Tidal deformation of a slowly rotating black hole

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(Received 17 November 2014; published 3 February 2015)

In the first part of this article I determine the geometry of a slowly rotating black hole deformed by generic tidal forces created by a remote distribution of matter. The metric of the deformed black hole is obtained by integrating the Einstein field equations in a vacuum region of spacetime bounded by $r < r_{\max}$, with $r_{\max}$ a maximum radius taken to be much smaller than the distance to the remote matter. The tidal forces are assumed to be weak and to vary slowly in time, and the metric is expressed in terms of generic tidal quadrupole moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ that characterize the tidal environment. The metric incorporates couplings between the black hole's spin vector and the tidal moments, and captures all effects associated with the dragging of inertial frames. In the second part of the article I determine the tidal moments by immersing the black hole in a larger post-Newtonian system that includes an external distribution of matter; while the black hole's internal gravity is allowed to be strong, the mutual gravity between the black hole and the external matter is assumed to be weak. The post-Newtonian metric that describes the entire system is valid when $r > r_{\min}$, where $r_{\min}$ is a minimum distance that must be much larger than the black hole's radius. The black-hole and post-Newtonian metrics provide alternative descriptions of the same gravitational field in an overlap $r_{\min} < r < r_{\max}$, and matching the metrics determine the tidal moments, which are expressed as post-Newtonian expansions carried out through one-and-a-half post-Newtonian order. Explicit expressions are obtained in the specific case in which the black hole is a member of a post-Newtonian two-body system.

DOI: 10.1103/PhysRevD.91.044004 PACS numbers: 04.20.-q, 04.25.-g, 04.25.Nx, 04.70.-s

I. INTRODUCTION AND SUMMARY

A. This work and its context

The theory of tidal deformation and dynamics of compact bodies in general relativity has recently been the subject of vigorous development. While work on this topic goes back several decades, the origin of the recent burst of activity can be traced to Flanagan and Hinderer [1,2], who pointed out that tidal effects can have measurable consequences on the gravitational waves emitted by a binary neutron star in the late stages of its orbital evolution. The gravitational waves therefore carry information regarding the internal structure of each body, from which one can extract useful constraints on the equation of state of dense nuclear matter. Their study was followed up with more detailed analyses [3–14], with the conclusion that tidal effects might indeed be accessible to measurement by the current generation of gravitational-wave detectors.

Another astrophysical context for relativistic tidal dynamics comes from extreme-mass-ratio inspirals, which implicate a solar-mass compact body in a tight orbit around a supermassive black hole. Such systems have been identified as promising sources of gravitational waves for an eventual space-borne interferometric detector, and it was demonstrated [15–20] that the tide raised on the large black hole by the small body can lead to a significant transfer of angular momentum from the black hole to the orbital motion.

These observations have motivated the formulation of a relativistic theory of tidal deformation and dynamics that can be applied to neutron stars and black holes. The description of tidal deformations, featuring the relativistic generalization of the Newtonian Love numbers [21–23], is now mature, and the relativistic Love numbers have been computed for realistic neutron-star models constructed from tabulated equations of state [3,7,9,24]; the gravitational Love numbers of a black hole were shown to be precisely zero [22]. The Love numbers of neutron stars have been implicated in a remarkable set of nearly universal relations—the I-Love-Q relations [25–31]—involving the moment of inertia I, the Love number k_2, and the rotational quadrupole moment Q of a neutron star.

In another development, the tidal deformation of compact bodies has been incorporated in an effective description in terms of point particles; the description involves a point-particle action that includes invariants constructed from the tidal multipole moments to be introduced below. In Ref. [32], Bini, Damour, and Faye expressed the tidal invariants as a post-Newtonian expansion carried out through second post-Newtonian order. In a recent update [33], Bini and Damour pushed the expansion to seven-and-a-half post-Newtonian order (restricted to small mass ratios), and showed that their results compare well with numerical calculations of tidal invariants obtained from the gravitational self-force [34]; the work by Bini and Damour extends a previous study by Chakrabarti, Delsate, and Steinhoff [35] (see also Ref. [36]).

This work is an additional contribution to the development of a relativistic theory of tidal deformations. I specifically consider the tidal deformation of a black hole, and in the first part of the paper I determine the geometry of the deformed black hole in terms of generic tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$; in the second part I determine the tidal moments by immersing the black hole in a post-Newtonian environment. The novelty of this work lies in the fact that I allow the black hole to be slowly rotating, so as to capture all effects associated with the dragging of inertial frames, which are included in couplings between the black-hole spin vector and the tidal moments. The assumption of slow rotation allows me to neglect the rotational deformation of the black hole.

B. Geometry of a slowly rotating, tidally deformed black hole

My goal in the first part of this article is to determine the geometry of a slowly rotating black hole deformed by generic tidal forces exerted by remote matter. I assume that the tidal forces are weak, that they vary slowly in time, and use perturbative methods to integrate the Einstein field equations for this situation. I focus my attention on a domain $\mathcal{N}$ bounded by $r < r_{\max}$ (see Fig. 1), with r denoting the distance to the black hole, and $r_{\max} \ll b$ a maximum distance that is taken to be much smaller than b, the distance to the external matter; the domain is assumed to be empty of matter. I construct the metric of the deformed black hole in $\mathcal{N}$, and express it as an expansion in powers of r/b. Because $\mathcal{N}$ does not include the external matter, the metric is expressed in terms of tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ that cannot be determined by the vacuum field equations restricted to $\mathcal{N}$.

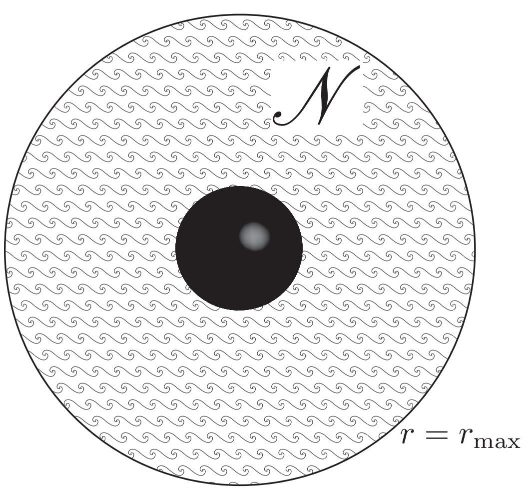

FIG. 1. The domain $\mathcal{N}$ around the black hole, bounded by $r < r_{\max}$, is shown in a wavy pattern. The external matter, situated at a distance $b \gg r_{\max}$ far outside the domain, is not shown.

Such constructions originate in the seminal work of Manasse [37], which was revived by Alvi [38,39], Detweiler [40,41], and this author [42,43]. The geometry of a tidally deformed, rapidly rotating black hole was described by Yunes and Gonzalez [44], and more recently by O'Sullivan and Hughes [45]. While Yunes and Gonzalez obtain an analytical expression for the metric in $\mathcal{N}$ by integrating the Teukolsky equation (for weak and slow tides) and exploiting metric-reconstruction techniques, O'Sullivan and Hughes focus on the intrinsic geometry of the event horizon, which they determine numerically for the rapidly changing tidal forces produced by an orbiting body. Because I take the black hole to be rotating slowly, the calculations presented here are a simplification of the work carried out by Yunes and Gonzalez. What is gained from this simplification is an explicit expression for the metric that can be displayed in a few lines, a metric cast in a coordinate system that possesses a clear geometrical meaning, and much more insight into the coupling between rotational and tidal effects.

It is helpful to introduce the various scales that enter the description of the black hole and its tidal environment; I work in geometrized units, in which $G = c = 1$. The black-hole mass is m_1, and its spin vector S_1^a is related to a dimensionless quantity χ_1^a by $S_1^a = \chi_1^a m_1^2$. The Kerr solution describes a black hole when $\chi_1 \coloneqq |\chi_1^a| \le 1$, but here I demand that $\chi_1 \ll 1$, so that the black hole is rotating slowly. The external matter is characterized by a mass scale m_2 and its distance to the black hole is comparable to the length scale b. The time scale associated with changes in the tidal environment is $\tau \sim \sqrt{b^3/m}$, in which $m \coloneqq m_1 + m_2$ is a scale for the total mass of the system. A velocity scale is then $u \sim b/\tau = \sqrt{m/b}$. The tidal environment is characterized by an electric-type tidal quadrupole moment $\mathcal{E}_{ab}$ that scales as m_2/b^3, and a magnetic-type tidal quadrupole moment $\mathcal{B}_{ab}$ that scales as $m_2 u/b^3$. The leading tidal terms in the metric are proportional to $r^2\mathcal{E}_{ab} \sim (m_2/b)(r/b)^2$ and $r^2\mathcal{B}_{ab} \sim (m_2/b)u(r/b)^2$; these are small in $\mathcal{N}$ by virtue of the fact that $r \ll b$. The next-to-leading tidal terms involve the time derivative of the tidal quadrupole moments as well as tidal octupole moments, and those are suppressed by additional factors of $u(r/b)$ and (r/b), respectively. Here I assume that the coupling terms between χ_1^a and the tidal quadrupole moments, which are proportional to $\chi_1^a\mathcal{E}_{bc}$ and $\chi_1^a\mathcal{B}_{bc}$ and therefore suppressed relative to the leading terms by factors of order χ_1, nevertheless dominate over the next-to-leading terms. This requires $\chi_1 \gg (r/b)$, and taking r to be of the same order of magnitude as m in $\mathcal{N}$, we find that the spin parameter must be constrained by $u^2 \ll \chi_1 \ll 1$. Because $u^2 \ll 1$, the mutual gravity between the black hole and the external matter is required to be weak.

