Two-dimensional Horava-Lifshitz black hole solutions

D. Bazeia,[1,2] F. A. Brito,[1,2,*] and F. G. Costa[1,3]

[1]*Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil*

[2]*Departamento de Física, Universidade Federal de Campina Grande, Caixa Postal 10071, 58109-970 Campina Grande, Paraíba, Brazil*

[3]*Instituto Federal de Educação Ciência e Tecnologia da Paraíba (IFPB), Campus Picuí, Brazil*

(Received 4 September 2014; published 13 February 2015)

In this paper, we address the issue of black hole solutions in $(1+1)$-dimensional nonprojectable Horava-Lifshitz gravity. We consider several models by considering different potentials in the scalar matter sector. We also consider the gravitational collapse of a distribution of pressureless dust filling a region in one-dimensional space. The time of the collapse can be faster or slower depending on the parameter λ of the theory.

DOI: 10.1103/PhysRevD.91.044026 PACS numbers: 04.60.Kz, 04.50.Kd, 04.70.Bw

I. INTRODUCTION

The $(1+1)$-dimensional theory of gravity has been considered in some detail in several studies in the literature [1–6]. Remarkably, it has similarity to four-dimensional general relativity in many aspects. These include a Newtonian limit, cosmological solutions, interior solutions, gravitational waves, and the gravitational collapse of pressureless dust into black holes with event horizon structures, which is identical to the four-dimensional Schwarzschild solution. Since from the classical point of view the $(1+1)$-dimensional gravity structure is so close to $(3+1)$-dimensional gravity, it is expected that its quantization procedure should be quite similar to that in $(3+1)$-dimensional quantum gravity. Furthermore, its semiclassical properties also produces interesting effects such as Hawking radiation and also black hole condensation. This is because the nontrivial event horizon structure developed in $(1+1)$-dimensional theory of gravity has similarities with their $(3+1)$-dimensional general relativistic counterparts.

Recently, in Ref. [7], a new theory of gravity was put forward. This is now well known as the Horava-Lifshitz (HL) gravity. In the HL gravity, it is intended to obtain a renormalizable four-dimensional theory of gravity via power counting due to higher-order scaling on the 3-momentum at the UV scale. The price to pay is that space and time now scales in a different way via a dynamical critical exponent in the UV regime, and as a consequence, the Lorentz invariance is lost at the high-energy scale. Despite this, several studies have been considered in the literature [8], including modifications of the original theory in order to circumvent undesirable extra modes [9]. However, to our knowledge, in low-dimensional HL gravity, there have been few studies considered in the literature. To quote a few, very recently considerations appeared on black hole solutions in $2+1$ dimensions [10–12] and quantization of the $(1+1)$-dimensional projectable Horava-Lifshitz gravity [13].

In the present study, we investigate black hole solutions and gravitational collapse of a pressureless dust distribution in $1+1$ dimensions. We shall consider the nonprojectable version of HL gravity [9].

The paper is organized as follows. In Sec. II, we briefly introduce our setup. In Sec. III, we are able to find several black hole solutions by considering distinct models through several specific potentials in the scalar matter sector. In Sec. IV, we shall assume only dust in the matter sector to address the issue of gravitational collapse. In Sec. V, we present our final considerations.

II. LOWEST-DIMENSIONAL HL THEORY

In this section, we shall briefly review the nonprojectable HL gravity. In Horava-Lifshitz gravity, the spacetime decomposes as follows:

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (1)$$

For this theory, one finds

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (2)$$

and the $(D+1)$-dimensional action is written as

$$S = \frac{M_{\rm Pl}^2}{2}\int d^D x dt \sqrt{g}(K_{ij}K^{ij} + \lambda K^2 + \mathcal{V}), \quad (3)$$

where $\lambda > 1$ and the potential $\mathcal{V}$ is associated with the nonprojectable HL gravity defined as

$$\mathcal{V} = \xi R + \eta a^i a_i + \frac{1}{M_*^2}L_4 + \frac{1}{M_*^2}L_6, \quad (4)$$

