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Stress-energy tensor of the quantized massive fields in Schwarzschild-Tangherlini spacetimes: The backreaction

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We construct and study the approximate stress-energy tensor of the quantized massive scalar field in higher dimensional Schwarzschild-Tangherlini spacetimes. The stress-energy tensor is calculated within the framework of the Schwinger-DeWitt approach. It is shown that in N-dimensional spacetime the main approximation can be obtained from the effective action constructed form the coincidence limit of the Hadamard-DeWitt coefficient a_k, where $k-1$ is the integer part of $N/2$. The backreaction of the quantized field upon the black hole spacetime is analyzed and the quantum-corrected Komar mass and the Hawking temperature is calculated. It is shown that for the minimal and conformal coupling the increase of the Komar mass of the quantum corrected black hole leads to the decrease of its Hawking temperature. This is not generally true for more exotic values of the coupling parameter. The general formula describing the vacuum polarization, $\langle\phi^2\rangle$, is constructed and briefly examined.

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I. INTRODUCTION

The aim of this paper is to construct and discuss the regularized stress-energy tensor of the quantized massive scalar field in a large mass limit in the spacetime of N-dimensional static and spherically-symmetric Schwarzschild-Tangherlini black hole described by the line element [1]

$$ds^2 = -f^{(0)}(r)dt^2 + \frac{1}{f^{(0)}(r)}dr^2 + r^2 d\Omega^2_{N-2} \tag{1}$$

with

$$f^{(0)}(r) = 1 - \left(\frac{r_+}{r}\right)^{N-3}, \tag{2}$$

where $d\Omega^2_{N-2}$ is a metric on a unit $(N-2)$-dimensional sphere, and, subsequently, analyze its influence on the background geometry via the semiclassical Einstein field equations. One can associate the mass with the solution simply by comparing its asymptotic behavior with the solutions of N-dimensional linearized gravity. The mass, M, of the classical Schwarzschild-Tangherlini black hole is given by

$$M = \frac{\pi^{(N-3)/2}(N-2)}{8\Gamma(\frac{N-1}{2})} r_+^{N-3}, \tag{3}$$

where r_+ is the radial coordinate of the event horizon. The Hawking temperature calculated in the standard way is always inversely proportional to the radius of the event horizon

$$T_H = \frac{N-3}{4\pi r_+}. \tag{4}$$

The quantum-corrected solution is, of course, characterized by a different radius of the event horizon and Eqs. (3) and (4) do not hold.

The Schwarzschild-Tangherlini black holes are classically stable with respect to the linear perturbations. Moreover, it can be demonstrated that there are no static scalar perturbations that is regular everywhere outside the event horizon [2]. The latter means that if such perturbations exist it would be possible to construct an asymptotically flat vacuum black hole solutions with nonspherical event horizons of topology S^{N-2}. The nonexistence of such solutions confirms the uniqueness of the N-dimensional spherically-symmetric static vacuum black holes.

The stress-energy tensor of the quantized field employed in this paper is constructed within a generalized Schwinger-DeWitt framework [3–8]. In this approach one assumes that for sufficiently massive quantized fields the vacuum polarization effects can be separated from the particle creation. Since the vacuum polarization is local and for a given type of field it depends solely on the spacetime geometry, it is possible to construct the general expression describing the one-loop effective action. The stress-energy tensor can be obtained by differentiating the effective action with respect to the metric and the result is a linear combination of the purely geometric terms constructed form curvature. Moreover, as the particle creation is negligible in this regime, the geometric approximation based on the Schwinger-DeWitt method is expected to be quite good. Indeed, extensive numerical analyses carried out in Ref. [9] indicate that for $N = 4$ black holes, the relative error of the approximation is

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below 2%, provided $Mm > 2$. It is a very important result as it explicitly demonstrates the usefulness of the method. The general criterion for applicability of the approximation is that the length scale associated with the quantized field should be much smaller than the characteristic scale of the curvature of the spacetime.

The paper is organized as follows. In Sec. II we construct the general expression describing the stress-energy tensor of the quantized massive scalar fields in a large mass limit in N-dimensional spacetime. Subsequently, the general formulas are used in $N = 4, 5, 6$ and 7-dimensional Schwarzschild-Tangherlini spacetimes. The semiclassical Einstein field equations are investigated in Sec. III, where the backreaction of the quantized fields upon the spacetime metric is examined. Section IV concludes the paper with some final remarks, putting our results in a somewhat broader perspective. Also in that section the field fluctuation, $\langle\phi^2\rangle$, is constructed and briefly examined.

Throughout the paper the natural system of unit is used. The signature of the metric is "mainly positive" $(-,+,\ldots,+)$ and our conventions for curvature are $R^a{}_{bcd} = \partial_c\Gamma^a_{bd}\ldots$ and $R^a{}_{bac} = R_{bc}$.

II. THE STRESS-ENERGY TENSOR

Let us start with the massive scalar field, ϕ, propagating on N-dimensional spacetime, satisfying the covariant Klein-Gordon equation. The associated Green function is the solution of the equation

$$(\Box - m^2 - \xi R)G(x,x') = -\delta(x,x') \equiv -\frac{\delta(x-x')}{|g|^{1/2}}, \tag{5}$$

where m is the mass of the field, ξ is the parameter of the curvature coupling and R is the curvature scalar.

