-dilaton gravity in the presence of a power-law Maxwell Lagrangian

which is coupled to the dilaton field as $[-\exp(-4\alpha\Phi/(n-1))F_{\mu\nu}F^{\mu\nu}]^p$. Then, we constructed a new class of $(n+1)$ -dimensional ($n \geq 3$) topological black hole solutions of this theory in the presence of Liouville-type potentials for the dilaton field. In contrast to the topological black holes of EMD gravity [10] which exists for the Liouville-type potentials with two terms, here we found that the solutions exist provided we assume three Liouville-type potentials for the dilaton field. In the limiting case where $p = 1$, one of the Liouville potentials vanishes. Because of the presence of the dilaton field, the obtained solutions are neither asymptotically flat nor (A)dS. Besides, for the cases of $k = \pm 1$ the solutions do not exist for the string case where $\alpha = 1$. When $p = 1$, all results of the topological black holes of EMD gravity are recovered [10].

These facts that (i) the gauge potential of electromagnetic field A_t is finite, (ii) the terms contain mass in the metric function (15) should be disappeared in the spacial infinity, imply that the parameters p and α should be restricted as follows. For $1/2 < p < n/2$, we should have $\alpha^2 < n - 2$, while for $n/2 < p < n - 1$, we should have $2p - n < \alpha^2 < n - 2$. For these allowed ranges, the dilaton potential $V(\Phi)$ has always a finite lower limit. Requiring the fact that our solutions should be positive in spacial infinity leads to another restriction on α in the case of $k = 1$, namely, $\alpha < 1$. Our solutions are well defined in the permitted ranges of p and α , while in the case of a linear Maxwell field, the solutions are ill defined for $\alpha = \sqrt{n}$. We showed that our solution cannot represent black holes with a single event horizon. However, they can represent black holes with two horizons, extreme black holes and naked singularity depending on the model parameters. We also calculated the charge, mass, temperature, entropy and electric potential of the topological dilaton black holes and found that the first law of thermodynamics is satisfied

on the black hole horizon. By calculating the Smarr-type formula, $M(S, Q)$, we analyzed thermal stability of the solutions in both canonical and grand canonical ensembles. We showed that for $\alpha < 1$, there are stable black holes in the cases of $k = 0, -1$ provided $q < q_{\text{ext}}$ in the canonical ensemble, whereas in the grand canonical ensemble, black holes may be unstable in the range of $2p - 1 < \alpha^2 \leq 1$. In the cases of (i) $k = 0, -1$, and $\alpha > 1$ and (ii) $k = 1$ and $\alpha < 1$, there is a maximum value for the dilaton coupling constant α_{max} for which the obtained solutions are thermally unstable, provided $\alpha > \alpha_{\text{max}}$. For $k = -1, \alpha > 1$ and $k = 1, \alpha < 1$, there is a Hawking-Page phase transition between small and large black holes, while for $k = 0, \alpha > 1$, large black holes are unstable. This fact can be understood from Figs. 4, 7 and 11. We also found that there is a p_{min} for which we have stable black holes, provided $p > p_{\text{min}}$. Finally, we discussed the dynamical stability of the obtained solutions in the absence and presence of the nonlinear electromagnetic source, separately.

Note that the $(n+1)$ -dimensional charged topological dilaton black holes obtained here are static. Thus, it would be interesting if one could construct charged rotating black holes or branes in $(n+1)$ dimensions in the presence of the dilaton and the power-law Maxwell field. These issues are now under investigation, and the results will appear elsewhere.

ACKNOWLEDGMENTS

We thank the referee for constructive comments which helped us improve the paper significantly. We also thank the Research Council of Shiraz University. This work has been financially supported by the Research Institute for Astronomy & Astrophysics of Maragha (RIAAM), Iran.

