

**Binary system of unequal counterrotating Kerr-Newman sources**I. Cabrera-Munguia<sup>\*</sup>*Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa,  
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Stationary axisymmetric binary systems of unequal counterrotating Kerr-Newman sources with a massless strut in between are studied. By means of the choice of a suitable parametrization, the axis conditions and the absence of individual magnetic charges are fulfilled; thus, the entire metric reduces to a six-parametric asymptotically flat exact solution. Later on, with the purpose to describe interacting black holes, the analytic functional form of the horizon half-length parameter  $\sigma_k$  is obtained explicitly in terms of physical Komar parameters: mass  $M_k$ , electric charge  $Q_k$ , angular momentum  $J_k$ , and coordinate distance  $R$ , where the seven physical parameters satisfy a simple algebraic relation. Finally, in the limit of extreme black holes, the full metric is derived in a closed analytical form, and a study on the absence or appearance of naked singularities off the axis is presented.

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**I. INTRODUCTION**

In stationary axisymmetric spacetimes, the study of binary systems composed by Reissner-Norström (RN) sources [1,2] began to receive attention since the well-known Weyl family of solutions was presented in 1917 [3]; it relates the masses  $M_k$  and electric charges  $Q_k$ , for  $k = 1, 2$ , through the relation  $M_1 Q_2 - M_2 Q_1 = 0$ . In the context of black hole sources, a particular case of Weyl's family is the Majumdar-Papapetrou solution [4,5], which describes two extreme black holes in equilibrium when the charges are equal to the masses according to  $Q_k = \pm M_k$ , regardless of the distance  $R$  between sources. Many years later, Varzugin and Chyistiakov studied intensively these binary configurations in the black hole sector [6], and, after solving the corresponding Riemann-Hilbert problem, they found explicitly the formulas for the event horizons of length  $2\sigma_{kE}$ , which are given by

$$\begin{aligned}\sigma_{1E} &= \sqrt{M_1^2 - Q_1^2 + 2\mu Q_1}, \\ \sigma_{2E} &= \sqrt{M_2^2 - Q_2^2 - 2\mu Q_2}, \\ \mu &:= \frac{M_2 Q_1 - M_1 Q_2}{R + M_1 + M_2}.\end{aligned}\quad (1)$$

The above set of equations (1) is quite important to understand better the physical and thermodynamical properties of such configurations, which were used in the years following its discovery. For instance, by applying soliton techniques the equilibrium problem without strut in between the black holes is studied in Ref. [7], while the interaction force associated with the strut and the full metric

are derived in Ref. [8]. The strut prevents the sources from falling onto each other; it represents a line source of pressure computed through the conical singularity (angle's deficit) [9,10]. The counterpart of the aforementioned charged solutions is the rotating case, which has been extensively studied since the double-Kerr-NUT solution was developed by Kramer and Neugebauer in 1980 [11], but nowadays in the vacuum case there exist no analogous formulas like those given by Eqs. (1). It is worth mentioning that charged or rotating binary systems develop naked singularities off the axis as singular surfaces (SS) [12,13] or ring singularities (RS) [14], respectively, if at least one of the masses results to be negative [15,16].

On the other hand, the electrovacuum sector seems to be impossible to treat due to the technical difficulties provided by the electromagnetic field into rotating systems. Therefore, only equal cases have been studied after taking into account the advantages of their symmetry properties in which the axis conditions are automatically fulfilled. A first study of a binary system of identical Kerr-Newman (KN) sources [17] in equilibrium under their mutual electromagnetic and gravitational interactions was given by Parker, Rufinni, and Wilkins [18] through the Perjés-Israel-Wilson method [19,20]. The sources are two thin disks lying on the equatorial plane, whose charges are equal to their respective masses and contain opposite spin (a counterrotating system), and both charges are equal in magnitude and sign. Recently, a more general description of such a problem was developed in Ref. [21], where the functional form of the horizon half-length parameter  $\sigma$  is introduced in terms of physical Komar parameters [22]. Furthermore, if the charges have opposite sign, the solution represents a counterrotating black dihole system [23,24], whose magnetic dipole moment is generated by the rotation of

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electrically charged black holes [25,26]. Additionally, it has been shown that individual magnetic charges can be created by the rotation of these diholes [26,27], and now the system turns out to be dyonic [28], where the Smarr mass formula [29] is generalized in order to include the contribution of the magnetic charge into the mass [30].

Since, in balancing configurations (without the supporting strut) [14–16], the interpretation of such binary systems as describing two black holes is spoiled, this motivates the search on stationary configurations of interacting black hole sources with a strut in between, with the purpose to use this kind of solution and provide novel evidence on the interactions and their mathematical properties, which may be used to construct initial data in numerical simulations like the *momentary stationary data* performed in Ref. [31] for two extreme Kerr sources. The present paper aims at the construction of a model for unequal counterrotating KN black hole sources interacting by means of a massless strut, in which the full metric and all its geometrical properties are given in a more physical way through the derivation of  $\sigma_k$  as a function of Komar's physical parameters.

To accomplish our goal, first we solve analytically the axis conditions in order to describe a binary system of unequal counterrotating KN sources with a supporting strut. We will use the same idea given in Ref. [25] to eliminate the individual magnetic charges as well as the total monopolar magnetic charge; thus, the Smarr formula for the mass is ensured. Afterward, we will be able to derive the nontrivial expressions for  $\sigma_k$ ,  $k = 1, 2$ , which include the rotation parameter and generalize the aforementioned formulas given by Eq. (1). In fact, this paper deals with the unequal case of the solutions already discussed in Refs. [21,25], where now the seven physical parameters satisfy an algebraic relation which might be understood as a dynamic scenario between sources, since the physical properties of one body are affected by the presence of the other one. Later on, all the thermodynamical features of the system will be determined in a concise explicit form. The second objective pursued in this paper is the derivation of the extreme limit case in a closed analytical form, by means of Perjés' representation [32]. We will prove that the model saturates the inequality for interacting black holes with struts discovered by Gabach Clement [33]. Additionally, a numerical study on *ring singularities* off the axis is given. In the extreme double-Reissner-Norström (DRN) sector an easy analytical proof is also provided on the absence or appearance of *singular surfaces* as a complement of the well-known positive mass theorem [34,35].

## II. THE ASYMPTOTICALLY FLAT EXACT SOLUTION

Let us start the section by introducing the Ernst equations [36] as follows:

$$\begin{aligned} (\text{Re}\mathcal{E} + |\Phi|^2)\Delta\mathcal{E} &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\nabla\mathcal{E}, \\ (\text{Re}\mathcal{E} + |\Phi|^2)\Delta\Phi &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\nabla\Phi, \end{aligned} \quad (2)$$

which describe stationary axisymmetric electrovacuum spacetimes through the complex potentials  $\mathcal{E} = f - |\Phi|^2 + i\Psi$  and  $\Phi = -A_4 + iA_3'$ , where  $\nabla$  and  $\Delta$  are the gradient and Laplace operators, respectively, defined in Weyl-Papapetrou cylindrical coordinates  $(\rho, z)$ . The electrovacuum exact solution of Eq. (2) describing a binary system composed by KN sources can be obtained by means of the Sibgatullin method (SM) [37,38], which is based on the soliton theory and is helpful to construct the Ernst potentials  $(\mathcal{E}, \Phi)$  in the whole spacetime as well as the metric functions  $f(\rho, z)$ ,  $\omega(\rho, z)$ , and  $\gamma(\rho, z)$  of the stationary axisymmetric line element [39]

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2. \quad (3)$$

In this approach, the Ernst potentials on the symmetry axis  $\mathcal{E}(\rho = 0, z) := e(z)$  and  $\Phi(\rho = 0, z) := f(z)$  have the following form:

$$e(z) = 1 + \sum_{j=1}^2 \frac{e_j}{z - \beta_j}, \quad f(z) = \sum_{j=1}^2 \frac{f_j}{z - \beta_j}, \quad (4)$$

where  $\{e_j, f_j, \beta_j\}$  are complex constants which can be related with Simon's multipolar moments [40]. The specific choice (4) of the axis data is motivated by two facts: the mentioned relation of the parameters  $e_j, f_j$ , and  $\beta_j$  with the multipolar terms and that the SM deals with poles on the complex plane. Therefore, this specific representation turns out to work very well in the framework of the SM. With the purpose to describe a binary system of KN sources, the explicit solution is worked out after setting  $N = 2$  in the last formulas of Sec. III in [38].