[*]Corresponding author.
fabrito@df.ufcg.edu.br

with a_i being a vector that describes the proper acceleration of the vector field of unit normals to the foliation surfaces [9] given by

$$a_i = \partial_i \ln N, \tag{5}$$

where $i = 1, 2, 3$ in $3+1$ dimensions, though we shall concentrate ourselves in $1+1$ dimensions. In $1+1$ dimensions, the theory turns out to be much simpler, such that

$$K = \frac{1}{2N}(\dot{g}_{11} - 2\nabla_1 N_1), \tag{6}$$

with $z = D = 1$, $g_{ij} = g_{11}$, and $a_i = a_1$. In this form, the HL action coupled to matter fields

$$S = S_{\rm HL} + S_\phi \tag{7}$$

becomes written in terms of

$$S_{\rm HL} = \frac{M_{\rm Pl}^2}{2}\int dxdt\sqrt{g}[(1-\lambda)K^2 + \eta g^{11}a_1 a_1] \tag{8}$$

and

$$S_\phi = \int dxdtN\sqrt{g}\left[\frac{1}{2N}(\partial_t\phi - N^1\nabla_1)^2 - \alpha(\nabla_1\phi)^2 - V(\phi) - \beta\phi\nabla^1 a_1 - \gamma\phi a^1\nabla_1\phi\right], \tag{9}$$

with α, β, and γ being dimensionless coupling constants. In the relativistic limit, we have $\alpha = 1$ and $\beta = \gamma = 0$ [13].

III. LOWEST-DIMENSIONAL BLACK HOLES IN HL THEORY

Notice that from Eq. (8) the λ parameter will be irrelevant for the static black hole solutions of the present section because, in this case, $K = 0$, whereas the η parameter will be present in most of the solutions. One should mention that in next section, where we shall address the problem of gravitational collapse, the opposite will occur because, in that case, $a_1 \to 0$ (projectability).

Now, rewriting the complete action using the fact that $K = 0$, by admitting $N_1(x) = 0$ in our case, we have

$$S = \frac{M_{\rm Pl}^2}{2}\int dxdt\sqrt{-g}\left(\eta g^{11}a_1^2 - \frac{2}{M_{\rm Pl}^2}\alpha g^{11}\phi'^2 - \frac{2}{M_{\rm Pl}^2}V(\phi)\right), \tag{10}$$

or simply

$$S = \frac{M_{\rm Pl}^2}{2}\int dxdt\left(-\eta N^2 a_1^2 + \frac{2}{M_{\rm Pl}^2}\alpha N^2\phi'^2 - \frac{2}{M_{\rm Pl}^2}V(\phi)\right) \tag{11}$$

since $\sqrt{-g} = 1$ in the present study—see below. Now, varying this action with respect to N, it is easy to get the important condition

$$-\eta a_1^2 + \frac{2}{M_{\rm Pl}^2}\alpha\phi'^2 = 0, \tag{12}$$

whereas by varying S with respect to the scalar field ϕ, we find

$$\frac{d}{dx}(N^2\phi') = \frac{1}{2\alpha}\frac{\partial V}{\partial\phi}. \tag{13}$$

Notice from Eq. (12) that for $\alpha = \eta M_{\rm Pl}^2/2$ we can find $a_1 = |\phi'|$. This gives us an important relation between the matter scalar field and a_1. From the nonprojectable HL theory, we can take advantage of Eq. (5), which in $1+1$ dimensions simply becomes

$$a_1 \equiv \frac{d\ln N}{dx} = |\phi'| \to N = e^{\pm\phi}, \tag{14}$$

where we have used the aforementioned condition between a_1 and ϕ' and integrated out the equation for N.

