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Born-Infeld gravity with a massless graviton in four dimensions

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We construct Born-Infeld (BI) type gravity theories which describe tree-level unitary (nonghost and nontachyonic) massless spin-2 modes around their maximally symmetric vacua in four dimensions. Building unitary BI actions around flat vacuum is straightforward, but this is a complicated task around (anti)-de Sitter backgrounds. In this work, we solve the issue and give details of constructing perturbatively viable determinantal BI theories. It is interesting that the Gauss-Bonnet combination, which is a total derivative in four dimensions, plays an important role in the construction of viable BI theories.

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I. INTRODUCTION

It is well known that Einstein's gravity, otherwise an excellent theory in the "intermediate" scales, needs to be modified both at large and small scales. At large distances, the galaxy rotation curves and the accelerated expansion of the Universe are somewhat urgent problems which could be provisionally solved by keeping Einstein's theory intact but introducing large amounts of dark matter and dark energy or, alternatively, by modifying gravity at long distances such as upgrading it to massive gravity. (Of course, it could happen that one may need to both modify gravity and also add dark matter and dark energy to solve these long distance problems. This possibility should not be ruled out.) At small scales or at high energies, the problem is more complicated, even if phenomenologically less urgent, due to lack of data. It is more complicated because what one really needs is a quantum theory of gravity whose basic degrees of freedom, symmetries, and even principles as applied to spacetime are unknown. Namely, the geometric nature of spacetime, even its number of dimensions at small distances, is not clear. Even though there are candidates such as string theory, loop quantum gravity, or asymptotically safe theories, it is fair to say that we are still far away from a consistent theory of quantum gravity.

In the absence of guiding principles for a renormalizable theory of gravity, one is forced to introduce effective theories which work better than Einstein's gravity at small distances and hopefully also at large distances. One such attempt is, emulating pre-quantum-electrodynamics era electromagnetism, to write Born-Infeld (BI) type gravity theories [1] which were inspired by the work of Eddington [2] who used the idea of "generalized volume" suggested actions of the form

$$I = \int d^4x \sqrt{\det R_{\mu\nu}(\Gamma)}, \tag{1}$$

and assumed the metric and the connection to be independent variables. (A note about history: Eddington's work in gravity precedes the works of Born and Infeld [3] in electrodynamics, but it is actually difficult to find this action in Eddington's book in one compact form even though the discussion is scattered in the book. Schrodinger attributes this theory to Eddington on page 113 of his book [4].) After all, good ideas never disappear: Eddington's idea was resuscitated recently in a number of works [5–7], which led to interesting results such as singularity-free cosmology.

In analogy with the *minimal* electromagnetic BI theory, in the current work, we take the more conventional path of assuming the metric to be the only independent variable, following Deser and Gibbons [1] who gave a jumpstart to the BI gravity theories. This line of reasoning recently [8–11] bore much fruit in the lower dimensional setting where we have found a BI-type action which reads

$$I_{\mathrm{BINMG}} = -\frac{4m^2}{\kappa^2}\int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{m^2}G_{\mu\nu}\right)} - \left(\frac{\Lambda_0}{2m^2} + 1\right)\sqrt{-\det g}\right], \tag{2}$$

where $G_{\mu\nu}$ is the Einstein tensor without a cosmological constant. This theory is called the Born-Infeld new massive gravity (BINMG) theory with the following remarkable properties:

(1) For $\Lambda_0 \neq 0$, unlike any generic finite order theory besides the cosmological Einstein's theory, it has a

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unique maximally symmetric vacuum with an effective cosmological constant $\Lambda = \Lambda_0(1+\frac{\Lambda_0}{4m^2})$ for $\Lambda_0 > -2m^2$ [9,12]. Flat space is the unique vacuum when $\Lambda_0 = 0$.

(2) It has a unitary spin-2 massive degree of freedom with $M^2 = m^2 + \Lambda$ about the flat ($\Lambda = 0$) and AdS backgrounds. This provides an infinite order extension of the quadratic NMG[1] [14–16].

(3) It reproduces, up to desired order in the curvature expansion, the extended NMG theories that are consistent with the AdS/CFT duality and that have a c-function [9,17,18].

(4) The BINMG action appears as a counterterm in AdS_4 [13].

In addition to these properties, the existence of a supersymmetric extension to the cubic order truncation of the theory suggests that a supersymmetric extension presumably exists for the full theory [19].

All these virtues of the three-dimensional BI gravity led us to search for similar theories beyond three dimensions and especially in the more relevant $3+1$ dimensions. In this work, this is the task that we take on. Some of our computations, especially in the context of general formalism, will be in generic n dimensions, but in most of the current work we focus on $n = 3+1$ dimensions since it has rather distinctive features compared to the $n > 4$ cases. The generic n dimensional theory is somewhat more complicated and deserves separate attention [20]. As we shall see, the set of viable Lagrangians is larger in four dimensions compared to the three dimensional case, where there are only two theories as mentioned above, since vanishing of the Weyl tensor and the linear theory having no propagating degrees of freedom by itself in three dimensions make $n = 2+1$ rather simple and unique. One should not expect such a simplicity in four dimensions and beyond.

In constructing viable BI-type gravity theories, the important point is to find the physical constraints that one imposes on the theory. Here, the constraints we assume are as follows:

(1) In small curvature expansion, the theory at the lowest order reduces to (cosmological) Einstein's gravity.

(2) The theory admits flat or (A)dS vacuum.

(3) The theory describes only massless spin-2 excitations around its flat vacuum or the (A)dS vacuum, and these excitations are nonghost and nontachyonic (namely, the theory is tree-level unitary) as a full theory (thus, infinitely many terms in the curvature expansion contribute to the propagator of the theory).

(4) On top of the previous condition, the theory is tree-level unitary at any finite *truncated* order in the curvature expansion.

Let us briefly explain why these conditions are imperative for a healthy theory. We require that in small curvature expansion the theory reduces to the (cosmological) Einstein theory which is a natural condition to reproduce the plethora of data explained by Einstein's theory. The second requirement is sort of self-explanatory since one needs a maximally symmetric vacuum with vanishing conserved quantities such as energy and angular momentum. The third requirement is also somewhat obvious both in the context of the stability of the vacuum and perturbative viability of the quantum version of the theory. The fourth condition is extremely difficult to satisfy in the curvature expansion for (A)dS backgrounds. Observe that we require not only that the theory is tree-level unitary as a whole (condition 3) but that it is also tree-level unitary at any truncated order in the curvature expansion (condition 4). To the best of our knowledge, the theories that we present are the only ones that satisfy this requirement in four dimensions. (There are Lovelock theories [21] in higher dimensions that also satisfy this requirement, but they reduce to Einstein's theory in four dimensions.)

As we shall find out, these conditions still leave a large set of viable theories. Of course, one can additionally impose that there be no dimensionless or dimensionful parameters, save the Newton's constant and, perhaps, the BI parameter, which highly constrains the viable theories. As we shall see, the most "minimal" BI theory also has a unique vacuum. This is actually quite important since, once Einstein's theory is augmented with additional powers of curvature, immediately one undesired feature arises that is the nonuniqueness of the maximally symmetric vacuum. Since asymptotic structures of spacetimes with different cosmological constants are not the same, their energy properties are not comparable. Therefore, there is no way to choose one vacuum over the other if there is more than one viable vacuum. Hence, it would be highly desirable to have a theory with a unique vacuum.

To see that four dimensional BI theories are somewhat special, let us start with the following n dimensional generic action:

$$I = \frac{2}{\kappa\gamma}\int d^n x\left[\sqrt{-\det(g_{\mu\nu}+\gamma A_{\mu\nu})} - (\gamma\Lambda_0+1)\sqrt{-\det g}\right], \tag{3}$$

[1]There are in fact two extensions of NMG, the second one being

$$I = -\frac{4m^2}{\kappa^2}\int d^3x\left\{\sqrt{-\det\left[g_{\mu\nu}+\frac{1}{m^2}\left(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\right)\right]} - \left(\frac{\Lambda_0}{2m^2}+1\right)\sqrt{-\det g}\right\},$$

which has the same perturbative properties. It was conjectured in [13] that this action could appear as a dS_4 counterterm. This theory is yet to be explored further.

where κ is the modified Newton's constant which in four dimensions reads $\kappa = 16\pi G$, G is the Newton's constant, and γ is a dimensionful BI parameter with mass dimension -2 in four dimensions. To stick to the idea of obtaining *minimal* theories, we will find the simplest two tensor $A_{\mu\nu}$ which does not have derivatives of the Riemann tensor and which has as small powers and contractions of the Riemann tensor as possible. The most naive approach would be to take $A_{\mu\nu} = G_{\mu\nu} + \beta g_{\mu\nu} R$, similar to the $2+1$ dimensional case. As shown in Appendix A, upon small curvature expansion, this theory will generate quadratic terms which have massless spin-2, massive spin-0 and massive spin-2 modes, the last one being a ghost even around flat spacetime.[2] Hence, this too optimistic guess does not lead to a perturbatively viable theory. In the small curvature expansion ($|\gamma A_{\mu\nu}| \ll 1$) of the action (3), either quadratic terms must be eliminated or they must appear in the benign Gauss-Bonnet combination to get rid of the massive modes; therefore, in four dimensions and beyond, to build viable BI-gravity theories, one has to take $A_{\mu\nu}$ to be up to *at least quadratic* order in the curvature, which of course leads to an eight order theory in the curvature under the square root when the determinant is explicitly written in terms of the traces.

Upon inspection, one can see that the most general two-tensor up to and including quadratic order can be written as

$$
\begin{aligned}
A_{\mu\nu} = {} & R_{\mu\nu} + \beta S_{\mu\nu} + \gamma (a_1 C_{\mu\rho\sigma\lambda} C_{\nu}{}^{\rho\sigma\lambda} + a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + a_3 R_{\mu\rho} R^{\rho}_{\nu} + a_4 S_{\mu\rho} S^{\rho}_{\nu}) \\
& + \frac{\gamma}{n} g_{\mu\nu} (b_1 C_{\rho\sigma\lambda\gamma} C^{\rho\sigma\lambda\gamma} + b_2 R_{\rho\sigma} R^{\rho\sigma} + b_3 S_{\rho\sigma} S^{\rho\sigma}),
\end{aligned}
\tag{4}
$$

where $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ is the traceless-Ricci tensor and $C_{\mu\alpha\nu\beta}$ is the Weyl tensor, and β, a_i, and b_i are dimensionless constants. Observe that there is no $R_{\mu\nu} S^{\mu\nu}$ cross term because of the following relation:

$$
R_{\mu\rho} S^{\rho}_{\nu} = \frac{1}{2} R_{\mu\rho} R^{\rho}_{\nu} + \frac{1}{2} S_{\mu\rho} S^{\rho}_{\nu} - \frac{1}{2n} g_{\mu\nu} (R_{\rho\sigma} R^{\rho\sigma} - S_{\rho\sigma} S^{\rho\sigma}). \tag{5}
$$

Suppose $\bar{g}_{\mu\nu}$ is a maximally symmetric vacuum of the theory and we would like to study excitations ($h_{\mu\nu}$) about this vacuum. If this vacuum is flat, then our task is easy since all we need is to expand the action up to quadratic order in the curvature, then expand the resultant action up to $O(h^2_{\mu\nu})$ and check the propagating modes in the theory. But, if this vacuum is an (A)dS space, then in principle all the terms in the curvature expansion contribute to the free theory [that is the vacuum and the $O(h^2_{\mu\nu})$ theory], and hence one has a highly complicated task. Therefore, in building viable BI actions for gravity the main hurdle is to satisfy the tree-level unitarity around nonflat maximally symmetric backgrounds, but, fortunately, we have built the necessary formalism to carry out this task in [10]. (The Ph.D thesis [24] of one of the authors is devoted to these issues and expounds upon many of the discussions in the published papers.)

It was shown in these works (and we shall give another argument in this paper) that in *four dimensions* no terms beyond $O(A^2_{\mu\nu})$ expansion around $A_{\mu\nu} = 0$ contribute to the free theory, namely the vacuum and the excitations. Therefore, to study the excitations of (3) about its maximally symmetric vacua in four dimensions, all one needs to study is the following theory:

$$
I = \frac{1}{\kappa\gamma} \int d^4 x \sqrt{-g} \left[A - 2\gamma \Lambda_0 + \frac{1}{4} A^2 - \frac{1}{2} A_{\mu\nu} A^{\mu\nu} \right], \tag{6}
$$

with $A \equiv A^{\mu}_{\mu}$, and note that this is a fourth order theory in the curvature. For generic even n dimensions, one needs to expand up to $O(A^{n/2}_{\mu\nu})$, and for odd dimensions, all the powers contribute. In four dimensions, because of the identity

$$
C_{\mu\rho\sigma\lambda} C_{\nu}{}^{\rho\sigma\lambda} = \frac{1}{4} g_{\mu\nu} C_{\alpha\rho\sigma\lambda} C^{\alpha\rho\sigma\lambda}, \tag{7}
$$

we can also eliminate a_1 or b_1; without loss of generality, we choose $a_1 = 0$. Note that instead of this basis (namely the Weyl, Ricci and traceless-Ricci tensors), one can use the Riemann and Ricci tensors and the scalar curvature, which we do in Appendix B for the purpose of comparison. There, we also give formulas relating one basis to the other. The $A_{\mu\nu}$ tensor with these seven dimensionless parameters looks cumbersome, but in what follows unitarity of theory with only massless spin-2 excitations about the (A)dS vacua will eliminate three (or four depending on the theory) of these parameters and in addition, conforming to the notion of minimality, will lead to a theory without free dimensionless parameters. Note that we do not count the dimensionful BI parameter γ which can be constrained by experiments: As long as γR is small, any γ is viable in our analysis. Hence, it should be considered as a new dimensionful parameter. Of course, not to introduce a new dimensionful parameter, one

[2]Nevertheless, it is remarkable that instead of the square root Lagrangian, if one considers a different power in n dimensions $[\det(g + \gamma G)]^{1/(n-1)}$ then one has a massive gravity theory without the nonlinear Boulware-Deser ghost [22,23].

can choose $\gamma = \kappa$ since they are of the same dimensions.[3] (Observe that, since $\kappa = 4\pi\ell_p^2$ with ℓ_p being the Planck length, the condition $\kappa R \ll 1$ is satisfied as long as we are far away from the Planck regime: $R \ll \frac{1}{\ell_p^2}$.)

The layout of the paper is as follows: In Sec. II, we recall that the "free theory" of BI gravity should be the same as the free theory of Einstein–Gauss-Bonnet (EGB) theory that describes unitary massless spin-2 excitations around flat and (A)dS spaces. In Sec. III, we give details of finding the maximally symmetric vacua of generic gravity theories, including the BI gravity, with the help of equivalent linear actions (ELAs) which circumvent the complicated task of deriving the field equations. In that section, we also derive the equivalent quadratic curvature action (EQCA) that has the same free theory, including the vacuum of the original generic gravity, specifically the BI theory. In Sec. IV, we determine the vacua of the BI gravity. In Sec. V, we impose that the BI gravity describes unitary massless spin-2 gravitons around its flat background. In Sec. VI, we study the unitarity of the BI theory around its unique viable (A)dS vacuum and impose the condition that only the massless spin-2 particle is allowed. In the appendixes, we give details of the computations relevant to the results in the text.

II. CONSTRUCTING THE BORN-INFELD ACTION

The most general quadratic theory in n dimensions that describes *only* massless spin-2 excitations around its flat or (A)dS vacuum is the EGB theory with the Lagrangian

$$\mathcal{L} = \frac{1}{\kappa}(R - 2\Lambda_0 + \gamma\chi_{\mathrm{GB}}), \tag{8}$$

where the GB combination is given as

$$\chi_{\mathrm{GB}} \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{9}$$

In four dimensions, the GB part is a total derivative, and hence does not contribute to the field equations and plays no role in the particle spectrum or the vacuum of the theory. But as we shall see here, it plays a major role in constructing BI-type actions: Namely, we will see that at the quadratic level BI gravity reduces to the EGB theory instead of the Einstein's theory even though classically they are equivalent. In some sense, the dimensionful parameter γ in front of the GB term plays the role of the BI parameter.

It is clear that flat space is a vacuum for $\Lambda_0 = 0$, and if $\Lambda_0 \neq 0$ (A)dS is the vacuum with $\Lambda = \Lambda_0$. In the basis discussed in the Introduction, we can recast the EGB action as

$$\mathcal{L} = \frac{1}{\kappa}\left[R - 2\Lambda_0 + \gamma\left(C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{2}{3}R^{\mu\nu}R_{\mu\nu} - \frac{8}{3}S^{\mu\nu}S_{\mu\nu}\right)\right], \tag{10}$$

where we have used the *four* dimensional identity

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2. \tag{11}$$

It is not difficult to see that (10) describes only massless spin-2 excitations in flat and AdS vacua. There are many ways to show this but because this is almost common knowledge let us briefly sketch the proof without going into further details: Linearization of the field equations derived from (10) about its (A)dS vacuum yields

$$\frac{1}{\kappa}\mathcal{G}_{\mu\nu} = 0, \tag{12}$$

where $\mathcal{G}_{\mu\nu}$ is the linearized Einstein tensor, which in the transverse-traceless gauge for perturbations $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ reads

$$\frac{1}{\kappa}\mathcal{G}_{\mu\nu} = -\frac{1}{2\kappa}\left(\bar{\Box} - \frac{2\Lambda_0}{3}\right)h_{\mu\nu} = 0. \tag{13}$$

Despite the appearance of a masslike term, Eq. (13), together with the transversality and the tracelessness conditions, describes a massless spin-2 excitation. This can be easily seen if one writes the AdS metric in its conformal to flat coordinates $\bar{g}_{\mu\nu} = \Omega^2\eta_{\mu\nu}$ with $\Omega = (1 - \frac{3}{4}\Lambda_0 x^2)^{-1}$ where $x^2 = \eta_{\mu\nu}x^\mu x^\nu$ which reduces (13) to the massless wave equation in flat space $\partial^2 h_{\mu\nu} = 0$ [25]. For this massless spin-2 excitation to be unitary, there is only one condition that is the positivity of the Newton's constant, $\kappa > 0$; namely, gravity is attractive or, similarly, the massless spin-2 field has a positive kinetic energy as seen from the action (in the mostly positive signature convention)

$$I = \int d^4x\sqrt{-\bar{g}}\left\{\frac{1}{4\kappa}h^{\mu\nu}\left(\bar{\Box} - \frac{2\Lambda_0}{3}\right)h_{\mu\nu} + h^{\mu\nu}T_{\mu\nu}\right\}. \tag{14}$$

If one requires a generic gravity theory of the form $\mathcal{L} \equiv \sqrt{-g}f(R^{\mu\nu}_{\rho\sigma})$, namely with a Lagrangian density built from arbitrary powers of the Riemann tensor and its contractions but not its derivatives to propagate only unitary massless spin-2 excitations, it should have the same propagator structure as the EGB theory (or equivalently the cosmological Einstein's theory). It appears to be a highly cumbersome task to find the propagator of a generic gravity theory or a BI-type gravity theory in constant curvature backgrounds because, in principle, infinitely many terms

[3] In Born-Infeld electrodynamics, one necessarily introduces a dimensionful BI parameter, but in BI gravity one can simply recycle Newton's constant and no new parameter is introduced.

contribute to the propagator. Fortunately, there is a highly useful shortcut which works by constructing an *equivalent quadratic curvature action* that has the same propagator structure and the vacua as the generic theory under study. We work this out in the next section.