Now, making use of the (formal) definition of the one-loop effective action $W^{(1)}$ in the standard form

$$W^{(1)} = -\frac{i}{2}\ln \mathrm{Tr}G \tag{6}$$

and the Schwinger-DeWitt representation of the Green function

$$G^F(x,x') = \frac{i\Delta^{1/2}}{(4\pi)^{n/2}}\int_0^\infty ids\frac{1}{(is)^{n/2}} \times \exp\left[-im^2 s + \frac{i\sigma(x,x')}{2s}\right]A(x,x';is), \tag{7}$$

expressed in terms of the Hadamard-DeWitt coefficients, $a_k(x,x')$, where Δ is the vanVleck-Morette determinant constructed form the word function σ (a biscalar equal to one half the square of the geodesic distance between x and x') and

$$A(x,x';is) = \sum_{k=0}^{\infty}(is)^k a_k(x,x'), \tag{8}$$

one obtains

$$W^{(1)} = \lim_{x'\to x}\int d^N x(-g)^{1/2}\frac{\Delta^{1/2}}{2(4\pi)^{N/2}} \times \int_0^\infty \frac{ids}{(is)^{N/2+1}}\exp\left[-im^2 s + \frac{i\sigma}{2s}\right]A(x,x';is). \tag{9}$$

Consequently, the effective Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2(4\pi)^{N/2}}\int\frac{ids}{(is)^{N/2+1}}e^{-im^2 s}\sum_{k=0}^{\infty}a_k(is)^k, \tag{10}$$

where a_k is the coincidence limit of $a_k(x,x')$, i.e., $a_k = \lim_{x'\to x}a_k(x,x')$.

Let $\lfloor x\rfloor$ denote the floor function, i.e., it gives the largest integer less than or equal to x. Since the first $\lfloor\frac{N}{2}\rfloor + 1$ terms of the series (counting from the zeroth-term) lead to the divergent integrals, let us substitute A in (7) by its "regularized" counterpart

$$A_{\mathrm{reg}}(x,x';is) = \sum_{k=\lfloor\frac{N}{2}\rfloor+1}^{n'} a_k(x,x';is)(is)^k. \tag{11}$$

The upper limit n' reflects the fact that only a first few Hadamard-DeWitt coefficients are known.

Assuming that m^2 has a small imaginary part ($i\varepsilon$, $\varepsilon < 0$) and integrating over s gives

$$\mathcal{L}_{\mathrm{reg}} = \frac{1}{2(4\pi)^{N/2}}\sum_{k=\lfloor\frac{N}{2}\rfloor+1}^{n'}\frac{a_k}{(m^2)^{k-N/2}}\Gamma\left(k-\frac{N}{2}\right). \tag{12}$$

The (regularized) stress-energy tensor can be calculated from the standard definition

$$T^{ab} = \frac{2}{(-g)^{1/2}}\frac{\delta}{\delta g_{ab}}W^{(1)}_{\mathrm{reg}}, \tag{13}$$

where $W^{(1)}_{\mathrm{reg}}$ is given by

$$W^{(1)}_{\mathrm{reg}} = \int d^N x(-g)^{1/2}\mathcal{L}_{\mathrm{reg}}. \tag{14}$$

This result may be thought of as a generalization of the Frolov-Zel'nikov formula to the N-dimensional case.

In what follows we restrict ourselves to the first-order approximation, i.e., for a given N we retain only the lowest regular term of the expansion (12) and denote resulting Lagrangian density by L^N. Inspection of (12)–(14) shows

that to calculate the approximate stress-energy tensor in the spacetimes of dimension 4 and 5 the coincidence limit of the fourth coefficient, a_3, is needed. Similarly, the coefficient a_4 allows calculations in $N = 6$ and 7, and so on. Unfortunately, the Hadamard-DeWitt coefficients, except for simple geometries with a high degree of symmetry, are very hard to calculate as they are constructed from the differential and algebraic curvature invariants. The differential invariants involve the covariant derivatives of the curvature tensor (and their contractions) up to $(n-2)$-order [8,10–13]. The problem at hand is even more complicated, since what we need is the result of the functional differentiation of the (integrated) coefficient a_k with respect to the metric tensor rather than the coefficient itself. To make things worse, we have to apply the thus obtained formulas in a concrete spacetime, what is usually associated with large-scale calculations.

Before going any further, let us summarize what has been done so far. Here we limit ourselves almost exclusively to literature on the regularized stress-energy tensor calculated within the framework of the Schwinger-DeWitt approximation. Assuming that the Compton length associated with the mass of the field is much less than the gravitational radius of the black hole, Frolov and Zel'nikov [14] constructed the stress-energy tensor of the massive scalar field in the Hartle-Hawking state in the Schwarzschild spacetime. The large mass limit allows separation of the vacuum polarization effects and the final result can be calculated from the (coincidence limit) of the Ricci-flat version of the coefficient a_3. The scalar results have been extended to spin $1/2$ and spin 1 fields in the Kerr spacetime [7,15]. The Forolv-Zel'nikov results (for all mentioned spins) have been generalized to *arbitrary* spacetime in Refs. [16,17]. This has been achieved by constructing the functional derivatives of 10 curvature (algebraic and differential) invariants of the background dimensionality 6 (i.e. having the dimension of length^{-6}) with respect to the metric tensor. In the $N = 4$ case, the resulting stress-energy tensor consists of almost 100 geometric terms constructed from the curvature and metric. The interested reader is referred to Refs. [16,17]. Identical results for the static spherically-symmetric asymptotically-flat geometries have been obtained using different methods in Ref. [18]. The analysis of the functional derivatives of the curvature invariants have been also carried out by Decanini and Folacci in Refs. [19,20]. A natural question that appears in this context is the problem of the quality of the approximation. A detailed numerical study carried out in Ref. [9] indicates that the Schwinger-DeWitt approximation, when employed in its domain of applicability, is reliable.