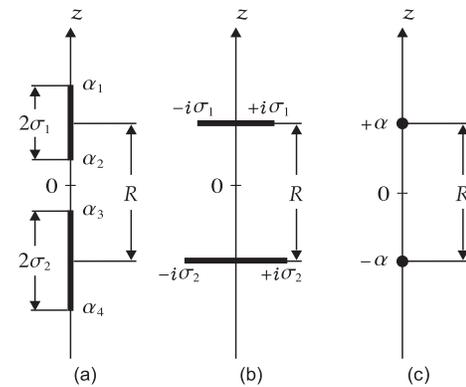


FIG. 1. Location of two unequal KN sources on the symmetry axis: (a) black hole configuration; (b) hyperextreme sources after making  $\sigma_k \rightarrow i\sigma_k$ ; (c) the extreme limit case  $\sigma_k = 0$ .

Then, the two-body system is depicted by twelve algebraic parameters  $\{\alpha_n, f_j, \beta_j\}$ , for  $n = \overline{1,4}$  and  $j = 1, 2$ , where  $\alpha_n$  define the location of the sources on the symmetry axis (see Fig. 1). It should be pointed out that SM provides a mathematical recipe to construct the Ernst potentials and metric functions at whole spacetime, but generally the solution is not asymptotically flat at spatial infinity, due to the presence of NUT sources [41] and the global monopolar magnetic charge. Hence, to eliminate the gravitomagnetic monopole (NUT parameter) and disconnect the region between sources, one needs to solve the following axis conditions [27,42]:

$$\begin{aligned} \text{Im}[\bar{\mathbf{a}}_-(\mathbf{g}_- + \mathbf{h}_-)] &= 0, & \text{Im}[\bar{\mathbf{a}}_+(\mathbf{g}_+ + \mathbf{h}_+)] &= 0, \\ \mathbf{g}_\pm &= \begin{pmatrix} 0 & 2 & 2 & 1 \pm 1 & 1 \pm 1 \\ 1 \\ 1 & (\mathbf{a}_\pm) \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{h}_\pm &= \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 \\ 1 & (\mathbf{a}_\pm) \\ \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}, \\ \mathbf{a}_\pm &= \begin{pmatrix} \pm\gamma_{11} & \pm\gamma_{12} & \gamma_{13} & \gamma_{14} \\ \pm\gamma_{21} & \pm\gamma_{22} & \gamma_{23} & \gamma_{24} \\ M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{pmatrix}, \\ M_{jn} &= [\bar{e}_j + 2\bar{f}_j f(\alpha_n)](\alpha_n - \bar{\beta}_j)^{-1}, \\ f(\alpha_n) &= \sum_{j=1}^2 f_j \gamma_{jn}, & \gamma_{jn} &= (\alpha_n - \beta_j)^{-1}, \\ e_1 &= \frac{2 \prod_{n=1}^4 (\beta_1 - \alpha_n)}{(\beta_1 - \beta_2)(\beta_1 - \bar{\beta}_1)(\beta_1 - \bar{\beta}_2)} - \sum_{k=1}^2 \frac{2f_1 \bar{f}_k}{\beta_1 - \bar{\beta}_k}, \\ e_2 &= \frac{2 \prod_{n=1}^4 (\beta_2 - \alpha_n)}{(\beta_2 - \beta_1)(\beta_2 - \bar{\beta}_1)(\beta_2 - \bar{\beta}_2)} - \sum_{k=1}^2 \frac{2f_2 \bar{f}_k}{\beta_2 - \bar{\beta}_k}. \end{aligned} \quad (5)$$

The parameters  $\alpha_n$  can be written in terms of the relative distance  $R$  and the half-length  $\sigma_k$  of each rod as follows:

$$\begin{aligned} \alpha_1 &= \frac{R}{2} + \sigma_1, & \alpha_2 &= \frac{R}{2} - \sigma_1, \\ \alpha_3 &= -\frac{R}{2} + \sigma_2, & \alpha_4 &= -\frac{R}{2} - \sigma_2, \end{aligned} \quad (6)$$

where  $\sigma_k$  can take real positive or pure imaginary values as shown in Fig. 1. In order to solve the above set of algebraic equations (5), one notes that the first Simon's multipolar terms [40] as the total mass  $\mathcal{M}$ , total electric charge  $\mathcal{Q}$ , and total magnetic charge  $\mathcal{B}$  can be calculated from the Ernst potentials (4) on the symmetry axis, leading us to

$$\beta_1 + \beta_2 + \bar{\beta}_1 + \bar{\beta}_2 = -2\mathcal{M}, \quad f_1 + f_2 = \mathcal{Q} + i\mathcal{B}. \quad (7)$$

By choosing  $\beta_1 + \beta_2 = -\mathcal{M} := -M$  and  $\mathcal{B} := 0$ , one may describe a two-body system of counterrotating KN sources apart by a massless strut, where the upper constituent is endowed with mass  $M_1$ , electric and magnetic charge  $Q_1$  and  $B_1$ , respectively, and angular momentum  $J_1$ , while the lower one contains  $M_2$ ,  $Q_2$ ,  $B_2$ , and  $J_2$ , respectively. The individual magnetic charges are equal in magnitude but opposite in sign, i.e.,  $B_1 = -B_2 = Q_B$ . The angular momenta have opposite spin, and both bodies are separated by the coordinate distance  $R$ . Moreover, if we want to remove also the individual magnetic charges ( $B_k = 0$ ), the following condition should be satisfied [30]:

$$\begin{aligned} A_4(\rho = 0, z = \alpha_{2k-1}) - A_4(\rho = 0, z = \alpha_{2k}) &= 0, \\ k &= 1, 2, \end{aligned} \quad (8)$$

where  $A_4$  is the electric potential computed from the real part of  $\Phi$ . Therefore, the explicit solution of the axis conditions (5), together with an absence of magnetic charges Eq. (8) is given by

$$\begin{aligned} f_{1,2} &= \pm \frac{Q\beta_{1,2} + q_o + ib_o}{\beta_1 - \beta_2}, & \beta_{1,2} &= \frac{-M \pm \sqrt{p + iq_+}}{2}, \\ p &= R^2 + M^2 - 2\Delta_o + 2\left(\epsilon_1 - \frac{\epsilon_2 R}{M}\right) - \frac{4Qq_o}{M}, & q_\pm &= 2\left(\frac{\beta_o(MR \mp \epsilon_2) \mp Q\beta}{M\alpha_o}\right)\delta, & b_o &= \frac{\beta\delta}{2\alpha_o}, \\ \delta &:= \sqrt{\alpha_o[\beta_o[M^2(\Delta_o - 2\epsilon_1) + \epsilon_2^2 + 4q_o^2] - \beta^2]}, & \alpha_o &:= \beta_o[\beta_o(M^2 R^2 - \epsilon_2^2) - 2\epsilon_2 Q\beta] + \Delta_o \beta^2, \\ \beta &:= 2q_o R - \epsilon_2 Q, & \beta_o &:= R^2 - \Delta_o, & \Delta_o &:= M^2 - Q^2, & \epsilon_{1,2} &:= \sigma_1^2 \pm \sigma_2^2. \end{aligned} \quad (9)$$

After cumbersome calculations the Ernst potentials and metric functions which define a six-parametric asymptotically flat exact solution assume the form