III. EQUIVALENT LINEAR ACTION AND EQUIVALENT QUADRATIC CURVATURE ACTION

The first step in finding the particle spectrum of a given gravity theory about its maximally symmetric vacuum is to show that the theory in fact admits such a vacuum, and if it does admit such a solution, one must find the effective cosmological constant of the vacuum. The most direct way to find the maximally symmetric vacuum is to derive the field equations first and then solve these equations. But, for the determinantal actions of the form that we study in this work or for higher derivative theories with many powers of curvature, finding the field equations is by itself a difficult task. As a demonstration of the complication, the reader could check the field equations of the action we study in this work in Appendix F.

In this section, we shall give a method to find the maximally symmetric vacuum or vacua of a given theory which circumvents the procedure of deriving the field equations. The method involves constructing an ELA that has the same vacuum or vacua as the original action, and it is so powerful that it pays to lay out some details here.

Consider a generic action of the form[4]

$$I = \int d^n x\sqrt{-g} f(g^{\alpha\beta}, R^\mu{}_{\nu\rho\sigma}, \nabla_\rho R^\mu{}_{\nu\rho\sigma}, \ldots, \nabla_{\rho_1}\nabla_{\rho_2}\ldots\nabla_{\rho_m} R^\mu{}_{\nu\rho\sigma}), \tag{15}$$

for which we ask if it admits a maximally symmetric vacuum, and if it does, what is the effective cosmological constant? Here we work in generic n dimensions. At this stage, it is clear that the derivative terms will not contribute to the maximally symmetric vacuum since they will yield covariant derivatives of the metric tensor at the level of the field equations which vanish by metric compatibility. Therefore, for notational simplicity let us denote the action as

$$I = \int d^n x\sqrt{-g} f(R^{\mu\nu}_{\alpha\beta}), \tag{16}$$

where we have also gotten rid of the inverse metric without loss of generality and taken the independent variable to be $R^{\mu\nu}_{\alpha\beta}$ which could stand for the Riemann tensor, or if once contracted to the Ricci tensor, and if twice contracted to the scalar curvature. For example, the Einstein-Hilbert action in this language reads $\int d^n x\sqrt{-g}\delta^\alpha_\mu\delta^\beta_\nu R^{\mu\nu}_{\alpha\beta}$. To find the field equations for the maximally symmetric spacetime, one varies the action as

$$\delta I = \int d^n x\left(\delta\sqrt{-g} f(R^{\mu\nu}_{\alpha\beta}) + \sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\delta R^{\mu\nu}_{\rho\sigma}\right). \tag{17}$$

Needless to say, this procedure will not yield the full equations of the most general theory (15) but only the part relevant for the maximally symmetric spacetime. On the other hand, if f does not depend on the derivatives of the Riemann tensor, as will be the case in this work, it will yield the full equations. We can write the variation of the Riemann tensor as

$$\begin{aligned}\delta R^{\mu\nu}_{\rho\sigma} = {}& \frac{1}{2}(g_{\alpha\rho}\nabla_\sigma\nabla^\nu - g_{\alpha\sigma}\nabla_\rho\nabla^\nu)\delta g^{\mu\alpha} \\ & -\frac{1}{2}(g_{\alpha\rho}\nabla_\sigma\nabla^\mu - g_{\alpha\sigma}\nabla_\rho\nabla^\mu)\delta g^{\alpha\nu} \\ & -\frac{1}{2}R_{\rho\sigma}{}^\nu{}_\alpha\delta g^{\mu\alpha} + \frac{1}{2}R_{\rho\sigma}{}^\mu{}_\alpha\delta g^{\alpha\nu},\end{aligned} \tag{18}$$

which was obtained from

$$\delta R^\mu{}_{\nu\rho\sigma} = \nabla_\rho\delta\Gamma^\mu_{\nu\sigma} - \nabla_\sigma\delta\Gamma^\mu_{\nu\rho}. \tag{19}$$

In calculating the derivative $\frac{\partial f}{\partial R^{\mu\nu}_{\alpha\beta}}$ in (17), one may try to symmetrize it in such a way that it satisfies the symmetries of the Riemann tensor. However, this is not required since at the end, it is multiplied with $\delta R^{\mu\nu}_{\rho\sigma}$ which kills the parts of $\frac{\partial f}{\partial R^{\mu\nu}_{\alpha\beta}}$ that do not obey the symmetries of the Riemann tensor. Then, inserting the variation of the Riemann tensor (18) into the varied action (17) leads to a bunch of terms

$$\begin{aligned}\delta I = {}& \int d^n x\left(-\frac{1}{2}g_{\mu\nu}\sqrt{-g} f(R^{\alpha\beta}_{\rho\sigma})\delta g^{\mu\nu}\right) \\ & +\frac{1}{2}\int d^n x\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}(g_{\alpha\rho}\nabla_\sigma\nabla^\nu - g_{\alpha\sigma}\nabla_\rho\nabla^\nu)\delta g^{\mu\alpha} \\ & -\frac{1}{2}\int d^n x\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}(g_{\alpha\rho}\nabla_\sigma\nabla^\mu - g_{\alpha\sigma}\nabla_\rho\nabla^\mu)\delta g^{\alpha\nu} \\ & -\frac{1}{2}\int d^n x\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}(R_{\rho\sigma}{}^\nu{}_\alpha\delta g^{\mu\alpha} - R_{\rho\sigma}{}^\mu{}_\alpha\delta g^{\alpha\nu}).\end{aligned} \tag{20}$$

After integration by parts and dropping the boundary terms, one arrives at the field equations

[4]Of course one can work in the basis introduced in the Introduction, but here we work with the Riemann tensor, as this basis is more common in many other applications.

$$\frac{1}{2}(g_{\nu\rho}\nabla^\lambda\nabla_\sigma - g_{\nu\sigma}\nabla^\lambda\nabla_\rho)\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} - \frac{1}{2}(g_{\mu\rho}\nabla^\lambda\nabla_\sigma - g_{\mu\sigma}\nabla^\lambda\nabla_\rho)\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}} - \frac{1}{2}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu} - \frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\mu}\right) - \frac{1}{2}g_{\mu\nu}f(R^{\alpha\beta}_{\rho\sigma}) = 0. \tag{21}$$

For the maximally symmetric spacetimes, the first line of the field equations just yields zero. Therefore, the relevant part of the field equations that determines the effective cosmological constant is the second line

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\nu} - \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\mu} + g_{\mu\nu}f(\bar{R}^{\alpha\beta}_{\rho\sigma}) = 0, \tag{22}$$

where the barred quantities are evaluated at the maximally symmetric value of the Riemann tensor given as

$$\bar{R}^{\mu\lambda}_{\rho\sigma} = \frac{2\Lambda}{(n-1)(n-2)}(\delta^\mu_\rho\delta^\lambda_\sigma - \delta^\mu_\sigma\delta^\lambda_\rho). \tag{23}$$

Equation (22) is the vacuum field equation, and the information on the functional form of the Lagrangian enters the field equation through *only two* background-evaluated quantities

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \qquad f(\bar{R}^{\alpha\beta}_{\rho\sigma}). \tag{24}$$

Therefore, Eq. (22) tells us that if these two quantities are the same for any given two gravity theories, then those two gravity theories have the same maximally symmetric vacua; namely their effective cosmological constants are equal. Then, for a theory defined by a given $f(R^{\mu\nu}_{\alpha\beta})$, one can determine the vacua of the theory by performing a first order Taylor series expansion around a yet to be determined maximally symmetric background as

$$I = \int d^n x\sqrt{-g}\left\{f(\bar{R}^{\mu\nu}_{\alpha\beta}) + \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}(R^{\lambda\nu}_{\rho\sigma} - \bar{R}^{\lambda\nu}_{\rho\sigma})\right\}, \tag{25}$$

which from now on will be called an equivalent linearized action (ELA). At the risk of being a little pedantic, let us reiterate the above observation: Considered as another generic gravity theory, Eq. (25) has the same vacua as (16). As a result, to get the effective cosmological constant of the vacuum (or vacua) of the most general gravity theory, all one needs to do is a first order Taylor series expansion of the generic theory in the Riemann tensor and construct the equivalent linear action.

Furthermore, let us show that (25) reduces to a cosmological Einstein-Hilbert action. Let us define ζ which satisfies

$$\left[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R. \tag{26}$$

Here, the term $[\partial f/\partial R^{\mu\nu}_{\rho\sigma}]_{\bar{R}^{\mu\nu}_{\rho\sigma}}$ is made up of the Kronecker deltas such as $\delta^\rho_\mu\delta^\sigma_\nu$, and it should satisfy the symmetries of the Riemann tensor, so antisymmetrizing $\delta^\rho_\mu\delta^\sigma_\nu$ yields $\delta^{[\rho}_\mu\delta^{\sigma]}_\nu$.[5] Considering this together with (26) yields the background-evaluated first derivative of $f(R^{\mu\nu}_{\alpha\beta})$ as

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} = \zeta\delta^{[\rho}_\mu\delta^{\sigma]}_\nu. \tag{27}$$

Using these, one can recast the equivalent linear action (25) in a more explicit form such that it becomes the usual cosmological Einstein-Hilbert action as

$$I_{\text{ELA}} = \frac{1}{\kappa_l}\int d^n x\sqrt{-g}(R - 2\Lambda_{0,l}), \tag{28}$$

where the subindex l denotes the equivalent linear action values. The effective Newton's constant and the effective "bare" cosmological constant are

$$\frac{1}{\kappa_l} = \zeta, \qquad \frac{\Lambda_{0,l}}{\kappa_l} = -\frac{1}{2}\bar{f} + \frac{n\Lambda}{n-2}\zeta, \tag{29}$$

where we have used $\bar{R} = \frac{2n\Lambda}{n-2}$ and defined $\bar{f} \equiv f(\bar{R}^{\alpha\beta}_{\rho\sigma})$. Then, the field equation for the maximally symmetric background is simply $\Lambda = \Lambda_{0,l}$ which yields $\Lambda = \frac{n-2}{4\zeta}\bar{f}$. Note that this is definitely the field equation that one gets after putting (23) and (27) in (22).

This construction implies that the maximally symmetric vacua of a generic gravity theory can also be found by expanding the original action in the metric perturbation $h_{\mu\nu}$ up to the first order $O(h_{\mu\nu})$ and taking the variation with respect to $h_{\mu\nu}$.

Once the vacuum of the theory is established, one can move on to discuss the particle spectrum around this

[5]Note that the total antisymmetrization in the up indices implies the total antisymmetrization in the down indices, that is, $\delta^{[\rho}_\mu\delta^{\sigma]}_\lambda = \delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\lambda]}$.

vacuum by expanding the action up to $O(h_{\mu\nu}^2)$ in the metric perturbation. Directly expanding the action in powers of $h_{\mu\nu}$ is a highly complicated task, but again, fortunately, a similar method to the one described above exists [26]. The method amounts to finding an EQCA that has the same degrees of freedom around the same vacua as the original theory. EQCA can be found by expanding the action in Taylor series up to quadratic order in the Riemann tensor as we show below. Here we shall assume that the action does not depend on the derivatives of the Riemann tensor.

For an action that does not depend on the derivatives of the Riemann tensor, the field equations are (21). To analyze the spectrum of the $f(R^{\alpha\beta}_{\rho\sigma})$ theory, that is, the excitations around a given background, one needs the linearized field equations from which one can identify the excitations by decoupling the linearized field equations into a set of individual wave equations for each excitation. As we discussed above, one way to obtain this linearized field equation is to expand the action in $h_{\mu\nu}$ up to second order and perform variation with respect to $h_{\mu\nu}$. On the other hand, naturally, one can also directly linearize the field equations (21). To obtain the linearized field equations, one needs the following two linearized tensors:

$$[g_{\mu\nu} f(R^{\mu\nu}_{\alpha\beta})]_L = h_{\mu\nu} f(\bar{R}^{\mu\nu}_{\alpha\beta}) + \bar{g}_{\mu\nu}\left[\frac{\partial f}{\partial R^{\alpha\beta}_{\rho\sigma}}\right]_{\bar{R}^{\alpha\beta}_{\rho\sigma}} (R^{\alpha\beta}_{\rho\sigma})_L \tag{30}$$

and

$$\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} R^{\lambda}_{\rho\sigma\nu}\right)_L = \left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} (R^{\eta\theta}_{\alpha\tau})_L \bar{R}_{\rho\sigma}{}^{\lambda}{}_{\nu} + \left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} (R_{\rho\sigma}{}^{\lambda}{}_{\nu})_L, \tag{31}$$

and

$$\left(g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = \bar{g}_{\nu\rho}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \bar{\nabla}^{\lambda}\bar{\nabla}_{\sigma}(R^{\eta\theta}_{\alpha\tau})_L + \bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\alpha}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \bar{\nabla}^{\lambda}(\Gamma^{\sigma}_{\sigma\alpha})_L - \bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\alpha\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \bar{\nabla}^{\lambda}(\Gamma^{\alpha}_{\sigma\mu})_L - \bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\mu\alpha}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \bar{\nabla}^{\lambda}(\Gamma^{\alpha}_{\sigma\lambda})_L, \tag{32}$$

where the subindex L means that the quantity is expanded up to $O(h_{\mu\nu})$. The linearization of the other terms in (21) follows from these terms upon symmetrization and antisymmetrization. Notice that the information on the functional form of the Lagrangian enters the linearized field equations through the following three background-evaluated quantities:

$$\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \qquad \left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \qquad f(\bar{R}^{\alpha\beta}_{\rho\sigma}). \tag{33}$$

Therefore, if these three quantities are the same for any two given gravity theories, then those two theories have the same spectrum around the same vacua. Then, for a theory defined by a given $f(R^{\mu\nu}_{\alpha\beta})$, one can determine the spectrum of the theory through the quadratic gravity defined by the up-to-second-order Taylor series expansion of $f(R^{\mu\nu}_{\alpha\beta})$ around the maximally symmetric background as

$$I = \int d^n x\sqrt{-g}\left\{f(\bar{R}^{\mu\nu}_{\alpha\beta}) + \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} (R^{\lambda\nu}_{\rho\sigma} - \bar{R}^{\lambda\nu}_{\rho\sigma}) + \frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} (R^{\eta\theta}_{\alpha\tau} - \bar{R}^{\eta\theta}_{\alpha\tau})(R^{\mu\lambda}_{\rho\sigma} - \bar{R}^{\mu\lambda}_{\rho\sigma})\right\}, \tag{34}$$

which from now on will be called the equivalent quadratic curvature action (EQCA). Note that this action not only has the same spectrum but also has the same vacua as the original $f(R^{\mu\nu}_{\alpha\beta})$ theory.

Now, let us further recast (34) in the form of a quadratic gravity theory. To do this, first let us define the quadratic curvature parameters α, β, and γ as

$$\frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\eta\theta}_{\alpha\tau} R^{\mu\lambda}_{\rho\sigma} \equiv \alpha R^2 + \beta R^{\lambda}_{\sigma} R^{\sigma}_{\lambda} + \gamma(R^{\eta\lambda}_{\rho\sigma} R^{\rho\sigma}_{\eta\lambda} - 4R^{\lambda}_{\sigma}R^{\sigma}_{\lambda} + R^2). \tag{35}$$

Since the background-evaluated second order derivative of $f(R^{\mu\nu}_{\alpha\beta})$ just involves Kronecker deltas and obeys the symmetries of the Riemann tensors $R^{\eta\theta}_{\alpha\tau}$ and $R^{\mu\lambda}_{\rho\sigma}$, one has

$$\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = 2\alpha\delta^{[\alpha}_{\eta}\delta^{\tau]}_{\theta}\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\lambda} + \beta(\delta^{\alpha}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\tau|}_{[\mu}\delta^{\sigma]}_{\lambda]} - \delta^{\tau}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\alpha|}_{[\mu}\delta^{\sigma]}_{\lambda]}) + 12\gamma\delta^{[\alpha}_{\eta}\delta^{\tau}_{\theta}\delta^{\rho}_{\mu}\delta^{\sigma]}_{\lambda}, \tag{36}$$

where the last term, clearly, should have the totally antisymmetric form since the Gauss-Bonnet combination is the quadratic Lovelock term.[6] Using these together with (26), one can put (34) in a more explicit form as a quadratic gravity theory F[27]:

$$I_{\mathrm{EQCA}} = \int d^n x \sqrt{-g}\left[\frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0) + \alpha R^2 + \beta R^{\lambda}_{\sigma} R^{\sigma}_{\lambda} + \gamma \chi_{\mathrm{GB}}\right], \tag{37}$$

where the effective Newton's constant is given as

$$\frac{1}{\tilde{\kappa}} = \zeta - \frac{4\Lambda}{n-2}\left[n\alpha + \beta + \gamma\frac{(n-2)(n-3)}{(n-1)}\right], \tag{38}$$

and the effective "bare" cosmological constant reads

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} = -\frac{1}{2} f(\bar{R}^{\alpha\beta}_{\rho\sigma}) + \frac{n\Lambda}{n-2}\zeta - \frac{2\Lambda^2 n}{(n-2)^2}\left[n\alpha + \beta + \gamma\frac{(n-2)(n-3)}{(n-1)}\right]. \tag{39}$$

The maximally symmetric solution of (37) satisfies [28]

$$\frac{\Lambda - \tilde{\Lambda}_0}{2\tilde{\kappa}} + \left[(n\alpha + \beta)\frac{(n-4)}{(n-2)^2} + \gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}\right]\Lambda^2 = 0, \tag{40}$$

which certainly is the same vacuum equation as that of the $f(R^{\mu\nu}_{\alpha\beta})$ theory and its equivalent linearized version (28). We made the equivalence between the linearized field equations of the $f(R^{\mu\nu}_{\alpha\beta})$ theory and (37) more explicit in Appendix D.

A. ELA and EQCA construction for Born-Infeld gravity

The above discussion was for generic $f(R^{\mu\nu}_{\alpha\beta})$ theories; let us now focus on the BI-type theories. To calculate the EQCA, one basically needs (A)dS background calculated values for the matrix function $\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})}$ and its first and second derivatives. (Note that we are pulling out a factor of $\sqrt{-\det g}$ so that we can work with the Kronecker delta δ^{ν}_{μ} whose variation is zero.) First, the background value of $\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})}$ is given as

$$\sqrt{\det(\delta^{\rho}_{\nu} + \gamma \bar{A}^{\rho}_{\nu})} = (1 + \bar{a})^{\frac{n}{2}}, \tag{41}$$

where $\bar{a}$ is defined via $\gamma \bar{A}^{\rho}_{\nu} = \bar{a}\delta^{\rho}_{\nu}$. Then, by using $\det N = \exp(\mathrm{Tr}(\ln N))$, the first and second order differentials of $\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})}$ can be, respectively, expressed as

$$\partial\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})} = \frac{\gamma}{2}\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})} B^{\lambda}_{\gamma}\partial A^{\gamma}_{\lambda} \tag{42}$$

and

$$\partial^2\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})} = \frac{\gamma}{2}\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})}\left[B^{\lambda}_{\gamma}\partial^2 A^{\gamma}_{\lambda} - \gamma B^{\lambda}_{\theta} B^{\tau}_{\gamma}(\partial A^{\theta}_{\tau})\partial A^{\gamma}_{\lambda} + \frac{\gamma}{2}(B^{\lambda}_{\gamma}\partial A^{\gamma}_{\lambda})^2\right], \tag{43}$$

where B^{λ}_{γ} represents the inverse of the matrix $(\delta^{\lambda}_{\gamma} + \gamma A^{\lambda}_{\gamma})$ and for the differential of B we use $\partial B = -\gamma B(\partial A)B$. Note that one may not be able to find the explicit form of the B matrix for a given A matrix, and in fact, even for the simple case of $A_{\mu\nu} = R_{\mu\nu}$ it is not possible to find the explicit form of B. However, this is not needed since just the (A)dS background value of the B matrix is required to calculate the background values for the first and second derivatives of $\sqrt{\det(\delta^{\rho}_{\nu} + \gamma A^{\rho}_{\nu})}$. One can calculate it as

$$\bar{B}^{\lambda}_{\gamma} = (1 + \bar{a})^{-1}\delta^{\lambda}_{\gamma}. \tag{44}$$

Note that the matrix $(I + A)$ becomes singular for $\bar{a} = -1$, so we assume that $\bar{a} \neq -1$. In the absence of the specific definition for the A tensor, there is no need to further study the background values of (42) and (43) by employing (41) and (44).