The equation of motion, Eq. (13), can be now written in the simpler form

$$\frac{d}{dx}(\eta e^{2\phi}\phi') = V_\phi, V_\phi \to \frac{1}{M_{\rm Pl}^2}\frac{\partial V}{\partial\phi}, \tag{15}$$

where we used the above definition of α and the solution for N given in Eq. (14). The Planck mass appeared here because we kept $2/\kappa^2 \equiv M_{\rm Pl}^2/2$ in the Lagrangian (for later convenience—see next section), although $[\kappa] = \frac{z-D}{2} = 0$ since in our case $z = D = 1$. Thus, for the moment, we can indeed consider $M_{\rm Pl}^2 = 1$ for simplicity and consistence.

Let us now focus on the following model with $V(\phi) = 0$. By using the equation of motion (13), we find

$$N^2\phi' = M, \tag{16}$$

where M is an integration constant. Now, using Eq. (14), we find

$$N\frac{dN}{dx} = \pm M, \tag{17}$$

that integrating for $N(x)$ we find the general solution

$$N(x) = \sqrt{2(M|x| + C)}. \tag{18}$$

For an integration constant chosen as $C = -1/2$, we find

$$N(x)^2 = 2M|x| - 1. \tag{19}$$

Consequently, the scalar field can also be found via relation $N = e^\phi$ given in Eq. (14) such that

$$\phi(x) = \ln\sqrt{2M|x| - 1}. \tag{20}$$

This scalar solution can be thought of as a dilatonic solution. Its diverging behavior near the horizon is in accord with two other well-known places where the same phenomenon develops. Whereas our present study of black holes in low-dimensional (two-dimensional) Horava gravity has some resemblance with low-dimensional nonextremal black Dp branes (e.g., $p < 3$) in string theory [14], the fact that Horava gravity is in general a higher-derivative theory of gravity is somehow connected with higher-derivative gravity in arbitrary dimensions—see, for instance, the dilatonic Einstein–Gauss–Bonnet in Ref. [15]. In both these situations, the dilatonic solutions diverge on the black hole horizon. In the following, we shall focus on the black hole solutions.

Thus, we obtain the following simplest solution of black hole in two-dimensional HL gravity:

$$ds^2 = -(2M|x| - 1)dt^2 + \frac{1}{(2M|x| - 1)}dx^2. \tag{21}$$

This solution has previously appeared in Ref. [1]. Of course, since we have a scalar potential $V(\phi)$ that in general does not vanish, it is very natural to look for other solutions. However, as we shall see, it is not possible to find analytical solutions in some interesting cases. Despite this, we shall consider the following models.

First, let us consider the model with $V(\phi) = \Lambda\phi$. By using the equation of motion (15) and the fact that $N = e^{\phi}$, we get the following equation:

$$\eta(NN'' + N'^2) - V_\phi = 0. \tag{22}$$

This equation can be solved analytically, for which the solution for $N(x)$ and $\phi(x)$ is given, respectively, by

$$N(x)^2 = (\Lambda/\eta)x^2 - 2C_1x + 2C_2 \tag{23}$$

and

$$\phi(x) = \ln[(\Lambda/\eta)x^2 - 2C_1x + 2C_2]^{1/2}. \tag{24}$$

Now, taking $C_1 = -M$ and $\epsilon = 2C_2$, we find

$$\phi(x) = \ln[(\Lambda/\eta)x^2 + 2Mx - \epsilon]^{1/2} \tag{25}$$

and also

$$N(x)^2 = (\Lambda/\eta)x^2 + 2Mx - \epsilon. \tag{26}$$

Thus, in the present model, the new solution of the black hole in two-dimensional HL gravity is

$$ds^2 = -((\Lambda/\eta)x^2 + 2Mx - \epsilon)dt^2 + \frac{1}{((\Lambda/\eta)x^2 + 2Mx - \epsilon)}dx^2. \tag{27}$$