The tidal potentials that result from the couplings between the spin vector and the tidal moments are introduced in Sec. II. The background spacetime of a slowly rotating black hole in isolation is described in Sec. III, and its metric is cast in coordinates (v, r, θ, ϕ) that are tied to the behavior of incoming null geodesics that are tangent to converging null cones. In Sec. IV I introduce the tidal deformation, characterized by tidal moments $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$, and I integrate the Einstein field equations to find the metric of the deformed black hole in the domain $\mathcal{N}$. I adopt a gauge—the light-cone gauge [46]—that preserves the geometrical meaning of the (v, r, θ, ϕ) coordinates, so that v continues to be constant on converging null cones, θ and ϕ continue to be constant on each generator, and $-r$ continues to be an affine parameter on each generator. The light-cone gauge does not fully specify the coordinate system, and the metric depends on six arbitrary constants that parametrize the residual gauge freedom. In Secs. V and VI I introduce the corotating frame of the deformed black hole, show that the precise definition of this frame determines the six constants of the light-cone gauge, and express the metric in the corotating coordinates $(v, r, \vartheta, \varphi)$, in which $\vartheta = \theta$ and $\varphi = \phi - \omega_{\rm H} v$, with $\omega_{\rm H} := \chi_1/(4m_1)$ denoting the black hole's angular velocity. The deformed event horizon admits a particularly simple description in the corotating coordinates: its radial position stays at $r = 2m_1$, and its null generators move with constant values of ϑ and φ. This allows me, in Sec. VII, to provide a simple and transparent description of the horizon's intrinsic geometry.

As I show in Sec. VII, the intrinsic geometry of a slowly rotating, tidally deformed event horizon is captured by the following expression for the Ricci scalar $\mathcal{R}$ of a two-dimensional cross-section of the horizon,

$$(2m_1)^2\mathcal{R} = 2 - 8m_1^2\mathcal{E}_{ab}(v)\Omega_\dagger^a\Omega_\dagger^b + \frac{40}{3}m_1^2\chi_{1\langle a}\mathcal{B}_{bc\rangle}(v)\Omega^a\Omega^b\Omega^c, \quad (1.1)$$

where angular brackets indicate the symmetric-tracefree operation—completely symmetrize over indices and remove all traces—and

$$\Omega^a := [\sin\vartheta\cos(\varphi + \omega_{\rm H} v), \sin\vartheta\sin(\varphi + \omega_{\rm H} v), \cos\vartheta] \quad (1.2)$$

specifies the direction to a given point on the horizon, while

$$\Omega_\dagger^a := \left[\sin\vartheta\cos\left(\varphi + \omega_{\rm H} v + \frac{2}{3}\chi_1\right), \sin\vartheta\sin\left(\varphi + \omega_{\rm H} v + \frac{2}{3}\chi_1\right), \cos\vartheta\right] \quad (1.3)$$

is a shifted version of the position vector. Except for this azimuthal shift, the term proportional to $\mathcal{E}_{ab}$ in Eq. (1.1) is identical to the one describing the tidal deformation of a nonrotating black hole. The shift is a manifestation of the dragging of inertial frames that accompanies the black hole's rotation. This effect was first identified by Hartle in his 1974 seminal article [47], but it should be noted that Hartle's calculation contains a sign error that was previously documented by Fang and Lovelace [48]; Hartle obtains a shift of $\frac{5}{3}\chi$ instead of $\frac{2}{3}\chi$. The dragging of inertial frames also produces the octupolar deformation proportional to $\chi_{1\langle a}\mathcal{B}_{bc\rangle}$; to the best of my knowledge, this octupole component was never noticed before. My calculation, carried out to first order in χ_1, cannot reveal the azimuthal shift of the octupole terms.

A concrete description of the tidal deformation of a slowly rotating black hole requires the determination of the tidal moments $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$, and this task is undertaken in the second part of this article. An important point is that the tidal moments are determined as functions of advanced time v in the local rest frame of the black hole, and that they do not make a direct reference to the spatial positions of the external matter. To illustrate this point, and to provide a concrete example of Eq. (1.1) as a description of tidal deformation on a slowly rotating black hole, I consider a tidal environment in a state of rigid rotation of angular frequency $\bar{\omega}$ around the black hole's rotation axis. This could be produced, for example, by a companion body on a circular orbit in the black hole's equatorial plane. The scaling relations introduced previously imply that $m_1\bar{\omega} \sim m_1/\tau \sim u^3 \ll u^2 \ll \chi_1$, which means that $\bar{\omega}$ is necessarily much smaller than $\omega_{\rm H}$. For such situations, the nonvanishing components of $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ can be shown to be given by [refer to Eqs. (11.14) and (11.15) below]

$$\mathcal{E}_{11} = \mathcal{E}_0 + \mathcal{E}_2\cos 2\bar{\phi}, \qquad \mathcal{E}_{12} = \mathcal{E}_2\sin 2\bar{\phi},$$
$$\mathcal{E}_{22} = \mathcal{E}_0 - \mathcal{E}_2\cos 2\bar{\phi}, \qquad \mathcal{E}_{33} = -2\mathcal{E}_0 \quad (1.4)$$

and

$$\mathcal{B}_{13} = \mathcal{B}_1\cos\bar{\phi}, \qquad \mathcal{B}_{23} = \mathcal{B}_1\sin\bar{\phi}, \quad (1.5)$$

where $\mathcal{E}_0$, $\mathcal{E}_2$, and $\mathcal{B}_1$ are constants, and $\bar{\phi} = \bar{\omega}(v - v_0)$ is the phase of the tidal field, with v_0 an arbitrary constant that specifies the phase at $v = 0$. As we see from the displayed expressions, the tidal moments depend on v through the phase $\bar{\phi}$. In a Newtonian context $\bar{\phi}$ could be identified with the azimuthal position of the companion body, but no such interpretation is directly available in general relativity. While $\bar{\phi}$ could still be related to the position of the companion body, this would require a mapping between the body's position and a corresponding position on the event horizon. A number of mappings were examined by Fang and Lovelace [48] (see also Ref. [45]), with the general conclusion that these constructions bring arbitrariness and ambiguity in the description of the tidal deformation. The main point that I wish to make is that a relation between $\bar{\phi}$ and the position of the companion body is not required; there is no arbitrariness nor ambiguity in the description of the tidal deformation when $\bar{\phi}$ is properly viewed as a phase function instead of an azimuthal position.

Making the substitutions, we find that the deformation terms entering Eq. (1.1) are given by

$$\mathcal{E}_{ab}\Omega_{\dagger}^{a}\Omega_{\dagger}^{b} = \mathcal{E}_0(1-3\cos^2\vartheta) + \mathcal{E}_2\sin^2\vartheta\cos 2\left(\varphi + \omega_{\rm H}v + \frac{2}{3}\chi_1 - \bar{\phi}\right) \tag{1.6}$$

and

$$\chi_{1\langle a}\mathcal{B}_{bc\rangle}\Omega^a\Omega^b\Omega^c = -\frac{2}{5}\chi_1\mathcal{B}_1\sin\vartheta(1-5\cos^2\vartheta) \times\cos(\varphi+\omega_{\rm H}v-\bar{\phi}). \tag{1.7}$$

Equation (1.6) indicates that the azimuthal position of the quadrupole bulge is given by

$$\varphi_{\rm bulge} = (\bar{\phi}-\omega_{\rm H}v) - \frac{2}{3}\chi_1; \tag{1.8}$$

the combination $\bar{\phi}-\omega_{\rm H}v$ is the phase of the tidal field as measured in the black hole's corotating frame (in which the horizon's generators are stationary), and $-\frac{2}{3}\chi_1$ is a phase lag produced by the dragging of inertial frames. It is interesting to note that in the Newtonian theory of tidal interactions, a condition $\bar{\omega} \ll \omega_{\rm H}$ would imply a tidal lead instead of a lag; as Hartle first observed [47], a rotating black hole does not respect the Newtonian relation, in spite of the fact that the condition $\bar{\omega} \ll \omega_{\rm H}$ still leads to a decrease of the black hole's spin (see, for example, Sec. IV D of Ref. [49]). On the other hand, Eq. (1.7) indicates that the octupole bulge is directly in phase with the tidal field; an eventual phase shift could only be revealed in a calculation carried out to second order in χ_1.

C. Determination of the tidal moments through 1.5PN order

My goal in the second part of this paper is to determine the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ that characterize the geometry of a slowly rotating, tidally deformed black hole. This requires immersing the black hole in a patch of spacetime that extends well beyond the domain $\mathcal{N}$, and specifying the conditions in this larger spacetime. I assume that the mutual gravity between the black hole and the external matter is sufficiently weak that it can be adequately represented as a post-Newtonian (PN) expansion. In terms of the scaling quantities introduced previously, I assume that $u^2 \sim m/b \ll 1$; the black hole, of course, is still taken to have strong internal gravity. I therefore aim to obtain the tidal moments as post-Newtonian expansions, with the specific goal to compute $\mathcal{E}_{ab}$ through order u^3 beyond the Newtonian expression of order m_2/b^3, and to compute $\mathcal{B}_{ab}$ through order u beyond its leading-order expression of order m_2u/b^3. Overall this shall give us 1.5PN accuracy for the tidal moments.