The stress-energy tensor have been calculated in numerous, physically interesting geometries, such as exterior and interior regions of black holes [21,22], wormholes [23] and cosmology [24,25]. Interesting results have been obtained in the geometries with maximally symmetric subspaces, such as the Bertotti-Robinson solution [26–28]. Recently, there is a growing interest in the higher dimensional calculations, (see e.g., [29,30] and the references cited therein), that reflects the view that the physical world has more than the familiar four dimensions.

Now, let us return to our main problem. To construct the first-order approximation to the stress-energy tensor one has to calculate the variational derivatives of the effective action expressed in terms of the coincidence limit of the heat kernel coefficients for arbitrary dimensions. Here we shall limit ourselves to coefficients a_3 and a_4. Using FORM, which is particularly suited for large scale calculations [31–33], we have constructed the coincidence limit of the coefficients a_3 and a_4 and subsequently the functional derivatives of the effective action with respect to the metric tensor. After some simplifications we have obtained the general expressions (stored in FORM format) describing the stress-energy tensor of the quantized massive scalar field in $N = 4, 5, 6$ and 7-dimensional geometries, respectively. Unfortunately, the general results are very complicated, and, except for the geometries with a high degree of symmetry, hard to use.

In light of the foregoing discussion, to shorten the presentation and minimize efforts, here we will follow a less general approach.[1] The static spherically symmetric solution of the Einstein field equations, written in the standard curvature coordinates, has the form

$$ds^2 = g_{00}(r)dt^2 + g_{11}(r)dr^2 + r^2 d\Omega^2_{N-2}, \tag{15}$$

where $d\Omega^2_{N-2}$ is the line element on a unit sphere S^{N-2}. To simplify notation, let us introduce two functions $f(r)$ and $h(r)$ defined as $f(r) = g_{00}(r)$, and $h(r) = g_{11}(r)$, respectively. Calculating the Hadamard-DeWitt coefficient for the line element one obtains the Lagrangian density, L^N, which can be schematically written in the form

$$L^N = \mathcal{L}^N(f(r), \ldots, f^{(i_N)}(r), h(r), \ldots, h^{(j_N)}(r), r)\sqrt{g_{S_{N-2}}}, \tag{16}$$

where and $g_{S_{N-2}}$ is the determinant of the metric tensor on a unit S_{N-2} sphere, $f^{(k)}$ and $h^{(k)}$ denote a kth derivative of $f(r)$ and $h(r)$, respectively. Note that the numerical coefficient, the mass and the factor $\sqrt{f(r)h(r)}$ have been absorbed into the definition of $\mathcal{L}^N$. Now the stress-energy tensor can be obtained from the Euler-Lagrange equations

[1] It should be noted however, that all calculations of the stress-energy tensor presented in this paper have been checked using this more general approach.

$$T_t^{(N)t} = 2\left(\frac{f}{h}\right)^{1/2}\left[\frac{\partial}{\partial f}\mathcal{L}^N + \sum_{k=1}^{p(N)}(-1)^k\frac{d^k}{dr^k}\left(\frac{\partial}{\partial f^{(k)}}\mathcal{L}^N\right)\right] \tag{17}$$

and

$$T_r^{(N)r} = 2\left(\frac{h}{f}\right)^{1/2}\left[\frac{\partial}{\partial h}\mathcal{L}^N + \sum_{k=1}^{s(N)}(-1)^k\frac{d^k}{dr^k}\left(\frac{\partial}{\partial h^{(k)}}\mathcal{L}^N\right)\right], \tag{18}$$

where $p(N)$ and $s(N)$ can easily be inferred form the Lagrangian density. The angular components can be obtained from the covariant conservation equation $\nabla_a T^{ab} = 0$, which, for the line element (15), reduces to

$$\begin{aligned} T_{\alpha_1}^{(N)\alpha_1} &= \ldots = T_{\alpha_{N-2}}^{(N)\alpha_{N-2}} \\ &= -\frac{r}{2f(N-2)}\left(T_t^{(N)t} - T_r^{(N)r}\right)\frac{d}{dr}f \\ &\quad + \frac{r}{N-2}\frac{d}{dr}T_r^{(N)r} + T_r^{(N)r}, \end{aligned} \tag{19}$$

where $T_{\alpha_i}^{(N)\alpha_i}$ is any angular component of the stress-energy tensor. The coordinates $\{\alpha_1, \ldots, \alpha_{N-2}\}$ cover the $N-2$-dimensional sphere. Note that once the time and radial components of the stress-energy tensor are known the angular components can be obtained at practically no expense.