To find the EQCA for a specific BI gravity theory, one needs to find $\bar{a}$ and calculate the first and second derivatives of the A tensor with respect to the Riemann tensor, $R^{\mu\nu}_{\rho\sigma}$. Then, the formulas (41)–(44) are enough to work out the EQCA for the BI gravity theory.

1. Even-dimensional EQCA and ELA

In order to calculate the EQCA and ELA of BI gravity, one needs to calculate the following three (A)dS background-evaluated quantities as explicitly seen from (34):

$$\sqrt{\det(\delta^{\beta}_{\nu} + \gamma \bar{A}^{\beta}_{\nu})}, \qquad \left[\frac{\partial}{\partial R^{\lambda\nu}_{\rho\sigma}}\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}},$$

$$\left[\frac{\partial}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}. \tag{45}$$

[6] More explicitly, the Lovelock Lagrangian density can be written as $\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4} = 4\chi_{\mathrm{GB}}$ where $\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} = \epsilon_{abcd}\delta^{\mu_1}_{\nu_a}\delta^{\mu_2}_{\nu_b}\delta^{\mu_3}_{\nu_c}\delta^{\mu_4}_{\nu_d} = 4!\delta^{\mu_1}_{\nu_{[a}}\delta^{\mu_2}_{\nu_b}\delta^{\mu_3}_{\nu_c}\delta^{\mu_4}_{\nu_{d]}}$.

Using (41)–(44) and $\gamma\bar{A}^{\rho}_{\nu} = \bar{a}\delta^{\rho}_{\nu}$, the (A)dS-evaluated value of the Lagrangian can be calculated as

$$\sqrt{\det(\delta^{\beta}_{\nu} + \gamma\bar{A}^{\beta}_{\nu})} = (1+\bar{a})^{\frac{n}{2}}, \tag{46}$$

and its first derivative reads

$$\begin{aligned}\left[\frac{\partial}{\partial R^{\lambda\nu}_{\rho\sigma}}\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} &= \frac{\gamma}{2}\sqrt{\det(\delta^{\beta}_{\nu} + \gamma\bar{A}^{\beta}_{\nu})}\bar{B}^{\kappa}_{\gamma}\left[\frac{\partial A^{\gamma}_{\kappa}}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \\ &= \frac{\gamma}{2}(1+\bar{a})^{\frac{(n-2)}{2}}\delta^{\kappa}_{\gamma}\left[\frac{\partial A^{\gamma}_{\kappa}}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}},\end{aligned} \tag{47}$$

and finally its second derivative boils down to

$$\begin{aligned}\left[\frac{\partial^2}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} &= \frac{\gamma}{2}(1+\bar{a})^{\frac{(n-2)}{2}}\delta^{\kappa}_{\gamma}\left[\frac{\partial^2 A^{\gamma}_{\kappa}}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \frac{\gamma^2}{2}(1+\bar{a})^{\frac{(n-4)}{2}}\delta^{\kappa}_{\xi}\delta^{\zeta}_{\gamma}\left[\frac{\partial A^{\xi}_{\zeta}}{\partial R^{\eta\theta}_{\alpha\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\left[\frac{\partial A^{\gamma}_{\kappa}}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \\ &\quad + \frac{\gamma^2}{4}(1+\bar{a})^{\frac{(n-4)}{2}}\delta^{\kappa}_{\gamma}\delta^{\zeta}_{\xi}\left[\frac{\partial A^{\gamma}_{\kappa}}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\left[\frac{\partial A^{\xi}_{\zeta}}{\partial R^{\eta\theta}_{\alpha\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}.\end{aligned} \tag{48}$$

Here, a difference between the odd and even dimensional cases arises: It is important to notice that only finite integer powers of $\bar{a}$ appear in these expansions for even dimensions. But, for odd dimensions, infinite powers of $\bar{a}$ appear. This observation is crucial, because the same second order expansion in curvature around an (A)dS background can be obtained by first performing an infinite order expansion in $A_{\mu\nu}$ "around $A_{\mu\nu} = 0$," and then carrying out the second order expansion in curvature by using this infinite order series in $A_{\mu\nu}$. *A priori*, all orders in $A_{\mu\nu}$ contribute to the second order expansion in curvature in (A)dS backgrounds; however, as we just observed for even dimensions, only a finite number of terms in the $A_{\mu\nu}$ expansion contribute to the EQCA. More specifically for even n dimensions expansion up to $O(A^{\frac{n}{2}}_{\mu\nu})$ is necessary and sufficient. In four dimensions, we need to expand up to $O(A^2_{\mu\nu})$, as given in (I). On the other hand, one needs all the powers of $A^i_{\mu\nu}$ for odd dimensions.

Since there are nontrivial cancellations, it pays to make this observation more explicit in four dimensions. To construct the EQCA of the $O(A^2_{\mu\nu})$ which is

$$\left[\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{O(A^2)} = 1 + \frac{\gamma}{2}A^{\mu}_{\mu} + \frac{\gamma^2}{8}A^{\mu}_{\mu}A^{\nu}_{\nu} - \frac{\gamma^2}{4}A^{\nu}_{\mu}A^{\mu}_{\nu}, \tag{49}$$

one first needs the (A)dS background-evaluated value of the $O(A^2_{\mu\nu})$ Lagrangian which can be calculated by putting $\gamma\bar{A}^{\rho}_{\nu} = \bar{a}\delta^{\rho}_{\nu}$ in (49) as

$$\left[\sqrt{\det(\delta^{\beta}_{\nu} + \gamma\bar{A}^{\beta}_{\nu})}\right]_{O(A^2)} = 1 + 2\bar{a} + 2\bar{a}^2 - \bar{a}^2 = (1+\bar{a})^2, \tag{50}$$

which is an exact expression represented by a finite number of terms in the $A_{\mu\nu}$ expansion, and it matches (46) when $n = 4$. Moving on to the first and second derivatives (49), one gets, respectively,

$$\begin{aligned}\frac{\partial}{\partial R^{\lambda\nu}_{\rho\sigma}}\left[\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{O(A^2)} &= \frac{\gamma}{2}\frac{\partial A^{\kappa}_{\kappa}}{\partial R^{\lambda\nu}_{\rho\sigma}} + \frac{\gamma^2}{4}A^{\beta}_{\beta}\frac{\partial A^{\kappa}_{\kappa}}{\partial R^{\lambda\nu}_{\rho\sigma}} \\ &\quad - \frac{\gamma^2}{2}A^{\beta}_{\kappa}\frac{\partial A^{\kappa}_{\beta}}{\partial R^{\lambda\nu}_{\rho\sigma}}\end{aligned} \tag{51}$$

and

$$\frac{\partial^2}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\left[\sqrt{\det(\delta^{\beta}_{\nu} + \gamma A^{\beta}_{\nu})}\right]_{O(A^2)} = \frac{\gamma}{2}\frac{\partial^2 A^{\kappa}_{\kappa}}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\lambda\nu}_{\rho\sigma}} + \frac{\gamma^2}{4}A^{\beta}_{\beta}\frac{\partial^2 A^{\kappa}_{\kappa}}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\lambda\nu}_{\rho\sigma}} - \frac{\gamma^2}{2}A^{\beta}_{\kappa}\frac{\partial^2 A^{\kappa}_{\beta}}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\lambda\nu}_{\rho\sigma}} + \frac{\gamma^2}{4}\frac{\partial A^{\beta}_{\beta}}{\partial R^{\eta\theta}_{\alpha\tau}}\frac{\partial A^{\kappa}_{\kappa}}{\partial R^{\lambda\nu}_{\rho\sigma}} - \frac{\gamma^2}{2}\frac{\partial A^{\beta}_{\kappa}}{\partial R^{\eta\theta}_{\alpha\tau}}\frac{\partial A^{\kappa}_{\beta}}{\partial R^{\lambda\nu}_{\rho\sigma}}. \tag{52}$$

These derivatives can be evaluated for the (A)dS background, respectively, as

$$\left[\frac{\partial}{\partial R^{\lambda\nu}_{\rho\sigma}}\left[\sqrt{\det(\delta^\beta_\nu+\gamma A^\beta_\nu)}\right]_{O(A^2)}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}=\frac{\gamma}{2}\left[\frac{\partial A^\kappa_\kappa}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}+\gamma\bar{a}\left[\frac{\partial A^\kappa_\kappa}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}-\frac{\gamma}{2}\bar{a}\delta^\beta_\kappa\left[\frac{\partial A^\kappa_\beta}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}$$
$$=\frac{\gamma}{2}(1+\bar{a})\delta^\beta_\kappa\left[\frac{\partial A^\kappa_\beta}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}, \tag{53}$$

$$\left[\frac{\partial^2}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\left[\sqrt{\det(\delta^\beta_\nu+\gamma A^\beta_\nu)}\right]_{O(A^2)}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}=\frac{\gamma}{2}(1+\bar{a})\left[\frac{\partial^2 A^\kappa_\kappa}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}+\frac{\gamma^2}{4}\left[\frac{\partial A^\beta_\beta}{\partial R^{\eta\theta}_{\alpha\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\left[\frac{\partial A^\kappa_\kappa}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}-\frac{\gamma^2}{2}\left[\frac{\partial A^\beta_\kappa}{\partial R^{\eta\theta}_{\alpha\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\left[\frac{\partial A^\kappa_\beta}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \tag{54}$$

which matches (47) and (48), respectively, when $n=4$. Hence, we have achieved our goal of showing that in four dimensions, to get the vacuum and the spectrum of a determinantal BI gravity theory, one needs to expand only up to $O(A^2_{\mu\nu})$ around $A_{\mu\nu}=0$. It is a remarkable fact about this determinantal action that at every order $O(A^{2+i}_{\mu\nu})$, contributions of the various terms to the EQCA cancel among each other; that is, they do not contribute to the free theory around the maximally symmetric background. Moreover, this cancellation works in such a way that $O(\gamma^0)$, $O(\gamma^1)$ and $O(\beta)$ terms cancel among each other. This fact also means that when $A_{\mu\nu}$ is at most quadratic in curvature as in (4), one can also get the same result by performing a Taylor series expansion in small curvature, that is, around $R^{\mu\nu}_{\rho\sigma}=0$, up to $O(R^4)$ in four dimension; hence, we end up with a quartic gravity theory. This also immediately leads to the fact that *a priori* there will be four possible maximally symmetric vacua of the theory which we study next.

IV. DETERMINING THE VACUA OF THE BI THEORY

Let us find the maximally symmetric vacua (generically there will be four different vacua in four dimensions as noted above) of our theory,

$$\kappa\mathcal{L}=\frac{2}{\gamma}\left[\sqrt{\det(\delta^\rho_\sigma+\gamma A^\rho_\sigma)}-(\lambda_0+1)\right], \tag{55}$$

with $A_{\mu\nu}$ as (4), and we have defined a dimensionless cosmological parameter $\lambda_0\equiv\gamma\Lambda_0$ which we shall use from now on. We resort to the equivalent linear action formalism described above which in this basis follows from the zeroth and the first order Taylor series expansion of the action (55),

$$\kappa\mathcal{L}_{\rm ELA}=\frac{2}{\gamma}\left[\sqrt{\det(\delta^\rho_\sigma+\gamma\bar{A}^\rho_\sigma)}-(\lambda_0+1)\right]+\left[\frac{\partial\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}C^{\mu\nu}_{\alpha\beta}$$
$$+\left[\frac{\partial\mathcal{L}}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}S^\mu_\nu+\left[\frac{\partial\mathcal{L}}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}(R^\mu_\nu-\bar{R}^\mu_\nu). \tag{56}$$

Note that the background values $\bar{C}^{\mu\nu}_{\alpha\beta}$ and $\bar{S}^\mu_\nu$ are zero, and $\bar{R}^{\mu\nu}_{\rho\sigma}$ is given in (23) with $n=4$ and $\bar{R}^\mu_\nu=\Lambda\delta^\mu_\nu$. The background values of the Lagrangian density and its first order derivatives are calculated in Appendix E, and using these results, the equivalent linearized action of the BI theory given in (3) becomes

$$\mathcal{L}_{\rm ELA}=\frac{1}{\kappa_l}\left(R-\frac{2}{\gamma}\lambda_{0,l}\right). \tag{57}$$

The Newton's constant and the cosmological constant, upon using the computations in Appendix E, turn out to be

$$\frac{1}{\kappa_l}=(1+\bar{a})[1+2\lambda(a_3+b_2)],$$
$$\bar{a}=\lambda+\lambda^2(a_3+b_2),$$
$$\lambda_{0,l}=\kappa_l(1+\lambda_0-(1+\bar{a})^2)+2\lambda, \tag{58}$$

where we assumed $1/\kappa_l$ is not zero; otherwise the theory would not reproduce Einstein's gravity. Then, since the vacua of the equivalent linear theory is determined by

$$\lambda=\lambda_{0,l}, \tag{59}$$

one arrives at the quartic equation that gives the four possible maximally symmetric vacua

$$c^2\lambda^4+c\lambda^3-\lambda+\lambda_0=0, \tag{60}$$

where we have defined $c\equiv a_3+b_2$. Let us not depict the solutions in their general form since they are not needed, but let us note some specific points. If $c=0$, then we have a unique vacuum with $\lambda=\lambda_0$. If $c\neq 0$, then there are real and complex solutions depending on the values of c and λ_0. It is possible to have four real solutions if $0<c<\frac{1}{4}$ and if a complicated condition on λ_0 is satisfied. As an example, let us take $c=\frac{1}{8}$ and $\lambda_0=\frac{3}{4}$; then we have $\lambda=(-6,-2(1\pm\sqrt{2}),2)$. But, as we shall see later on, unitarity will demand that $c=\frac{1}{4}$ and $\lambda<1$, and the non-vanishing of the effective Newton's constant demands that

$\lambda \neq -2$. These conditions are satisfied if $\lambda_0 < \frac{11}{16}$ and $\lambda_0 \neq -1$. One can compute the discriminant to be $\Delta = \frac{1}{256}(1+\lambda_0)^2(-11+16\lambda_0)$ which is always negative in the allowed region. This says that there are two real and two complex-conjugate roots. One of the real roots does not satisfy the $\lambda < 1$ condition, but the other one always satisfies this condition. Therefore, we have a unique *viable* vacuum.

V. UNITARITY AROUND FLAT BACKGROUNDS

First, let us start with the flat space $\lambda = 0$ for which we must take the bare cosmological parameter to be $\lambda_0 = 0$. From (60) we see that flat space is the vacuum of the theory but it is not the only vacuum since the equation reduces to

$$\lambda(c^2\lambda^3 + c\lambda^2 - 1) = 0, \tag{61}$$

with at least one more real solution with a nonzero λ with the exception that $c = 0$. For example, for $c = \frac{1}{4}$, whose relevance will appear below, one has $\lambda \approx 1.679$.

Let us focus on the flat vacuum. In this case our job is not complicated at all: All we need to do is expand (3) up to $O(R^2)$ and demand that it matches either with the Einstein-Hilbert action or with the Einstein–Gauss-Bonnet action.

The determinant $[\det(1+M)]^{1/2}$ can be expanded up to $O(M^4)$ as

$$[\det(1+M)]^{1/2} = 1 + \frac{1}{2}\mathrm{Tr}M + \frac{1}{8}(\mathrm{Tr}M)^2 - \frac{1}{4}\mathrm{Tr}(M^2) + \frac{1}{6}\mathrm{Tr}(M^3) - \frac{1}{8}\mathrm{Tr}M\mathrm{Tr}(M^2) + \frac{1}{48}(\mathrm{Tr}M)^3, \tag{62}$$

where $\mathrm{Tr}M = g^{\mu\nu}M_{\mu\nu}$. Using this expression, the $O(R^2)$ expansion of (3) yields the quadratic Lagrangian

$$\kappa\mathcal{L}_{O(R^2)} = R + \gamma b_1 C_{\mu\rho\sigma\lambda}C^{\mu\rho\sigma\lambda} + \gamma\left(a_3 + b_2 + \frac{1}{2}\right)R_{\mu\rho}R^{\mu\rho} + \gamma\left(a_4 + b_3 - 1 - \frac{\beta(\beta+2)}{2}\right)S_{\mu\rho}S^{\mu\rho}. \tag{63}$$

Here, note that a_2 does not appear in the $O(R^2)$ expansion, so unitarity constraints around the flat background do not put any condition on the $C_{\mu\rho\nu\sigma}R^{\rho\sigma}$ term. As already noted, there are two unitary theories that (63) can reduce to: the Einstein theory and the EGB theory, which need separate attention even though they are classically equivalent in four dimensions. Let us start with the reduction to Einstein's theory.

A. Reduction to the Einstein theory

We will compare (63) with

$$\kappa\mathcal{L} = R, \tag{64}$$

which yields the elimination of three parameters,

$$b_1 = 0, \qquad a_3 = -\frac{1}{2} - b_2, \qquad a_4 = \frac{\beta(\beta+2)}{2} + 1 - b_3, \tag{65}$$

leaving a theory with four dimensionless parameters that can be built from

$$A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \frac{\gamma}{4}g_{\mu\nu}(b_2 R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}) + \gamma\left[a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \left(\frac{1}{2} + b_2\right)R_{\mu\rho}R^{\rho}_{\nu} + \left(\frac{\beta(\beta+2)}{2} + 1 - b_3\right)S_{\mu\rho}S^{\rho}_{\nu}\right]. \tag{66}$$

It is important to understand that the BI gravity defined with this $A_{\mu\nu}$ describes a massless, unitary spin-2 graviton about its flat vacuum in all finite orders in the curvature expansion as well as the full theory, namely in the infinite order in the curvature expansion. This is because in flat backgrounds, only terms up to quadratic in curvature contribute to the propagator of the theory. However, this theory does not have quadratic terms, and when expanded in curvature it symbolically reads $\mathcal{L} = R + R^3 + \ldots$.