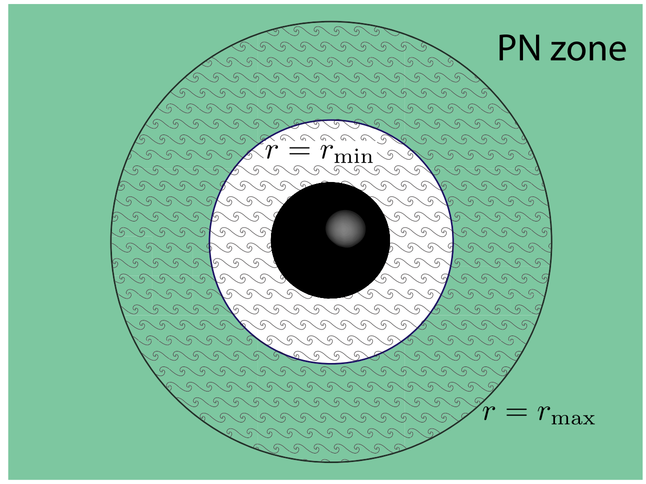

FIG. 2 (color online). The domain $\mathcal{N}$, shown in a wavy pattern, is bounded by $r < r_{\rm max} \ll b$. The post-Newtonian zone, shaded (shown in green online), is bounded by $r > r_{\rm min} \gg m_1$. When $m_1 \ll b$ there is an overlap $r_{\rm min} < r < r_{\rm max}$ between $\mathcal{N}$ and the post-Newtonian zone.

The calculation is based on the asymptotic matching of two metrics in a common region of validity (see Fig. 2). We have already encountered the black-hole metric, which is restricted to the domain $\mathcal{N}$ bounded by $r < r_{\rm max} \ll b$. I now introduce a post-Newtonian metric to describe the mutual gravity between the black hole and the external matter. This metric is restricted to a post-Newtonian zone bounded by $r > r_{\rm min} \gg m_1$, beyond which the black hole's gravity is too strong to be adequately described by a post-Newtonian expansion. (The post-Newtonian zone is also bounded externally by $r < \lambda$, with λ a length scale for the wavelength of the gravitational waves emitted by the entire system; this is the boundary of the near zone, which plays no role in the following developments.) When $m_1 \ll b$, the domain $\mathcal{N}$ and the post-Newtonian zone overlap when $r_{\rm min} < r < r_{\rm max}$, and in this overlap both metrics are solutions to the Einstein field equations that describe the same gravitational field; the metrics must therefore agree once they are cast in the same coordinate system. This observation delivers the tidal moments: Matching the metrics in the overlap determines $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ in terms of information provided by the post-Newtonian metric. Information flows in the other direction as well: Because the post-Newtonian zone is truncated at $r = r_{\rm min}$, the post-Newtonian metric contains unknown functions that must be determined in terms of information provided by the black-hole metric.

The asymptotic matching of a black-hole metric to a post-Newtonian metric was performed before [50–53], with the specific objective of constructing initial data for the numerical simulation of a binary black hole inspiral [54]. These works neglected the spin of each black hole, except for Galloin, Nakano, Yunes, and Campanelli [52], who calculated the tidal moments of a spinning black hole through 1PN order. The calculation presented here is an improvement on this work, but I make no attempt to construct a global metric that could be used as initial data for numerical evolutions. My interest is entirely on the tidal moments, and I calculate them by exploiting the methods introduced by Taylor and Poisson [55], who computed $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ through 1PN order for a nonspinning black hole.

I begin in Sec. VIII by transforming the black-hole metric from the light-cone coordinates (v, r, θ, ϕ) to local harmonic coordinates $(\bar{t}, \bar{x}^a)$ that facilitate the matching to the post-Newtonian metric; this system of harmonic coordinates defines what I call the black-hole frame. In Sec. IX I construct the post-Newtonian metric that describes the mutual gravity between the black hole and the external matter. The metric is first presented in global harmonic coordinates (t, x^a) that define the post-Newtonian barycentric frame, and expressed in terms of a Newtonian potential U, a vector potential U^a, a post-Newtonian potential ψ, and a superpotential X. It is next transformed to the black-hole frame and matched to the black-hole metric. The matching determines $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, which are given in terms of the potentials $U_{\rm ext}$, $U^a_{\rm ext}$, $\psi_{\rm ext}$, and $X_{\rm ext}$ that are sourced by the external matter. In Sec. X the tidal moments are evaluated for the specific situation in which the black hole is a member of a spinning two-body system, and in Sec. XI these results are further specialized to circular orbital motion (with the spins either aligned or antialigned with the orbital angular momentum). Finally, in Sec. XII I complete the calculation of the tidal moments though 1.5PN order by incorporating previously neglected terms in the black-hole metric that involve $\dot{\mathcal{E}}_{ab}$, the time derivative of the electric-type tidal moments.

With the notation introduced in Eqs. (1.4) and (1.5), the tidal moments of a slowly rotating black hole of mass m_1 and dimensionless spin χ_1 in circular motion around a companion body of mass m_2 and dimensionless spin χ_2 are given by

$$\mathcal{E}_0 = -\frac{m_2}{2b^3}\left[1 + \frac{m_1}{2m}u^2 - 6\frac{m_2}{m}\chi_2 u^3 + O(u^4)\right], \quad (1.9a)$$

$$\mathcal{E}_2 = -\frac{3m_2}{2b^3}\left[1 - \frac{3m_1 + 4m_2}{2m}u^2 - 2\frac{m_2}{m}\chi_2 u^3 + O(u^4)\right], \quad (1.9b)$$

$$\mathcal{B}_1 = -\frac{3m_2}{b^3}u\left[1 - \frac{m_2}{m}\chi_2 u + O(u^2)\right], \quad (1.9c)$$

where b is the orbital radius in harmonic coordinates, and u is the orbital velocity, related to b by the post-Newtonian relation

$$m/b = u^2[1 + (3-\eta)u^2 + \tilde{\chi}u^3 + O(u^4)], \quad (1.10)$$

with $m := m_1 + m_2$ denoting the total mass, $\eta := m_1 m_2/m^2$ the symmetric mass ratio, and

$$\tilde{\chi} := \frac{m_1(2m_1 + 3m_2)}{m^2}\chi_1 + \frac{m_2(2m_2 + 3m_1)}{m^2}\chi_2. \quad (1.11)$$

The phase of the tidal moments is given by

$$\bar{\phi} = \bar{\omega}(v - v_0) + \frac{8}{3}\frac{m_1}{m}u^3 + O(u^4), \quad (1.12)$$

where

$$\bar{\omega} = \sqrt{\frac{m}{b^3}}\left[1 - \frac{1}{2}(3+\eta)u^2 - \frac{1}{2}\bar{\chi}u^3 + O(u^4)\right] \quad (1.13)$$

is the tidal angular frequency, expressed in terms of the mass-weighted averaged spin

$$\bar{\chi} := \frac{m_1}{m^2}(2m_1 + m_2)\chi_1 + 3\eta\chi_2. \quad (1.14)$$

The tidal frequency $\bar{\omega}$ is distinct from the orbital frequency $\omega := u/b$, because ω is defined in terms of time t in the post-Newtonian barycentric frame, while $\bar{\omega}$ is defined in terms of advanced-time v in the black hole's moving frame. The frequencies also differ because the black-hole frame is slowly precessing relative to the barycentric frame.

Equations (1.9), (1.12), and (1.13) are the final outcomes of the work carried out in the second part of this article. While the 1.5PN expressions for the tidal quadrupole moments and tidal frequency are derived under the assumption that the black hole is rotating slowly ($\chi_1 \ll 1$), it is important to observe that terms at 1.5PN order in post-Newtonian expansions of the metric and equations of motion are known to be linear in the spins. The 1.5PN expressions, therefore, are *valid to all orders in the spins*.

The methods exploited here can be extended to provide a description of the tidal deformation of a slowly rotating material body such as a neutron star. The background metric outside such a body is still the one examined in Sec. III, but the perturbation constructed in Sec. IV must be generalized to account for the body's gravitational Love numbers $k_2^{\rm el}$ and $k_2^{\rm mag}$ [21,22]. Additional Love numbers may be required to describe the external geometry of the deformed body, and the metric inside the body must also be obtained and joined with the external metric at the body's surface. These details are unlikely to influence the results obtained in the second part of the paper: The tidal moments of Eqs. (1.9) are expected to apply to both black holes and material bodies. Work toward this generalization is now underway, and will be reported in a forthcoming publication [56].

To conclude, I note that the expressions of Eqs. (1.9) and (1.13) are ready to be inserted into the flux formulas obtained in Ref. [20] to describe the 1.5PN tidal heating and torquing of a black hole, whether it is slowly or rapidly rotating. While expressions that were purported to be accurate through 1.5PN order were already presented in this work, these were in fact incomplete, because they did not incorporate the spin terms in the tidal moments, nor the spin terms in the tidal frequency. This situation will be remedied in a separate publication [57].

II. IRREDUCIBLE POTENTIALS

In this section we construct the irreducible potentials that will appear in the metric of a slowly rotating, tidally deformed black hole. The construction is based on the methods introduced in Poisson and Vlasov [43], hereafter referred to as PV. In this and the following sections, the black-hole mass will be denoted M instead of m_1, and its dimensionless spin will be denoted χ^a instead of χ_1^a. The notation employed in Sec. I will be resumed in Sec. X, when the black hole specifically becomes a member of a two-body system.