$$\begin{aligned}
\mathcal{E} &= \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, & \Phi &= \frac{\chi}{\Lambda - \Gamma}, & f &= \frac{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2}{|\Lambda - \Gamma|^2}, & \omega &= \frac{\text{Im}[(\Lambda - \Gamma)\bar{\mathcal{G}} - \chi\bar{\mathcal{I}}]}{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2}, & e^{2\gamma} &= \frac{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2}{256\sigma_1^2\sigma_2^2\alpha_o^2r_1r_2r_3r_4}, \\
\Lambda &= 4\sigma_1\sigma_2(\alpha_o - M^2\beta_o^3)(r_1r_2 + r_3r_4) + [2M^2\beta_o^4 + (\Delta_o - \epsilon_1 - \beta_o)(\alpha_o - M^2\beta_o^3)](r_1 - r_2)(r_3 - r_4) \\
&\quad + 2\sigma_1\sigma_2(\alpha_o + M^2\beta_o^3)(r_1 + r_2)(r_3 + r_4) - 2i\beta_o\delta[\sigma_1(r_1 + r_2)(r_3 - r_4) - \sigma_2(r_1 - r_2)(r_3 + r_4)], \\
\Gamma &= (2/MR)\{\sigma_2[2\sigma_1M^2(a_o\beta_o^2 - \alpha_o)(r_1 + r_2) + \epsilon_+(r_1 - r_2)] - \sigma_1[2\sigma_2M^2(a_o\beta_o^2 + \alpha_o)(r_3 + r_4) + \epsilon_-(r_3 - r_4)]\}, \\
\chi &= (2/R)\{\sigma_2[2\sigma_1R(\alpha_oQ - M^2\beta_o^2)(r_1 + r_2) + \epsilon_+(r_1 - r_2)] + \sigma_1[2\sigma_2R(\alpha_oQ + M^2\beta_o^2)(r_3 + r_4) + \epsilon_-(r_3 - r_4)]\}, \\
\mathcal{G} &= 2z\Gamma + (1/R^2)\{4\sigma_1\sigma_2\epsilon_o(r_1r_2 - r_3r_4) + \sigma_1\kappa_-(r_1 + r_2)(r_3 - r_4) + \sigma_2\kappa_+(r_1 - r_2)(r_3 + r_4) - 2i(a_o + \epsilon_2\beta) \\
&\quad \times \delta R^2(r_1 - r_2)(r_3 - r_4)\} + (2/MR)\{\sigma_2[2\sigma_1R\nu_+(r_1 + r_2) + v_+(r_1 - r_2)] + \sigma_1[2\sigma_2R\nu_-(r_3 + r_4) - v_-(r_3 - r_4)]\}, \\
\mathcal{I} &= (1/MR^2)\{4\sigma_1\sigma_2R[M^2(a_+r_1r_2 - a_-r_3r_4) + 2c_oR(\alpha_o + i\delta R)] + c_oR^2[c_+(r_1r_4 + r_2r_3) - c_-(r_1r_3 + r_2r_4)] + i\delta R^2 \\
&\quad \times [(c_oR - 2M^2\beta)(r_1 - r_2)(r_3 - r_4) + c_o[(\sigma_1 + \sigma_2)(r_1r_4 - r_2r_3) - (\sigma_1 - \sigma_2)(r_1r_3 - r_2r_4)]] - \sigma_1d_-(r_1 + r_2)(r_3 - r_4) \\
&\quad + \sigma_2d_+(r_1 - r_2)(r_3 + r_4)\} + (1/R)\{\sigma_2[2\sigma_1\lambda_+(r_1 + r_2) + \mu_+(r_1 - r_2)] + \sigma_1[2\sigma_2\lambda_-(r_3 + r_4) + \mu_-(r_3 - r_4)]\}, \\
\epsilon_{\pm} &:= M^2(\alpha_o\Delta_o - \epsilon_2a_o\beta_o^2) \pm R(\alpha_o\nu_o - a_oM^2\beta_o^2R) - i(M^2\beta_o \mp a_o)\delta R, & a_o &:= Q\beta + \epsilon_2\beta_o, \\
\epsilon_{\pm} &:= M^2\beta_o^2(R^2 \pm \epsilon_2) - \alpha_o(2q_oR \pm \Delta_oQ) - i(\beta \mp Q\beta_o)\delta R, & \epsilon_o &:= 2(\alpha_o - M^2\beta_o^2)R^2 + \alpha_oQ^2 - M^2\beta_o\beta^2, \\
\kappa_{\pm} &:= 2\alpha_o(\Delta_o + M^2)R^2 - \epsilon_oR^2 \pm [2\epsilon_2(\alpha_o + M^2\beta_o^3 + 2M^2q_o^2\beta_o)R^2 + Q(2\beta + \epsilon_2Q)(\alpha_o - \epsilon_2^2M^2\beta_o)], \\
\nu_{\pm} &:= \alpha_o[\nu_o\beta_o \pm M^2R(R^2 - 2\beta_o - 2\epsilon_1 \mp \epsilon_2)] + M^2\beta_o[2\epsilon_1a_o\beta_oR - \epsilon_2\beta c_o - \beta_o(a_o \mp M^2\beta_o)R^3] + i\beta_o(\nu_o \mp M^2R)\delta R, \\
\nu_{\pm} &:= \alpha_o\nu_o \pm M^4\beta_o^2R + i(a_o \pm M^2\beta_o)\delta, & a_{\pm} &:= \alpha_oQ - a_o\beta\beta_o \mp c_o\beta_o^2R, & c_o &:= 2\Delta_oq_o - \epsilon_2QR, \\
\mu_{\pm} &:= \alpha_o[Q(R^2 + \beta_o - 2\epsilon_1 \mp \epsilon_2) \pm \beta] \pm 2c_o^2\beta \pm M^2\beta_o^2[\beta(R^2 - 2\beta_o + 2\epsilon_1) \mp 2Q\beta_o(R^2 \pm \epsilon_2) \pm 3\epsilon_2\beta] \\
&\quad - i[(QR \pm 4q_o)\beta_o \mp 3R\beta]\delta R, & \lambda_{\pm} &:= \alpha_o(2q_oR \mp 3Q\beta_o) \pm M^2\beta_o^2(2QR^2\beta_o - 3\epsilon_2\beta) + i(\beta \mp Q\beta_o)\delta R, \\
c_{\pm} &:= \alpha_o - M^2\beta_o^2[R^2 - (\sigma_1 \pm \sigma_2)^2], & d_{\pm} &:= \alpha_o[M^2R(2q_o \pm QR) + a_oQ] \pm M^2a_o\beta\beta_o(R^2 \pm \epsilon_2), \tag{10}
\end{aligned}$$

where

$$\begin{aligned}
r_{1,2} &= \sqrt{\rho^2 + (z - R/2 \mp \sigma_1)^2}, \\
r_{3,4} &= \sqrt{\rho^2 + (z + R/2 \mp \sigma_2)^2}, \tag{11}
\end{aligned}$$

and now the Ernst potentials on the symmetry axis given by Eq. (4) reduce to

$$\begin{aligned}
e(z) &= \frac{e_+}{e_-}, & f(z) &= \frac{Qz + q_o + ib_o}{e_-}, \\
e_{\pm} &= z^2 \mp Mz + \frac{M(\Delta_o - \epsilon_1 - R^2/2) \mp \nu_o}{2M} - i\frac{q_{\mp}}{4}, \\
\nu_o &:= \epsilon_2R + 2Qq_o. \tag{12}
\end{aligned}$$

At this point, it is worth mentioning that the procedure to eliminate the magnetic charges of the system was considered first in Ref. [25] for a binary system of identical counterrotating KN black holes endowed with opposite electric charges, i.e., stationary black diholes. This particular solution [25] has the advantage that naturally

satisfies the axis conditions, due to its symmetry properties. Nevertheless, as was already mentioned in Ref. [26], a more general procedure to determine new exact solutions has to do with the choice of a suitable parametrization in order to solve the axis conditions. In fact, the algebraic equations (5) represent a generalization of the axis conditions introduced in Ref. [43] for vacuum solutions.

With respect to the total angular momentum of the system  $J$ , which can be obtained asymptotically from the Ernst potentials on the symmetry axis Eq. (12), it reads

$$J = \frac{\epsilon_2(R^2 - M^2) + 2Qq_oR}{2M\alpha_o}\delta, \tag{13}$$

which under the transformation  $\epsilon_2 \rightarrow -\epsilon_2$ ,  $z \rightarrow -z$ ,  $q_o \rightarrow -q_o$  changes its global sign as well as the metric function  $\omega$ . This fact means that the metric function  $\omega$  and the total angular momentum  $J$  change their global signs if one exchanges the physical properties and the position of the constituents; i.e.,  $J_{(1 \leftrightarrow 2)} = -J$  and  $\omega(\rho, -z)_{(1 \leftrightarrow 2)} = -\omega(\rho, z)$ . Therefore, the full metric describes a two-body system of counterrotating KN sources. If  $Q = 0$  and

$q_o = 0$ , Eq. (10) is reduced to the vacuum solution presented in Ref. [43]. Moreover, in the absence of rotation from Eq. (13) the following result is obtained:

$$q_o = \frac{\epsilon_2 QR \pm M \sqrt{(R^2 - \Delta_o)(\Delta_o^2 - 2\epsilon_1 \Delta_o + \epsilon_2^2)}}{2\Delta_o}, \quad (14)$$

where now Eq. (10) defines a binary system of RN sources. The case concerning black holes [7,8] is straightforwardly obtained after using the Varzugin-Chyistiakov parametrization [6], which is described by Eqs. (1). However, the reduced metric given by Eq. (14) can be useful also to describe relativistic disks (hyperextreme sources) under the transformation  $\sigma_k \rightarrow i\sigma_k$ . In what follows in this paper, we are interested only in the description of a two-body system of unequal counterrotating KN black holes, by means of an explicit derivation of the formulas for the two horizons  $\sigma_k$  as a function of physical Komar parameters.