See in Ref. [4] (the first reference) a similar solution. Before presenting more examples, some comments are in order. The cases studied previously are the simplest ones in which we can choose a scalar potential and obtain explicit solutions. For further generalized potentials, we cannot in general obtain explicit solutions analytically. In this sense, one can choose, instead, not a scalar potential itself but its derivative as a function of an implicit scalar field, which in turns depends on the spatial coordinate. So, we shall now consider forms of $V_\phi(\phi(x)) \equiv V_\phi(x)$ as follows:

$$V_\phi(x) = A + \frac{B}{x^3} + \frac{C}{x^4}. \tag{28}$$

Now, substituting Eq. (28) into the equation of motion (15), we find the general solution for $N(x)$ and $\phi(x)$ given explicitly by

$$N(x)^2 = 2C_2 + \frac{A}{\eta}x^2 - 2C_1x + \frac{B}{\eta x} + \frac{C}{3\eta x^2} \tag{29}$$

and

$$\phi(x) = \ln\sqrt{2C_2 + \frac{A}{\eta}x^2 - 2C_1x + \frac{B}{\eta x} + \frac{C}{3\eta x^2}}. \tag{30}$$

The only problem with this procedure is finding the potential back in terms of the scalar field ϕ because, in most cases, one cannot invert the solutions in order to obtain $x = f(\phi)$. Aside from this fact, we can find several interesting solutions for $N(x)$ and $\phi(x)$ given explicitly, as we can see below. By properly choosing the parameters, the spacetime may represent a black hole, a white hole, a naked singularity, or other more complicated structures. As stated in Ref. [1], this spacetime can also be used to easily extended it to multiple point sources.

Some special cases are in order:

(i) For $C_1 = -M$, $C_2 = -1/2$, $\eta = 1$, and $A = B = C = 0$, we simply have $V_\phi = 0$ ($V = \text{const}$), which is equivalent to the case with $V = 0$ for which the solution is given by Eqs. (20)–(21).

(ii) Another similar example is $C_2 = -\epsilon/2$, $C_1 = -M$, $A = \Lambda$, and $B = C = 0$ for which we have $V_\phi = A$, which is also equivalent to the case $V(\phi) = \Lambda\phi$. The solution for this case is given by Eqs. (26)–(27). If one wants to leave the solution in the same form presented in Ref. [1], one can still consider $\eta = 1$ and $A = -\Lambda/2$.

In the following, we shall consider two-dimensional Schwarzschild and Reissner–Nordström-like solutions. These types of solutions have previously appeared, e.g., in Refs. [5] and [6], respectively.

A. Schwarzschild-like solution

In this case, one considers $C_2 = 1/2$, $B = -2M$, $\eta = 1$, and $A = C = C_1 = 0$. This leaves

$$V_\phi(x) = -\frac{2M}{x^3}. \tag{31}$$

This gives the following explicit solution for $N(x)$ and $\phi(x)$:

$$N(x)^2 = 1 - \frac{2M}{x} \tag{32}$$

and

$$\phi(x) = \ln\sqrt{1 - \frac{2M}{x}}. \tag{33}$$

Interestingly, in this case, we can invert (31) and integrate on ϕ to obtain the scalar potential

$$V(\phi) = \frac{B}{8M^3}\phi - \frac{3B}{16M^3}e^{2\phi} + \frac{3B}{32M^3}e^{4\phi} - \frac{B}{48M^3}e^{6\phi}. \tag{34}$$

This will be quite easy anytime the polynomial form of $V_\phi(x)$ is restricted to a unique term.

B. Reissner–Nordström-like solution

One can expect that the last term in $V_\phi(x)$ given in Eq. (28) is associated with the effect of an electrical *charge* Q. This is more evident through the use of the general solution (29)–(30) and making a suitable choice of the parameters, that is, for $C_2 = 1/2, B = -2M,\ C = 3Q^2$, $\eta = 1$, and $A = C_1 = 0$,

$$V_\phi(x) = -\frac{2M}{x^3} + \frac{3Q^2}{x^4}. \tag{35}$$

Again, as in the previous cases, this gives the following explicit solution for $N(x)$ and $\phi(x)$:

$$N(x)^2 = 1 - \frac{2M}{x} + \frac{Q^2}{x^2} \tag{36}$$

and

$$\phi(x) = \ln\sqrt{1 - \frac{2M}{x} + \frac{Q^2}{x^2}}. \tag{37}$$

Differently from the previous case, now one cannot easily invert (31) and integrate on ϕ to obtain a scalar potential.