The most primitive ingredients featured in the potentials are the dimensionless spin pseudovector χ_a, the electric-type tidal quadrupole moment $\mathcal{E}_{ab}$, and the magnetic-type tidal quadrupole moment $\mathcal{B}_{ab}$. These quantities are defined in the rest frame of the black hole, in a local asymptotic region in which the gravitational field is dominated by the tidal field of the external spacetime. In this region the black hole's own gravitational field can be neglected, and the black hole can be viewed as a test body moving on a (potentially accelerated) world line γ; this test body possesses a mass M and a spin tensor $S^{\mu\nu}$ as measured in the local asymptotic rest frame.

The definitions of χ_a, $\mathcal{E}_{ab}$, and $\mathcal{B}_{ab}$ rely on a triad of vectors e^α_a on γ; these are mutually orthonormal, orthogonal to γ's tangent vector u^α (the black hole's velocity vector), and Fermi-Walker transported on the world line. The dimensionless spin pseudovector is defined by $\chi_a \coloneqq S_a/M^2$, where $S_a \coloneqq \frac{1}{2}\epsilon_{apq} e^p_\mu e^q_\nu S^{\mu\nu}$ is the spin pseudovector, ϵ_{apq} is the completely antisymmetric permutation symbol, and latin indices are lowered and raised freely with the Euclidean metric. The dimensionless spin χ_a is taken to be numerically small, to reflect the assumption that the black hole is rotating slowly. The tidal quadrupole moments are defined in terms of the Weyl tensor of the external spacetime evaluated on γ; we have $\mathcal{E}_{ab} \coloneqq C_{a0b0}$ and $\mathcal{B}_{ab} \coloneqq \frac{1}{2}\epsilon_{apq} C^{pq}{}_{b0}$, where $C_{a0b0} \coloneqq C_{\alpha\mu\beta\nu} e^\alpha_a u^\mu e^\beta_b u^\nu$ and $C_{abc0} \coloneqq C_{\alpha\beta\gamma\mu} e^\alpha_a e^\beta_b e^\gamma_c u^\mu$. By virtue of the symmetries of the Weyl tensor, $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ are both symmetric and tracefree.

In the context of this work, a parity transformation is a reflection of the triad described by $e^\alpha_a \to -e^\alpha_a$; the transformation keeps ϵ_{abc} unchanged. Under a parity transformation the spin pseudovector and tidal moments change according to

$$\chi_a \to \chi_a, \qquad \mathcal{E}_{ab} \to \mathcal{E}_{ab}, \qquad \mathcal{B}_{ab} \to -\mathcal{B}_{ab}. \tag{2.1}$$

We observe that while $\mathcal{E}_{ab}$ transforms as an ordinary Cartesian tensor under a parity transformation, $\mathcal{B}_{ab}$ transforms as a pseudotensor; χ_a transforms as a pseudovector, as expected of a quantity describing spin. We shall say that $\mathcal{E}_{ab}$ has even parity, while χ_a and $\mathcal{B}_{ab}$ have odd parity.

The potentials are constructed by following the rules described in Sec. II and Appendix A of PV. We consider first a construction involving Cartesian coordinates x^a, and describe next a construction involving a related system of spherical polar coordinates (r, θ^A), in which $\theta^A = (\theta, \phi)$. The relation between the two systems is the usual $x^a = r\Omega^a(\theta^A)$, in which Ω^a is the unit vector x^a/r expressed in terms of the angles θ^A; explicitly $\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

The potentials are obtained by combining χ_a, $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, and Ω^a in various irreducible ways, with each potential carrying a specific multipole order ℓ and a specific parity label (even or odd). For our purposes here (see Sec. I) we construct potentials that are linear in χ_a, $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, as well as bilinear potentials that couple χ_a to $\mathcal{E}_{ab}$ and χ_a to $\mathcal{B}_{ab}$. The nonlinear couplings between $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ were considered by PV, and we neglect couplings between χ^a and itself—all expressions will be linearized with respect to the dimensionless spin.

Because χ_a is of odd parity, it is to be involved in the construction of an odd-parity rotational potential. Following the general prescription of Eq. (A9) of PV, we introduce

$$\chi^{\mathsf{d}}_a \coloneqq \epsilon_{abc}\Omega^b\chi^c; \tag{2.2}$$

this is a vectorial, dipolar ($\ell = 1$) potential, as indicated by the label d. The tidal potentials associated with $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ are constructed in Sec. II of PV. We have

$$\mathcal{E}^{\mathsf{q}} \coloneqq \mathcal{E}_{ab}\Omega^a\Omega^b, \qquad \mathcal{E}^{\mathsf{q}}_a \coloneqq \gamma_a{}^e\mathcal{E}_{ec}\Omega^c,$$
$$\mathcal{E}^{\mathsf{q}}_{ab} \coloneqq 2\gamma_a{}^e\gamma_b{}^f\mathcal{E}_{ef} + \gamma_{ab}\mathcal{E}^{\mathsf{q}}, \tag{2.3}$$

where $\gamma_a{}^b \coloneqq \delta_a{}^b - \Omega_a\Omega^b$ is a projector to the subspace transverse to Ω_a; these are even-parity potentials of quadrupole order ($\ell = 2$), as indicated by the label q. We also have

$$\mathcal{B}^{\mathsf{q}}_a \coloneqq \epsilon_{aef}\Omega^e\mathcal{B}^f{}_c\Omega^c,$$
$$\mathcal{B}^{\mathsf{q}}_{ab} \coloneqq \epsilon_{aef}\Omega^e\mathcal{B}^f{}_c\gamma^c{}_b + \epsilon_{bef}\Omega^e\mathcal{B}^f{}_c\gamma^c{}_a, \tag{2.4}$$

and these are odd-parity potentials, also of quadrupole order.

The coupling of χ_a and $\mathcal{E}_{ab}$ produces the pseudotensors

$$\mathcal{F}_a := \mathcal{E}_{ab}\chi^b, \qquad \mathcal{F}_{abc} := \mathcal{E}_{\langle ab}\chi_{c\rangle}, \tag{2.5}$$

in which the angular brackets designate the operation of symmetrization and trace removal, so that $\mathcal{F}_{abc}$ is a symmetric-tracefree tensor. These give rise to the dipolar ($\ell = 1$), odd-parity potential

$$\mathcal{F}_a^{\mathsf{d}} := \epsilon_{abc}\Omega^b\mathcal{F}^c \tag{2.6}$$

and the octupolar ($\ell = 3$), odd-parity potentials

$$\mathcal{F}_a^{\mathsf{o}} := \epsilon_{aef}\Omega^e\mathcal{F}^f{}_{rs}\Omega^r\Omega^s,$$
$$\mathcal{F}_{ab}^{\mathsf{o}} := (\epsilon_{aef}\Omega^e\mathcal{F}^f{}_{cr}\gamma^c{}_b + \epsilon_{bef}\Omega^e\mathcal{F}^f{}_{cr}\gamma^c{}_a)\Omega^r. \tag{2.7}$$

On the other hand, the coupling of χ_a and $\mathcal{B}_{ab}$ produces the tensors

$$\mathcal{K}_a := \mathcal{B}_{ab}\chi^b, \qquad \mathcal{K}_{abc} := \mathcal{B}_{\langle ab}\chi_{c\rangle}, \tag{2.8}$$

the dipolar, even-parity potentials

$$\mathcal{K}^{\mathsf{d}} := \mathcal{K}_a\Omega^a, \qquad \mathcal{K}_a^{\mathsf{d}} := \gamma_a{}^e\mathcal{K}_e, \tag{2.9}$$

and the octupolar, even-parity potentials

$$\mathcal{K}^{\mathsf{o}} := \mathcal{K}_{abc}\Omega^a\Omega^b\Omega^c, \qquad \mathcal{K}_a^{\mathsf{o}} := \gamma_a{}^e\mathcal{K}_{ebc}\Omega^b\Omega^c,$$
$$\mathcal{K}_{ab}^{\mathsf{o}} := 2\gamma_a{}^e\gamma_b{}^f\mathcal{K}_{efc}\Omega^c + \gamma_{ab}\mathcal{K}^{\mathsf{o}}. \tag{2.10}$$

We next convert the Cartesian potentials introduced previously into angular-coordinate potentials. The relations $x^a = r\Omega^a(\theta^A)$ give rise to the transformation matrix $\Omega^a_A := \partial\Omega^a/\partial\theta^A$, and the transformation of the potentials is described by

$$\mathcal{E}_A^{\mathsf{q}} := \mathcal{E}_a^{\mathsf{q}}\Omega^a_A, \qquad \mathcal{E}_{AB}^{\mathsf{q}} := \mathcal{E}_{ab}^{\mathsf{q}}\Omega^a_A\Omega^a_B, \tag{2.11}$$

with similar relations defining χ_A^{d}, $\mathcal{F}_{AB}^{\mathsf{o}}$, and so on.