Making use of the coincidence limit of the Hadamard-DeWitt coefficient $a_3(x, x')$ in the $N = 4$ case, one has

$$T_t^{(4)t} = \frac{1}{m^2\pi^2 r^6}\left[\frac{1237x^3}{40320} - \frac{25x^2}{896} + \left(\frac{x^2}{8} - \frac{11x^3}{80}\right)\xi\right] \tag{20}$$

and

$$T_r^{(4)r} = \frac{1}{m^2\pi^2 r^6}\left[-\frac{47x^3}{5760} + \frac{7x^2}{640} + \left(\frac{3x^3}{80} - \frac{x^2}{20}\right)\xi\right], \tag{21}$$

where $x = r_+/r$. Similarly, for $N = 5$ one obtains

$$T_t^{(5)t} = \frac{1}{m\pi^2 r^6}\left[\frac{841x^6}{5040} - \frac{81x^4}{560} + \left(\frac{3x^4}{5} - \frac{7x^6}{10}\right)\xi\right] \tag{22}$$

and

$$T_r^{(5)r} = \frac{1}{m\pi^2 r^6}\left[-\frac{37x^6}{1008} + \frac{33x^4}{560} + \left(\frac{x^6}{5} - \frac{3x^4}{10}\right)\xi\right]. \tag{23}$$

The calculations of the stress-energy tensor in the spacetime of the higher-dimensional black holes require the knowledge of the higher-order Hadamard-DeWitt coefficients. Indeed, making use of the coincidence limit of the coefficient $a_4(x, x')$ in 6-dimensional Schwarzschild-Tangherlini spacetime gives

$$\begin{aligned} T_t^{(6)t} = \frac{1}{m^2\pi^3 r^8}\Bigg[&-\frac{73973x^{12}}{5040} + \frac{40457x^9}{2016} - \frac{387x^6}{64} \\ &+ \xi\left(\frac{59985x^{12}}{896} - \frac{19945x^9}{224} + \frac{405x^6}{16}\right)\Bigg] \end{aligned} \tag{24}$$

and

$$\begin{aligned} T_r^{(6)r} = \frac{1}{m^2\pi^3 r^8}\Bigg[&\frac{26969x^{12}}{10080} - \frac{103x^9}{18} + \frac{153x^6}{64} \\ &+ \xi\left(-\frac{33325x^{12}}{2688} + \frac{18055x^9}{672} - \frac{45x^6}{4}\right)\Bigg], \end{aligned} \tag{25}$$

whereas for $N = 7$ one obtains

$$\begin{aligned} T_t^{(7)t} = \frac{1}{m\pi^3 r^8}\Bigg[&-\frac{4713x^{16}}{128} + \frac{387x^{12}}{8} - \frac{217x^8}{16} \\ &+ \xi\left(\frac{1188x^{16}}{7} - 216x^{12} + \frac{225x^8}{4}\right)\Bigg] \end{aligned} \tag{26}$$

and

$$\begin{aligned} T_r^{(7)r} = \frac{1}{m\pi^3 r^8}\Bigg[&\frac{30549x^{16}}{4480} - \frac{8261x^{12}}{560} + \frac{237x^8}{40} \\ &+ \xi\left(-\frac{891x^{16}}{28} + \frac{3915x^{12}}{56} - \frac{225x^8}{8}\right)\Bigg]. \end{aligned} \tag{27}$$

The components of the stress-energy tensor in 4-dimensional Schwarzschild spacetime have been calculated earlier in Refs. [14,15]. As the angular components of the stress-energy tensor can easily be calculated from the covariant conservation equation (19) we shall not display them here.

The intermediate calculations of $T_b^{(N)a}$ are rather complicated but the final result is surprisingly simple, with only a weak increase of its complexity with dimension. It should be noted that in general, the coincidence limit of a_k is a kth degree polynomial in ξ, with the (geometric) coefficients of ξ^i, for $i > 1$, involving products of the Ricci tensor, its contractions and covariant derivatives. Additionally, there is a term $\Box^{k-1}R$, which, being a total divergence, does not contribute to the final result. That explains why the stress-energy tensor in the Schwarzschild-Tangherlini spacetime is always linear in ξ. The same is true for the more general Ricci-flat metrics. This behavior can be easily traced back to the recurrence equation for the general Hadamard-DeWitt coefficient $a_k(x, x')$.

The stress-energy tensor is regular in a physical sense if it is regular in a freely-falling frame of reference. To demonstrate that the components of the stress-energy tensor (20)–(27) do satisfy this requirement let us introduce the vectors of the frame defined as follows. For radial motion

the frame consists of the N-velocity vector $e^a_{(0)} = u^a$ and a unit length spacelike vector $e^a_{(1)} = n^a$. (The remaining vectors of the frame are unimportant for our purposes). Now, using the geodesic equations, one has

$$e^a_{(0)} = u^a = \left(\frac{E_0}{f}, -\sqrt{\left(\frac{E_0^2}{f} - 1\right)\frac{1}{h}}, 0, \ldots, 0\right) \tag{28}$$

and

$$e^a_{(1)} = n^a = \left(-\frac{1}{f}\sqrt{E_0^2 - f}, \frac{E_0}{\sqrt{fh}}, 0, \ldots, 0\right), \tag{29}$$

where E_0 is the constant of motion. The components of the stress-energy tensor in the frame can be written in the form:

$$T_{(0)(0)} = -\frac{E_0^2(T_0^0 - T_1^1)}{f} - T_1^1 \tag{30}$$

$$T_{(1)(1)} = -\frac{E_0^2(T_0^0 - T_1^1)}{f} + T_1^1 \tag{31}$$

$$T_{(0)(1)} = \frac{E_0\sqrt{E_0^2 - f}(T_0^0 - T_1^1)}{f}, \tag{32}$$

and, consequently, the stress-energy tensor in a freely-falling frame is regular as $r \to r_+$ if

$$|T^b_a| < \infty \quad \text{and} \quad |(T_0^0 - T_1^1)/f| < \infty. \tag{33}$$