### A. Physical representation for $\sigma_k$

As was already pointed out,  $\sigma_k$  defines the half-length of the rod representing the  $k$ th black hole, whose event horizon is defined as a null hypersurface  $H_k = \{\alpha_{2k} \leq z \leq \alpha_{2k-1}, 0 \leq \varphi \leq 2\pi, \rho \rightarrow 0\}$ . A starting point for the physical representation is to use the well-known Tomimatsu formulas [30]

$$\begin{aligned} M_k &= -\frac{1}{8\pi} \int_{H_k} \omega \Psi_{,z} d\varphi dz, & Q_k &= \frac{1}{4\pi} \int_{H_k} \omega A'_{3,z} d\varphi dz, \\ J_k &= -\frac{1}{8\pi} \int_{H_k} \omega \left[ 1 + \frac{\omega \Psi_{,z}}{2} - \tilde{A}_3 A'_{3,z} - (A'_3 A_3)_{,z} \right] d\varphi dz, \end{aligned} \quad (15)$$

with  $\tilde{A}_3 := A_3 + \omega A_4$  and  $\Psi = \text{Im}(\mathcal{E})$ . The magnetic potential  $A_3$  is obtained after taking the real part of Kinnersley's potential  $\Phi_2$  [44], whose expression is performed within the framework of SM [38] and has the form

$$\begin{aligned} \Omega_k &= \frac{\sqrt{(R^2 - M^2 + Q^2)(\sigma_{kE}^2 - \sigma_k^2)} P_0}{(R + M)[2P_{1k}(M_k + \sigma_k) - Q_k P_{2k}]}, & \Phi_k^H &= \frac{(R + M)P_{2k}(M_k + \sigma_k) - 2Q_k P_1}{(R + M)[2P_{1k}(M_k + \sigma_k) - Q_k P_{2k}]}, & k &= 1, 2, \\ P_0 &:= [(R + M_1)^2 - M_2^2][(R + M_2)^2 - M_1^2] + [(Q_1 - Q_2)R + (M_1 - M_2)Q]^2, \\ P_1 &:= [Q_1(R + M_1 - M_2) + M_1 Q_2][Q_2(R - M_1 + M_2) + M_2 Q_1] - M_1 M_2 Q_1 Q_2, \\ P_{11} &:= M_1[(R + M_2)^2 - M_1^2] - Q_1[Q_2 R - (M_1 - M_2)Q], & P_{12} &= P_{11(1 \leftrightarrow 2)}, \\ P_{21} &:= 2(M_1 - M_2)(M_2 Q_1 - M_1 Q_2) + R(Q_1 R + 2M_1 Q_2) - Q_1(\sigma_{1E}^2 - \sigma_{2E}^2), & P_{22} &= P_{21(1 \leftrightarrow 2)}. \end{aligned} \quad (20)$$

By placing these last expressions in the right-hand side of the Smarr mass formula Eq. (19), one obtains  $\sigma_k$  as a function of the physical Komar parameters and the coordinate distance

$$A_3 = \text{Re}(\Phi_2) = \text{Re}\left(-i \frac{I}{E_-}\right) = -zA'_3 + \text{Im}\left(\frac{\mathcal{I}}{\Lambda - \Gamma}\right). \quad (16)$$

The corresponding masses  $M_k$  and electric charges  $Q_k$  are given, respectively, by

$$\begin{aligned} M_{1,2} &= \frac{M}{2} \pm \frac{2Qq_o R + \epsilon_2(R^2 - M^2)}{2M(R^2 - M^2 + Q^2)}, \\ Q_{1,2} &= \frac{Q}{2} \pm \frac{2q_o(MR + M^2 - Q^2) - \epsilon_2 Q(R + M)}{2M(R^2 - M^2 + Q^2)}, \end{aligned} \quad (17)$$

and it can be shown from Eqs. (17) that  $M = M_1 + M_2$  and  $Q = Q_1 + Q_2$ . Additionally, the following relations arise:

$$\begin{aligned} q_o &= \frac{1}{2} Q_1(R - 2M_2) - \frac{1}{2} Q_2(R - 2M_1), \\ \epsilon_2 &= \sigma_{1E}^2 - \sigma_{2E}^2, \end{aligned} \quad (18)$$

where  $\sigma_{1E}$  and  $\sigma_{2E}$  are the expressions of Eq. (1) for the case of two electrostatic black hole horizons corresponding to the DRN problem, given in Ref. [6]. Because of the fact that there are no individual magnetic charges, the last term which defines the angular momentum  $J_k$  does not make any contribution to the individual mass  $M_k$ ; therefore, the Smarr formula for the mass [29] is fulfilled:

$$\begin{aligned} M_k &= \frac{\kappa_k S_k}{4\pi} + 2\Omega_k J_k + \Phi_k^H Q_k \\ &= \sigma_k + 2\Omega_k J_k + \Phi_k^H Q_k, \quad k = 1, 2, \end{aligned} \quad (19)$$

where  $\Phi_k^H = -A_{4k}^H - \Omega_k A_{3k}^H$  is the electric potential measured over the black hole horizon  $H_k$ ,  $\Omega_k := 1/\omega^{H_k}$  is the angular velocity, and  $\omega^{H_k}$  is the metric function  $\omega$  evaluated at the horizon. Straightforward calculations lead us to the following expressions for  $\Omega_k$  and  $\Phi_k^H$ :

$$\sigma_1 = \sqrt{M_1^2 - Q_1^2 + 2\mu Q_1 - \frac{J_1^2(R^2 - \Delta_o)[(R + M_1)^2 - M_2^2][(R + M_2)^2 - M_1^2] + [(Q_1 - Q_2)R + (M_1 - M_2)Q]^2}{(R + M)^2(M_1[(R + M_2)^2 - M_1^2] - Q_1[Q_2R - (M_1 - M_2)Q])^2}},$$

$$\sigma_2 = \sqrt{M_2^2 - Q_2^2 - 2\mu Q_2 - \frac{J_2^2(R^2 - \Delta_o)[(R + M_1)^2 - M_2^2][(R + M_2)^2 - M_1^2] + [(Q_1 - Q_2)R + (M_1 - M_2)Q]^2}{(R + M)^2(M_2[(R + M_1)^2 - M_2^2] - Q_2[Q_1R + (M_1 - M_2)Q])^2}}, \quad (21)$$

whereas the second equation of (18) implies the following relation between the seven physical parameters:

$$M_1M_2(R + M_1 + M_2) \left[ J_1 + J_2 + R \left( \frac{J_1}{M_1} + \frac{J_2}{M_2} \right) - M_1M_2 \left( \frac{J_1}{M_1^2} + \frac{J_2}{M_2^2} \right) \right] \\ + (M_1 - M_2)(Q_1 + Q_2)(Q_1J_2 - Q_2J_1) - Q_1Q_2(J_1 + J_2)R = 0. \quad (22)$$

For completeness, it can be shown that the total angular momentum of the system defined by Eq. (13) accounts exactly for the sum of the individual angular momenta, i.e.,

$$J = J_1 + J_2. \quad (23)$$

An interesting physical property of this kind of configuration is the interaction force associated with the strut in between the black holes [10], which can be calculated via the formula [10,45]

$$\mathcal{F} = \frac{1}{4}(e^{-\gamma_0} - 1) = \frac{M_1M_2 - (Q_1 - \mu)(Q_2 + \mu)}{R^2 - (M_1 + M_2)^2 + (Q_1 + Q_2)^2}, \quad (24)$$

where  $\gamma_0$  is the value of the metric function  $\gamma$  on the region of the strut. One should observe that Eq. (24) shows no explicit dependence of the angular momentum into this force. Indeed, this formula was derived in Ref. [8], and it

corresponds to the DRN problem. Nonetheless, the contribution of the angular momentum arises from Eq. (22), which relates the seven physical parameters.