It is helpful to decompose the angular-coordinate potentials in scalar, vector, and tensor spherical harmonics. The methods to achieve this are described in Sec. II and Appendix A of PV. The decomposition involves the even-parity harmonics

$$Y_A^{\ell\mathsf{m}} := D_A Y^{\ell\mathsf{m}},$$
$$Y_{AB}^{\ell\mathsf{m}} := \left[D_A D_B + \frac{1}{2}\ell(\ell+1)\Omega_{AB}\right]Y^{\ell\mathsf{m}} \tag{2.12}$$

and the odd-parity harmonics

$$X_A^{\ell\mathsf{m}} := -\epsilon_A{}^B D_B Y^{\ell\mathsf{m}},$$
$$X_{AB}^{\ell,\mathsf{m}} := -\frac{1}{2}(\epsilon_a{}^C D_B + \epsilon_B{}^C D_A)D_C Y^{\ell\mathsf{m}}. \tag{2.13}$$

Here $\Omega_{AB} = \mathrm{diag}(1, \sin^2\theta)$ is the metric on a unit two-sphere, and D_A is the covariant-derivative operator compatible with this metric; ϵ_{AB} is the Levi-Civita tensor on the unit two-sphere ($\epsilon_{\theta\phi} = \sin\theta$), and its index is raised with Ω^{AB}, the matrix inverse to Ω_{AB}. It should be noted that the tensorial harmonics are tracefree, in the sense that $\Omega^{AB}Y_{AB}^{\ell\mathsf{m}} = \Omega^{AB}X_{AB}^{\ell\mathsf{m}} = 0$. It will be a consistent convention in this work that tensorial operations on spherical harmonics refer to the unit two-sphere (instead of a sphere of radius r).

The starting point for the decomposition are the identities

$$\chi_a\Omega^a = \sum_{\mathsf{m}}\chi_{\mathsf{m}}^{\mathsf{d}}Y^{1\mathsf{m}}, \qquad \mathcal{F}_a\Omega^a = \sum_{\mathsf{m}}\mathcal{F}_{\mathsf{m}}^{\mathsf{d}}Y^{1\mathsf{m}},$$
$$\mathcal{K}_a\Omega^a = \sum_{\mathsf{m}}\mathcal{K}_{\mathsf{m}}^{\mathsf{d}}Y^{1\mathsf{m}}, \tag{2.14a}$$

$$\mathcal{E}_{ab}\Omega^a\Omega^b = \sum_{\mathsf{m}}\mathcal{E}_{\mathsf{m}}^{\mathsf{q}}Y^{2\mathsf{m}}, \qquad \mathcal{B}_{ab}\Omega^a\Omega^b = \sum_{\mathsf{m}}\mathcal{B}_{\mathsf{m}}^{\mathsf{q}}Y^{2\mathsf{m}}, \tag{2.14b}$$

$$\mathcal{F}_{abc}\Omega^a\Omega^b\Omega^c = \sum_{\mathsf{m}}\mathcal{F}_{\mathsf{m}}^{\mathsf{o}}Y^{3\mathsf{m}},$$
$$\mathcal{K}_{abc}\Omega^a\Omega^b\Omega^c = \sum_{\mathsf{m}}\mathcal{K}_{\mathsf{m}}^{\mathsf{o}}Y^{3\mathsf{m}}, \tag{2.14c}$$

which involve the (scalar) spherical-harmonic functions $Y^{\ell\mathsf{m}}(\theta^A)$. The identities provide a packaging of the 3 independent components of χ_a, $\mathcal{F}_a$, and $\mathcal{K}_a$ in the coefficients $\chi_{\mathsf{m}}^{\mathsf{d}}$, $\mathcal{F}_{\mathsf{m}}^{\mathsf{d}}$, and $\mathcal{F}_{\mathsf{m}}^{\mathsf{o}}$, a packaging of the 5 independent components of $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ in $\mathcal{E}_{\mathsf{m}}^{\mathsf{q}}$ and $\mathcal{B}_{\mathsf{m}}^{\mathsf{q}}$, and a packaging of the 7 independent components of $\mathcal{F}_{abc}$ and $\mathcal{K}_{abc}$ in $\mathcal{F}_{\mathsf{m}}^{\mathsf{o}}$ and $\mathcal{K}_{\mathsf{m}}^{\mathsf{o}}$. The precise definition of the various coefficients in Eq. (2.14) depend on the conventions adopted for the spherical harmonics. These are spelled out in Table I, and the coefficients are defined in Table II. To simplify all expressions we align χ_a with the polar axis, so that $\chi_a = (0, 0, \chi)$.

The decomposition of the potentials in spherical harmonics is described by

$$\chi_A^{\mathsf{d}} = \sum_{\mathsf{m}}\chi_{\mathsf{m}}^{\mathsf{d}}X_A^{1\mathsf{m}}, \tag{2.15a}$$

$$\mathcal{F}_A^{\mathsf{d}} = \sum_{\mathsf{m}}\mathcal{F}_{\mathsf{m}}^{\mathsf{d}}X_A^{1\mathsf{m}}, \tag{2.15b}$$

$$\mathcal{K}^{\mathsf{d}} = \sum_{\mathsf{m}}\mathcal{K}_{\mathsf{m}}^{\mathsf{d}}Y^{1\mathsf{m}}, \qquad \mathcal{K}_A^{\mathsf{d}} = \sum_{\mathsf{m}}\mathcal{K}_{\mathsf{m}}^{\mathsf{d}}Y_A^{1\mathsf{m}}, \tag{2.15c}$$

TABLE I. Spherical-harmonic functions $Y^{\ell \mathsf{m}}$. The functions are real, and they are listed for the relevant modes $\ell = 1$ (dipole), $\ell = 2$ (quadrupole), and $\ell = 3$ (octupole). The abstract index m describes the dependence of these functions on the angle ϕ; for example $Y^{\ell,2s}$ is proportional to $\sin 2\phi$. To simplify the expressions we write $C := \cos\theta$ and $S := \sin\theta$.

$$\begin{aligned} Y^{1,0} &= C \\ Y^{1,1c} &= S\cos\phi \\ Y^{1,1s} &= S\sin\phi \\ Y^{2,0} &= 1 - 3C^2 \\ Y^{2,1c} &= 2SC\cos\phi \\ Y^{2,1s} &= 2SC\sin\phi \\ Y^{2,2c} &= S^2\cos 2\phi \\ Y^{2,2s} &= S^2\sin 2\phi \\ Y^{3,0} &= C(3 - 5C^2) \\ Y^{3,1c} &= \tfrac{3}{2}S(1 - 5C^2)\cos\phi \\ Y^{3,1s} &= \tfrac{3}{2}S(1 - 5C^2)\sin\phi \\ Y^{3,2c} &= 3S^2C\cos 2\phi \\ Y^{3,2s} &= 3S^2C\sin 2\phi \\ Y^{3,3c} &= S^3\cos 3\phi \\ Y^{3,3s} &= S^3\sin 3\phi \end{aligned}$$

$$\mathcal{E}^{\mathsf{q}} = \sum_{\mathsf{m}} \mathcal{E}^{\mathsf{q}}_{\mathsf{m}} Y^{2\mathsf{m}}, \quad \mathcal{E}^{\mathsf{q}}_A = \frac{1}{2}\sum_{\mathsf{m}} \mathcal{E}^{\mathsf{q}}_{\mathsf{m}} Y^{2\mathsf{m}}_A, \quad \mathcal{E}^{\mathsf{q}}_{AB} = \sum_{\mathsf{m}} \mathcal{E}^{\mathsf{q}}_{\mathsf{m}} Y^{2\mathsf{m}}_{AB}, \tag{2.15d}$$

$$\mathcal{B}^{\mathsf{q}}_A = \frac{1}{2}\sum_{\mathsf{m}} \mathcal{B}^{\mathsf{q}}_{\mathsf{m}} X^{2\mathsf{m}}_A, \qquad \mathcal{B}^{\mathsf{q}}_{AB} = \sum_{\mathsf{m}} \mathcal{B}^{\mathsf{q}}_{\mathsf{m}} X^{2\mathsf{m}}_{AB}, \tag{2.15e}$$

$$\mathcal{F}^{\mathsf{o}}_A = \frac{1}{3}\sum_{\mathsf{m}} \mathcal{F}^{\mathsf{o}}_{\mathsf{m}} X^{3\mathsf{m}}_A, \qquad \mathcal{F}^{\mathsf{o}}_{AB} = \frac{1}{3}\sum_{\mathsf{m}} \mathcal{F}^{\mathsf{o}}_{\mathsf{m}} X^{3\mathsf{m}}_{AB}, \tag{2.15f}$$

$$\mathcal{K}^{\mathsf{o}} = \sum_{\mathsf{m}} \mathcal{K}^{\mathsf{o}}_{\mathsf{m}} Y^{3\mathsf{m}}, \qquad \mathcal{K}^{\mathsf{o}}_A = \frac{1}{3}\sum_{\mathsf{m}} \mathcal{K}^{\mathsf{o}}_{\mathsf{m}} Y^{3\mathsf{m}}_A,$$

$$\mathcal{K}^{\mathsf{o}}_{AB} = \frac{1}{3}\sum_{\mathsf{m}} \mathcal{K}^{\mathsf{o}}_{\mathsf{m}} Y^{3\mathsf{m}}_{AB}. \tag{2.15g}$$

It should be noted that tensorial potentials are not defined when $\ell = 1$. The origin of the numerical coefficients that appear in front of the sums is explained in Eqs. (A7), (A8), (A14), and (A15) of PV.