Inspection of (20)–(27) shows that

$$T_r^{(N)r} - T_t^{(N)t} = F(r)\left[1 - \left(\frac{r_+}{r}\right)^{N-3}\right], \tag{34}$$

where $F(r)$ is a simple polynomial in r_+/r, and, consequently, the components $T_{(0)(0)}$ $T_{(1)(1)}$ and $T_{(0)(1)}$ are regular. Moreover, by the same argument, the components $T^{(N)\alpha_i}_{\alpha_i}$ given by Eq. (19) are regular also. We would like to emphasize that as the tensors have been calculated using various computational strategies, the regularity of the angular components has been established independently.

Although interesting in its own right, the main role played by the stress-energy tensor is to provide the source term to the semiclassical Einstein field equation. The backreaction of the quantized fields upon the classical background is the main theme of the next section.

III. THE BACKREACTION

In their simplest form the semiclassical Einstein field equations can be written as

$$G^a_b = 8\pi T^{(N)a}_b, \tag{35}$$

where, in general, the total stress energy tensor describes both classical and quantum matter. Ideally, the stress-energy tensor of the quantized field should functionally depend on a general metric or at least on the wide class of metrics. This allows, in principle, to construct the solution of the semiclassical Einstein field equations in a self-consistent way. On the other hand, one can follow a simpler approach, in which the stress-energy tensor is calculated in a concrete spacetime and the backreaction on the metric is treated perturbatively. In the black hole context the semiclassical Einstein field equations have been studied for the first time by York [34] more than thirty years ago (see also Ref. [35]). Since then various aspects of the backreaction problem have been studied in a number of papers, see e.g., [36–42] and the references cited therein.

In order to construct the semiclassical Einstein field equations, let us start with the line element

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2_{N-2}, \tag{36}$$

where

$$f(r) = e^{2\psi(r)}\left(1 - \frac{2M(r)}{r^{N-3}}\right)$$
$$\text{and} \quad h(r) = \left(1 - \frac{2M(r)}{r^{N-3}}\right)^{-1}. \tag{37}$$

The main reason for introducing the new functions $M(r)$ and $\psi(r)$ is to simplify the resulting equations. With such a substitution, the semiclassical Einstein field equations read

$$\frac{dM}{dr} = -\varepsilon\frac{8\pi r^{N-2}}{N-2}T_t^{(N)t} \tag{38}$$

and

$$\frac{d\psi}{dr} = \varepsilon\frac{8\pi r}{N-2}\frac{T_r^{(N)r} - T_t^{(N)t}}{1 - \frac{2M}{r^{N-3}}}, \tag{39}$$

where to simplify the calculations and to keep control of the order of terms in the complicated series expansions we have introduced the dimensionless parameter, ε, substituting $T^{(N)a}_b \to \varepsilon T^{(N)a}_b$. We have to put $\varepsilon = 1$ at the final stage of calculations.

The quantum corrections to the Schwarzschild-Tangherlini metric can be calculated making use of the expansion

$$M(r) = \frac{r_+^{N-3}}{2}[1 + \varepsilon(N-3)\mu(r)] \tag{40}$$

in (38) and (39), and integrating the linearized equation with the initial condition $\mu(r_+) = C_1$. This condition means that the function $\mu(r)$ can be written as $\mu(r) = \mu_0(r) + C_1$ with $\mu_0(r_+) = 0$. The second equation can

easily be integrated with the natural condition $\psi(\infty) = 0$. Note that with such a choice $\psi(r) \sim \mathcal{O}(\varepsilon)$. Putting this all together one has

$$f(r) = 1 - \left(\frac{r_+}{r}\right)^{N-3}(1 + \varepsilon(N-3)C_1) - \varepsilon(N-3)\left(\frac{r_+}{r}\right)^{N-3}\mu_0(r), \tag{41}$$

where

$$\mu_0(r) = -\frac{16\pi}{r_+^{N-3}(N-3)(N-2)}\int_{r_+}^{r} r^{N-2}T_t^{(N)t}dr. \tag{42}$$

The integration constant C_1 can be absorbed into the definition of the radius of the event horizon r_H as follows

$$r_H = r_+(1 + \varepsilon C_1) \tag{43}$$

in the process of the finite renormalization. The physical radius of the event horizon, r_H, is measurable as opposed to the unphysical (bare) r_+. Since μ_0 depends on r and r_+ and the third term on the right-hand side of Eq. (41) is $\mathcal{O}(\varepsilon)$, in the linearized calculations, one can use r_H instead of r_+ both in $\mu(r)$ and $\psi(r)$. With such a substitution one introduces $\mathcal{O}(\varepsilon^2)$ error. Let us return to the second equation of the system. Since Eq. (34) holds, the problem reduces to the two simple quadratures.