## B. Physical and geometrical properties of the solution

The thermodynamical characteristics of the solution are contained in the Smarr mass formula Eq. (19), in which the surface gravity  $\kappa_k$  and area of the horizon  $S_k$  of the  $k$ th black hole are related to each other by means of  $\sigma_k$ . The area of the horizon  $S_k$  is calculated via the formulas [30,46]

$$S_k = \frac{4\pi\sigma_k}{\kappa_k}, \quad \kappa_k = \sqrt{-\Omega_k^2 e^{-2\gamma^{H_k}}}, \quad (25)$$

where  $\gamma^{H_k}$  is the metric function  $\gamma$  evaluated at the corresponding horizon  $H_k$ . The area of the horizons, surface gravities, and angular velocities acquire the final form

$$S_1 = \frac{4\pi(R + M)^2[2P_{11}(M_1 + \sigma_1) - Q_1P_{21}]}{P_0} = 4\pi \frac{P_{11}^2[(R + M)(M_1 + \sigma_1) - Q_1Q]^2 + [J_1(R - M_1 + M_2)(R^2 - \Delta_o)]^2}{P_{11}^2[(R + \sigma_1)^2 - \sigma_2^2]},$$

$$\kappa_1 = \frac{\sigma_1 P_0}{(R + M)^2[2P_{11}(M_1 + \sigma_1) - Q_1P_{21}]} = \frac{\sigma_1 P_{11}^2[(R + \sigma_1)^2 - \sigma_2^2]}{P_{11}^2[(R + M)(M_1 + \sigma_1) - Q_1Q]^2 + [J_1(R - M_1 + M_2)(R^2 - \Delta_o)]^2},$$

$$\Omega_1 = \frac{J_1(R^2 - \Delta_o)P_0}{(R + M)^2P_{11}[2P_{11}(M_1 + \sigma_1) - Q_1P_{21}]} = \frac{P_{11}[(R + \sigma_1)^2 - \sigma_2^2]J_1(R^2 - \Delta_o)}{P_{11}^2[(R + M)(M_1 + \sigma_1) - Q_1Q]^2 + [J_1(R - M_1 + M_2)(R^2 - \Delta_o)]^2},$$

$$S_2 = S_{1(1 \leftrightarrow 2)}, \quad \kappa_2 = \kappa_{1(1 \leftrightarrow 2)}, \quad \Omega_2 = \Omega_{1(1 \leftrightarrow 2)}. \quad (26)$$

The above formulas (26) are expressed in two equivalent forms, where the left-hand side of  $S_k$  and  $\kappa_k$  can be obtained directly after combining Eq. (10) with Eq. (17), and without any previous knowledge of the explicit form of  $\sigma_k$ . The electric and magnetic dipole moments expressed in physical parameters are reduced to

$$\begin{aligned}
 q_o &= \frac{1}{2}Q_1(R - 2M_2) - \frac{1}{2}Q_2(R - 2M_1), \\
 b_o &= \left[ \frac{Q_1 J_1 (R + M_1 - M_2)}{P_{11}} + \frac{Q_2 J_2 (R - M_1 + M_2)}{P_{12}} \right] \\
 &\quad \times \left( \frac{R^2 - \Delta_o}{R + M_1 + M_2} \right), \quad (27)
 \end{aligned}$$

and it follows that our solution derives the identical cases already discussed in Refs. [21,25]. For instance, the Bretón-Manko solution [21] is obtained after setting  $M_1 = M_2 = m$ ,  $Q_1 = Q_2 = q$ , and  $J_1 = -J_2 = j$ , where the unique  $\sigma$  reads

$$\sigma = \sqrt{m^2 - q^2 - \frac{j^2(R^2 - 4m^2 - 4q^2)}{[m(R + 2m) - q^2]^2}}. \quad (28)$$

Moreover, if  $M_1 = M_2 = m$ ,  $Q_1 = -Q_2 = q$ , and  $J_1 = -J_2 = j$ , the corresponding solution for stationary black diholes [25] is given by

$$\sigma = \sqrt{m^2 - \left[ q^2 + \frac{j^2[(R + 2m)^2 + 4q^2]}{[m(R + 2m) + q^2]^2} \right] \frac{R - 2m}{R + 2m}}. \quad (29)$$

In both cases, Eq. (22) is satisfied automatically. It should be noticed that in the limit  $R \rightarrow \infty$  the strut in between vanishes and formulas (21) reduce to  $\sigma_k = \sqrt{M_k^2 - Q_k^2 - J_k^2/M_k^2}$ ; therefore, the uniqueness theorem holds for the isolated KN black hole [46]. The

individual magnetic charges  $B_k$  could be included after applying a *duality rotation*, as was performed in Ref. [25] for stationary black diholes.

On the other hand, magnetic charges of equal magnitude but opposite sign could be included from the beginning of the construction of the solution, in order to eliminate the global monopolar magnetic charge of the system ( $\mathcal{B} = 0$ ). As Tomimatsu proposed [30], this mechanism will include contributions from the Dirac string to the electromagnetic part of the individual angular momentum; thus, the Smarr formula for the mass does not hold [26]. Naturally, this violates the uniqueness theorem when sources are far away from each other, due to the fact that in the limit  $R \rightarrow \infty$  one should recover the horizon for one isolated KN black hole joined to a monopolar magnetic charge [26]; i.e., the isolated KN black hole has a monopole hair.

Turning now to our binary system, if  $R \rightarrow \sqrt{M^2 - Q^2}$ , the angular velocities stop, since both horizons are touching each other. For such a situation, the interaction force given by Eq. (24) tends its value to infinity, and the system collapses and evolves as one single RN black hole [1,2] of mass  $M$  and charge  $Q$ .

### III. THE EXTREME LIMIT CASE AND SOME GEOMETRICAL PROPERTIES

The extreme limit case arises as a four-parametric exact solution after setting  $\sigma_k = 0$  in Eq. (10). In that case, a simple representation for the metric functions  $f$ ,  $\omega$ , and  $\gamma$  is written down in terms of four basic polynomials  $\mu_o$ ,  $\sigma_o$ ,  $\pi_o$ , and  $\tau_o$ :

$$\begin{aligned}
 \mathcal{E} &= \frac{A - B}{A + B}, & \Phi &= \frac{F}{A + B}, & \Phi_2 &= \frac{-iI}{A + B}, & f &= \frac{D}{N}, & \omega &= \frac{R(y^2 - 1)W}{2D}, & e^{2\gamma} &= \frac{D}{M^4 R^8 (x^2 - y^2)^4}, \\
 A &= M^2 R^2 [\beta_o (x^2 - y^2)^2 + \Delta_o (x^4 - 1)] + 4\Delta_o (q_o^2 + b_o^2) (1 - y^4) + 2i\delta_o M^2 R^2 (x^2 + y^2 - 2x^2 y^2), \\
 B &= 2M [M^2 R x \Gamma_+ + 2Q(q_o + ib_o) y \Gamma_-], & F &= 2M^2 [Q R x \Gamma_+ + 2(q_o + ib_o) y \Gamma_-], \\
 I &= M R^2 Q \{ M y [\Gamma_+ + 2(M R x + \Delta_o - i\delta_o)] + 4Q(q_o y^2 + ib_o) \} (1 - x^2) + 2(q_o + ib_o) (M R x + \Delta_o) \\
 &\quad \times \{ M [\Gamma_- + 2(M R x + \Delta_o + i\delta_o)] + 4Q(q_o - ib_o) y \} (1 - y^2), \\
 \Gamma_{\pm} &= (\sqrt{\Delta_o - b} \mp i\sqrt{\beta_o}) [\sqrt{\Delta_o - b} (x^2 - 1) \pm i\sqrt{\beta_o} (x^2 - y^2)] + b(x^2 - 1), \\
 D &= \mu_o^2 + (x^2 - 1)(y^2 - 1)\sigma_o^2, & N &= D + \mu_o \pi_o + (1 - y^2)\sigma_o \tau_o, & W &= (x^2 - 1)\sigma_o \pi_o - \mu_o \tau_o, \\
 \mu_o &= M^2 R^2 [\beta_o (x^2 - y^2)^2 + \Delta_o (x^2 - 1)^2] - \Delta_o (q_o^2 + b_o^2) (y^2 - 1)^2, & \sigma_o &= 4M^2 R^2 \delta_o x y, \\
 \pi_o &= (4/M) \{ M^2 R (M x [(R x + M)(M R x + \Delta_o) - M \beta_o y^2] + 2Q q_o y [R(x^2 - y^2) + 2M x]) \\
 &\quad + 4(q_o^2 + b_o^2) y [M(M^2 - 2\Delta_o) y + 2Q \Delta_o (q_o / \beta_o) (1 + y^2)] \}, & \tau_o &= 4M R \delta_o (x^2 - 1) [M^2 y - 2Q(q_o / \beta_o) (R x + M)], \\
 \delta_o &:= \frac{\sqrt{\Delta_o [M^2 \beta_o - 4(q_o^2 + b_o^2)]}}{M}, & b &:= \frac{4\Delta_o (q_o^2 + b_o^2)}{M^2 \beta_o}, & b_o &:= q_o \sqrt{\frac{\Delta_o (M^2 \beta_o - 4q_o^2)}{M^2 \beta_o^2 + 4\Delta_o q_o^2}}, \quad (30)
 \end{aligned}$$