Let us also note that if we require the uniqueness of the vacuum, namely that the flat space is the unique vacuum, then one cannot reduce the theory to Einstein's gravity since a unique vacuum condition is achieved with $c = 0$ but Einsteinian reduction is achieved with $c = -\frac{1}{2}$ as seen from the second equation of (65), hence the contradiction. Therefore, the theory (66) has two vacua, one with $\lambda = 0$ and the other with $\lambda \approx 2.594$. Of course, to have a consistent theory we must check its unitarity about the second vacuum. As we shall see in the next section, $c = -\frac{1}{2}$ is excluded. Before that discussion, let us consider some specific theories by taking the undetermined dimensionless parameters to be zero.

Unitarity about its flat vacuum does not constrain this theory any further: Let us use the notion of minimality and fix the undetermined parameters. There could be many ways to define minimal theories here: For example, if we set $\beta = a_2 = a_3 = a_4 = 0$ we arrive at a unitary theory around its flat background with the action

$$I = \frac{2}{\kappa\gamma}\int d^4x\left\{\sqrt{-\det\left[g_{\mu\nu} + \gamma R_{\mu\nu} + \frac{\gamma^2}{8} g_{\mu\nu} R_{\sigma\rho} G^{\sigma\rho}\right]} - \sqrt{-\det g}\right\}, \tag{67}$$

which was already given in [1]. Another option is choosing $\beta = b_2 = b_3 = a_2 = 0$ which yields another theory

$$I = \frac{2}{\kappa\gamma}\int d^4x\left\{\sqrt{-\det\left[g_{\mu\nu} + \gamma R_{\mu\nu} + \frac{\gamma^2}{2}\left(G_{\mu\rho}G^{\rho}_{\nu} - \frac{1}{8} g_{\mu\nu} R^2\right)\right]} - \sqrt{-\det g}\right\}. \tag{68}$$

Of course, with four free parameters there are many other options, but in any case the most general theory that has a unitary massless spin-2 excitation around its flat vacuum is constructed with (66).

B. Reduction to the Einstein–Gauss-Bonnet theory

The next possible option is to try to reduce (63) to the EGB theory which has the same spectrum, field equations, etc., in four dimensions with Einstein's gravity. We will compare (63) with (10) for $\lambda_0 = 0$ which yields the following relations between the parameters:

$$a_3 = \frac{2}{3} b_1 - b_2 - \frac{1}{2}, \qquad a_4 = \frac{\beta(\beta+2)}{2} - \frac{8}{3} b_1 - b_3 + 1, \tag{69}$$

eliminating two of them and yielding the following $A_{\mu\nu}$ tensor with five parameters:

$$\begin{aligned} A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma\left(a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + \left(\frac{2}{3} b_1 - b_2 - \frac{1}{2}\right) R_{\mu\rho} R^{\rho}_{\nu} + \left(\frac{\beta(\beta+2)}{2} - \frac{8}{3} b_1 - b_3 + 1\right) S_{\mu\rho} S^{\rho}_{\nu}\right) \\ + \frac{\gamma}{4} g_{\mu\nu} (b_1 C_{\rho\sigma\lambda\gamma} C^{\rho\sigma\lambda\gamma} + b_2 R_{\rho\sigma} R^{\rho\sigma} + b_3 S_{\rho\sigma} S^{\rho\sigma}). \end{aligned} \tag{70}$$

This defines the most general theory that has a massless unitary graviton about its flat vacuum. Unlike the previous case, we can further require that the flat vacuum is the *unique vacuum*, that is, $c = 0$; then we arrive at the relations

$$a_3 = -b_2, \qquad b_1 = \frac{3}{4}, \qquad a_4 = \frac{\beta(\beta+2)}{2} - b_3 - 1, \tag{71}$$

which reduces the general $A_{\mu\nu}$ tensor to

$$\begin{aligned} A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma(a_2 C_{\mu\rho\nu\sigma} R^{\rho\sigma} + a_3 R_{\mu\rho} R^{\rho}_{\nu} + a_4 S_{\mu\rho} S^{\rho}_{\nu}) \\ + \frac{\gamma}{4} g_{\mu\nu}\left[\frac{3}{4} C_{\rho\sigma\lambda\gamma} C^{\rho\sigma\lambda\gamma} - a_3 R_{\rho\sigma} R^{\rho\sigma} + \left(\frac{\beta(\beta+2)}{2} - a_4 - 1\right) S_{\rho\sigma} S^{\rho\sigma}\right]. \end{aligned} \tag{72}$$

Hence, the unitarity and the unique vacuum conditions give us a four parameter theory. By judiciously choosing some of these parameters to vanish, we can define various minimal theories. The first choice can be to set $\beta = a_2 = b_2 = b_3 = 0$ yielding

$$A_{\mu\nu} = R_{\mu\nu} + \frac{3\gamma}{16} g_{\mu\nu} \chi_{\mathrm{GB}} + \frac{3\gamma}{8} g_{\mu\nu} R_{\sigma\rho} G^{\sigma\rho} - \gamma R_{\mu\rho} G^{\rho}_{\nu}. \tag{73}$$

Another minimal theory option is obtained after setting $\beta = a_2 = a_3 = a_4 = 0$ yielding an $A_{\mu\nu}$ in terms of the Ricci tensor and the metric tensor multiplied with specific quadratic terms as

$$A_{\mu\nu} = R_{\mu\nu} + \frac{\gamma}{8} g_{\mu\nu}\left(\frac{3}{2}\chi_{\mathrm{GB}} + R_{\rho\sigma} G^{\rho\sigma}\right), \tag{74}$$

where we have made use of the GB identity. The second option leads to the action

$$I = \frac{2}{\kappa\gamma}\int d^4x\left\{\sqrt{\text{-det}\left[g_{\mu\nu} + \gamma R_{\mu\nu} + \frac{\gamma^2}{8}g_{\mu\nu}\left(\frac{3}{2}\chi_{\text{GB}} + R_{\sigma\rho}G^{\sigma\rho}\right)\right]} - \sqrt{-\det g}\right\}, \tag{75}$$

which should be considered as an exact theory for all values of the curvature: At any order in the curvature expansion the flat vacuum is the unique vacuum solution and the theory describes a unitary massless graviton. Now, let us see in small curvature expansion what kind of theory we get up to $O(R^3)$. For this purpose we use (62) and get from (75) the following effective theory:

$$I = \frac{1}{\kappa}\int d^4x\sqrt{-\det g}\left\{R + \frac{3\gamma}{4}\chi_{\text{GB}} + \frac{\gamma^2}{48}(9RR_{\mu\sigma\nu\rho}R^{\mu\sigma\nu\rho} + 16R^{\alpha}_{\mu}R^{\beta}_{\alpha}R^{\mu}_{\beta} - 42RR_{\mu\nu}R^{\mu\nu} + 8R^3)\right\}. \tag{76}$$

The Gauss-Bonnet term does not contribute to the field equations. In case it is not apparent that this theory has a unique vacuum and a unitary massless spin-2 excitation from our construction above, let us show this here in a different way. In fact, these can be seen either from the field equations or from the equivalent quadratic curvature action that the flat space is the unique vacuum. Let us follow the second path and find the vacuum and the excitations for this BI-generated cubic curvature modification of Einstein's theory. Unitarity and the particle spectrum of all cubic curvature gravity theories based on the Riemann tensor and its contractions were studied in [29]. The most general cubic curvature gravity is defined with the action as

$$I = \frac{1}{\kappa}\int d^4x\sqrt{-g}[R - 2\Lambda_0 + \alpha R^2 + \beta R^{\mu\nu}R_{\mu\nu} + \gamma\chi_{\text{GB}} + F(R^{\mu\nu}_{\rho\sigma})], \tag{77}$$

where $F(R^{\mu\nu}_{\rho\sigma})$ represents the eight possible cubic curvature terms with no derivatives,

$$\begin{aligned} F(R^{\mu\nu}_{\rho\sigma}) \equiv\ & c_1 R^{\mu\nu}_{\rho\sigma}R^{\rho\beta}_{\mu\alpha}R^{\sigma\alpha}_{\nu\beta} + c_2 R^{\mu\nu}_{\rho\sigma}R^{\alpha\beta}_{\mu\nu}R^{\rho\sigma}_{\alpha\beta} + c_3 R^{\mu}_{\nu}R^{\rho\sigma}_{\alpha\mu}R^{\alpha\nu}_{\rho\sigma} + c_4 RR^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu} \\ & + c_5 R^{\mu}_{\nu}R^{\rho}_{\sigma}R^{\nu\sigma}_{\mu\rho} + c_6 R^{\mu}_{\nu}R^{\rho}_{\mu}R^{\nu}_{\rho} + c_7 RR^{\mu}_{\nu}R^{\nu}_{\mu} + c_8 R^3. \end{aligned} \tag{78}$$

The EQCA of (77) was calculated in [29] as

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0) + \tilde{\alpha}R^2 + \tilde{\beta}R^2_{ab} + \tilde{\gamma}(R^2_{abcd} - 4R^2_{ab} + R^2)\right], \tag{79}$$

with effective parameters

$$\begin{aligned} \frac{1}{\tilde{\kappa}} &\equiv \frac{1}{\kappa} - \frac{\Lambda^2}{3\kappa}[c_1 + 4c_2 + 6(c_3 + 4c_4) + 9(c_5 + c_6 + 4c_7 + 16c_8)], \\ \tilde{\Lambda}_0 &\equiv \frac{\tilde{\kappa}}{\kappa}\Lambda_0 + \frac{2\Lambda}{3}\left(1 - \frac{\tilde{\kappa}}{\kappa}\right), \\ \tilde{\alpha} &\equiv \frac{\alpha}{\kappa} + \frac{\Lambda}{3\kappa}[3c_1 - 6c_2 - 8c_4 + c_5 + 3(-c_3 + 2c_7 + 12c_8)], \\ \tilde{\beta} &\equiv \frac{\beta}{\kappa} + \frac{\Lambda}{3\kappa}[-9c_1 + 24c_2 + 16c_3 + 5c_5 + 3(16c_4 + 3c_6 + 4c_7)], \\ \tilde{\gamma} &\equiv \frac{\gamma}{\kappa} + \frac{\Lambda}{\kappa}[-c_1 + 2c_2 + (c_3 + 4c_4)]. \end{aligned} \tag{80}$$

Comparing (77) with (76), one obtains the following parameters for the EQCA from (80):

$$\frac{1}{\tilde{\kappa}} = \frac{1}{\kappa}, \qquad \tilde{\Lambda}_0 = 0, \qquad \tilde{\alpha} = -\frac{\Lambda}{4\kappa}\gamma^2, \qquad \tilde{\beta} = \frac{\Lambda}{2\kappa}\gamma^2, \qquad \tilde{\gamma} = \frac{3\gamma}{4\kappa}(1 + \Lambda\gamma), \tag{81}$$

which give the equivalent quadratic curvature action of (76),

$$I = \frac{1}{\kappa}\int d^4x\sqrt{-\det g}\left\{R + \frac{3}{4}\gamma\chi_{\text{GB}} + \frac{\gamma^2\Lambda}{4}(3\chi_{\text{GB}} + 2R_{\mu\nu}R^{\mu\nu} - R^2)\right\}. \tag{82}$$

Here, we still have to find Λ which corresponds to the maximally symmetric vacuum. There are two ways to do this: One can either derive the field equations of the cubic theory (82) and get the vacuum from those equations, or one can find the field equations of the equivalent quadratic theory (82) instead. Of course the second method is easier, and in fact these field equations were given in [30]; hence, there is no need to repeat them here. Inserting $R_{\mu\sigma\nu\rho} = \frac{\Lambda}{3}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})$ to the field equations, one finds that $\Lambda = 0$. Therefore, flat space is the unique vacuum. In fact, more importantly, $\Lambda = 0$ also kills the ghost term $(R_{\mu\nu}R^{\mu\nu})$ in the action. As expected (76) has a unitary massless spin-2 excitation just like its exact "mother" (75). Happily, this state of affairs is intact for any $O(R^i)$ truncation of the exact theory: That is, at any order the vacuum is *uniquely* flat and the theory has a massless unitary graviton.

VI. UNITARITY AROUND (A)DS BACKGROUNDS

Let us now study the unitarity of the BI gravity around its (A)dS background. It is important to establish what we mean by the tree-level unitarity of the BI theory in (A)dS backgrounds: As we noted in the Introduction, we require that the theory is tree-level unitary at any finite order in the curvature expansion and at infinite order in the curvature expansion which is the full theory. Namely, the full theory or any truncated version of the theory, for example the linear Einstein theory, quadratic gravity or, in general, $O(R^i)$ theory, should be unitary. Note that this condition on unitarity is stronger than the unitarity condition in string theory generated effective gravity models. For example, the full string theory is unitary, yet $O(R^3)$ effective theory is nonunitary for bosonic string theory [31] as shown in [29]. In (A)dS backgrounds, unlike the flat space case, infinitely many terms contribute to the propagator and to the free theory, i.e., the vacuum etc. Therefore, as explained above, we need the equivalent quadratic curvature theory of

$$\kappa\mathcal{L} = \frac{2}{\gamma}\left[\sqrt{\det\left(\delta^\rho_\nu + \gamma A^\rho_\nu\right)} - (\lambda_0 + 1)\right], \tag{83}$$

which upon use of (34) in the Weyl–traceless-Ricci–Ricci (CSR) basis reads as

$$\begin{aligned}\kappa\mathcal{L}_{\text{EQCA}} = {}& \frac{2}{\gamma}\left[\sqrt{\det\left(\delta^\rho_\nu + \gamma\bar{A}^\rho_\nu\right)} - (\lambda_0 + 1)\right] + \left[\frac{\partial\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta} + \left[\frac{\partial\mathcal{L}}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} S^\mu_\nu + \left[\frac{\partial\mathcal{L}}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} (R^\mu_\nu - \bar{R}^\mu_\nu)\\ &+ \frac{1}{2}\left[\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau} + \frac{1}{2}\left[\frac{\partial^2\mathcal{L}}{\partial S^\mu_\nu \partial S^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} S^\mu_\nu S^\alpha_\beta + \frac{1}{2}\left[\frac{\partial^2\mathcal{L}}{\partial R^\mu_\nu \partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} (R^\mu_\nu - \bar{R}^\mu_\nu)(R^\alpha_\beta - \bar{R}^\alpha_\beta)\\ &+ \left[\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial S^\eta_\theta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}S^\eta_\theta + \left[\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial R^\eta_\theta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}(R^\eta_\theta - \bar{R}^\eta_\theta) + \left[\frac{\partial^2\mathcal{L}}{\partial S^\mu_\nu \partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} S^\mu_\nu (R^\alpha_\beta - \bar{R}^\alpha_\beta),\end{aligned} \tag{84}$$

where the bracketed and barred quantities denote the maximally symmetric background values for the corresponding expressions. Note again that $\bar{C}^{\mu\nu}_{\alpha\beta} = 0$ and $\bar{S}^\mu_\nu = 0$. The terms up to quadratic order are just the ELA given in (57), so we just need the quadratic contributions which are again given in Appendix E. By using these results, the equivalent quadratic curvature action of (3) can be compactly written as

$$\kappa\mathcal{L}_{\text{EQCA}} = \frac{1}{\tilde{\kappa}}\left(R - \frac{2}{\gamma}\tilde{\lambda}_0 + \alpha_1 C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu} + \alpha_2 R^\nu_\mu R^\mu_\nu + \alpha_3 S^\nu_\mu S^\mu_\nu\right), \tag{85}$$

where the effective Newton's constant and the effective "bare" cosmological constant are given as

$$\frac{1}{\tilde{\kappa}} = 1 + \bar{a} - \lambda(2\lambda c + 1)^2, \tag{86}$$

$$\tilde{\lambda}_0 = \tilde{\kappa}[\lambda(1 + \bar{a})(2\lambda c + 1) - \bar{a}(2 + \bar{a}) + \lambda_0] + \lambda, \tag{87}$$

and the quadratic curvature parameters read as

$$\alpha_1 = \gamma b_1 \tilde{\kappa}(1 + \bar{a}), \tag{88}$$

$$\alpha_2 = \frac{\gamma}{2\lambda}[\tilde{\kappa}(1 + \bar{a})(2\lambda c + 1) - 1], \tag{89}$$

$$\begin{aligned}\alpha_3 = {}& \frac{\gamma}{2\lambda}[\tilde{\kappa}((1 + \bar{a})(2\lambda(a_4 + b_3) - 1) - \lambda(2a_3\lambda + \beta + 1)^2)\\ &+ 1].\end{aligned} \tag{90}$$

Here, $\bar{a}$ represents the combination

$$\bar{a} = \lambda + \lambda^2 c. \tag{91}$$

Note that in the $\lambda \to 0$ limit, the equivalent quadratic action of the full theory (85) reduces to the second order of the full theory in small curvature expansion (63) as expected.

Let us list the conditions that our full theory should satisfy:

(1) It should reduce to the cosmological Einstein or Einstein–Gauss-Bonnet theory at the lowest order.

(2) It should describe unitary massless spin-2 excitations at any finite order in the curvature expansion and infinite order in the curvature expansion. We have shown that if the theory is unitary at $O(R^4)$ it is unitary at any order of the form $O(R^{4+i})$ including $i \to \infty$. Therefore, together with the first condition, once exact unitarity of the theory is checked, all that is required is to check the unitarity at $O(R^3)$.

A. Reduction to cosmological Einstein theory

To reduce (85) to Einstein's theory one should set $\alpha_1 = \alpha_2 = \alpha_3 = 0$. These conditions together with the condition that the theory is unitary at $O(R^2)$, which are

$$b_1 = 0, \qquad a_3 = -\frac{1}{2} - b_2, \qquad a_4 = \frac{\beta(\beta+2)}{2} + 1 - b_3, \tag{92}$$

lead to the following relation:

$$0 = \tilde{\kappa}(1+\bar{a})(-\lambda+1) - 1, \tag{93}$$

which is obtained from $\alpha_2 = 0$, and

$$0 = \tilde{\kappa}\left\{(1+\bar{a})\left[2\lambda\left(\frac{\beta(\beta+2)}{2} + 1\right) - 1\right] - \lambda\left[-2\left(\frac{1}{2} + b_2\right)\lambda + \beta + 1\right]^2\right\} + 1, \tag{94}$$

which is obtained from $\alpha_3 = 0$. Note that for $c = -\frac{1}{2}$, $\frac{1}{\tilde{\kappa}}$ and $\bar{a}$ take the forms

$$\frac{1}{\tilde{\kappa}} = 1 - \lambda^3 + \frac{3\lambda^2}{2}, \qquad \bar{a} = \lambda - \frac{\lambda^2}{2}. \tag{95}$$

With these results, the $\alpha_2 = 0$ condition becomes

$$\lambda(\lambda - 2) = 0, \tag{96}$$

which is consistent only if $\lambda_0 = 2$ or $\lambda_0 = 0$, namely $\lambda = \lambda_0$. We have studied the $\lambda = 0$ case before. For the other case, that is, $\lambda = 2$, the theory is not unitary since $\tilde{\kappa} = -1$ as follows from (95). This means that we cannot reduce our theory to the cosmological Einstein theory.