The same result can be obtained solving the semiclassical Einstein field equations with the stress-energy tensor depending on a general metric and with the quantum-corrected "exact" event horizon, r_H, as the initial condition from the very beginning.[2] Let us employ the second method and construct the semiclassical Einstein field equations for $M(r)$ and $\psi(r)$ with the initial conditions

$$M(r_H) = \frac{1}{2}r_H^{N-3} \quad \text{and} \quad \psi(\infty) = 0. \tag{44}$$

Assuming

$$M(r) = M_0(r) + \varepsilon M_1(r) + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \psi(r) = \varepsilon\psi_1(r) + \mathcal{O}(\varepsilon^2) \tag{45}$$

one obtains differential equations which can be solved with the conditions

$$M_0(r_H) = \frac{1}{2}r_H^{N-3}, \qquad M_1(r_H) = 0 \quad \text{and} \quad \psi_1(\infty) = 0. \tag{46}$$

The zeroth-order equation for a general N gives

$$M_0(r) = \frac{1}{2}r_H^{N-3}, \tag{47}$$

whereas the functions $M_1(r)$ and $\psi_1(r)$ assume more complicated, dimension-dependent form. After some algebra, one has

$$M_1(r) = \frac{1}{\pi m^2}\left[\frac{1237r_H^3}{60480r^6} - \frac{5r_H^2}{224r^5} + \frac{113}{60480r_H^3} + \xi\left(-\frac{11r_H^3}{120r^6} + \frac{r_H^2}{10r^5} - \frac{1}{120r_H^3}\right)\right], \tag{48}$$

$$\psi_1(r) = \frac{1}{\pi m^2}\left(\frac{7r_H^2\xi}{60r^6} - \frac{29r_H^2}{1120r^6}\right) \tag{49}$$

and

$$M_1(r) = \frac{1}{\pi m}\left[\frac{841r_H^6}{15120r^8} - \frac{9r_H^4}{140r^6} + \frac{131}{15120r_H^2} + \xi\left(-\frac{7r_H^6}{30r^8} + \frac{4r_H^4}{15r^6} - \frac{1}{30r_H^2}\right)\right], \tag{50}$$

$$\psi_1(r) = -\frac{1}{\pi m}\left(\frac{19r_H^4}{280r^8} - \frac{3r_H^4}{10r^8}\xi\right), \tag{51}$$

respectively, for $N = 4$ and $N = 5$. The analogous calculations in higher dimensional spacetimes are slightly more involved and for $N = 6$ give

$$M_1(r) = \frac{1}{\pi^2 m^2}\left[-\frac{73973r_H^{12}}{37800r^{15}} + \frac{40457r_H^9}{12096r^{12}} - \frac{43r_H^6}{32r^9} - \frac{13291}{302400r_H^3} + \xi\left(\frac{3999r_H^{12}}{448r^{15}} - \frac{19945r_H^9}{1344r^{12}} + \frac{45r_H^6}{8r^9} + \frac{97}{336r_H^3}\right)\right] \tag{52}$$

and

$$\psi_1(r) = \frac{1}{\pi^2 m^2}\left[\frac{3887r_H^9}{1680r^{15}} - \frac{45r_H^6}{32r^{12}} + \xi\left(\frac{195r_H^6}{32r^{12}} - \frac{1333r_H^9}{126r^{15}}\right)\right]. \tag{53}$$

Similarly, for $N = 7$ one has

$$M_1(r) = \frac{1}{\pi^2 m}\left[-\frac{1571r_H^{16}}{480r^{18}} + \frac{387r_H^{12}}{70r^{14}} - \frac{217r_H^8}{100r^{10}} - \frac{1439}{16800r_H^2} + \xi\left(\frac{528r_H^{16}}{35r^{18}} - \frac{864r_H^{12}}{35r^{14}} + \frac{9r_H^8}{r^{10}} + \frac{3}{5r_H^2}\right)\right] \tag{54}$$

and

[2]To cross-check the calculations we have employed both methods.

$$\psi_1(r) = \frac{1}{\pi^2 m}\left[\frac{4073 r_H^{12}}{1050 r^{18}} - \frac{1559 r_H^8}{700 r^{14}} + \xi\left(\frac{135 r_H^8}{14 r^{14}} - \frac{627 r_H^{12}}{35 r^{18}}\right)\right]. \tag{55}$$

Having established the form of the quantum corrected metric the correction to the temperature of the Schwarzschild-Tangherlini black hole can be calculated. First, observe that for the static and spherically symmetric black hole the Euclidean version of the line element has no conical singularity, provided the complexified time coordinate is periodic with a period β given by

$$\beta = \lim_{r\to r_H} 4\pi (g_{00} g_{11})^{1/2}\left(\frac{d}{dr} g_{00}\right)^{-1}. \tag{56}$$