where  $(x, y)$  are prolate spheroidal coordinates defined as

$$\begin{aligned} x &= \frac{r_+ + r_-}{2\alpha}, & y &= \frac{r_+ - r_-}{2\alpha}, \\ r_{\pm} &= \sqrt{\rho^2 + (z \pm \alpha)^2}, & \alpha &:= \frac{R}{2}, \end{aligned} \quad (31)$$

and thereby one gets a simple representation for the metric functions which is analogous to the one proposed by Perjés [32] in relation to the well-known Tomimatsu-Sato spacetimes [47]. The above Eq. (30) shows also the Kinnersley potential  $\Phi_2$  [44] with the purpose to get straightforwardly the magnetic potential  $A_3$ . The individual angular momentum  $J_k$  is obtained whether the extremality condition is achieved, whose expression yields from Eq. (21)

$$J_k = \varepsilon_k \frac{\sigma_{kE} P_{1k} (R + M)}{\sqrt{P_0 (R^2 - \Delta_o)}}, \quad k = 1, 2, \quad (32)$$

where  $\varepsilon_1 = -\varepsilon_2 = \varepsilon = \pm 1$ . The masses  $M_k$  and charges  $Q_k$  are related by means of the second formula of Eq. (18), as follows:

$$\varepsilon_2 := M_1^2 - Q_1^2 + 2\mu Q_1 - (M_2^2 - Q_2^2 - 2\mu Q_2) = 0. \quad (33)$$

On the other hand, Gabach Clement has recently discovered [33] a geometrical inequality for black holes with struts, which is given by

$$\sqrt{1 + 4\mathcal{F}} \geq \frac{\sqrt{(8\pi J_k)^2 + (4\pi Q_k^2)^2}}{S_k}, \quad k = 1, 2, \quad (34)$$

and, therefore, it can be shown that the binary black hole system saturates the aforementioned inequality; i.e., the equality sign is reached in the extreme limit case previously

defined by Eq. (30). In order to prove the equality into the geometrical inequality between extreme black holes with struts, we have that the area of the horizon is now reduced to

$$S_k = \frac{4\pi(R + M)^2(2M_k P_{1k} - Q_k P_{2k})}{P_0}, \quad k = 1, 2. \quad (35)$$

Since the interaction force  $\mathcal{F}$  does not show explicit dependence of the angular momentum, the binary system saturates the inequality, namely,

$$\sqrt{1 + 4\mathcal{F}} = \frac{\sqrt{(8\pi J_k)^2 + (4\pi Q_k^2)^2}}{S_k}, \quad k = 1, 2. \quad (36)$$

## IV. SINGULARITIES OFF THE AXIS

### A. Ring singularities

According to the well-known positive mass theorem [34], a regular solution which is free of singularities allows only positive values for the total Arnowitt-Deser-Misner (ADM) mass of the system [48]. However, the theorem is not enough proof to provide regularity in the solution, since a positive ADM mass given by the inequality  $M = M_1 + M_2 > 0$  can be ensured even if one of the masses is negative. Therefore, one should add analytic conditions that guarantee the regularity of the solution. In order to analyze whether the solution Eq. (30) is regular outside the axis, we need to look at the denominator of the Ernst potentials, where a singular solution occurs if

$$A + B = F_R + iF_I = 0, \quad (37)$$

for which

$$\begin{aligned} F_R &= M^2 R^2 [(R^2 - \Delta_o)(x^2 - y^2)^2 + \Delta_o(x^4 - 1)] + 4\Delta_o(q_o^2 + b_o^2)(1 - y^4) \\ &\quad + 2M\{(M^2 R x + 2Q q_o y)[(R^2 - \Delta_o)(x^2 - y^2) + \Delta_o(x^2 - 1)] + 2Q\delta_o b_o y(1 - y^2)\} = 0, \\ F_I &= 2\delta_o M^2 R^2 (x^2 + y^2 - 2x^2 y^2) + 4MQR^2 b_o y(x^2 - y^2) + 2M(1 - y^2)[\delta_o(M^2 R x - 2Q q_o y) - 2Q\Delta_o b_o y] = 0. \end{aligned} \quad (38)$$

Any solution of Eq. (38) represents a single point outside the symmetry axis named *ring singularity*. However, due to the high order polynomials of Eq. (38), it is not possible to concrete an analytical study on the appearance or absence of naked singularities; therefore, one must resort to numerical analysis. With regard to Eq. (31), now the cylindrical coordinates  $(\rho, z)$  are given by

$$\rho = \alpha \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \alpha xy, \quad (39)$$

and it shows that the region  $x > 1, |y| < 1$  defines the values that Eq. (30) can take in the plane  $(x, y)$  to develop RS. After assigning a wide range of numerical values for the parameters of the solution, the curves depicted by Eq. (38) present no intersections in such a region if both masses are positive. Furthermore, if at least one of the masses is negative, an intersection occurs in this region as a consequence of the presence of a negative mass. Table I shows a set of numerical values for the physical parameters of the solution, Eq. (30).

TABLE I. Numerical values for the extreme limit case of the double-Kerr-Newman problem.

$M_1$	$M_2$	$Q_1$	$Q_2$	$J_1$	$J_2$	$R$
1.5	1	1.4	0.46	-2.275	1.951	2.8
1	0.5	1.1	-0.35	0.886	-0.67	2.5
1	2	0.5	2.143	7.574	-7.789	1.5
1	-0.6	0.7	-0.1	0.306	0.855	2
-1	-1.1	0.1	-0.191	-0.148	0.179	2.4

It has been shown for identical constituents that the presence of the electromagnetic field displaces the RS to the right side of its corresponding stationary limit surface (SLS) [26,27]. Nevertheless, in the unequal case, the RS not only is displaced to the right side, it also can move downward as shown in Fig. 2; apparently, this is due to the difference between the values of both electric charges. We mention also that if one of the masses is negative, the system becomes corotating (see Table I).

### B. Singular surfaces

On the other hand, the easiest analytical proof on the regularity of the solution can be performed in the DRN sector, since the curves defined by Eq. (38) are now reduced in two subfamilies [12]. The first one of them is the Majumdar-Papapetrou (MP) solution [4,5], whose masses and electric charges are related by  $Q_k = \pm M_k$  and both charges have the same sign. The curves given by Eq. (38) are reduced to the equation of the hyperbola,

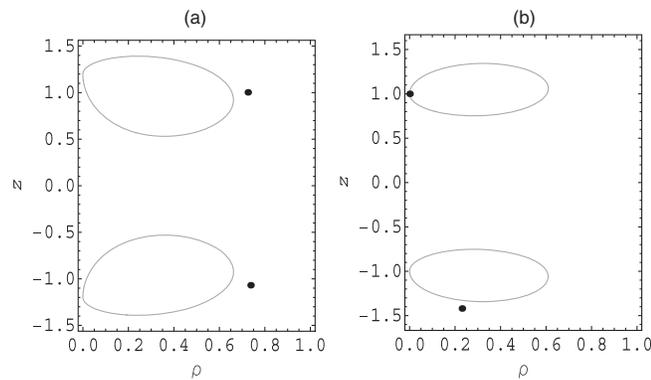


FIG. 2. (a) The SLS ( $f = 0$ ) in the extreme limit case and their corresponding RS for the values  $M_1 = -1$ ,  $M_2 = -1.1$ ,  $Q_1 = 0.1$ ,  $Q_2 = -0.191$ ,  $J_1 = -0.148$ ,  $J_2 = 0.179$ , and  $R = 2.4$ . The RS are located at  $\rho \approx 0.72$ ,  $z \approx 1.0$  and  $\rho \approx 0.74$ ,  $z \approx -1.07$ . (b) If one of the masses is negative, the system becomes corotating, for  $M_1 = 1$ ,  $M_2 = -0.6$ ,  $Q_1 = 0.7$ ,  $Q_2 = -0.1$ ,  $J_1 = 0.306$ ,  $J_2 = 0.855$ , and  $R = 2$ . The RS is located at  $\rho \approx 0.23$ ,  $z \approx -1.42$ .