III. BACKGROUND SPACETIME

The metric of a slowly rotating black hole can be obtained from the exact Kerr metric by neglecting all terms beyond linear order in the spin parameter $\chi := S/M^2$, in which S is the black hole's angular momentum and M is its mass. In the original Boyer-Lindquist coordinates $(t, r, \theta, \tilde{\phi})$, the metric is given by

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\tilde{\phi}^2) - 2\frac{2\chi M^2}{r}\sin^2\theta dt d\tilde{\phi}, \tag{3.1}$$

TABLE II. Spherical-harmonic coefficients of Eq. (2.14). The relations between $\mathcal{K}^{\mathsf{d}}_{\mathsf{m}}$, $\mathcal{K}_a$, and $\chi\mathcal{B}^{\mathsf{q}}_{\mathsf{m}}$ are identical to those between $\mathcal{F}^{\mathsf{d}}_{\mathsf{m}}$, $\mathcal{F}_a$, and $\chi\mathcal{E}^{\mathsf{q}}_{\mathsf{m}}$. Similarly, the relations between $\mathcal{K}^{\mathsf{o}}_{\mathsf{m}}$, $\mathcal{K}_{abc}$, and $\chi\mathcal{B}^{\mathsf{q}}_{\mathsf{m}}$ are identical to those between $\mathcal{F}^{\mathsf{o}}_{\mathsf{m}}$, $\mathcal{F}_{abc}$, and $\chi\mathcal{E}^{\mathsf{q}}_{\mathsf{m}}$.

$$\begin{aligned} \chi^{\mathsf{d}}_0 &= \chi_3 = \chi \\ \chi_{1c} &= \chi_1 = 0 \\ \chi_{1s} &= \chi_2 = 0 \\ \mathcal{E}^{\mathsf{q}}_0 &= \tfrac{1}{2}(\mathcal{E}_{11} + \mathcal{E}_{22}) \\ \mathcal{E}^{\mathsf{q}}_{1c} &= \mathcal{E}_{13} \\ \mathcal{E}^{\mathsf{q}}_{1s} &= \mathcal{E}_{23} \\ \mathcal{E}^{\mathsf{q}}_{2c} &= \tfrac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22}) \\ \mathcal{E}^{\mathsf{q}}_{2s} &= \mathcal{E}_{12} \\ \mathcal{F}^{\mathsf{d}}_0 &= \mathcal{F}_3 = -2\chi\mathcal{E}^{\mathsf{q}}_0 \\ \mathcal{F}^{\mathsf{d}}_{1c} &= \mathcal{F}_1 = \chi\mathcal{E}^{\mathsf{q}}_{1c} \\ \mathcal{F}^{\mathsf{d}}_{1s} &= \mathcal{F}_2 = \chi\mathcal{E}^{\mathsf{q}}_{1s} \\ \mathcal{F}^{\mathsf{o}}_0 &= \tfrac{1}{2}(\mathcal{F}_{113} + \mathcal{F}_{223}) = \tfrac{3}{5}\chi\mathcal{E}^{\mathsf{q}}_0 \\ \mathcal{F}^{\mathsf{o}}_{1c} &= \tfrac{1}{2}(\mathcal{F}_{111} + \mathcal{F}_{122}) = -\tfrac{4}{15}\chi\mathcal{E}^{\mathsf{q}}_{1c} \\ \mathcal{F}^{\mathsf{o}}_{1s} &= \tfrac{1}{2}(\mathcal{F}_{112} + \mathcal{F}_{222}) = -\tfrac{4}{15}\chi\mathcal{E}^{\mathsf{q}}_{1s} \\ \mathcal{F}^{\mathsf{o}}_{2c} &= \tfrac{1}{2}(\mathcal{F}_{113} - \mathcal{F}_{223}) = \tfrac{1}{3}\chi\mathcal{E}^{\mathsf{q}}_{2c} \\ \mathcal{F}^{\mathsf{o}}_{2s} &= \mathcal{F}_{123} = \tfrac{1}{3}\chi\mathcal{E}^{\mathsf{q}}_{2s} \\ \mathcal{F}^{\mathsf{o}}_{3c} &= \tfrac{1}{4}(\mathcal{F}_{111} - 3\mathcal{F}_{122}) = 0 \\ \mathcal{F}^{\mathsf{o}}_{2s} &= \tfrac{1}{4}(3\mathcal{F}_{112} - F_{222}) = 0 \end{aligned}$$

where $f := 1 - 2M/r$. The Boyer-Lindquist coordinates are singular at the event horizon, and we replace them with light-cone coordinates (v, r, θ, ϕ) defined by

$$dv = dt + f^{-1}dr, \qquad d\phi = d\tilde{\phi} + \frac{2\chi M^2}{r^3 f}dr. \tag{3.2}$$

The coordinate transformation produces

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2 - 2\frac{2\chi M^2}{r}\sin^2\theta dv d\phi, \tag{3.3}$$

where $d\Omega^2 := \Omega_{AB}d\theta^A d\theta^B := d\theta^2 + \sin^2\theta d\phi^2$. The transformation from $\tilde{\phi}$ to ϕ was introduced to remove a term proportional to $dr d\tilde{\phi}$ in the metric, produced when dt is replaced by $dv = dt + f^{-1}dr$.

The coordinates (v, r, θ, ϕ) differ from the coordinates $(\hat{v}, r, \theta, \hat{\phi})$ attached to the incoming principal null congruence of the Kerr spacetime (see, for example, Sec. 5.3.6 of Ref. [58]). While $\hat{v} = v + O(\chi^2)$, we have that $d\hat{\phi} = d\tilde{\phi} + (\chi M/r^2 f)dr + O(\chi^2) = d\phi + (\chi M/r^2)dr + O(\chi^2)$, so that

$$\hat{\phi} = \phi - \chi\frac{M}{r} + O(\chi^2); \tag{3.4}$$

the constant of integration was selected to ensure that $\hat{\phi} = \phi$ when $r = \infty$.

The coordinates (v, r, θ, ϕ) are tied to the behavior of incoming null geodesics that are tangent to converging null cones. It is easy to show that $\ell_\alpha = -\partial_\alpha v$ is null, so that each surface $v = \text{constant}$ is a null hypersurface. The vector ℓ^α is tangent to its null generators, and the only nonvanishing component is $l^r = -1$; this indicates that v, θ, and ϕ are all constant on the generators, and that $-r$ is an affine parameter. We note that the null geodesics have zero angular momentum, because the component of ℓ_α along the azimuthal Killing vector ϕ^α vanishes. It can also be noted that $\tilde{\phi}$ increases on the geodesics, while $\hat{\phi}$ decreases.

In the notation introduced in Sec. II, the components of the background metric can be expressed as

$$g_{vv} = -f, \qquad g_{vr} = 1, \qquad g_{vA} = \frac{2M^2}{r}\chi^{\mathsf{d}}_A,$$
$$g_{AB} = r^2\Omega_{AB}, \tag{3.5}$$

where χ^{d}_A is the rotational potential of Eq. (2.15a).

IV. PERTURBED SPACETIME

Our goal in this section is to add a tidal perturbation to the background metric of Eq. (3.5). As explained in Sec. I, we focus our attention on a domain $\mathcal{N}$ empty of matter, bounded by $r < r_{\max} \ll b$, with b denoting the distance scale to the external matter. We follow the general methods detailed in Poisson and Vlasov ([43]: PV). In particular, we continue to work in light-cone coordinates, and insist that the coordinates (v, r, θ, ϕ) keep their geometrical meaning in the perturbed spacetime: v shall continue to be constant on converging null cones, θ and ϕ shall continue to be constant on each generator, and $-r$ shall continue to be an affine parameter on each generator. As shown in Sec. V A of PV, these requirements imply that $g_{vr} = 1$, $g_{rr} = 0 = g_{rA}$, so that g_{vv}, g_{vr}, g_{vA}, and g_{AB} are the only nonvanishing components of the metric. In addition, we shall see that the field equations allow us to preserve the meaning of r as an areal radius, in the sense that $4\pi r^2$ measures the area of a two-surface of constant v and r.

At a large distance from the black hole (where r is such that $2M \ll r < r_{\max}$), the metric is dominated by the tidal field, which is characterized by the tidal moments $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$. As shown in Eq. (3.4) of PV, the asymptotic form of the metric is given by

$$g_{vv} \sim -r^2\mathcal{E}^{\mathsf{q}}, \qquad g_{vA} \sim -\frac{2}{3}r^3(\mathcal{E}^{\mathsf{q}}_A - \mathcal{B}^{\mathsf{q}}_A),$$
$$g_{AB} \sim -\frac{1}{3}r^4(\mathcal{E}^{\mathsf{q}}_{AB} - \mathcal{B}^{\mathsf{q}}_{AB}), \tag{4.1}$$

in addition to the exact statement $g_{vr} = 1$. These expressions are accurate to leading order in an expansion in powers of r/b, and we shall maintain this degree of accuracy throughout this work. When r becomes comparable to $2M$ the information contained in Eq. (3.5) becomes important, and what is required is a perturbed metric that incorporates this information in addition to the asymptotic behavior captured by Eq. (4.1). As indicated previously, we wish the metric to contain terms that are linear in χ, $\mathcal{E}_{ab}$, $\mathcal{B}_{ab}$, as well as terms that scale as $\chi\mathcal{E}_{ab}$ and $\chi\mathcal{B}_{ab}$; we are satisfied with the neglect of higher-order terms in χ and higher-order terms in the tidal moments.