B. Reduction to Einstein–Gauss-Bonnet theory

The next possible option is to try to reduce it to the EGB theory. Unitarity of the theory at $O(R^2)$ yields

$$a_3 = \frac{2}{3}b_1 - b_2 - \frac{1}{2}, \qquad a_4 = \frac{\beta(\beta+2)}{2} - \frac{8}{3}b_1 - b_3 + 1, \tag{97}$$

and using (86) and (87), one gets the effective Newton's constant and the effective "bare" cosmological constant as

$$\frac{1}{\tilde{\kappa}} = 1 + \bar{a} - \lambda\left(2\lambda\left(\frac{2}{3}b_1 - \frac{1}{2}\right) + 1\right)^2, \tag{98}$$

$$\tilde{\lambda}_0 = \tilde{\kappa}\left[\lambda(1+\bar{a})\left(2\lambda\left(\frac{2}{3}b_1 - \frac{1}{2}\right) + 1\right) - \bar{a}(2+\bar{a}) + \lambda_0\right] + \lambda. \tag{99}$$

In addition, the quadratic curvature parameters of EQCA also become

$$\alpha_1 = \gamma b_1 \tilde{\kappa}(1+\bar{a}), \tag{100}$$

$$\alpha_2 = \frac{\gamma}{2\lambda}\left[\tilde{\kappa}(1+\bar{a})\left(2\lambda\left(\frac{2}{3}b_1 - \frac{1}{2}\right) + 1\right) - 1\right], \tag{101}$$

$$\alpha_3 = \frac{\gamma}{2\lambda}\left[\tilde{\kappa}\left((1+\bar{a})\left(2\lambda\left(\frac{\beta(\beta+2)}{2} - \frac{8}{3}b_1 + 1\right) - 1\right) - \lambda(2a_3\lambda + \beta + 1)^2\right) + 1\right]. \tag{102}$$

Here, $\bar{a}$ represents the combination

$$\bar{a} = \lambda + \lambda^2\left(\frac{2}{3}b_1 - \frac{1}{2}\right). \tag{103}$$

To reduce our theory to the EGB theory, we must impose two conditions:

$$\frac{\alpha_2}{\alpha_1} = \frac{2}{3}, \qquad \frac{\alpha_3}{\alpha_1} = -\frac{8}{3},$$

which, respectively, lead to the following two equations:

$$\frac{3\gamma}{2\lambda}\left[\tilde{\kappa}(1+\bar{a})\left(2\lambda\left(\frac{2}{3}b_1 - \frac{1}{2}\right) + 1\right) - 1\right] = 2\gamma b_1 \tilde{\kappa}(1+\bar{a}), \tag{104}$$

$$\frac{3\gamma}{2\lambda}\left[\tilde{\kappa}\left((1+\bar{a})\left(2\lambda\left(\frac{\beta(\beta+2)}{2} - \frac{8}{3}b_1 + 1\right) - 1\right) - \lambda(2a_3\lambda + \beta + 1)^2\right) + 1\right] = -8\gamma b_1 \tilde{\kappa}(1+\bar{a}). \tag{105}$$

Simplification of (104) yields

$$\left(b_1 - \frac{9}{8}\right)\left(b_1 - \frac{3}{4}\right)\lambda = -\frac{3}{2}\left(b_1 - \frac{9}{8}\right). \tag{106}$$

Note that it is immediately clear that $b_1 \neq \frac{3}{4}$. Here, the discussion bifurcates: Either $b_1 \neq \frac{9}{8}$ or $b_1 = \frac{9}{8}$. We have to study both cases.

C. Case 1: $b_1 \neq \frac{9}{8}$

Then, from (106), one gets

$$\lambda = \frac{-3}{2(b_1 - \frac{3}{4})}, \tag{107}$$

and inserting this in (103) leads to $\bar{a} = 0$ which also gives $\frac{1}{\tilde{\kappa}} = 1 - \lambda$; therefore, one has the constraint $\lambda < 1$ for the unitarity of the theory. From (107) $\lambda < 1$ yields $|b_1| > \frac{3}{4}$. Note that, this condition on b_1 also guarantees that $\lambda \neq 1$. Now, let us look at the second constraint (105) which simplifies to

$$-4a_3\lambda(a_3\lambda + \beta + 1) = 0. \tag{108}$$

Since we are studying the $\lambda \neq 0$ case, this equation is satisfied when either $a_3 = 0$ or $a_3\lambda + \beta + 1 = 0$. We must consider these subclasses separately.

1. Case 1a: $a_3 = 0$

Using (69) b_2 can be determined as

$$b_2 = \frac{2}{3}\left(b_1 - \frac{3}{4}\right). \tag{109}$$

Making use of (107) one obtains $b_2 = -\frac{1}{\lambda}$. Since $b_2 = c$ in this case the vacuum equation (60) leads to $\lambda = \lambda_0$. Then we have the following $A_{\mu\nu}$ tensor:

$$A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma\left(a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} + \left(\frac{\beta(\beta+2)}{2} + \frac{4}{\lambda_0} - b_3 - 1\right)S_{\mu\rho}S^{\rho}_{\nu}\right) + \frac{\gamma}{4}g_{\mu\nu}\left(\left(\frac{3}{4} - \frac{3}{2\lambda_0}\right)C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} - \frac{1}{\lambda_0}R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}\right). \tag{110}$$

2. Case 1b: $a_3\lambda + \beta + 1 = 0$

Together with (107) one has

$$a_3 = \frac{2}{3}(\beta + 1)\left(b_1 - \frac{3}{4}\right). \tag{111}$$

Then from (69) b_2 can be determined as

$$b_2 = -\frac{2}{3}\beta\left(b_1 - \frac{3}{4}\right), \tag{112}$$

which leads to

$$c = \frac{2}{3}\left(b_1 - \frac{3}{4}\right). \tag{113}$$

From (60) these lead to $\lambda = \lambda_0$, $b_1 = \frac{3}{4} - \frac{3}{2\lambda_0}$, $b_2 = \frac{\beta}{\lambda_0}$ and $a_3 = -\frac{\beta+1}{\lambda_0}$ yielding

$$A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} + \gamma\left(a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{\beta+1}{\lambda_0}R_{\mu\rho}R^{\rho}_{\nu} + \left(\frac{\beta(\beta+2)}{2} + \frac{4}{\lambda_0} - b_3 - 1\right)S_{\mu\rho}S^{\rho}_{\nu}\right) + \frac{\gamma}{4}g_{\mu\nu}\left(\left(\frac{3}{4} - \frac{3}{2\lambda_0}\right)C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} + \frac{\beta}{\lambda_0}R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}\right). \tag{114}$$

Note that, since λ_0 appears in the inverse power there is no $\lambda_0 \to 0$ limit for (110) and (114). Therefore, we will not study these theories anymore even though they describe unitary massless spin-2 excitations at all orders in the curvature expansion about their (A)dS vacuum. Let us study the second case.

D. Case 2: $b_1 = \frac{9}{8}$

In this case, λ is not determined from (106). This choice reduces (97) and (98) to

$$a_3 = \frac{1}{4} - b_2, \qquad a_4 = \frac{\beta(\beta+2)}{2} - 2 - b_3, \tag{115}$$

$$\frac{1}{\tilde{\kappa}} = (1 - \lambda)\left(1 + \frac{\lambda}{2}\right)^2. \tag{116}$$

Again, positivity of the Newton's constant leads to $\lambda < 1$, and we demand that $\lambda \neq -2$, so that the Newton's constant does not vanish. The vacuum equation (60) boils down to

$$\frac{\lambda^4}{16} + \frac{\lambda^3}{4} - \lambda + \lambda_0 = 0. \tag{117}$$

The solutions of this equation were discussed in Sec. IV; hence, we do not repeat them here, but just note that there is a unique viable solution with $\lambda < 1$ as long as $\lambda_0 < \frac{11}{16}$. Note also that for $\lambda \neq -2$, one must have $\lambda_0 \neq -1$. The second condition (105) gives

$$\frac{1}{2}(\lambda + 2)(\beta + 1) = \pm (2a_3\lambda + \beta + 1). \tag{118}$$

We must study both signs separately.

Let us consider the minus sign case which yields

$$a_3 = -\frac{(\lambda + 4)(\beta + 1)}{4\lambda}. \tag{119}$$

Since we would like to have a smooth $\lambda \to 0$ limit, we must have $\beta = -1$ and $a_3 = 0$. Then, the theory is

$$\begin{aligned} A_{\mu\nu} = R_{\mu\nu} - S_{\mu\nu} &+ \gamma\left(a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} - \left(b_3 + \frac{5}{2}\right)S_{\mu\rho}S_\nu^\rho\right) \\ &+ \frac{\gamma}{4}g_{\mu\nu}\left(\frac{9}{8}C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} + \frac{1}{4}R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}\right), \end{aligned} \tag{120}$$

which will also appear as a subcase below.

Let us consider the final case, choosing the plus sign in (118), which leads to $a_3 = \frac{\beta+1}{4}$. The $A_{\mu\nu}$ tensor reads

$$\begin{aligned} A_{\mu\nu} = R_{\mu\nu} + \beta S_{\mu\nu} &+ \gamma\left(a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} + \frac{\beta+1}{4}R_{\mu\rho}R_\nu^\rho \right. \\ &\left. + \left(\frac{\beta(\beta+2)}{2} - 2 - b_3\right)S_{\mu\rho}S_\nu^\rho\right) \\ &+ \frac{\gamma}{4}g_{\mu\nu}\left(\frac{9}{8}C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} - \frac{\beta}{4}R_{\rho\sigma}R^{\rho\sigma} + b_3 S_{\rho\sigma}S^{\rho\sigma}\right), \end{aligned} \tag{121}$$

where a_2, b_3 and β are arbitrary real parameters.

Let us summarize the properties of this theory:

(1) With a given $\lambda_0 < \frac{11}{16}$, it has a unique viable maximally symmetric vacuum with a cosmological parameter $\lambda < 1$, and an effective Newton's constant $\frac{1}{\tilde{\kappa}} = (1-\lambda)(1+\frac{\lambda}{2})^2$.

(2) It describes a unitary massless spin-2 excitation around this vacuum for any value of $\lambda_0 < \frac{11}{16}$ including $\lambda_0 = 0$, except $\lambda_0 = -1$, which yields $\lambda = -2$, and so it is ruled out by the requirement of a nonzero effective Newton's constant. This statement means that the theory has the same propagator structure as Einstein's gravity in (A)dS and flat backgrounds.

(3) It provides an infinite order unitary extension of Einstein's gravity.

All these features are quite attractive but we still have to show that the theory is also healthy at the truncated orders $O(R^2)$ and $O(R^3)$. At $O(R^2)$ since the theory is equivalent to the Einstein-Gauss-Bonnet theory it is unitary as long as κ is positive. Let us now check the $O(R^3)$ theory. Expanding the Lagrangian density built with (121) up to $O(R^3)$, we arrive at

$$\begin{aligned} \kappa\mathcal{L}_{O(R^3)} = R - 2\Lambda_0 &+ \gamma\left(-\frac{2b_1}{3} - \frac{a_4 + b_3}{4} + \frac{(\beta+1)^2}{8} + \frac{1}{8}\right)R^2 \\ &+ \gamma\left(2b_1 + b_2 + a_3 + b_3 + a_4 - \frac{(\beta+1)^2}{2}\right)R_{\mu\nu}R^{\mu\nu} + \gamma b_1\chi_{\rm GB} \\ &+ \gamma^2\frac{b_1}{4}RR_{\mu\rho\nu\sigma}R^{\mu\rho\nu\sigma} - \gamma^2(\beta+1)a_2R_{\mu\nu}R^{\mu\sigma\nu\rho}R_{\sigma\rho} \\ &+ \gamma^2\left(\frac{7(\beta+1)a_2}{6} + \frac{b_2 + b_3 - 2b_1}{4} + \frac{(3\beta+4)a_4}{4} + \frac{(\beta+2)a_3}{4} - \frac{(\beta+1)^3}{4}\right)RR_{\mu\nu}R^{\mu\nu} \\ &+ \gamma^2\left(\frac{b_1}{12} - \frac{(2\beta+3)a_4}{16} - \frac{b_3}{16} - \frac{(\beta+1)a_2}{6} + \frac{(\beta+1)^3}{24}\right)R^3 \\ &+ \gamma^2(\beta+1)\left(-a_2 - a_3 - a_4 + \frac{(\beta+1)^2}{3}\right)R_{\mu\nu}R_\rho^\nu R^{\rho\mu}. \end{aligned} \tag{122}$$

The question is if one takes this theory as the full theory what kind of excitations will it have? We can answer this question with the methods we have employed several times in this work. Namely we can construct an equivalent quadratic action that has the same vacuum and excitations as this theory. Using the above cubic curvature parameters in (80) yields the EQCA parameters for (122) as

$$\frac{1}{\tilde{\kappa}} \equiv \frac{1}{\kappa}[1 - 3\lambda^2(a_3 + b_2)], \tag{123}$$

$$\tilde{\lambda}_0 \equiv \frac{\tilde{\kappa}}{\kappa}\lambda_0 + \frac{2\lambda}{3}\left(1-\frac{\tilde{\kappa}}{\kappa}\right), \tag{124}$$

$$\kappa\tilde{\alpha} \equiv \gamma\left(-\frac{2b_1}{3}-\frac{a_4+b_3}{4}+\frac{(\beta+1)^2}{8}+\frac{1}{8}\right)+\frac{\lambda\gamma}{3}\left(-2b_1+\frac{3b_2}{2}-\frac{3b_3}{4}+\frac{3(\beta+2)a_3}{2}-\frac{3a_4}{4}\right), \tag{125}$$

$$\kappa\tilde{\beta} \equiv \gamma\left(2b_1+b_2+a_3+b_3+a_4-\frac{(\beta+1)^2}{2}\right)+\lambda\gamma(2b_1+b_2-(2\beta+1)a_3+b_3+a_4), \tag{126}$$

$$\kappa\tilde{\gamma} \equiv \gamma(1+\lambda)b_1. \tag{127}$$

For this theory to describe unitary massless spin-2 excitations we must set $\tilde{\alpha}=0=\tilde{\beta}$. These conditions are automatically satisfied because of the conditions (104) and (105) of the full unitary theory. We only need to show that the effective Newton's constant remains positive. Thus we have

$$\frac{1}{\kappa}\left(1-\frac{3\lambda^2}{4}\right)>0, \tag{128}$$

which is satisfied only if $-\frac{2}{\sqrt{3}}<\lambda<\frac{2}{\sqrt{3}}$. The upper bound is weaker than $\lambda<1$ but a lower bound is introduced. Thus, unitarity of our theory at $O(R^3)$ is achieved if $-\frac{2}{\sqrt{3}}<\lambda<1$. Of course now the vacuum equation should allow such a solution. Here the vacuum equation at this order is

$$\lambda^3-4\lambda+4\lambda_0=0, \tag{129}$$

and if $-\frac{4}{3\sqrt{3}}<\lambda_0<\frac{4}{3\sqrt{3}}$ then there is such a real λ. Observe that the upper bound is larger than $\frac{11}{16}$. Hence, the condition on λ_0 is $-\frac{4}{3\sqrt{3}}<\lambda_0<\frac{11}{16}$.

Therefore, with these constraints coming from the unitarity of the theory at $O(R^3)$, we can now summarize the properties of the theory (121) as follows: It describes a unitary massless spin-2 excitation about its unique viable vacuum (with $-\frac{2}{\sqrt{3}}<\lambda<1$) at every order in the curvature expansion including the infinite order expansion as long as $\kappa>0$, $-\frac{4}{3\sqrt{3}}<\lambda_0<\frac{11}{16}$ for arbitrary real β, a_2 and b_3. What is fascinating is that no new condition arises at any $O(R^{4+i})$ expansion. Namely, at every such order, contributions to the effective parameters vanish among each other; therefore, for example, the effective Newton's constant or the vacuum equation does not receive any corrections from the terms of the $O(R^{4+i})$ theory. This is the first known theory in four dimensions which is unitary at every order in the curvature expansion in its (A)dS vacuum.

Having three arbitrary parameters at our disposal, we can define various minimal theories out of which one is particularly interesting: For $\beta=-1$, $a_2=0$ and $b_3=-\frac{5}{2}$, one has the BI action

$$I=\frac{2}{\kappa\gamma}\int d^4x\left\{\sqrt{\text{-det}\left[g_{\mu\nu}+\frac{\gamma}{4}g_{\mu\nu}R+\frac{9\gamma^2}{32}g_{\mu\nu}\left(\chi_{\text{GB}}-\frac{1}{9}R^2\right)\right]}-(\lambda_0+1)\sqrt{-\det g}\right\}, \tag{130}$$

which actually can be recast as

$$I=\frac{2}{\kappa\gamma}\int d^4x\sqrt{-\det g}\left\{\left[1+\frac{\gamma}{4}R+\frac{9\gamma^2}{32}\left(\chi_{\text{GB}}-\frac{1}{9}R^2\right)\right]^2-(\lambda_0+1)\right\}, \tag{131}$$

or more explicitly

$$I=\frac{1}{\kappa}\int d^4x\sqrt{-\det g}\left\{R-2\Lambda_0-\frac{\gamma^2}{32}R^3+\frac{9\gamma^2}{32}R\chi_{\text{GB}}+\frac{\gamma^3}{512}R^4-\frac{9\gamma^3}{256}R^2\chi_{\text{GB}}+\frac{81\gamma^3}{512}\chi_{\text{GB}}^2\right\}, \tag{132}$$

where we dropped the boundary term. The important point here is that as an $O(R^4)$ theory, this describes massless unitary excitations about its (A)dS vacuum, but it also describes massless unitary excitations at order $O(R^i)$ for $i\leq 4$, when expanded in small curvature.

VII. CONCLUSION AND FURTHER DISCUSSIONS

Using physical requirements such as the existence of a unique viable maximally symmetric vacuum with a zero or a nonzero curvature, unitary massless spin-2 excitations about this vacuum at tree level, and the reduction to the cosmological Einstein theory for weak field gravity, we have constructed Born-Infeld gravity actions with the metric being the only independent variable following the route of [1]. To the best of our knowledge, the theory we have constructed is the only known theory in four dimensions that is unitary at *every order* in the curvature expansion about its (A)dS vacuum.

One interesting observation is that the four dimensional Gauss-Bonnet term, being a total derivative, which has no classical effect, plays an important role in the construction of the actions: Namely, at the lowest order BI gravity reduces to the Einstein-Gauss-Bonnet theory and not to its classically equivalent partner, the Einstein's theory. In addition to the above-mentioned physical requirements, we have also employed the notion of minimality which is essentially constructing determinantal actions that are as simple as possible and that do not involve many powers of curvature and derivatives of curvature. This leads to a quadratic theory inside the determinant. In the most general form, the set of such theories has three dimensionless and one dimensionful parameter which is the BI parameter that comes from the coefficient of the Gauss-Bonnet term. To further restrict the viable BI theories, one must turn to their phenomenological applications. By construction, the theory matches Einstein's gravity for small curvature; hence, deviations from the results of Einstein's theory should be expected at the strong gravity regime.

In this work, we have concentrated on pure gravity and have not worried about matter couplings which can either be done by the usual way of assuming a $\int d^4x\sqrt{-g}g^{\mu\nu}T_{\mu\nu}$ type interaction in the action or in the nonminimal way by inserting matter fields into the determinant. As an example of the latter case, one can couple Maxwell theory by simply taking $A_{\mu\nu} \rightarrow A_{\mu\nu} + \alpha F_{\mu\nu}$, with $F_{\mu\nu}$ being the field strength tensor. Conformally invariant versions of the actions can also be found following [32,33].