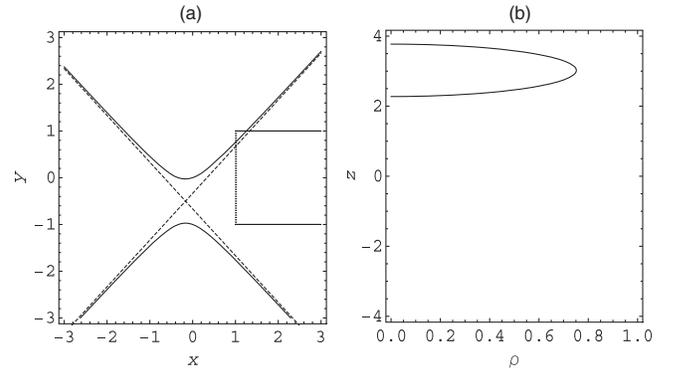


FIG. 3. (a) Crossing inside the region  $x > 1$ ,  $|y| < 1$ , if one of the masses is negative (in this case  $M_1 < 0$ ), for the values  $\alpha = 3$ ,  $M_1 = -1$ , and  $M_2 = 2$ . (b) Appearance of the corresponding SS if  $M_1 < 0$  in the MP sector.

$$F_R \equiv F_{MP} = \left(x + \frac{M_1 + M_2}{2\alpha}\right)^2 - \left(y - \frac{M_1 - M_2}{2\alpha}\right)^2 - \frac{M_1 M_2}{\alpha^2} = 0, \quad (40)$$

which contains two asymptotes described by

$$y = \pm \left(x + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha}. \quad (41)$$

In this respect, the conditions  $x = 1$  and  $|y| < 1$  are enough to prove that at least one asymptote is crossing inside the region  $x > 1$ ,  $|y| < 1$ ; therefore, naked singularities arise as SS. Let us suppose that the straight line with positive slope crosses this region, but the other one with negative slope does not (see Fig. 3), and then we get

$$\begin{aligned} + \left(1 + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha} < 1 &\Rightarrow M_1 < 0, \\ - \left(1 + \frac{M_1 + M_2}{2\alpha}\right) + \frac{M_1 - M_2}{2\alpha} < -1 &\Rightarrow M_2 > 0. \end{aligned} \quad (42)$$

Besides, if the electric charges have opposite sign, a second electrostatic subfamily can be also obtained from Eq. (30) where the charges and masses are related by means of

$$\begin{aligned} Q_1 &= \epsilon M_1 \sqrt{\frac{(2\alpha + M_2)^2 - M_1^2}{(2\alpha - M_2)^2 - M_1^2}}, \\ Q_2 &= -\epsilon M_2 \sqrt{\frac{(2\alpha + M_1)^2 - M_2^2}{(2\alpha - M_1)^2 - M_2^2}}, \quad \epsilon = \pm 1, \end{aligned} \quad (43)$$

and it follows that electric charges are greater than their corresponding positive masses, according to  $|Q_k| > M_k$

[12]. As was already proved in Ref. [12], it represents the electrostatic analogue of the well-known Kerr-NUT space-time [49] obtainable via Bonnor's procedure of a complex continuation of the parameters [50], whereby the subfamily is called Bonnor's solution (BS) [51]. In BS, Eq. (38) reduces to

$$F_R \equiv F_B = D^2 \left( x + \frac{M_1 + M_2}{2\alpha} \right)^2 - \left( y - \frac{M_1 - M_2}{2\alpha} \right)^2 = 0, \quad (44)$$

$$D := \sqrt{\frac{4\alpha^2 - (M_1 - M_2)^2}{4\alpha^2 - (M_1 + M_2)^2}}, \quad (44)$$

which represents the geometric locus of two straight lines intersecting at an angle of  $\theta = 2 \arctan D$ , where the straight lines are given by

$$y = \pm D \left( x + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha}. \quad (45)$$

Let us suppose now that the straight line with positive slope is not crossing inside the region (the one associated with  $M_1$ ) while the other one does (see Fig. 4). Similarly to the MP case, we have

$$+D \left( 1 + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha} > 1 \Rightarrow \sqrt{\frac{(2\alpha + M_1)^2 - M_2^2}{(2\alpha - M_1)^2 - M_2^2}} > 1 \Rightarrow M_1 > 0, \quad (46)$$

$$-D \left( 1 + \frac{M_1 + M_2}{2\alpha} \right) + \frac{M_1 - M_2}{2\alpha} > -1 \Rightarrow \sqrt{\frac{(2\alpha + M_2)^2 - M_1^2}{(2\alpha - M_2)^2 - M_1^2}} < 1 \Rightarrow M_2 < 0.$$

Therefore, one concludes that SS appear as a consequence of the presence of negative masses in the solution even if the total ADM mass is positive. Finally, it should be pointed out that the values for the individual masses should fulfill the Smarr mass formula  $M_k \geq 2\Omega_k J_k + \Phi_k^H Q_k > 0$ , in order to avoid the appearance of naked singularities off the axis in the solutions given by Eqs. (10) and (30), in agreement with the positive mass theorem [34,35].

## V. CONCLUDING REMARKS

In this paper, we developed a six-parametric exact solution which is used as a basis to describe a binary

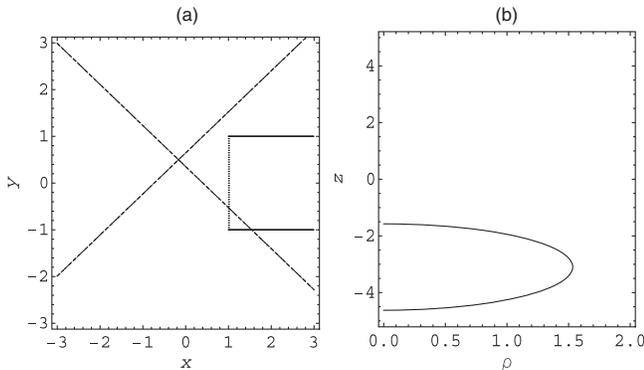


FIG. 4. (a) Crossing inside the region  $x > 1$ ,  $|y| < 1$ , due to  $M_2 < 0$ , for the values  $\alpha = 3$ ,  $M_1 = 2$ , and  $M_2 = -1$ . (b) Emergence of the corresponding SS if  $M_2 < 0$  in Bonnor's sector.

system of KN black hole sources separated by a massless strut, in which the seven physical Komar parameters are related by means of a simple algebraic equation. This model generalizes the known DRN problem [6–8] and the double-Kerr problem [43]. It is worthwhile to stress the fact that the paper uses the idea provided in Ref. [25] to remove first the individual magnetic charges with the purpose of solving analytically the axis conditions given in Refs. [26,27]. Since our model contains unequal constituents, we provide also a physical parametrization for the two half-length parameters  $\sigma_k$  associated with the black hole horizons which characterize the solution in a more transparent way. Using the advantages that provide the physical parametrization, we show that our model saturates the Gabach Clement inequality for interacting black holes with struts [33]. The extreme limit case is also derived by using Perjés' representation, and, later on, we give a numerical analysis on the presence of ring singularities off the axis and their location depending on the values of the electric charges. The numerical analysis reveals that the ring singularity located outside the ergosurface moves downward depending on the difference between values of the electric charges. Furthermore, we give a simple analytical proof in the DRN problem on the conditions to avoid the appearance of singular surfaces, which in our opinion can be used as a complement of the well-known positive mass theorem [34,35].