To construct this metric it is helpful to alter our perspective and take the background spacetime to have the metric

$$g^{\text{back}}_{vv} = -f, \qquad g^{\text{back}}_{vr} = 1, \qquad g^{\text{back}}_{AB} = r^2\Omega_{AB} \tag{4.2}$$

of a nonrotating black hole. In this new perspective, the missing rotational term $p_{vA} = (2M^2/r)\chi^{\mathsf{d}}_A$ is treated as a perturbation, which can be added to the tidal perturbation

$$p_{vv} = -r^2 e^{\mathsf{q}}_1\mathcal{E}^{\mathsf{q}}, \qquad p_{vA} = -\frac{2}{3}r^3(e^{\mathsf{q}}_4\mathcal{E}^{\mathsf{q}}_A - b^{\mathsf{q}}_4\mathcal{B}^{\mathsf{q}}_A),$$
$$p_{AB} = -\frac{1}{3}r^4(e^{\mathsf{q}}_7\mathcal{E}^{\mathsf{q}}_{AB} - b^{\mathsf{q}}_7\mathcal{B}^{\mathsf{q}}_{AB}) \tag{4.3}$$

first constructed in Ref. [42] (see also Refs. [40,41]). The perturbed metric is expressed as $g_{\alpha\beta} = g^{\text{back}}_{\alpha\beta} + p_{\alpha\beta}$, in which $g^{\text{back}}_{\alpha\beta}$ is the new background metric of Eq. (4.2), and $p_{\alpha\beta}$ is a sum of rotational and tidal perturbations. The radial functions e^{q}_n and b^{q}_n are displayed Table III. While the

TABLE III. Radial functions appearing in the metric of Eq. (4.6); $f \coloneqq 1 - 2M/r$.

$e^{\mathsf{q}}_1 = f^2$
$e^{\mathsf{q}}_4 = f$
$e^{\mathsf{q}}_7 = 1 - 2\frac{M^2}{r^2}$
$b^{\mathsf{q}}_4 = f$
$b^{\mathsf{q}}_7 = 1 - 6\frac{M^2}{r^2}$
$\hat{e}^{\mathsf{q}}_1 = \frac{M^2}{r^2} - 4\frac{M^3}{r^3} + 2\frac{M^4}{r^4} + \gamma^{\mathsf{q}}\frac{M^4}{r^4}$
$\hat{e}^{\mathsf{q}}_4 = \frac{8}{9}\frac{M^2}{r^2} - \frac{4}{9}\frac{M^4}{r^4} - \frac{2}{3}\gamma^{\mathsf{q}}\frac{M^3}{r^3}(1 + \frac{M}{r})$
$\hat{e}^{\mathsf{q}}_7 = \frac{7}{9}\frac{M^2}{r^2} - \frac{1}{3}\gamma^{\mathsf{q}}\frac{M^3}{r^3}$
$\hat{b}^{\mathsf{q}}_4 = \frac{8}{9}\frac{M^2}{r^2} - \frac{4}{9}\frac{M^3}{r^3} - \frac{16}{9}\frac{M^4}{r^4}$
$\hat{b}^{\mathsf{q}}_7 = c^{\mathsf{q}}\frac{M^2}{r^2} - \frac{2}{9}\frac{M^3}{r^3}$
$k^{\mathsf{d}}_1 = 2\frac{M}{r} - \frac{34}{5}\frac{M^2}{r^2} + \frac{32}{5}\frac{M^3}{r^3} + c^{\mathsf{d}}\frac{M^4}{r^4}$
$k^{\mathsf{d}}_4 = \frac{M}{r} - \frac{8}{5}\frac{M^2}{r^2} - \frac{24}{5}\frac{M^4}{r^4} - c^{\mathsf{d}}\frac{M^4}{r^4}$
$f^{\mathsf{d}}_4 = \gamma^{\mathsf{d}}\frac{M}{r} + \frac{8}{5}\frac{M^2}{r^2}$
$k^{\mathsf{o}}_1 = 2\frac{M^2}{r^2} - \frac{8}{3}\frac{M^3}{r^3} + c^{\mathsf{o}}\frac{M^4}{r^4}$
$k^{\mathsf{o}}_4 = \frac{4}{3}\frac{M^2}{r^2} + 4\frac{M^4}{r^4} + \frac{1}{4}c^{\mathsf{o}}\frac{M^3}{r^3}(5 + 2\frac{M}{r})$
$k^{\mathsf{o}}_7 = \frac{4}{3}\frac{M^2}{r^2} + \frac{1}{2}c^{\mathsf{o}}\frac{M^3}{r^3}$
$f^{\mathsf{o}}_4 = \frac{4}{3}\frac{M^2}{r^2} - \frac{10}{3}\frac{M^3}{r^3}$
$f^{\mathsf{o}}_7 = \gamma^{\mathsf{o}}\frac{M^2}{r^2} + \frac{4}{3}\frac{M^3}{r^3}$

expressions given here incorporate the leading rotational and tidal deformations, we wish to go beyond the leading order and obtain improved expressions that capture the couplings between the rotational and tidal terms. In the new perspective adopted here, this requires us to go beyond first-order perturbation theory.

The expansions of Eqs. (2.15) imply that the first-order perturbation $p_{\alpha\beta}$ admits the usual decomposition in spherical harmonics. In the even-parity sector we have

$$p_{vv} = \sum_{\ell \mathsf{m}} h_{vv}^{\ell \mathsf{m}} Y^{\ell \mathsf{m}}, \qquad p_{vA} = \sum_{\ell \mathsf{m}} j_v^{\ell \mathsf{m}} Y_A^{\ell \mathsf{m}},$$
$$p_{AB} = r^2 \sum_{\ell \mathsf{m}} G^{\ell \mathsf{m}} Y_{AB}^{\ell \mathsf{m}}, \tag{4.4}$$

in which the coefficients $h_{vv}^{\ell \mathsf{m}}$, $j_v^{\ell \mathsf{m}}$, and $G^{\ell \mathsf{m}}$ depend on r only. The decompositions, in fact, imply that the even-parity perturbation is made up entirely of a tidal deformation created by $\mathcal{E}_{ab}$, which is purely quadrupolar ($\ell = 2$). The decompositions omit a term $r^2 K^{2\mathsf{m}} \Omega_{AB} Y^{2\mathsf{m}}$ that could also be included in p_{AB}; this refinement of the light-cone gauge is allowed by the field equations (refer to Ref. [46]), and it ensures that r continues to be an areal radius in the perturbed spacetime. In the odd-parity sector we have

$$p_{vv} = 0, \qquad p_{vA} = \sum_{\ell \mathsf{m}} h_v^{\ell \mathsf{m}} X_A^{\ell \mathsf{m}}, \qquad p_{AB} = \sum_{\ell \mathsf{m}} h_2^{\ell \mathsf{m}} X_{AB}^{\ell \mathsf{m}}, \tag{4.5}$$

in which $h_v^{\ell \mathsf{m}}$ and $h_2^{\ell \mathsf{m}}$ depend on r only. In this case the decompositions imply that the odd-parity perturbation consists of a dipolar ($\ell = 1$) rotational perturbation and a quadrupolar tidal deformation created by $\mathcal{B}_{ab}$.

When the first-order perturbation is used as a seed for a second-order calculation of the perturbed metric, the $\ell = 1$ terms associated with the rotational perturbation couple to the $\ell = 2$ tidal terms. The composition of the relevant spherical harmonics implies that the resulting perturbation will have contributions at multipole orders $\ell = 1$, $\ell = 2$, and $\ell = 3$. The metric can then be obtained by creating an ansatz that incorporates all possible contributions at these multipole orders, substituting it into the Einstein field equations, and solving for the unknown functions of r. The ansatz must respect the parity rules implied by the composition of spherical harmonics, and after some thought it becomes clear that it will implicate the irreducible potentials introduced in Sec. II. Our metric ansatz is given by

$$g_{vv} = -f - r^2 e_1^{\mathsf{q}} \mathcal{E}^{\mathsf{q}} - r^2 \hat{e}_1^{\mathsf{q}} \chi \partial_\phi \mathcal{E}^{\mathsf{q}} + r^2 k_1^{\mathsf{d}} \mathcal{K}^{\mathsf{d}} - r^2 k_1^{\mathsf{o}} \mathcal{K}^{\mathsf{o}}, \tag{4.6a}$$

$$g_{vr} = 1, \tag{4.6b}$$

$$\begin{aligned} g_{vA} = {} & \frac{2M^2}{r} \chi_A^{\mathsf{d}} - \frac{2}{3} r^3 (e_4^{\mathsf{q}} \mathcal{E}_A^{\mathsf{q}} - b_4^{\mathsf{q}} \mathcal{B}_A^{\mathsf{q}}) \\ & - r^3 \chi \partial_\phi (\hat{e}_4^{\mathsf{q}} \mathcal{E}_A^{\mathsf{q}} - \hat{b}_4^{\mathsf{q}} \mathcal{B}_A^{\mathsf{q}}) - r^3 (f_4^{\mathsf{d}} \mathcal{F}_A^{\mathsf{d}} - k_4^{\mathsf{d}} \mathcal{K}_A^{\mathsf{d}}) \\ & + r^3 (f_4^{\mathsf{o}} \mathcal{F}_A^{\mathsf{o}} + k_4^{\mathsf{o}} \mathcal{K}_A^{\mathsf{o}}), \end{aligned} \tag{4.6c}$$