Thus, as in the classical Schwarzschild-Tangherlini spacetime, a quantum-corrected black hole has a natural temperature associated with it. The Hawking temperature is given by $T_H = \beta^{-1}$ and to $\mathcal{O}(\varepsilon)$, one has

$$T_H^{(N)} = \frac{N-3}{4\pi r_H} + \varepsilon \Delta T_H^{(N)}, \tag{57}$$

where

$$\Delta T_H^{(4)} = \frac{1}{\pi^2 m^2 r_H^5}\left(\frac{\xi}{240} - \frac{37}{40320}\right), \tag{58}$$

$$\Delta T_H^{(5)} = \frac{1}{\pi^2 m r_H^5}\left(\frac{\xi}{60} - \frac{13}{3024}\right), \tag{59}$$

$$\Delta T_H^{(6)} = \frac{1}{\pi^3 m^2 r_H^7}\left(\frac{47}{1920} - \frac{97\xi}{672}\right), \tag{60}$$

$$\Delta T_H^{(7)} = \frac{1}{\pi^3 m r_H^7}\left(\frac{767}{16800} - \frac{3}{10}\xi\right). \tag{61}$$

The corrections $\Delta T_H^{(N)}$ are linear functions of ξ and one expects that this behavior persists also in the back-reaction on a more general (classical) Ricci-flat black hole geometries.

Now, let us analyze the mass of the black hole as seen by a distant observer. It is evident that the mass as given by Eq. (3) is not the mass that would be measured at great distances from the corrected black hole. The coordinate independent Komar mass, M_∞, defined by [43]

$$\oint_\infty \nabla^a K^b_{(t)} d\sigma_{ab} = -16\pi \frac{N-3}{N-2} M_\infty, \tag{62}$$

where $K_{(t)}$ is the timelike Killing vector and the integrals are to be calculated over $(N-2)$-sphere at spatial infinity, is very useful in this regard. Here, the Komar mass is the total mass energy of the black hole and the vacuum polarization of the quantized massive field. Making use of this definition, one has

$$M_\infty = \frac{\pi^{(N-3)/2}(N-2)}{8\Gamma(\frac{N-1}{2})} r_H^{N-3} + \Delta M^{(N)}, \tag{63}$$

where

$$\Delta M^{(4)} = \frac{1}{\pi m^2 r_H^3}\left(\frac{113}{60480} - \frac{\xi}{120}\right), \tag{64}$$

$$\Delta M^{(5)} = \frac{1}{m r_H^2}\left(\frac{131}{20160} - \frac{\xi}{40}\right), \tag{65}$$

$$\Delta M^{(6)} = \frac{1}{\pi m^2 r_H^3}\left(\frac{97\xi}{252} - \frac{13291}{226800}\right), \tag{66}$$

and

$$\Delta M^{(7)} = \frac{1}{m r_H^2}\left(\frac{3\xi}{8} - \frac{1439}{26880}\right). \tag{67}$$

Precisely the same result can be easily calculated form

$$M_\infty = \frac{\pi^{(N-3)/2}(N-2)}{4\Gamma(\frac{N-1}{2})} \lim_{r\to\infty} M(r), \tag{68}$$

where $M(r)$ is given by (45).

It should be noted, however, that for $N = 4$, Eq. (64) does not coincide with the result obtained by Frolov and Zel'nikov in Ref. [15], although the Komar mass M_∞ is identical. It is simply because they used the equivalent representation for the Komar mass

$$-16\pi \frac{N-3}{N-2} M_\infty = 2\int_S R^a_b K^b_{(t)} dS_a + \oint_{\mathcal{H}} \nabla^a K^b_{(t)} d\sigma_{ab}, \tag{69}$$

where $\mathcal{H}$ is a spatial $(N-2)$-sphere on the event horizon and S is the region between $\mathcal{H}$ and space-like infinity, and interpreted (in 4-dimensional spacetime) the first term on the right-hand side of the above equation as $-8\pi\Delta M^{(4)}_{BH}$. Indeed, simple calculations reproduce the Frolov-Zel'nikov result

$$\Delta M^{(4)}_{BH} = \frac{1}{540\pi m^3 r_H^3}(2 - 9\xi). \tag{70}$$

On the other hand, the last term

$$M_H = -\frac{N-2}{16\pi(N-3)} \oint_{\mathcal{H}} \nabla^a K^b_{(t)} d\sigma_{ab} \tag{71}$$

interpreted as a horizon-defined black hole mass, when restricted to $N = 4$, gives

$$M_H = \frac{r_H}{2} + \frac{1}{\pi m^2 r_H^3}\left(\frac{\xi}{120} - \frac{36}{21160}\right). \tag{72}$$

It can easily be shown that the sum $\Delta M_{BH}^{(4)} + M_H$ is precisely the Komar mass, M_∞, of the 4-dimensional quantum-corrected Schwarzschild black hole. Both definitions of the mass correction terms have their merits and the calculation of ΔM_{BH} presents no problem, but, in our opinion, Eqs. (64)–(67) are better suited for further analysis.

Now, we shall analyze the influence of the quantized field on the black hole. To this end let us compare the classical and the quantum corrected black holes, both characterized by the *same* radius of the event horizon, r_H. Two particular values of ξ are of special interest: $\xi = 0$, which characterizes the minimal coupling and $\xi = (N-2)(4N-4)$ which characterizes the conformal coupling. Other values of the coupling parameter are of somewhat lesser interest. The corrections of the Hawking temperature caused by the quantum field depend on the dimension and the coupling parameter and are tabulated in Table I.