We shall study cosmological and black-hole-type solutions in a separate work, but here with the tools in our hands, we can find some exact solutions of the BI gravity (121). These solutions are the AdS-wave solutions of the cosmological Einstein's theory [34–36]. These solutions not only solve the exact cosmological Einstein's theory, but also its linearized version. These solutions remain intact in the BI gravity (121), and the only thing that one needs to change is the effective cosmological constant which can be found from the vacuum equation (117). This comes from the fact that the equivalent quadratic curvature action of a theory determines the linearized field equations which in turn determine the properties of its AdS-wave solutions [34,36], and here we have shown that for BI gravity this action is the Einstein–Gauss-Bonnet action whose linearized field equations are the same as the Einstein's theory. The same fact gives a way to construct the conserved charges of the BI theory which we now show.

A. Conserved charges in the BI gravity

The conserved charges of a given $f(R^{\mu\nu}_{\alpha\beta})$ theory can be written in terms of the conserved charges of cosmological Einstein's gravity as was shown in [27]. This follows from the linearized field equations of the generic $f(R^{\mu\nu}_{\alpha\beta})$ theory given in Appendix D and the charge construction in [28,30]. Without going into further details, let us recall the expression in [27]:

$$Q^0_f(\bar{\xi}) = \left(\frac{1}{\tilde{\kappa}} + \frac{4\Lambda n}{n-2}\alpha + \frac{4\Lambda}{n-2}\beta + \frac{4\Lambda(n-3)(n-4)}{(n-1)(n-2)}\gamma\right) Q^0_{\text{Einstein}}(\bar{\xi}), \tag{133}$$

where $\bar{\xi}$ is the background Killing vector which for energy reads $\bar{\xi}^\mu = (-1,0,0,0)$. Q^0_{Einstein} is the Abbott-Deser charge for asymptotically (A)dS spacetimes in cosmological Einstein's gravity [37]. For the viable BI gravity theory given in (121), $\alpha = 0$, $\beta = 0$, and $n = 4$; hence, the conserved charges of asymptotically (A)dS spacetimes read

$$Q^0_{\text{BI}}(\bar{\xi}) = (1-\lambda)\left(1+\frac{\lambda}{2}\right)^2 Q^0_{\text{Einstein}}(\bar{\xi}). \tag{134}$$

For example, for an asymptotically rotating (A)dS-Schwarzschild black hole,[7] the energy and the angular momentum read

$$E = (1-\lambda)\left(1+\frac{\lambda}{2}\right)^2 m, \qquad J = (1-\lambda)\left(1+\frac{\lambda}{2}\right)^2 ma,$$

where m is the mass parameter and a is the rotation parameter. It is also clear that the black hole has a positive mass when the graviton has a positive kinetic energy, that is, $\lambda < 1$.

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[7]By construction, this solution asymptotically exists in BI gravity since at large distances cosmological Einstein's theory is dominant.

APPENDIX A: NAIVE BI GRAVITY

Let us consider the following BI-gravity action:

$$I = -\frac{2}{\kappa\gamma}\int d^n x\left[\sqrt{-\det\left(g_{\mu\nu}+\gamma G_{\mu\nu}+\gamma\beta g_{\mu\nu}R\right)} - \sqrt{-\det g}\right]. \tag{A1}$$

Expanding to the $O(R^2)$ yields

$$\mathcal{L}_{O(R^2)} = \frac{n-2}{2\kappa}\left[R+\frac{\gamma}{n-2}\left(R_{\mu\nu}R^{\mu\nu}-\frac{n^2-6n+12}{8}R^2\right)\right] - \beta\frac{n}{\kappa}\left[R+\frac{n-2}{4}\gamma\beta R^2\right]. \tag{A2}$$

For $n=3$ and $\beta=0$ one gets the BINMG action which describes a unitary massive spin-2 graviton, but for any other dimension there is a massive spin-2 ghost due to the $R_{\mu\nu}R^{\mu\nu}$ term [40].

APPENDIX B: CONVERSIONS BETWEEN CSR BASIS AND RRR BASIS

In this appendix, we discuss the conversions between the Weyl–traceless-Ricci–Ricci (CSR) basis and Riemann–Ricci–curvature-scalar (RRR) basis.

The $A_{\mu\nu}$ tensor written in the CSR basis, which is

$$A_{\mu\nu} = R_{\mu\nu}+\beta S_{\mu\nu}+\gamma(a_1 C_{\mu\rho\sigma\lambda}C_\nu^{\rho\sigma\lambda}+a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma}+a_3 R_{\mu\rho}R_\nu^\rho+a_4 S_{\mu\rho}S_\nu^\rho)+\frac{\gamma}{4}g_{\mu\nu}(b_1 C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma}+b_2 R_{\rho\sigma}R^{\rho\sigma}+b_3 S_{\rho\sigma}S^{\rho\sigma}), \tag{B1}$$

can be converted to the RRR basis, which is

$$A_{\mu\nu} = (1+\tilde{\beta})R_{\mu\nu}-\frac{\tilde{\beta}}{4}g_{\mu\nu}R+c_1 g_{\mu\nu}R^2+c_2 RR_{\mu\nu}+c_3 g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}+c_4 R^\sigma{}_\mu R_{\nu\sigma}+c_5 R_{\mu\sigma\nu\rho}R^{\sigma\rho}+c_6 g_{\mu\nu}R_{\rho\sigma\lambda\gamma}R^{\rho\sigma\lambda\gamma}+c_7 R_\mu{}^{\sigma\rho\tau}R_{\nu\sigma\rho\tau}, \tag{B2}$$

by using $S_{\mu\nu}=R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R$ and the definition of the Weyl tensor in four dimensions,

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta}-g_{\mu[\nu}R_{\beta]\alpha}+g_{\alpha[\nu}R_{\beta]\mu}+\frac{R}{3}g_{\mu[\nu}g_{\beta]\alpha}, \tag{B3}$$

in place. Then, the coefficients in (B2) become

$$\tilde{\beta}=\beta,\quad c_1=\frac{\gamma}{48}(-8a_1+8a_2+3a_4+4b_1-3b_3),\quad c_2=\gamma\left(a_1-\frac{2}{3}a_2-\frac{1}{2}a_4\right),\quad c_3=\frac{\gamma}{4}(2a_1-2a_2-2b_1+b_2+b_3),$$
$$c_4=\gamma(-2a_1+a_2+a_3+a_4),\quad c_5=\gamma(-2a_1+a_2),\quad c_6=\frac{\gamma}{4}b_1,\quad c_7=\gamma a_1. \tag{B4}$$

Sometimes the inverse transformation is also needed; therefore, we shall give it here:

$$\beta=\tilde{\beta},\quad a_1=\frac{c_7}{\gamma},\quad a_2=\frac{1}{\gamma}(c_5+2c_7),\quad a_3=\frac{1}{\gamma}\left(2c_2+c_4+\frac{c_5}{3}+\frac{2c_7}{3}\right),\quad a_4=\frac{1}{\gamma}\left(-2c_2-\frac{4c_5}{3}-\frac{2c_7}{3}\right),$$
$$b_1=\frac{4c_6}{\gamma},\quad b_2=\frac{1}{\gamma}\left(16c_1+2c_2+4c_3+\frac{2c_5}{3}+\frac{8}{3}c_6\right)\quad b_3=\frac{1}{\gamma}\left(-16c_1-2c_2+\frac{4c_5}{3}+\frac{16c_6}{3}+2c_7\right). \tag{B5}$$

In the RRR basis, the EQCA takes the form

$$\mathcal{L}_{\mathrm{EQCA}} = -2\Lambda_0+\frac{2}{3}\left(2+\frac{l_1\Lambda}{\gamma}\right)l_1\Lambda^3+\left\{1-l_1\Lambda^2-\frac{4}{9\gamma}l_1^2\Lambda^3\right\}R$$
$$+\frac{1}{\gamma}\left\{(4c_1+c_2)\left(1+\gamma\Lambda+\frac{l_1}{3}\Lambda^2\right)+\frac{1}{8}(\gamma(\tilde{\beta}+1)+2l_2\Lambda)^2+\frac{1}{2}\left(\frac{\gamma}{2}+\frac{l_1}{3}\Lambda\right)^2\right\}R^2$$
$$+\frac{1}{\gamma}\left\{(4c_3+c_4+c_5)\left(1+\gamma\Lambda+\frac{l_1}{3}\Lambda^2\right)-\frac{1}{2}(\gamma(\tilde{\beta}+1)+2l_2\Lambda)^2\right\}R_{\mu\nu}^2+\frac{1}{\gamma}(4c_6+c_7)\left(1+\gamma\Lambda+\frac{l_1}{3}\Lambda^2\right)R_{\mu\sigma\nu\rho}^2, \tag{B6}$$

where the coefficients read

$$l_1 = 48c_1 + 12c_2 + 12c_3 + 3c_4 + 3c_5 + 8c_6 + 2c_7, \tag{B7}$$

$$l_2 = \frac{1}{3}(6c_2 + 3c_4 + c_5 + 2c_7). \tag{B8}$$

APPENDIX C: AN EXAMPLE ON EQCA CONSTRUCTION

The unitarity discussions using the EQCA construction involve various Taylor series expansions of functions depending on tensor quantities which sometimes complicate the inherent physical meaning. To understand the basic idea of the EQCA and the relation between various expansions, it may be worth considering analogue expansions for a function with a single scalar variable. First, remember that the EQCA of a gravity theory is given with the second order Taylor series expansion of the Lagrangian in the curvature around the maximally symmetric background, $\bar{R}^{\mu\nu}_{\rho\sigma}$, which is either already determined by using the ELA of the theory or will be determined by using the EQCA of the theory. Then, for a function $f(x)$, the analogue of EQCA is the following second order Taylor series expansion around $x = \bar{x}$:

$$f_{\text{EQCA}}(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2} f''(\bar{x})(x - \bar{x})^2, \tag{C1}$$

which can be recast in the form

$$f_{\text{EQCA}}(x) = f(\bar{x}) - f'(\bar{x})\bar{x} + \frac{1}{2} f''(\bar{x})\bar{x}^2 + [f'(\bar{x}) - f''(\bar{x})\bar{x}]x + \frac{1}{2} f''(\bar{x})x^2. \tag{C2}$$

Here, note that in the gravitational setting, the $O(x)$ term represents the Einstein-Hilbert piece and its coefficient is the effective Newton's constant of the theory whose positivity puts a constraint on the theory. The $O(1)$ term determines the effective "bare" cosmological constant while the $O(x^2)$ term is the quadratic curvature term.

In addition to this EQCA expansion, we also discussed the small curvature expansion of a gravitational theory which corresponds to the Taylor series expansion of $f(x)$ around $x = 0$ as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{C3}$$

Note that unless $f^{(n)}(0)$ is zero, each order in (C3) will contribute to EQCA as

$$(x^n)_{\text{EQCA}} = \left(1 - \frac{3n}{2} + \frac{n^2}{2}\right)\bar{x}^n + n^2 \bar{x}^{n-1} x + \frac{n(n-1)}{2} \bar{x}^{n-2} x^2. \tag{C4}$$

This result implies that to see the contributions to the EQCA of a gravity theory coming from the $O(R^i)$ terms in the small curvature expansion of the theory, one needs to look at the Λ^{i-2} terms at the quadratic curvature level, the Λ^{i-1} terms in the effective Newton's constant part $\frac{1}{\kappa}$, and Λ^i terms in the effective bare cosmological part $\frac{\tilde{\Lambda}_0}{\kappa}$.

Another implication of this result is that once the EQCA analogue of $f(x)$ is found, there is no need to calculate the EQCA analogue of any finite order truncation of (C3) separately. One just needs to have $O(\bar{x}^i)$, $O(\bar{x}^{i-1})$, and $O(\bar{x}^{i-2})$ expansions of $O(1)$, $O(x)$, and $O(x^2)$ terms in (C2), respectively, around $\bar{x} = 0$. For example, let us write the $O(x^3)$ truncation of (C3):

$$f_{x^3}(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f'''(0)x^3,$$

whose EQCA analogue expansion is

$$f_{x^3\text{–EQCA}}(x) = f(0) + \frac{1}{6} f'''(0)\bar{x}^3 + \left[f'(0) - \frac{1}{2} f'''(0)\bar{x}^2\right]x + \frac{1}{2}[f''(0) + f'''(0)\bar{x}]x^2. \tag{C5}$$

Up to $O(\bar{x}^3)$ expansion of the $O(1)$ term in (C2), that is, $f(\bar{x}) - f'(\bar{x})\bar{x} + \frac{1}{2} f''(\bar{x})\bar{x}^2$, around $\bar{x} = 0$ gives the first line of (C5). Then, up to $O(\bar{x}^2)$ expansion of the $O(x)$ term in (C2), that is, $f'(\bar{x}) - f''(\bar{x})\bar{x}$, around $\bar{x} = 0$ gives the coefficient of x in (C5). Finally, up to $O(\bar{x})$ expansion of the $O(x^2)$ term in (C2), that is, $\frac{1}{2} f''(\bar{x})$, around $\bar{x} = 0$ gives the coefficient of x^2 in (C5).

The same approach can be used in the gravitational setting. For example, the EQCA of the BI theory defined by (121) can be obtained by using (85) as

$$\kappa \mathcal{L}_{\text{EQCA}} = -2\frac{\lambda_0}{\gamma} + \frac{\lambda^3}{\gamma}\left(1 + \frac{3\lambda}{8}\right) + R(1 - \lambda)\left(1 + \frac{\lambda}{2}\right)^2 + \frac{9}{8}\gamma\left(1 + \frac{\lambda}{2}\right)^2 \chi_{\text{GB}}. \tag{C6}$$

Then, the EQCA of the $O(R^3)$ expansion of the BI theory defined by (121) can be obtained by taking up to $\frac{\lambda^3}{\gamma}$ order in the cosmological constant term, up to λ^2 order in the effective Newton's constant term, and up to λ order in the quadratic curvature parameters in (C6) as

$$\kappa\mathcal{L}_{\mathrm{EQCA}-O(R^3)} = -2\frac{\lambda_0}{\gamma}+\frac{\lambda^3}{\gamma}+R\left(1-\frac{3}{4}\lambda^2\right) + \frac{9}{8}\gamma(1+\lambda)\chi_{\mathrm{GB}}, \tag{C7}$$

which can also be obtained by using the EQCA result for $\kappa\mathcal{L}_{O(R^3)}$ given in (122). In addition, the vacuum equation for $O(R^3)$ truncation can be calculated from (C7) as

$$\lambda = \frac{\lambda_0 - \frac{\lambda^3}{2}}{(1-\frac{3}{4}\lambda^2)} \Rightarrow \lambda - \lambda_0 - \frac{1}{4}\lambda^3 = 0, \tag{C8}$$

which can again be obtained from the vacuum equation of the whole theory which is

$$\lambda - \lambda_0 - \frac{\lambda^3}{16}(4+\lambda) = 0, \tag{C9}$$

by eliminating the highest power coming from the $O(R^4)$ truncation.

APPENDIX D: LINEARIZATION OF THE FIELD EQUATIONS OF $f(R^{\mu\nu}_{\alpha\beta})$

In this appendix, we carry out the linearization of the field equations of an $f(R^{\mu\nu}_{\alpha\beta})$ theory, which is a gravity theory whose Lagrangian is constructed from the contractions of the Riemann tensor but not its derivatives, and we show that these linearized field equations are the same as those of a quadratic curvature gravity theory with redefined parameters. First, note that the field equations of an $f(R^{\mu\nu}_{\alpha\beta})$ theory are

$$\begin{aligned}&\frac{1}{2}(g_{\nu\rho}\nabla^\lambda\nabla_\sigma - g_{\nu\sigma}\nabla^\lambda\nabla_\rho)\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} \\ &\quad -\frac{1}{2}(g_{\mu\rho}\nabla^\lambda\nabla_\sigma - g_{\mu\sigma}\nabla^\lambda\nabla_\rho)\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}} \\ &\quad -\frac{1}{2}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu} - \frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\mu}\right) - \frac{1}{2}g_{\mu\nu}f(R^{\mu\nu}_{\alpha\beta}) \\ &= 0.\end{aligned} \tag{D1}$$

In Sec. III, we showed that the (A)dS spacetime solutions of this theory satisfy

$$-2\zeta\bar{R}_{\mu\nu} + g_{\mu\nu}f(\bar{R}^{\alpha\beta}_{\rho\sigma}) = 0, \tag{D2}$$

where ζ is defined in (26) as $[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R$. Let us linearize (D1) in the metric perturbation $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ is the (A)dS background solving (D2). Starting with the last term in (D1), which becomes

$$[g_{\mu\nu}f(R^{\mu\nu}_{\alpha\beta})]_L = h_{\mu\nu}f(\bar{R}^{\mu\nu}_{\alpha\beta}) + \bar{g}_{\mu\nu}\left[\frac{\partial f}{\partial R^{\alpha\beta}_{\rho\sigma}}\right]_{\bar{R}^{\alpha\beta}_{\rho\sigma}}(R^{\alpha\beta}_{\rho\sigma})_L, \tag{D3}$$

and using the equation defining ζ, one finds

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} = \zeta\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\nu}. \tag{D4}$$

As discussed in Sec. III, one gets

$$[g_{\mu\nu}f(R^{\mu\nu}_{\alpha\beta})]_L = h_{\mu\nu}f(\bar{R}^{\alpha\beta}_{\rho\sigma}) + \bar{g}_{\mu\nu}\zeta R_L. \tag{D5}$$

Moving to the first term in the third line of (D1), one has the linearization

$$\begin{aligned}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu}\right)_L &= \left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}(R^{\eta\theta}_{\alpha\tau})_L\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\nu} \\ &\quad + \left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}(R_{\rho\sigma}{}^{\lambda}{}_{\nu})_L.\end{aligned} \tag{D6}$$

Here, remember that $[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}$ has the following form as we discussed in Sec. III:

$$\begin{aligned}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} &= 2\alpha\delta^{[\alpha}_{\eta}\delta^{\tau]}_{\theta}\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\lambda} \\ &\quad + \beta(\delta^{\alpha}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\tau|}_{[\mu}\delta^{\sigma]}_{\lambda]} - \delta^{\tau}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\alpha|}_{[\mu}\delta^{\sigma]}_{\lambda]}) \\ &\quad + 12\gamma\delta^{[\alpha}_{\eta}\delta^{\tau}_{\theta}\delta^{\rho}_{\mu}\delta^{\sigma]}_{\lambda}.\end{aligned} \tag{D7}$$

Using this result together with (D4) and $\bar{R}^{\mu\lambda}_{\rho\sigma} = \frac{2\Lambda}{(n-1)(n-2)}(\delta^{\mu}_{\rho}\delta^{\lambda}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\lambda}_{\rho})$, one has

$$\begin{aligned}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu}\right)_L &= -\left(\alpha\frac{4\Lambda}{(n-2)} + \beta\frac{n\Lambda}{(n-1)(n-2)}\right)R_L\bar{g}_{\mu\nu} \\ &\quad + \left(\gamma\frac{8\Lambda(n-3)}{(n-1)(n-2)} - \beta\frac{2\Lambda}{n-1}\right) \\ &\quad \times\left(R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - \frac{2\Lambda}{n-2}h_{\mu\nu}\right) - \zeta R^L_{\mu\nu}.\end{aligned}$$