C. New black hole solution

The two cases presented above are well-known solutions in four dimensions with several issues addressed such as the number of horizons, Hawking temperature, entropy, and so on. The other cases with $B = C = 0$ are also well known in $1+1$-dimensional gravity. Thus, we do not need to say more about them. However, in the case that we will present here, we shall consider the Hawking temperature.

Just for maintaining the usual notation, let us rename the general solution (29) as follows $f(x) \equiv N(x)^2$, i.e.,

$$f(x) = 2C_2 + \frac{A}{\eta}x^2 - 2C_1 x + \frac{B}{\eta x} + \frac{C}{3\eta x^2}. \tag{38}$$

For $C_1 \neq 0$, $C_2 \neq 0$, $B \neq 0$, and $A = C = 0$, we have

$$f(x) = 2C_2 - 2C_1 x + \frac{B}{\eta x}. \tag{39}$$

This solution develops the following horizons:

$$x_h^\pm = \frac{C_2}{C_1} \pm \sqrt{\Delta}, \qquad \Delta = \frac{C_2^2}{C_1^2} + \frac{2B}{\eta C_1}. \tag{40}$$

As $\Delta = 0$, they degenerate, i.e., $x_h^+ = x_h^-$.

The Hawking temperature is given in terms of the outer (x_h^+) horizon as follows:

$$T_H = \left.\frac{f'(x)}{4\pi}\right|_{x=x_h^+}. \tag{41}$$

For the special case $C_2 = 0$, $C_1 = -M$ and $B = -2M$, the horizons are independent of the mass M:

$$x_h^\pm = \pm\frac{2}{\sqrt{\eta}}(\eta > 0). \tag{42}$$

The temperature is then given by

$$T_H = \frac{1}{4\pi}\left(-2C_1 - \frac{B}{(\frac{2}{\sqrt{\eta}})^2}\right), \tag{43}$$

or simply

$$T_H = \frac{1}{8\pi}(4+\eta)M. \tag{44}$$

This is a typical relation between the Hawking temperature and the mass of black holes in $1+1$ dimensions [1].

IV. GRAVITATIONAL COLLAPSE

In this section, we address the issue of the gravitational collapse of a certain mass of dust with negligible pressure confined into a region of the unidimensional space $[-r, r]$, of which the metric is given in comoving coordinates by

$$ds^2 = -N(\tau)^2 d\tau^2 + a(\tau)^2 d\rho^2. \tag{45}$$

In this case, the action we shall consider is that given by Eq. (7) with the matter sector not restricted only to scalar fields. Now, we have the action

$$S = \frac{M_{\rm Pl}^2}{2}\int d^2x N\sqrt{g_{11}}[(1-\lambda)K^2 + \eta g^{11}\phi'^2] + S_m, \tag{46}$$

which is explicitly given in terms of the metric (45) in the form

$$S = \frac{M_{\rm Pl}^2}{2}\int d^2x\left[\frac{(1-\lambda)a^3\dot{a}^2}{N}\right] + \frac{M_{\rm Pl}^2}{2}\int d^2x\left[\frac{N\eta\phi'}{a}\right] + \int d^2x N\sqrt{g_{11}}L_m. \tag{47}$$

The tensor energy momentum is given in terms of the matter Lagrangian through its usual definition

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g_{\mu\nu}}. \tag{48}$$

By varying the action with respect to N, i.e.,

$$\frac{\delta S}{\delta N} = 0, \tag{49}$$

we obtain the equation that relates the dynamics of the spacetime (45) with the energy density

$$\frac{(\lambda-1)M_{\rm Pl}^2 a^3\dot{a}^2}{2N^2} + \frac{M_{\rm Pl}^2\eta\phi'}{2a} = -\frac{1}{\sqrt{g_{11}}}\frac{\delta S_m}{\delta N} = \sigma. \tag{50}$$