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- [1] A. G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998); S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999); J. L. Tonry *et al.*, *Astrophys. J.* **594**, 1 (2003); A. T. Lee *et al.*, *Astrophys. J.* **561**, L1 (2001); C. B. Nettereld *et al.*, *Astrophys. J.* **571**, 604 (2002); N. W. Halverson *et al.*, *Astrophys. J.* **568**, 38 (2002); D. N. Spergel *et al.*, *Astrophys. J. Suppl. Ser.* **148**, 175 (2003).
- [2] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).
- [3] G. W. Gibbons and K. Maeda, *Nucl. Phys.* **B298**, 741 (1988); T. Koikawa and M. Yoshimura, *Phys. Lett. B* **189**, 29 (1987); D. Brill and J. Horowitz, *ibid.* **262**, 437 (1991).
- [4] D. Garfinkle, G. T. Horowitz, and A. Strominger, *Phys. Rev. D* **43**, 3140 (1991); R. Gregory and J. A. Harvey, *ibid.* **47**, 2411 (1993); M. Rakhmanov, *ibid.* **50**, 5155 (1994).
- [5] S. Mignemi and D. Wiltshire, *Classical Quantum Gravity* **6**, 987 (1989); D. L. Wiltshire, *Phys. Rev. D* **44**, 1100 (1991); S. Mignemi and D. L. Wiltshire, *ibid.* **46**, 1475 (1992).
- [6] S. J. Poletti and D. L. Wiltshire, *Phys. Rev. D* **50**, 7260 (1994).
- [7] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg, *J. High Energy Phys.* **10** (1998) 004.
- [8] S. W. Hawking and D. N. Page, *Commun. Math. Phys.* **87**, 577 (1983).
- [9] D. Birmingham, *Classical Quantum Gravity* **16**, 1197 (1999).
- [10] A. Sheykhi, *Phys. Rev. D* **76**, 124025 (2007).
- [11] J. P. S. Lemos, *Phys. Lett. B* **353**, 46 (1995).
- [12] R. G. Cai and Y. Z. Zhang, *Phys. Rev. D* **54**, 4891 (1996); L. Vanzo, *Phys. Rev. D* **56**, 6475 (1997).