Within the adopted approximation, the conformally coupled massive fields tend to lower the black hole temperature. On the other hand, under the influence of the minimally coupled fields the Hawking temperature increases for $N = 4$ and $N = 5$ and decreases for $N = 6$ and $N = 7$. Similarly, inspection of Table II shows that the correction to the black hole mass is always positive for the conformally coupled fields, whereas it is negative for the minimally coupled field in $N = 6$ and $N = 7$ dimensional quantum-corrected Schwarzschild-Tangherlini spacetime. Qualitatively, one has the following behavior for both values of the curvature coupling: Increase of the mass of the black hole due to quantum effects decreases the Hawking temperature. It should be noted however, that for more exotic values of the parameter ξ this observation may not necessarily be true. Finally, observe that the modifications of the characteristics of the black hole is bigger for minimally coupled fields, as can be easily seen in Table III. Once again, we observe that for other values of the coupling parameter corrections to the mass and the temperature can be quite significant.

TABLE I. The sign of $\Delta T^{(N)}$ for two physical choices of the coupling parameter $\xi = 0$ (minimal coupling) and $\xi_c = (N-2)/(4N-4)$ (conformal coupling).

	$N=4$	$N=5$	$N=6$	$N=7$
$\xi = 0$	−	−	+	+
$\xi = \xi_c$	−	−	−	−

TABLE II. The sign of $\Delta M^{(N)}$ for two physical choices of the coupling parameter $\xi = 0$ (minimal coupling) and $\xi_c = (N-2)/(4N-4)$ (conformal coupling).

	$N=4$	$N=5$	$N=6$	$N=7$
$\xi = 0$	+	+	−	−
$\xi = \xi_c$	+	+	+	+

TABLE III. The (absolute) value of the ratio of ΔT_0 to ΔT_c (the first row) and ΔM_0 to ΔM_c (the second row) for the quantum corrected Schwarzschild-Tangherlini black hole. The minimally coupled field leads to more prominent corrections.

	$N=4$	$N=5$	$N=6$	$N=7$
$\lvert\Delta T_0/\Delta T_c\rvert$	4.11	3.66	5.58	2.7
$\lvert\Delta M_0/\Delta M_c\rvert$	3.9	3.59	3.19	2.18

For s fields with masses m_i the main approximation to the one-loop effective action is still of the form (14) with $n' = \lfloor N/2 \rfloor + 1$, provided the following substitution is made

$$\frac{1}{(m^2)^{\lfloor N/2\rfloor - N/2+1}} \to \sum_i^s \frac{1}{(m_i^2)^{\lfloor N/2\rfloor - N/2+1}}. \tag{73}$$

Thus the quantum effects can be made arbitrary large by taking a large number of massive fields.

IV. FINAL REMARKS

We have constructed the approximate stress-energy tensor of the quantized massive scalar fields in the spacetimes of the Schwarzschild-Tangherlini black holes. The general expressions describing the stress-energy tensor constructed form the coefficient a_3 ($N = 4$ and $N = 5$) and from a_4 ($N = 6$ and $N = 7$) have been calculated using FORM. The coefficients a_k have been calculated within the framework of the manifestly covariant method. Unfortunately, the final results (which are valid in any spacetime provided the applicability conditions are satisfied) are rather complicated and their practical use may be limited to simple geometries of high symmetry. Although the Schwarzschild-Tangherlini black holes belong to the class of geometries for which such calculations can be performed in a reasonable time, here, for brevity, we followed a simplified approach and calculated the functional derivatives of the one-loop effective action with respect to the metric potentials of the general static and spherically symmetric metric.

Our general formulas have already been successfully tested. Indeed, recently we have calculated the stress-energy tensor of the quantized massive field in N-dimensional spatially-flat Friedman-Robertson-Walker spacetimes within the framework of the adiabatic approximation and it has been explicitly demonstrated that it coincides with the tensors obtained form the Schwinger-DeWitt method.

Finally observe, that as a by-product of the present calculations one can easily construct the field fluctuation. Indeed, from the formal definition

$$\langle \phi^2 \rangle_{\rm reg} = -i \lim_{x' \to x} G_{\rm reg}^{(N)}, \tag{74}$$

where $G_{\rm reg}^{(N)}$ is given by (7) with $A(x, x'; is)$ substituted by

$$A_{\rm reg}^{(N)}(x,x';is) = A(x,x';is) - \sum_{k=0}^{[\frac{N}{2}]-1} a_k(x,x')(is)^k, \tag{75}$$

one has

$$\langle\phi^2\rangle_{\rm reg} = \frac{1}{(4\pi)^{N/2}} \sum_{k=[N/2]}^{n'} \frac{a_k}{(m^2)^{k+1-N/2}} \Gamma\left(k+1-\frac{N}{2}\right). \tag{76}$$

This expression coincides with the result obtained in Ref. [29]. It should be noted, however, that the derivation presented here is simpler. The vacuum polarization can be calculated once the coincidence limits of the Hadamard-DeWitt coefficients in the concrete geometry are known. For example, the knowledge of the coefficients a_2, a_3 and a_4 in the Schwarzschild-Tangherlini spacetimes gives the field fluctuation for $4 \leq N \leq 9$.

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