Now, making $N=1$, and recalling that $N=e^{\phi}$ from our previous investigations, then consequently the scalar field $\phi = 0$ so that we have

$$(\lambda-1)M_{\rm Pl}^2 a^3\dot{a}^2 = 2\sigma. \tag{51}$$

A. Interior solution for the gravitational collapse

Since we are working with a pressureless fluid, then $T^{\mu\nu} = \sigma U^\mu U^\nu$, with $U^1 = 0$ and $U^t = 1$. Thus, the equation for conservation of energy and momentum now reads

$$\nabla_\mu T^{\mu\nu} = 0 \longrightarrow \frac{\partial}{\partial t}\left(\sigma\sqrt{a^2}\right) = 0, \tag{52}$$

which means that $\sigma\sqrt{a^2}$ is constant. Thus, in terms of the constants a_0 and ρ_0 defined at the initial time of the collapse, it simply gives

$$\sigma = \frac{\rho_0 a_0}{a}. \tag{53}$$

Now, substituting this equation into Eq. (51), we find the differential equation

$$a^4\dot{a}^2 = \frac{2\sigma_0 a_0}{M_{\rm Pl}^2(\lambda-1)} \tag{54}$$

that can still be recast in the form

$$a^2\dot{a} = \pm\sqrt{\frac{2\sigma_0 a_0}{M_{\rm Pl}^2(\lambda-1)}} = \pm\beta, \tag{55}$$

of which the solutions are

$$a = (\pm 3\beta\tau + C)^{1/3}. \tag{56}$$

Now, choosing $C=1$, $N=1$ and taking the solution with minus sign, we finally have the metric in the interior of the gravitational collapse:

$$ds^2 = -d\tau^2 + (1-3\beta\tau)^{2/3}d\rho^2. \tag{57}$$

See in Ref. [3] a similar solution. The density of the dust given by σ goes to infinity (singularity) as the scale factor approaches zero. This occurs in the finite time $\tau_c = 1/(3\beta)$, that is

$$\tau_c = \sqrt{\frac{(\lambda-1)M_{\rm Pl}^2}{18\sigma_0 a_0}}. \tag{58}$$

The square-root dependence on $\lambda - 1$ would be a problem in the projectable original HL gravity [7] where this parameter is allowed to be only $\lambda \le 1$. Fortunately, this is not the case in the *healthy* nonprojectable HL gravity developed in Ref. [9] where $\lambda > 1$. Notice that, given an initial density σ_0, the collapse can occur slower or faster depending on the parameter λ. As an example, for $\lambda \to 1$, the time $\tau_c \to 0$, which means a distribution of dust that collapses very quickly can otherwise live longer with a time $\tau_c \neq 0$ before collapsing for $\lambda > 1$.

B. Exterior solution for the gravitational collapse

Inspired in the Birkhoff theorem, which states that it is always possible to find a coordinate system in which the exterior solution of a spherical solution in 3 + 1 dimensions is time independent [16,17], we shall proceed in a similar way into 1 + 1 dimensions to connect our *interior time-dependent* solution previously obtained to an *exterior time-independent* solution [4]. Thus, we shall relate the coordinate x that describes a black hole, the static exterior solution, with a comoving coordinate ρ that describes the motion of the dust in the gravitational collapse, the interior solution, via

$$x(\tau,\rho) = \rho a(\tau) = \rho(1-3\beta\tau)^{2/3} \tag{59}$$

in such a way that, from the interior metric

$$ds^2 = -d\tau^2 + a^2(\tau,\rho)d\rho^2, \tag{60}$$

we should find the static exterior solution

$$ds^2 = -A(x)^2 dt^2 + A(x)^{-2}dx^2. \tag{61}$$

There is a Killing vector that corresponds to energy conservation satisfying

$$K_\mu \frac{dx^\mu}{d\tau} = K_t \frac{dt}{d\tau} + K_x \frac{dx}{d\tau} = \text{const}, \tag{62}$$