- [13] D. R. Brill, J. Louko, and P. Peldán, *Phys. Rev. D* **56**, 3600 (1997).
- [14] R. G. Cai and K. S. Soh, *Phys. Rev. D* **59**, 044013 (1999).
- [15] R. G. Cai, J. Y. Ji, and K. S. Soh, *Phys. Rev. D* **57**, 6547 (1998).
- [16] J. Crisóstomo, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **62**, 084013 (2000); R. Aros, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **63**, 084015 (2001); R. G. Cai, *Phys. Rev. D* **65**, 084014 (2002).
- [17] M. H. Dehghani, *Phys. Rev. D* **70**, 064019 (2004).
- [18] J. P. Lemos, *Classical Quantum Gravity* **12**, 1081 (1995); C. G. Huang and C. B. Liang, *Phys. Lett. A* **201**, 27 (1995); R. G. Cai, *Nucl. Phys.* **B524**, 639 (1998); S. Aminneborg, I. Bengtsson, S. Holst, and P. Peldán, *Classical Quantum Gravity* **13**, 2707 (1996); R. B. Mann, *Classical Quantum Gravity* **14**, L109 (1997); *Nucl. Phys.* **B516**, 357 (1998); D. Klemm, *Classical Quantum Gravity* **15**, 3195 (1998); D. Klemm, V. Moretti, and L. Vanzo, *Phys. Rev. D* **57**, 6127 (1998); **60**, 109902(E) (1999).
- [19] M. Bañados, A. Gomberoff, and C. Martínez, *Classical Quantum Gravity* **15**, 3575 (1998); R. G. Cai, *Phys. Rev. D* **65**, 084014 (2002); *Phys. Lett. B* **582**, 237 (2004); W. L. Smith and R. B. Mann, *Phys. Rev. D* **56**, 4942 (1997); Y. Wu, M. F. A. da Silva, N. O. Santos, and A. Wang, *Phys. Rev. D* **68**, 084012 (2003); Y. M. Cho and I. P. Neupane, *Phys. Rev. D* **66**, 024044 (2002); I. P. Neupane, *Phys. Rev. D* **69**, 084011 (2004); C. S. Peca and J. P. S. Lemos, *Phys. Rev. D* **59**, 124007 (1999); *J. Math. Phys. (N.Y.)* **41**, 4783 (2000); C. Martínez, R. Troncoso, and J. P. Staforelli, *Phys. Rev. D* **74**, 044028 (2006).
- [20] K. C. K. Chan, J. H. Horne, and R. B. Mann, *Nucl. Phys.* **B447**, 441 (1995).
- [21] R. G. Cai, J. Y. Ji, and K. S. Soh, *Phys. Rev. D* **57**, 6547 (1998); R. G. Cai and Y. Z. Zhang, *ibid.* **64**, 104015 (2001).
- [22] G. Clement, D. Gal'tsov, and C. Leygnac, *Phys. Rev. D* **67**, 024012 (2003); G. Clement and C. Leygnac, *ibid.* **70**, 084018 (2004).
- [23] T. Ghosh and P. Mitra, *Classical Quantum Gravity* **20**, 1403 (2003).
- [24] A. Sheykhi and N. Riazi, *Int. J. Theor. Phys.* **45**, 2453 (2006).
- [25] A. Sheykhi and N. Riazi, *Int. J. Mod. Phys. A* (unpublished).
- [26] M. H. Dehghani and N. Farhangkhah, *Phys. Rev. D* **71**, 044008 (2005).
- [27] M. H. Dehghani, *Phys. Rev. D* **71**, 064010 (2005).
- [28] A. Sheykhi, M. H. Dehghani, and N. Riazi, *Phys. Rev. D* **75**, 044020 (2007).
- [29] A. Sheykhi, M. H. Dehghani, N. Riazi, and J. Pakravan, *Phys. Rev. D* **74**, 084016 (2006).
- [30] S. S. Yazadjiev, *Phys. Rev. D* **72**, 044006 (2005).
- [31] S. S. Yazadjiev, *Classical Quantum Gravity* **22**, 3875 (2005).
- [32] A. Sheykhi, N. Riazi, and M. H. Mahzoon, *Phys. Rev. D* **74**, 044025 (2006).
- [33] A. Sheykhi and N. Riazi, *Phys. Rev. D* **75**, 024021 (2007).
- [34] M. H. Dehghani, S. H. Hendi, A. Sheykhi, and H. Rastegar Sedehi, *J. Cosmol. Astropart. Phys.* **02** (2007) 020.
- [35] M. Hassaine, *J. Math. Phys. (N.Y.)* **47**, 033101 (2006).
- [36] M. Hassaine and C. Martinez, *Phys. Rev. D* **75**, 027502 (2007); H. A. Gonzalez, M. Hassaine, and C. Martinez, *Phys. Rev. D* **80**, 104008 (2009); S. H. Hendi, *Eur. Phys. J. C* **69**, 281 (2010); *Classical Quantum Gravity* **26**, 225014 (2009); *Phys. Lett. B* **677**, 123 (2009); H. Maeda, M. Hassaine, and C. Martinez, *Phys. Rev. D* **79**, 044012 (2009).
- [37] A. Sheykhi, *Phys. Rev. D* **86**, 024013 (2012); A. Sheykhi and S. H. Hendi, *Phys. Rev. D* **87**, 084015 (2013).
- [38] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999); **83**, 4690 (1999).
- [39] G. Dvali, G. Gabadadze, and M. Porrati, *Phys. Lett. B* **485**, 208 (2000); G. Dvali and G. Gabadadze, *Phys. Rev. D* **63**, 065007 (2001).
- [40] S. H. Hendi and H. R. Rastegar-Sedehi, *Gen. Relativ. Gravit.* **41**, 1355 (2009).
- [41] J. Brown and J. York, *Phys. Rev. D* **47**, 1407 (1993); J. D. Brown, J. Creighton, and R. B. Mann, *Phys. Rev. D* **50**, 6394 (1994).
- [42] S. H. Hendi, A. Sheykhi, and M. H. Dehghani, *Eur. Phys. J. C* **70**, 703 (2010).
- [43] M. H. Dehghani and A. Bazrafshan, *Int. J. Mod. Phys. D* **19**, 293 (2010).
- [44] J. D. Beckenstein, *Phys. Rev. D* **7**, 2333 (1973); S. W. Hawking, *Nature (London)* **248**, 30 (1974); G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2738 (1977).
- [45] C. J. Hunter, *Phys. Rev. D* **59**, 024009 (1998); S. W. Hawking, C. J. Hunter, and D. N. Page, *ibid.* **59**, 044033 (1999); R. B. Mann *ibid.* **60**, 104047 (1999); **61**, 084013 (2000).
- [46] M. Cvetič and S. S. Gubser, *J. High Energy Phys.* **04** (1999) 024; M. M. Caldarelli, G. Cognola, and D. Klemm, *Classical Quantum Gravity* **17**, 399 (2000).
- [47] S. S. Gubser and I. Mitra, *J. High Energy Phys.* **08** (2001) 018.
- [48] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [49] T. Kobayashi, H. Motohashi, and T. Suyama, *Phys. Rev. D* **85**, 084025 (2012).
- [50] T. Kobayashi, H. Motohashi, and T. Suyama, *Phys. Rev. D* **89**, 084042 (2014).
- [51] C. Moreno and O. Sarbach, *Phys. Rev. D* **67**, 024028 (2003).