Now, let us linearize the first term in (D1), $g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}$, and to do this, first note that the linearization of the metric compatibility, $\nabla_\mu g_{\nu\rho} = 0$, yields

$$\bar{\nabla}_\mu h_{\nu\rho} = (\Gamma^\sigma_{\mu\nu})_L\bar{g}_{\sigma\rho} + (\Gamma^\sigma_{\mu\rho})_L\bar{g}_{\nu\sigma}. \tag{D8}$$

Then, for a two-tensor $A_{\mu\nu}$ with the background value $\bar{A}_{\mu\nu} = \bar{a}\bar{g}_{\mu\nu}$, the linearization of $\nabla_\mu A_{\nu\rho}$ yields

$$(\nabla_\mu A_{\nu\rho})_L = \bar{\nabla}_\mu A^L_{\nu\rho} - \bar{a}[(\Gamma^\sigma_{\mu\nu})_L\bar{g}_{\sigma\rho} + (\Gamma^\sigma_{\mu\rho})_L\bar{g}_{\nu\sigma}], \tag{D9}$$

and with (D8), one has

$$(\nabla_\mu A_{\nu\rho})_L = \bar{\nabla}_\mu A^L_{\nu\rho} - \bar{a}\bar{\nabla}_\mu h_{\nu\rho}. \tag{D10}$$

Next, let us consider the linearization of $(\nabla_\sigma\nabla_\mu A_{\nu\rho})_L$:

$$\begin{aligned}(\nabla_\sigma\nabla_\mu A_{\nu\rho})_L = {}& \bar{\nabla}_\sigma\bar{\nabla}_\mu A^L_{\nu\rho} - (\Gamma^\lambda_{\sigma\mu})_L\bar{\nabla}_\lambda\bar{A}_{\nu\rho} - (\Gamma^\lambda_{\sigma\nu})_L\bar{\nabla}_\mu\bar{A}_{\lambda\rho} \\ & - (\Gamma^\lambda_{\sigma\rho})_L\bar{\nabla}_\mu\bar{A}_{\nu\lambda} - \bar{\nabla}_\sigma((\Gamma^\lambda_{\mu\nu})_L\bar{A}_{\lambda\rho}) \\ & - \bar{\nabla}_\sigma((\Gamma^\lambda_{\mu\rho})_L\bar{A}_{\nu\lambda}),\end{aligned} \tag{D11}$$

which boils down to

$$(\nabla_\sigma\nabla_\mu A_{\nu\rho})_L = \bar{\nabla}_\sigma(\bar{\nabla}_\mu A^L_{\nu\rho} - \bar{a}\bar{\nabla}_\mu h_{\nu\rho}). \tag{D12}$$

Finally, we have

$$\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = \bar{g}_{\nu\rho}\bar{g}^{\lambda\beta}\bar{\nabla}_\beta\bar{\nabla}_\sigma\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L + \bar{g}_{\nu\rho}\bar{g}^{\lambda\beta}\bar{\nabla}_\beta\left[(\Gamma^\alpha_\sigma)_L\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\right], \tag{D13}$$

where the second term represents

$$(\Gamma^\alpha_\sigma)_L\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}} \equiv (\Gamma^\rho_{\sigma\alpha})_L\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\alpha\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}} + (\Gamma^\sigma_{\sigma\alpha})_L\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\alpha}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}} - (\Gamma^\alpha_{\sigma\mu})_L\left(\frac{\partial f}{\partial R^{\alpha\lambda}_{\rho\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}} - (\Gamma^\alpha_{\sigma\lambda})_L\left(\frac{\partial f}{\partial R^{\mu\alpha}_{\rho\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \tag{D14}$$

and using (D4), one has

$$(\Gamma^\alpha_\sigma)_L\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_{\bar{R}^{\mu\lambda}_{\rho\sigma}} = 0. \tag{D15}$$

On the other hand, $(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}})_L$ takes the following form by using (D7):

$$\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = \left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}(R^{\eta\theta}_{\alpha\tau})_L = 2\alpha\delta^{[\rho}_\mu\delta^{\sigma]}_\lambda R_L + 2\beta\delta^{[\rho}_\theta\delta^{|\tau|}_{[\mu}\delta^{\sigma]}_{\lambda]}(R^\theta_\tau)_L + 2\gamma\delta^{[\rho}_\mu\delta^{\sigma]}_\lambda R_L - 8\gamma\delta^{[\rho}_\theta\delta^{|\tau|}_{[\mu}\delta^{\sigma]}_{\lambda]}(R^\theta_\tau)_L + 2\gamma\delta^{[\rho}_\eta\delta^{\sigma]}_\theta(R^{\eta\theta}_{\mu\lambda})_L. \tag{D16}$$

Using these results, one arrives at

$$\begin{aligned}\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = {}& \alpha(\bar{g}_{\mu\nu}\bar{\nabla}^\lambda\bar{\nabla}_\lambda R_L - \bar{\nabla}_\nu\bar{\nabla}_\mu R_L) + \frac{\beta}{2}[\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\lambda(R^\rho_\mu)_L - \bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L - \bar{\nabla}_\nu\bar{\nabla}_\sigma(R^\sigma_\mu)_L + \bar{g}_{\mu\nu}\bar{\nabla}^\lambda\bar{\nabla}_\sigma(R^\sigma_\lambda)_L] \\ & + \gamma[\bar{g}_{\mu\nu}\bar{\nabla}^\lambda\bar{\nabla}_\lambda R_L - 2\bar{g}_{\nu\rho}\bar{\nabla}^\sigma\bar{\nabla}_\sigma(R^\rho_\mu)_L + 2\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L + 2\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\sigma(R^{\rho\sigma}_{\mu\lambda})_L] \\ & - \gamma[\bar{\nabla}_\nu\bar{\nabla}_\mu R_L - 2\bar{\nabla}_\nu\bar{\nabla}_\sigma(R^\sigma_\mu)_L + 2\bar{g}_{\mu\nu}\bar{\nabla}^\lambda\bar{\nabla}_\sigma(R^\sigma_\lambda)_L].\end{aligned} \tag{D17}$$

Let us recap the definitions of the linearized Ricci tensor $(R^\rho_\mu)_L$ and linearized Ricci scalar:

$$(R^\rho_\mu)_L = (g^{\rho\alpha}R_{\mu\alpha})_L = \bar{g}^{\rho\alpha}R^L_{\mu\alpha} - \frac{2\Lambda}{n-2}h^\rho_\mu, \qquad R_L = (R^\rho_\rho)_L, \tag{D18}$$

and the linearized Einstein tensor:

$$\mathcal{G}^L_{\mu\nu} \equiv R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - \frac{2\Lambda}{n-2}h_{\mu\nu}, \tag{D19}$$

which satisfies the linearized Bianchi identity $\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu} = 0$. With these two background tensors and R_L, one has

$$\begin{aligned}\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = {}& \alpha(\bar{g}_{\mu\nu}\bar{\Box}R_L - \bar{\nabla}_\mu\bar{\nabla}_\nu R_L) + \frac{\beta}{2}\left[\bar{\Box}\mathcal{G}^L_{\mu\nu} - \bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L - \frac{1}{2}\bar{\nabla}_\nu\bar{\nabla}_\mu R_L + \bar{g}_{\mu\nu}\bar{\Box}R_L\right] \\ & + \gamma\left[-2\bar{\Box}R^L_{\mu\nu} + \frac{4\Lambda}{n-2}\bar{\Box}h_{\mu\nu} + 2\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L + 2\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\sigma(R^{\rho\sigma}_{\mu\lambda})_L\right],\end{aligned} \tag{D20}$$

where $\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L$ can be calculated as

$$\bar{g}_{\nu\rho}\bar{\nabla}^\lambda\bar{\nabla}_\mu(R^\rho_\lambda)_L = \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu R_L + \frac{2n\Lambda}{(n-1)(n-2)}\mathcal{G}^L_{\mu\nu} + \frac{\Lambda}{n-1}\bar{g}_{\mu\nu}R_L. \tag{D21}$$

Finally, the last term in (D20) requires the linearized form of

$$\nabla^\mu\nabla_\nu R^{\nu\beta}_{\mu\alpha} = \Box R^\beta_\alpha - \nabla^\mu\nabla_\alpha R^\beta_\mu, \tag{D22}$$

which can be obtained from the once-contracted Bianchi identity

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu R_{\alpha\beta} - \nabla_\alpha R_{\mu\beta}, \tag{D23}$$

and the linearization yields

$$\bar{\nabla}^\mu\bar{\nabla}_\nu(R^{\nu\beta}_{\mu\alpha})_L = \bar{\Box}(R^\beta_\alpha)_L - \bar{\nabla}^\mu\bar{\nabla}_\alpha(R^\beta_\mu)_L. \tag{D24}$$

Then, putting the pieces together, one arrives at the desired expression

$$\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L = \frac{(2\alpha+\beta)}{2}(\bar{g}_{\mu\nu}\bar{\Box}R_L - \bar{\nabla}_\mu\bar{\nabla}_\nu R_L) + \frac{\beta}{2}\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{\beta}{2}\left(\frac{2n\Lambda}{(n-1)(n-2)}\mathcal{G}^L_{\mu\nu} + \frac{\Lambda}{n-1}\bar{g}_{\mu\nu}R_L\right). \tag{D25}$$

Now, let us start collecting terms in the linearization of the field equations (D1). Note that the linearization of the other three terms in the first line of (D1) yields the same contribution as $(g_{\nu\rho}\nabla^\lambda\nabla_\sigma\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}})_L$. In addition, the linearization of the first two terms in the second line of (D1) gives the same contribution. As a result, the linearized field equations become

$$\left[\zeta - \beta\frac{2\Lambda}{(n-1)(n-2)} - \gamma\frac{4\Lambda(n-3)}{(n-1)(n-2)}\right]\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box}R_L - \bar{\nabla}_\mu\bar{\nabla}_\nu R_L) + \beta\bar{\Box}\mathcal{G}^L_{\mu\nu} \tag{D26}$$

$$+\left(\alpha\frac{4\Lambda}{(n-2)} + \beta\frac{2\Lambda}{(n-1)(n-2)}\right)R_L\bar{g}_{\mu\nu} - h_{\mu\nu}\left[\frac{1}{2}f(\bar{R}^{\alpha\beta}_{\rho\sigma}) - \frac{2\Lambda}{n-2}\right] = 0. \tag{D27}$$

The last line vanishes because of the background equation (D2). The final equation can be recast in the form of the linearized field equations coming from the quadratic gravity theory

$$\mathcal{L} = \frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0) + \alpha R^2 + \beta R^\lambda_\sigma R^\sigma_\lambda + \gamma\chi_{\rm GB}, \tag{D28}$$

given in [28] as

$$\left[\frac{1}{\tilde{\kappa}} + \frac{4\Lambda n\alpha}{n-2} + \frac{4\Lambda\beta}{n-1} + \frac{4\Lambda\gamma(n-4)(n-3)}{(n-2)(n-1)}\right]\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}g_{\mu\nu}\right)R_L + \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}R_L\right) = 0. \tag{D29}$$

To match (D27) and (D29), one must have

$$\frac{1}{\tilde{\kappa}} = 2\zeta - \frac{4\Lambda}{n-2}\left[(n\alpha+\beta) + \gamma\frac{(n-2)(n-3)}{(n-1)}\right]. \tag{D30}$$

In addition, we have to require that both theories have the same vacua, which determines $\tilde{\Lambda}_0$ as

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} = -f(\bar{R}^{\alpha\beta}_{\rho\sigma}) + \frac{2n\Lambda}{n-2}\zeta - \frac{2\Lambda^2 n}{(n-2)^2}\left[(n\alpha+\beta) + \gamma\frac{(n-2)(n-3)}{(n-1)}\right], \tag{D31}$$

which follows from the vacuum field equation of (D28),

$$\frac{\Lambda - \tilde{\Lambda}_0}{2\tilde{\kappa}} + \Lambda^2\left[(n\alpha+\beta)\frac{(n-4)}{(n-2)^2} + \gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}\right] = 0, \tag{D32}$$

and (D2).

APPENDIX E: TERMS IN ELA AND EQCA

In order to calculate ELA and EQCA for the BI gravity theory defined by the Lagrangian density

$$\mathcal{L}(C^{\mu\nu}_{\alpha\beta}, R^\mu_\nu, R^\mu_\nu) = \frac{2}{\gamma}\left[\sqrt{\det(\delta^\rho_\sigma + \gamma A^\rho_\sigma)} - (\lambda_0 + 1)\right], \tag{E1}$$

one needs to calculate the background values of $\mathcal{L}$, and its first and second order derivatives. To find the background value of $\mathcal{L}$, one needs the background value of A^ρ_σ which can be found as

$$\gamma\bar{A}^\rho_\sigma \equiv \delta^\rho_\sigma\bar{a} = \delta^\rho_\sigma\lambda[1 + \lambda(a_3 + b_2)]. \tag{E2}$$

In calculating the first order derivatives of (E1), we use

$$\partial\left(\sqrt{\det(\delta^\mu_\nu + \gamma A^\mu_\nu)}\right) = \frac{\gamma}{2}\sqrt{\det(\delta^\mu_\nu + \gamma A^\mu_\nu)}B^\sigma_\rho\partial A^\rho_\sigma, \tag{E3}$$

where B^σ_ρ is defined as $B^\alpha_\rho(\delta^\rho_\beta + \gamma A^\rho_\beta) = \delta^\alpha_\beta$ with the background value $\bar{B}^\sigma_\rho = (1+\bar{a})^{-1}\delta^\sigma_\rho$. Thus, we just need the derivatives of A^ρ_σ which can be found as

$$\frac{\partial A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}} = \gamma a_1(C^{\alpha\beta}_{\sigma\nu}\delta^\rho_\mu + C^{\rho\beta}_{\mu\nu}\delta^\alpha_\sigma) + \frac{\gamma b_1}{2}C^{\alpha\beta}_{\mu\nu}\delta^\rho_\sigma + \gamma a_2 R^\beta_\nu\delta^\alpha_\sigma\delta^\rho_\mu, \tag{E4}$$

$$\frac{\partial A^\rho_\sigma}{\partial S^\mu_\nu} = \beta\delta^\nu_\sigma\delta^\rho_\mu + \gamma a_4(S^\nu_\sigma\delta^\rho_\mu + S^\rho_\mu\delta^\nu_\sigma) + \frac{\gamma b_3}{2}S^\nu_\mu\delta^\rho_\sigma, \tag{E5}$$

$$\frac{\partial A^\rho_\sigma}{\partial R^\mu_\nu} = \delta^\nu_\sigma\delta^\rho_\mu + \gamma a_2 C^{\rho\nu}_{\sigma\mu} + \gamma a_3(R^\nu_\sigma\delta^\rho_\mu + R^\rho_\mu\delta^\nu_\sigma) + \frac{\gamma b_2}{2}R^\nu_\mu\delta^\rho_\sigma, \tag{E6}$$

and their background values are

$$\left[\frac{\partial A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \lambda a_2\delta^\beta_\nu\delta^\alpha_\sigma\delta^\rho_\mu, \tag{E7}$$

$$\left[\frac{\partial A^\rho_\sigma}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \beta\delta^\nu_\sigma\delta^\rho_\mu, \tag{E8}$$

$$\left[\frac{\partial A^\rho_\sigma}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \delta^\nu_\sigma\delta^\rho_\mu + 2\lambda\left(a_3\delta^\rho_\mu\delta^\nu_\sigma + \frac{b_2}{4}\delta^\nu_\mu\delta^\rho_\sigma\right). \tag{E9}$$

Using these results, one can calculate the linear order terms in ELA and EQCA. However, prior to any calculation, it is clear that the Weyl term

$$\left[\frac{\partial \mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta} = \sqrt{\det(\delta^\rho_\nu + \gamma\bar{A}^\rho_\nu)}\,\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta} \tag{E10}$$

and the traceless-Ricci term

$$\left[\frac{\partial \mathcal{L}}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} S^\mu_\nu = \sqrt{\det(\delta^\rho_\nu + \gamma\bar{A}^\rho_\nu)}\,\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} S^\mu_\nu \tag{E11}$$

yield zero as they involve traces of $C^{\mu\nu}_{\alpha\beta}$ and S^μ_ν. The unique contribution comes from the Ricci term

$$\left[\frac{\partial \mathcal{L}}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}(R^\mu_\nu - \bar{R}^\mu_\nu) = \sqrt{\det(\delta^\rho_\nu + \gamma\bar{A}^\rho_\nu)}\,\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}(R^\mu_\nu - \bar{R}^\mu_\nu), \tag{E12}$$

which becomes

$$\left[\frac{\partial \mathcal{L}}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}(R^\mu_\nu - \bar{R}^\mu_\nu) = (\gamma R - 4\lambda)\frac{1}{2}(1+\bar{a})[1 + 2\lambda(a_3 + b_2)]. \tag{E13}$$

Adding the background value of $\mathcal{L}$,

$$\bar{\mathcal{L}} = \frac{2}{\gamma}[(1+\bar{a})^2 - (\lambda_0 + 1)], \tag{E14}$$

to this result yields the ELA given in (57).