where

$$K_\mu = (-A^2, 0). \tag{63}$$

Then, we find the following solution:

$$\frac{dt}{d\tau} = -\frac{E}{A^2}. \tag{64}$$

In addition, there is another constant of the motion along the geodesic

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{65}$$

that is

$$-\epsilon = -A^2\left(\frac{dt}{d\tau}\right)^2 + A^{-2}\left(\frac{dx}{d\tau}\right)^2, \tag{66}$$

or simply

$$-\epsilon A^2 = -A^4\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2, \tag{67}$$

$$\left(\frac{dx}{d\tau}\right)^2 + \epsilon A^2 - E^2 = 0. \tag{68}$$

If the particle is at rest ($dx/d\tau = 0$) in the limit $x \to r$, we have $\epsilon E^2 = C^2$ with $C \to A(r)$. Recall that r is the boundary of the dust in a one-dimensional region. In the case of massive particles, we can make $\epsilon = 1$, and then

$$\left(\frac{dx}{d\tau}\right)^2 + A^2 - C^2 = 0. \tag{69}$$

Notice also that from Eq. (64) we can now write

$$\frac{dt}{d\tau} = -\frac{C}{A^2}. \tag{70}$$

Imposing the conditions $x = x(\rho,\tau)$, $t = t(\rho,\tau)$, and $x = \rho a$, we have

$$dx = \frac{\partial x}{\partial \tau}d\tau + \frac{\partial x}{\partial \rho}d\rho = \rho\frac{\partial a}{\partial \tau}d\tau + a d\rho \tag{71}$$

$$dt = \frac{\partial t}{\partial \tau}d\tau + \frac{\partial t}{\partial \rho}d\rho = -\frac{C}{A^2}d\tau + \frac{\partial t}{\partial \rho}d\rho. \tag{72}$$

Substituting this into (61) and comparing with (60), we have the following conditions:

$$\frac{\partial t}{\partial \rho} = \frac{1}{C^2}\frac{1}{A^2}\rho a\frac{\partial a}{\partial \tau} = \frac{1}{C}\frac{1}{A^2}\rho\frac{\beta}{a} \tag{73}$$

$$A^2 = C^2 - \rho^2\left(\frac{\partial a}{\partial \tau}\right)^2 = C^2 - \frac{\beta^2}{a^4}\rho^2 \tag{74}$$

$$\left(\frac{\partial t}{\partial \rho}\right)^2 = \frac{a^2}{A^4} - \frac{a^2}{A^2}, \tag{75}$$

from which we conclude that

$$A^2 = 1 - \frac{\beta^2}{a^4}\rho^2. \tag{76}$$

Making $\rho = r$ and $x = ra$, we have

$$A^2 = 1 - \frac{\beta^2 r^6}{x^4} \tag{77}$$

$$ds^2 = -\left(1 - \frac{\beta^2 r^6}{x^4}\right)dt^2 + \left(1 - \frac{\beta^2 r^6}{x^4}\right)^{-1}dx^2. \tag{78}$$

This is precisely one of the solutions found in Ref. [3]. The scalar curvature is given by

$$R = \frac{20\beta^2 r^6}{x^6}. \tag{79}$$

Thus, $x = 0$ is truly a singularity, that is, a curvature singularity. Furthermore, the two-dimensional Schwarzschild radius is given by

$$x_H = r^{3/2}\beta^{1/2} = r^{3/2}\left[\frac{2\sigma_0 a_0}{M_{\rm Pl}^2(\lambda - 1)}\right]^{1/4}. \tag{80}$$

The previous analysis has many similarities with that considered long ago [3]. However, there are some peculiar points here. It seems the most striking difference is the one related to the Schwarzschild radius. In the $(1+1)$-dimensional Einstein gravity explored in Ref. [3], the Schwarzschild radius is not defined at $r = 1/\sqrt{4b}$, where $b = 2\pi G\rho_0$ and r is the boundary of the dust. On the other hand, the Schwarzschild radius in the present case is well defined for $\lambda > 1$, which is also naturally consistent for a collapse at finite time discussed above.