It should be pointed out that the binary model could include the magnetic charges (after their elimination), but it is not trivial to derive in a simple manner the procedure (duality

rotation) which allows us to include individual magnetic charges in the unequal charged case. For instance, as a particular case, if the electric charges are equal in magnitude but opposite in sign according to  $Q_1 = -Q_2 = Q_E$ , the

electric and magnetic dipole moments can be combined in such a way that the electric charge  $Q_E$  enters in the solution by means of the transformations  $Q_E \rightarrow Q_E + iQ_B$  and  $Q_E^2 \rightarrow |Q_E^2 + Q_B^2|$ . Then, the corresponding formulas for  $\sigma_k$  are

$$\begin{aligned}\sigma_1 &= \sqrt{M_1^2 - \left[ |Q_E^2 + Q_B^2| + \frac{J_1^2 [((R + M_1)^2 - M_2^2)((R + M_2)^2 - M_1^2) + 4|Q_E^2 + Q_B^2|R^2]}{[M_1((R + M_2)^2 - M_1^2) + |Q_E^2 + Q_B^2|R^2]} \right]} \frac{R - M_1 - M_2}{R + M_1 + M_2}, \\ \sigma_2 &= \sqrt{M_2^2 - \left[ |Q_E^2 + Q_B^2| + \frac{J_2^2 [((R + M_1)^2 - M_2^2)((R + M_2)^2 - M_1^2) + 4|Q_E^2 + Q_B^2|R^2]}{[M_2((R + M_1)^2 - M_2^2) + |Q_E^2 + Q_B^2|R^2]} \right]} \frac{R - M_1 - M_2}{R + M_1 + M_2},\end{aligned}\quad (47)$$

where now Eq. (22) leads us to the following relation between the seven physical parameters:

$$M_1 M_2 (R + M_1 + M_2) \left[ J_1 + J_2 + R \left( \frac{J_1}{M_1} + \frac{J_2}{M_2} \right) - M_1 M_2 \left( \frac{J_1}{M_1^2} + \frac{J_2}{M_2^2} \right) \right] + |Q_E^2 + Q_B^2| (J_1 + J_2) R = 0. \quad (48)$$

Finally, we must emphasize that the magnetic charges  $Q_B$  introduced in this particular model contain opposite signs and are equal in magnitude, where now the *new* electric and magnetic dipole moments are given by

$$\begin{aligned}\mathbf{q}_o &= \left[ Q_E - Q_B \left( \frac{J_1 (R + M_1 - M_2)}{M_1 [(R + M_2)^2 - M_1^2] + |Q_E^2 + Q_B^2|R} - \frac{J_2 (R - M_1 + M_2)}{M_2 [(R + M_1)^2 - M_2^2] + |Q_E^2 + Q_B^2|R} \right) \right] (R - M_1 - M_2), \\ \mathbf{b}_o &= \left[ Q_B + Q_E \left( \frac{J_1 (R + M_1 - M_2)}{M_1 [(R + M_2)^2 - M_1^2] + |Q_E^2 + Q_B^2|R} - \frac{J_2 (R - M_1 + M_2)}{M_2 [(R + M_1)^2 - M_2^2] + |Q_E^2 + Q_B^2|R} \right) \right] (R - M_1 - M_2),\end{aligned}\quad (49)$$

where  $Q_B = 0$  recovers the original electric and magnetic dipole moments obtainable from Eq. (27) after setting  $Q_1 = -Q_2 = Q_E$ . We hope to accomplish some extensions that include magnetic charges in the model under consideration.

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- [1] H. Reissner, *Ann. Phys. (Berlin)* **355**, 106 (1916).
  - [2] G. Nordström, *Proc. K. Ned. Akad. Wet.* **20**, 1238 (1918).
  - [3] H. Weyl, *Ann. Phys. (Berlin)* **359**, 117 (1917).
  - [4] S. D. Majumdar, *Phys. Rev.* **72**, 390 (1947).
  - [5] A. Papapetrou, *Proc. R. Irish Acad., Sect. A* **51**, 191 (1947).
  - [6] G. G. Varzugin and A. S. Chystiakov, *Classical Quantum Gravity* **19**, 4553 (2002).
  - [7] G. A. Alekseev and V. A. Belinski, *Phys. Rev. D* **76**, 021501 (R) (2007).
  - [8] V. S. Manko, *Phys. Rev. D* **76**, 124032 (2007).
  - [9] R. Bach and H. Weyl, *Math. Z.* **13**, 134 (1922).
  - [10] W. Israel, *Phys. Rev. D* **15**, 935 (1977).
  - [11] D. Kramer and G. Neugebauer, *Phys. Lett. A* **75**, 259 (1980).
  - [12] I. Cabrera-Munguia, V. S. Manko, and E. Ruiz, *Gen. Relativ. Gravit.* **43**, 1593 (2011).
  - [13] I. Cabrera-Munguia and A. Macías, *AIP Conf. Proc.* **1577**, 213 (2014).
  - [14] W. Dietz and C. Hoenselaers, *Ann. Phys. (N.Y.)* **165**, 319 (1985).
  - [15] V. S. Manko, E. Ruiz, and J. D. Sanabria-Gómez, *Classical Quantum Gravity* **17**, 3881 (2000).
  - [16] G. Neugebauer and J. Hennig, *Gen. Relativ. Gravit.* **41**, 2113 (2009).
  - [17] E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, *J. Math. Phys. (N.Y.)* **6**, 918 (1965).
  - [18] L. Parker, R. Rufinni, and D. Wilkins, *Phys. Rev. D* **7**, 2874 (1973).
  - [19] Z. Perjés, *Phys. Rev. Lett.* **27**, 1668 (1971).
  - [20] W. Israel and G. A. Wilson, *J. Math. Phys. (N.Y.)* **13**, 865 (1972).
  - [21] V. S. Manko, R. I. Rabadán, and E. Ruiz, *Classical Quantum Gravity* **30**, 145005 (2013).
  - [22] A. Komar, *Phys. Rev.* **113**, 934 (1959).

- [23] R. Emparan, *Phys. Rev. D* **61**, 104009 (2000).
- [24] R. Emparan and E. Teo, *Nucl. Phys.* **B610**, 190 (2001).
- [25] V. S. Manko, R. I. Rabadán, and J. D. Sanabria-Gomez, *Phys. Rev. D* **89**, 064049 (2014).
- [26] I. Cabrera-Munguia, C. Lämmerzahl, L. A. López, and A. Macías, *Phys. Rev. D* **90**, 024013 (2014).
- [27] I. Cabrera-Munguia, C. Lämmerzahl, L. A. López, and A. Macías, *Phys. Rev. D* **88**, 084062 (2013).
- [28] J. S. Schwinger, *Science* **165**, 757 (1969).
- [29] L. Smarr, *Phys. Rev. Lett.* **30**, 71 (1973).
- [30] A. Tomimatsu, *Prog. Theor. Phys.* **72**, 73 (1984).
- [31] S. Dain and O. E. Ortiz, *Phys. Rev. D* **80**, 024045 (2009).
- [32] Z. Perjés, *J. Math. Phys. (N.Y.)* **30**, 2197 (1989).
- [33] M. E. Gabach Clement, *Classical Quantum Gravity* **29**, 165008 (2012).
- [34] R. Schoen and S.-T. Yau, *Commun. Math. Phys.* **65**, 45 (1979).
- [35] R. Schoen and S.-T. Yau, *Commun. Math. Phys.* **79**, 231 (1981).
- [36] F. J. Ernst, *Phys. Rev.* **168**, 1415 (1968).
- [37] N. R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields* (Springer-Verlag, Berlin, 1991).
- [38] E. Ruiz, V. S. Manko, and J. Martín, *Phys. Rev. D* **51**, 4192 (1995).
- [39] A. Papapetrou, *Ann. Phys. (Berlin)* **447**, 309 (1953).
- [40] W. Simon, *J. Math. Phys. (N.Y.)* **25**, 1035 (1984).
- [41] E. Newman, L. Tamburino, and T. Unti, *J. Math. Phys. (N.Y.)* **4**, 915 (1963).
- [42] There is a misprint in formula (7) of Ref. [27]; a bar over the term  $\mathbf{a}$  is missed.
- [43] I. Cabrera-Munguia, C. Lämmerzahl, and A. Macías, *Classical Quantum Gravity* **30**, 175020 (2013).
- [44] W. Kinnersley, *J. Math. Phys. (N.Y.)* **18**, 1529 (1977).
- [45] G. Weinstein, *Commun. Pure Appl. Math.* **43**, 903 (1990).
- [46] B. Carter, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), p. 57.
- [47] A. Tomimatsu and H. Sato, *Phys. Rev. Lett.* **29**, 1344 (1972); *Prog. Theor. Phys.* **50**, 95 (1973).
- [48] R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* **122**, 997 (1961).
- [49] M. Demiański and E. T. Newman, *Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys.* **14**, 653 (1966).
- [50] W. B. Bonnor, *Z. Phys.* **161**, 439 (1961).
- [51] W. B. Bonnor, *J. Phys. A* **12**, 853 (1979).