The second order derivatives of $\mathcal{L}$ can be calculated by using

$$\partial^2\left(\sqrt{\det(\delta^\rho_\nu + \gamma A^\rho_\nu)}\right) = \frac{\gamma}{2}\sqrt{\det(\delta^\rho_\nu + \gamma A^\rho_\nu)}\Big[B^\lambda_\gamma\partial^2 A^\gamma_\lambda - \gamma B^\lambda_\theta B^\tau_\gamma(\partial A^\theta_\tau)\partial A^\gamma_\lambda + \frac{\gamma}{2}(B^\lambda_\gamma\partial A^\gamma_\lambda)^2\Big], \tag{E15}$$

where the second order derivatives of A^ρ_σ are needed. First, the second derivative of A^ρ_σ with respect to the Weyl tensor is

$$\left[\frac{\partial^2 A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}\partial C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \gamma a_1\delta^\alpha_\sigma\delta^\beta_\theta\delta^\lambda_\mu\delta^\tau_\nu\delta^\rho_\eta + \gamma\delta^\alpha_\eta\delta^\beta_\theta\delta^\tau_\nu\left(a_1\delta^\lambda_\sigma\delta^\rho_\mu + \frac{b_1}{2}\delta^\lambda_\mu\delta^\rho_\sigma\right), \tag{E16}$$

where the result does not have the symmetries of the Weyl tensor on the left-hand side. However, note that the result becomes symmetric accordingly when it is multiplied with $C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau}$ in finding the final contribution to the expression

$$\left[\frac{\partial^2 A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}\partial C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}\partial C^{\eta\theta}_{\lambda\tau}. \tag{E17}$$

Then, the other derivatives can be calculated as

$$\left[\frac{\partial^2 A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}\partial R^\eta_\theta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \gamma a_2\delta^\beta_\eta\delta^\theta_\nu\delta^\alpha_\sigma\delta^\rho_\mu,$$
$$\left[\frac{\partial^2 A^\rho_\sigma}{\partial S^\mu_\nu\partial S^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \gamma a_4(\delta^\nu_\sigma\delta^\beta_\mu\delta^\rho_\alpha + \delta^\nu_\alpha\delta^\beta_\sigma\delta^\rho_\mu) + \frac{\gamma b_3}{2}\delta^\nu_\alpha\delta^\beta_\mu\delta^\rho_\sigma, \tag{E18}$$

$$\left[\frac{\partial^2 A^\rho_\sigma}{\partial R^\mu_\nu\partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = \gamma a_3(\delta^\nu_\sigma\delta^\beta_\mu\delta^\rho_\alpha + \delta^\nu_\alpha\delta^\beta_\sigma\delta^\rho_\mu) + \frac{\gamma b_2}{2}\delta^\nu_\alpha\delta^\beta_\mu\delta^\rho_\sigma. \tag{E19}$$

It is clear that the remaining ones are just zero:

$$\frac{\partial^2 A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}\partial S^\eta_\theta} = 0, \qquad \frac{\partial^2 A^\rho_\sigma}{\partial S^\mu_\nu \partial R^\alpha_\beta} = 0. \tag{E20}$$

Using these results in (E15) let us compute the second order contributions to the EQCA term by term. First, the Weyl square term takes the form

$$\begin{aligned}\frac{1}{2}\left[\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta} C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau} = \frac{1}{2}\sqrt{\det(\delta^\mu_\nu + \gamma\bar{A}^\mu_\nu)}\Bigg\{\bar{B}^\sigma_\rho\left[\frac{\partial^2 A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}\partial C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^\sigma_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\rho\left[\frac{\partial A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \\ + \frac{\gamma}{2}\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\Bigg\}C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau},\end{aligned} \tag{E21}$$

which then yields

$$\frac{1}{2}\left[\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta} C^{\eta\theta}_{\lambda\tau}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} C^{\mu\nu}_{\alpha\beta}C^{\eta\theta}_{\lambda\tau} = \frac{1}{2}\gamma^2(1+\bar{a})(a_1+b_1)C^{\mu\nu}_{\rho\sigma}C^{\rho\sigma}_{\mu\nu}. \tag{E22}$$

Then, the term involving the square of the traceless-Ricci tensor has the form

$$\frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial S^\mu_\nu\partial S^\alpha_\beta}S^\mu_\nu S^\alpha_\beta = \frac{1}{2}\sqrt{\det(\delta^\mu_\nu + \gamma\bar{A}^\mu_\nu)}\Bigg\{\bar{B}^\sigma_\rho\left[\frac{\partial^2 A^\rho_\sigma}{\partial S^\mu_\nu\partial S^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^\sigma_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial S^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\rho\left[\frac{\partial A^\rho_\sigma}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} + \frac{\gamma}{2}\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial S^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial S^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\Bigg\}S^\mu_\nu S^\alpha_\beta, \tag{E23}$$

yielding

$$\frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial S^\mu_\nu\partial S^\alpha_\beta}S^\mu_\nu S^\alpha_\beta = \gamma^2\left[-\frac{1}{4}\beta^2 + \frac{1}{2}(1+\bar{a})(a_4+b_3)\right]S^\nu_\mu S^\mu_\nu. \tag{E24}$$

Moving to the Ricci square term which has the form

$$\begin{aligned}\frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial R^\mu_\nu\partial R^\alpha_\beta}(R^\mu_\nu - \bar{R}^\mu_\nu)(R^\alpha_\beta - \bar{R}^\alpha_\beta) = \frac{1}{2}\sqrt{\det(\delta^\mu_\nu + \gamma\bar{A}^\mu_\nu)}\Bigg\{\bar{B}^\sigma_\rho\left[\frac{\partial^2 A^\rho_\sigma}{\partial R^\mu_\nu\partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^\sigma_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\rho\left[\frac{\partial A^\rho_\sigma}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \\ + \frac{\gamma}{2}\bar{B}^\sigma_\rho\left[\frac{\partial A^\rho_\sigma}{\partial R^\mu_\nu}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^\epsilon_\zeta\left[\frac{\partial A^\zeta_\epsilon}{\partial R^\alpha_\beta}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\Bigg\}(R^\mu_\nu - \bar{R}^\mu_\nu)(R^\alpha_\beta - \bar{R}^\alpha_\beta),\end{aligned} \tag{E25}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial R^\mu_\nu\partial R^\alpha_\beta}(R^\mu_\nu - \bar{R}^\mu_\nu)(R^\alpha_\beta - \bar{R}^\alpha_\beta) = -\left(\gamma R - 2\lambda - \frac{\gamma^2}{2\lambda}R^\nu_\mu R^\mu_\nu + \frac{\gamma^2}{2\lambda}S^\nu_\mu S^\mu_\nu\right)\frac{\lambda}{2}[2(1+\bar{a})(a_3+b_2) + (2\lambda(a_3+b_2)+1)^2] \\ - \frac{\gamma^2((2a_3\lambda+1)^2 - 2(1+\bar{a})(a_3+b_2))}{4}S^\nu_\mu S^\mu_\nu,\end{aligned} \tag{E26}$$

after using

$$R^2 = 4(R^\nu_\mu R^\mu_\nu - S^\nu_\mu S^\mu_\nu). \tag{E27}$$

Then, the first two cross terms yield zero as

$$\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial S^{\eta}_{\theta}}C^{\mu\nu}_{\alpha\beta}S^{\eta}_{\theta} = \sqrt{\det(\delta^{\mu}_{\nu}+\gamma\bar{A}^{\mu}_{\nu})}\left\{\bar{B}^{\sigma}_{\rho}\left[\frac{\partial^2 A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}\partial S^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^{\sigma}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial S^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} + \frac{\gamma}{2}\bar{B}^{\sigma}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial S^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\right\}C^{\mu\nu}_{\alpha\beta}S^{\eta}_{\theta}, \tag{E28}$$

$$\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial S^{\eta}_{\theta}}C^{\mu\nu}_{\alpha\beta}S^{\eta}_{\theta} = 0, \tag{E29}$$

and

$$\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial R^{\eta}_{\theta}}C^{\mu\nu}_{\alpha\beta}(R^{\eta}_{\theta}-\bar{R}^{\eta}_{\theta}) = \sqrt{\det(\delta^{\mu}_{\nu}+\gamma\bar{A}^{\mu}_{\nu})}\left\{\bar{B}^{\sigma}_{\rho}\left[\frac{\partial^2 A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}\partial R^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^{\sigma}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial R^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} + \frac{\gamma}{2}\bar{B}^{\sigma}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial C^{\mu\nu}_{\alpha\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial R^{\eta}_{\theta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\right\}C^{\mu\nu}_{\alpha\beta}(R^{\eta}_{\theta}-\bar{R}^{\eta}_{\theta}), \tag{E30}$$

$$\frac{\partial^2\mathcal{L}}{\partial C^{\mu\nu}_{\alpha\beta}\partial R^{\eta}_{\theta}}C^{\mu\nu}_{\alpha\beta}(R^{\eta}_{\theta}-\bar{R}^{\eta}_{\theta}) = 0. \tag{E31}$$

Lastly, the nonzero cross term is

$$\frac{\partial^2\mathcal{L}}{\partial S^{\mu}_{\nu}\partial R^{\alpha}_{\beta}}S^{\mu}_{\nu}(R^{\alpha}_{\beta}-\bar{R}^{\alpha}_{\beta}) = \sqrt{\det(\delta^{\mu}_{\nu}+\gamma\bar{A}^{\mu}_{\nu})}\left\{\bar{B}^{\sigma}_{\rho}\left[\frac{\partial^2 A^{\rho}_{\sigma}}{\partial S^{\mu}_{\nu}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} - \gamma\bar{B}^{\sigma}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial S^{\mu}_{\nu}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} + \frac{\gamma}{2}\bar{B}^{\sigma}_{\rho}\left[\frac{\partial A^{\rho}_{\sigma}}{\partial S^{\mu}_{\nu}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\bar{B}^{\epsilon}_{\zeta}\left[\frac{\partial A^{\zeta}_{\epsilon}}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}\right\}S^{\mu}_{\nu}(R^{\alpha}_{\beta}-\bar{R}^{\alpha}_{\beta}), \tag{E32}$$

$$\frac{\partial^2\mathcal{L}}{\partial S^{\mu}_{\nu}\partial R^{\alpha}_{\beta}}S^{\mu}_{\nu}(R^{\alpha}_{\beta}-\bar{R}^{\alpha}_{\beta}) = -\frac{1}{2}\gamma^2\beta(1+2\lambda a_3)S^{\nu}_{\mu}S^{\mu}_{\nu}, \tag{E33}$$

after using

$$R^{\nu}_{\mu}S^{\mu}_{\nu} = S^{\nu}_{\mu}S^{\mu}_{\nu}. \tag{E34}$$

Adding all these second order contributions to the ELA yields the EQCA given in (85).

APPENDIX F: FIELD EQUATIONS

In this appendix we will derive the field equations by minimizing the action

$$I = \frac{2}{\kappa\gamma}\int d^4x\left[\sqrt{-\det(g_{\mu\nu}+\gamma A_{\mu\nu})} - (\lambda_0+1)\sqrt{-\det g}\right], \tag{F1}$$

where we work in the Riemann–Ricci–curvature-scalar curvature basis:

$$A_{\mu\nu} = (\beta+1)R_{\mu\nu} - \frac{\beta}{4}g_{\mu\nu}R + c_1 g_{\mu\nu}R^2 + c_2 RR_{\mu\nu} + c_3 g_{\mu\nu}R^2_{\sigma\rho} + c_4 R^{\sigma}{}_{\mu}R_{\nu\sigma} + c_5 R_{\mu\sigma\nu\rho}R^{\sigma\rho} + c_6 g_{\mu\nu}R^2_{\sigma\rho\alpha\beta} + c_7 R_{\mu}{}^{\sigma\rho\tau}R_{\nu\sigma\rho\tau}. \tag{F2}$$

One can eliminate c_6 or c_7 in favor of the other, but we will keep it this way.

The variation of the action is

$$\delta I = \frac{2}{\kappa\gamma}\int d^4x\sqrt{-\det g}\left\{\left(-\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\right)\left[\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)}-(\lambda_0+1)\right]+\left[\delta\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)}\right]\right\}. \tag{F3}$$

The first term is already in the desired form. On the other hand, the second term can be analyzed by using (42), that is,

$$\delta\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)} = \frac{\gamma}{2}\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)}B^\alpha_\beta\delta A^\beta_\alpha, \tag{F4}$$

where $B\equiv(\delta+\gamma A)^{-1}$. For notational convenience, let us define $\mathcal{E}^\alpha_\beta\equiv\frac{\gamma}{2}\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)}B^\alpha_\beta$. Then, after considering the variations of the curvature terms in δA^β_α, a lengthy computation yields the field equations as

$$\begin{aligned}
&-\frac{1}{2}g_{\alpha\beta}\left[\sqrt{\det\left(\delta_\mu^\nu+\gamma A_\mu^\nu\right)}-(\lambda_0+1)\right]\\
&+(\beta+1)\left(\mathcal{E}^\mu_\beta R_{\mu\alpha}-\nabla_\mu\nabla_\beta\mathcal{E}^\mu_\alpha+\frac{1}{2}\Box\mathcal{E}_{\beta\alpha}+\frac{1}{2}g_{\alpha\beta}\nabla_\mu\nabla_\nu\mathcal{E}^{\mu\nu}\right)-\frac{\beta}{4}\left(\mathcal{E}R_{\alpha\beta}-\nabla_\beta\nabla_\alpha\mathcal{E}+g_{\alpha\beta}\Box\mathcal{E}\right)\\
&+c_1\left(2\mathcal{E}RR_{\alpha\beta}-2\nabla_\alpha\nabla_\beta(\mathcal{E}R)+2g_{\alpha\beta}\Box(\mathcal{E}R)\right)+c_2\left(\mathcal{E}^\mu_\nu R^\nu_\mu R_{\alpha\beta}+\mathcal{E}^\mu_\beta RR_{\alpha\mu}\right)\\
&+c_2\left[-\nabla_\alpha\nabla_\beta(\mathcal{E}^\mu_\nu R^\nu_\mu)+g_{\alpha\beta}\Box(\mathcal{E}^\mu_\nu R^\nu_\mu)-\nabla_\mu\nabla_\beta(\mathcal{E}^\mu_\alpha R)+\frac{1}{2}\Box(\mathcal{E}_{\alpha\beta}R)+\frac{1}{2}\nabla_\mu\nabla_\lambda(\mathcal{E}^{\mu\lambda}R)g_{\alpha\beta}\right]\\
&+c_3\left(-2\nabla_\sigma\nabla_\beta(\mathcal{E}R^\sigma_\alpha)+\Box(\mathcal{E}R_{\alpha\beta})+\nabla_\rho\nabla_\sigma(\mathcal{E}R^{\sigma\rho})g_{\alpha\beta}+2\mathcal{E}R_{\alpha\rho}R^\rho_\beta\right)\\
&+c_4\Big[\left(\mathcal{E}^\mu_\nu R_{\alpha\mu}R^\nu{}_\beta+\mathcal{E}^\mu_\alpha R^\sigma{}_\mu R_{\beta\sigma}\right)-\nabla_\mu\nabla_\beta(\mathcal{E}_{\alpha\nu}R^{\nu\mu})-\nabla_\mu\nabla_\beta(\mathcal{E}^\mu_\nu R^\nu{}_\alpha)\\
&+\Box(\mathcal{E}_{\beta\nu}R^\nu{}_\alpha)+\frac{1}{2}\nabla_\mu\nabla_\sigma(\mathcal{E}^\mu_\nu R^{\nu\sigma})g_{\alpha\beta}+\frac{1}{2}\nabla_\sigma\nabla_\nu(\mathcal{E}^{\mu\nu}R^\sigma{}_\mu)g_{\alpha\beta}\Big]\\
&+c_5\Big[2\mathcal{E}^\mu_\nu R_\mu{}^{\sigma\nu}{}_\beta R_{\alpha\sigma}-\nabla_\sigma\nabla_\lambda(\mathcal{E}^\lambda_\alpha R^\sigma_\beta)+\frac{1}{2}\nabla_\lambda\nabla_\sigma(\mathcal{E}^{\lambda\sigma}R_{\alpha\beta})+\frac{1}{2}\nabla_\sigma\nabla_\lambda(\mathcal{E}_{\alpha\beta}R^{\lambda\sigma})\\
&-\nabla_\rho\nabla_\beta(\mathcal{E}^\mu_\nu R^{\nu\rho}{}_{\mu\alpha})+\frac{1}{2}\Box(\mathcal{E}^\mu_\nu R_{\mu\alpha}{}^\nu{}_\beta)+\frac{1}{2}\nabla_\rho\nabla_\gamma(\mathcal{E}^\mu_\nu R_\mu{}^{\gamma\nu\rho})g_{\alpha\beta}\Big]\\
&+2c_6\left[\mathcal{E}R_{\alpha\pi\sigma\tau}R_\beta{}^{\pi\sigma\tau}+\nabla_\lambda\nabla_\tau(\mathcal{E}R^\lambda{}_\alpha{}^\tau{}_\beta)+\nabla_\pi\nabla_\lambda(\mathcal{E}R_\beta{}^\pi{}_\alpha{}^\lambda)\right]\\
&+c_7\left[-\mathcal{E}_{\alpha\nu}R_{\beta\sigma\rho\tau}R^{\nu\sigma\rho\tau}+\mathcal{E}^\mu_\nu(R_{\beta\mu\rho\tau}R_\alpha{}^{\nu\rho\tau}+2R_{\alpha\rho\mu\sigma}R_\beta{}^{\rho\nu\sigma})+2\nabla_\lambda\nabla_\pi(\mathcal{E}^\mu_\alpha R_\mu{}^\lambda{}_\beta{}^\pi+\mathcal{E}^\mu_\beta R_{\mu\alpha}{}^{\lambda\pi}-\mathcal{E}^{\lambda\mu}R_{\mu\alpha\beta}{}^\pi)\right]=0,
\end{aligned} \tag{F5}$$

where $\mathcal{E}=g_{\mu\nu}\mathcal{E}^{\mu\nu}$.

For the sake of comparison with the equivalent linear action technique, let us find the maximally symmetric vacuum using the field equations. Note that $\bar{\mathcal{E}}^\alpha_\beta=\bar{e}\delta^\alpha_\beta$ with $\bar{e}=\frac{\gamma}{2}(1+\bar{a})$. In the calculations below, all the tensor quantities are evaluated at their background values.

c_7 contribution:

$$\begin{aligned}&-\mathcal{E}_{\alpha\nu}R_{\beta\sigma\rho\tau}R^{\nu\sigma\rho\tau}+\mathcal{E}^\mu_\nu R_{\mu\beta\rho\tau}R^\nu{}_\alpha{}^{\rho\tau}+\mathcal{E}^\mu_\nu R_{\mu\sigma\alpha\tau}R^{\nu\sigma}{}_\beta{}^\tau\\&+\mathcal{E}^\mu_\nu R_{\mu\sigma\rho\alpha}R^{\nu\sigma\rho}{}_\beta=\frac{4\Lambda^2}{3}\bar{e}\bar{g}_{\alpha\beta}.\end{aligned} \tag{F6}$$

c_6 contribution:

$$2\mathcal{E}R_{\alpha\pi\sigma\tau}R_\beta{}^{\pi\sigma\tau}=\frac{16\Lambda^2}{3}\bar{e}\bar{g}_{\alpha\beta}. \tag{F7}$$

c_5 contribution:

$$2\mathcal{E}^\mu_\nu R_\mu{}^{\sigma\nu}{}_\beta R_{\alpha\sigma}=2\Lambda^2\bar{e}\bar{g}_{\alpha\beta}. \tag{F8}$$

c_4 contribution:

$$\mathcal{E}^\mu_\nu R_{\alpha\mu}R^\nu{}_\beta+\mathcal{E}^\mu_\alpha R^\sigma{}_\mu R_{\beta\sigma}=2\bar{e}R_{\sigma\beta}R^\sigma_\alpha=2\Lambda^2\bar{e}\bar{g}_{\alpha\beta}. \tag{F9}$$

c_3 contribution:

$$8\bar{e}R_{\alpha\rho}R^{\rho}_{\beta} = 8\Lambda^2\bar{e}\bar{g}_{\alpha\beta}. \quad \text{(F10)}$$

c_2 contribution:

$$\mathcal{E}^{\mu}_{\nu}R^{\nu}_{\mu}R_{\alpha\beta} + \mathcal{E}^{\mu}_{\beta}RR_{\alpha\mu} = 2\bar{e}RR_{\alpha\beta} = 8\Lambda^2\bar{e}\bar{g}_{\alpha\beta}. \quad \text{(F11)}$$

c_1 contribution:

$$2\mathcal{E}RR_{\alpha\beta} = 32\Lambda^2\bar{e}\bar{g}_{\alpha\beta}. \quad \text{(F12)}$$

Using the conversion relations between the bases (B4) and

$$\gamma\bar{A}^{\rho}_{\sigma} \equiv \delta^{\rho}_{\sigma}\bar{a} = \delta^{\rho}_{\sigma}\lambda[1 + \lambda(a_3 + b_2)], \quad \text{(F13)}$$

and after defining $c \equiv a_3 + b_2$,

$$\bar{e} = \frac{\gamma}{2}(1 + \lambda + c\lambda^2), \quad \text{(F14)}$$

one arrives at

$$c^2\lambda^4 + c\lambda^3 - \lambda + \lambda_0 = 0, \quad \text{(F15)}$$

which is the same as the one found with ELA. Here $\bar{a}$ is defined in (58).

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