V. CONCLUSIONS

We have investigate black hole solutions in the two-dimensional HL gravity. The solutions are in principle the same obtained in $1+1$ general relativity but are controlled by the parameter η that controls the coupling of the vector associated with the nonprojectability of the theory. However, they do not depend on the coupling $\lambda > 1$. The opposite happens to the gravitational collapse of the pressureless dust. In this case, there is no dependence on η, but it depends on λ. This is due to the specific dependence of the solutions on the coordinates in each case. Whereas the black hole solutions are only spatial dependent, the obtained *interior* solution for the gravitational collapse of dust has only time dependence. The exterior solution is just obtained from the interior solution.

ACKNOWLEDGMENTS

We would like to thank to CNPq, PNPD-CAPES, and PROCAD-NF/2009-CAPES for partial financial support.

[1] R. B. Mann, A. Shiekh, and L. Tarasov, Nucl. Phys. **B341**, 134 (1990).
[2] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, R1005 (1992).
[3] A. E. Sikkema and R. B. Mann, Classical Quantum Gravity **8**, 219 (1991).
[4] D. Christensen and R. B. Mann, Classical Quantum Gravity **9**, 1769 (1992); C. Eling and T. Jacobson, Phys. Rev. D **74**, 084027 (2006).
[5] S. M. Christensen and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977).
[6] S. P. Trivedi, Phys. Rev. D **47**, 4233 (1993).
[7] P. Horava, Phys. Rev. D **79**, 084008 (2009).
[8] M. Visser, Phys. Rev. D **80**, 025011 (2009); T. Sotiriou, M. Visser, and S. Weinfurtner, J. High Energy Phys. 10 (2009) 033; C. Bogdanos and E. N. Saridakis, Classical Quantum Gravity **27**, 075005 (2010); A. Wang and R. Maartens, Phys. Rev. D **81**, 024009 (2010); Y.-Q. Huang, A. Wang, and Q. Wu, Mod. Phys. Lett. A **25**, 2267 (2010); C. Charmousis, G. Niz, A. Padilla, and P. M. Saffin, J. High Energy Phys. 08 (2009) 070; D. Blas, O. Pujolas, and S. Sibiryakov, J. High Energy Phys. 10 (2009) 029; K. Koyama and F. Arroja, J. High Energy Phys. 03 (2010) 061; I. Kimpton and A. Padilla, J. High Energy Phys. 07 (2010) 014; A. Wang and Q. Wu, Phys. Rev. D **83**, 044025 (2011); B. Chen and Q. G. Huang, Phys. Lett. B **683**, 108 (2010).
[9] D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010); J. High Energy Phys. 04 (2011) 018; T. Jacobson, Phys. Rev. D **81**, 101502 (2010); **82**, 129901(E) (2010).
[10] M. I. Park, Phys. Lett. B **718**, 1137 (2013).
[11] T. P. Sotiriou, I. Vega, and D. Vernieri, Phys. Rev. D **90**, 044046 (2014).
[12] K. Lin, E. Abdalla, R. G. Cai, and A. Wang, J. Mod. Phys. D **23**, 1443004 (2014).
[13] B. F. Li, A. Wang, Y. Wu, and Z. C. Wu, Phys. Rev. D **90**, 124076 (2014); J. Ambjorn, L. Glaser, Y. Sato, and Y. Watabiki, Phys. Lett. B **722**, 172 (2013); C. Anderson, S. Carlip, J. H. Cooperman, P. Horava, R. Kommu, and P. R. Zulkowski, Phys. Rev. D **85**, 044027 (2012).
[14] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory—A Modern Introduction* (Cambridge University Press, Cambridge, England, 2007).
[15] N. Ohta and T. Torii, Prog. Theor. Phys. **121**, 959 (2009).
[16] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[17] S. M. Carroll, arXiv:gr-qc/9712019.