$$\begin{aligned} g_{AB} = {} & r^2 \Omega_{AB} - \frac{1}{3} r^4 (e_7^{\mathsf{q}} \mathcal{E}_{AB}^{\mathsf{q}} - b_7^{\mathsf{q}} \mathcal{B}_{AB}^{\mathsf{q}}) \\ & - r^4 \chi \partial_\phi (\hat{e}_7^{\mathsf{q}} \mathcal{E}_{AB}^{\mathsf{q}} - \hat{b}_7^{\mathsf{q}} \mathcal{B}_{AB}^{\mathsf{q}}) \\ & - r^4 (f_7^{\mathsf{o}} \mathcal{F}_{AB}^{\mathsf{o}} - k_7^{\mathsf{o}} \mathcal{K}_{AB}^{\mathsf{o}}), \end{aligned} \tag{4.6d}$$

in which $\hat{e}_n^{\mathsf{q}}$, $\hat{b}_n^{\mathsf{q}}$, k_n^{d}, k_n^{o}, f_n^{d}, and f_n^{o} are functions of r to be determined by solving the Einstein field equations to second order in perturbation theory. Notice that the ansatz omits terms in p_{AB} that are proportional to Ω_{AB}; all the contributions are tracefree, in the sense that $\Omega^{AB} p_{AB} = 0$. This feature was also observed in the first-order perturbation, where it was identified as a refinement of the light-cone gauge. We have verified that adding terms $r^2 K^{\ell \mathsf{m}} \Omega_{AB} Y^{\ell \mathsf{m}}$ to the metric produces differential equations for $K^{\ell \mathsf{m}}(r)$ that are entirely decoupled from the remaining field equations. The simplest solution is $K^{\ell \mathsf{m}} = 0$. The freedom to refine the light-cone gauge is therefore preserved at second order, and r continues to be an areal radius in the metric of Eq. (4.6).

We note that expressions such as $\partial_\phi \mathcal{E}_A^{\mathsf{q}}$ and $\partial_\phi \mathcal{B}_{AB}^{\mathsf{q}}$ appear to break the covariance of the metric with respect to transformations of the angular coordinates θ^A. This violation, however, is only apparent. An equivalent covariant expression is easily obtained by introducing a vector ϕ^A on each two-sphere $(v, r) =$ constant that generates a rotation around the axis identified by the spin pseudovector χ_a. In our canonical coordinates in which χ_a is aligned with the polar axis, $\phi^A = (0, 1)$, and

$$\partial_\phi \mathcal{E}_A^{\mathsf{q}} = \mathcal{L}_\phi \mathcal{E}_A^{\mathsf{q}}, \qquad \partial_\phi \mathcal{B}_{AB}^{\mathsf{q}} = \mathcal{L}_\phi \mathcal{B}_{AB}^{\mathsf{q}}, \tag{4.7}$$

where $\mathcal{L}_\phi$ denotes the Lie derivative in the direction of the vector ϕ^A. The right-hand side of each equality is covariant, and ∂_ϕ can be freely replaced with $\mathcal{L}_\phi$ in the metric ansatz of Eq. (4.6).

The metric of Eq. (4.6) is used to calculate the Ricci tensor $R_{\alpha\beta}$, and time derivatives are neglected by virtue of our assumption that the tidal moments vary slowly. The Ricci tensor is then expanded in powers of χ, $\mathcal{E}_{ab}$, and $\mathcal{B}_{ab}$. The expansion includes contributions that are linear in these quantities, and these vanish when the appropriate substitutions are made for the radial functions e_n^{q} and b_n^{q}. The expansion also includes terms that scale as $\chi \mathcal{E}_{ab}$ and $\chi \mathcal{B}_{ab}$, but it neglects higher-order terms. The contributions to $R_{\alpha\beta}$ proportional to $\chi \mathcal{E}_{ab}$ and $\chi \mathcal{B}_{ab}$ are collected in a spherical-harmonic decomposition, and the vacuum field equations imply that each coefficient must vanish separately. This calculation gives rise to a number of differential equations for the remaining radial functions.

The solutions to these equations feature two types of integration constants. As an example, let us consider the radial function k_1^{o}, which appears in g_{vv} as the coefficient in front of $\mathcal{K}^{\mathsf{o}}$, a bilinear potential coupling χ_a to $\mathcal{B}_{ab}$. The equation satisfied by k_1^{o} involves a linear differential operator inherited from first-order perturbation theory, as well as a nonlinear source term generated by the first-order perturbation. (The actual situation is more complicated, because k_1^{o} is implicated with other radial functions in a system of coupled equations. But this complication has no bearing on this discussion, which for clarity we choose to frame in a simplified form.) The general solution to the equation is

$$k_1^{\mathsf{o}} = 2\frac{M^2}{r^2} - \frac{8}{3}\frac{M^3}{r^3} + c^{\mathsf{o}}\frac{M^4}{r^4} + d^{\mathsf{o}}\frac{r}{M}f^2\left(1 - \frac{M}{r}\right), \qquad (4.8)$$

and it depends on two arbitrary constants, c^{o} and d^{o}. The freedom to choose c^{o} represents a residual gauge freedom that keeps the metric in the light-cone gauge; this freedom is documented in Ref. [46] and Secs. VI A and VI C of PV. For the time being we shall keep the residual freedom of the light-cone gauge intact, and defer the task of making specific choices of integration constants to Sec. VII. On the other hand, the freedom to choose d^{o} corresponds to the freedom of adding to k_1^{o} a solution to the homogeneous differential equation. This is the equation that would govern an $\ell = 3$ perturbation in first-order perturbation theory, and such a perturbation would correspond to the presence of a tidal octupole moment $\mathcal{E}_{abc}$. The freedom to adjust d^{o} is therefore the freedom to shift the definition of $\mathcal{E}_{abc}$ by a multiple of the tensor $\mathcal{K}_{abc}$. This shift is meaningless, and we eliminate this freedom by discarding the solution to the homogeneous equation. Similar considerations apply to other radial functions, and in all cases we eliminate the freedom to add solutions to the homogeneous equations.

With this understood, we find that the field equations give rise to the radial functions listed in Table III. The residual gauge freedom is contained in the six arbitrary constants γ^{d}, γ^{q}, γ^{o}, c^{d}, c^{q}, and c^{o}.

Before moving on, it is worthwhile to examine the meaning of the gauge constant γ^{d}, which will make an appearance in later sections. A glance at Table III reveals that the constant is featured in the function f_4^{d} only, and that it gives rise to the (odd-parity) metric perturbation $p_{vA} = -\gamma^{\mathsf{d}} M r^2 \mathcal{F}_A^{\mathsf{d}}$, with all other components vanishing. If the perturbation is decomposed as in Eq. (4.5), then $h_v = -\gamma^{\mathsf{d}} M r^2 \mathcal{F}_{\mathsf{m}}^{\mathsf{d}}$ is the associated perturbation variable, and it is easy to show that it is produced by a gauge transformation generated by the vector $\Xi_A = \sum_{\mathsf{m}} \xi_{\mathsf{m}} X_A^{1,\mathsf{m}}$ with $\xi_{\mathsf{m}} = \gamma^{\mathsf{d}} M r^2 \int \mathcal{F}_{\mathsf{m}}^{\mathsf{d}} dv$. This represents a change of angular coordinates described by

$$\delta\theta^A = \gamma^{\mathsf{d}} M \Omega^{AB} \int \mathcal{F}_B^{\mathsf{d}} dv. \qquad (4.9)$$

To better understand this result, let us assume that the tidal field is in a state of rigid rotation about the black hole's rotation axis, so that $\mathcal{E}_0^{\mathsf{q}} = \text{constant}$ and $\mathcal{E}_{1c}^{\mathsf{q}} = \mathcal{E}_{1s}^{\mathsf{q}} = 0$. These assignments imply that $\mathcal{F}_0^{\mathsf{d}} = -2\chi\mathcal{E}_0^{\mathsf{q}} = -\chi(\mathcal{E}_{11} + \mathcal{E}_{22})$, $\mathcal{F}_{1c}^{\mathsf{d}} = \mathcal{F}_{1s}^{\mathsf{d}} = 0$, and Eq. (4.9) reduces to $\delta\theta = 0$ and $\delta\phi = \omega_{\text{gauge}} v$, where $\omega_{\text{gauge}} \coloneqq \gamma^{\mathsf{d}}\chi M(\mathcal{E}_{11} + \mathcal{E}_{22})$. We see that in this case, the gauge constant γ^{d} is associated with a uniform rotation around the black hole's rotation axis. More generally, Eq. (4.9) describes a small precession of the spatial coordinates around the rotation axis.

V. METRIC IN THE COROTATING FRAME

The background metric of Eqs. (3.3) or (3.5) is presented in light-cone coordinates (v, r, θ, ϕ) whose geometrical meaning was described in Sec. III. In these coordinates the event horizon is situated at $r = 2M$, and $k_\alpha \coloneqq \partial_\alpha(r - 2M)$ is normal (and tangent) to it. The components $k^\alpha = (1, 0, 0, \omega_{\mathrm{H}})$, with

$$\omega_{\mathrm{H}} \coloneqq \frac{\chi}{4M}, \qquad (5.1)$$

indicate that the horizon's null generators move with an angular velocity $d\phi/dv = \omega_{\mathrm{H}}$. We have that $\phi(v) = \phi(0) + \omega_{\mathrm{H}} v$ along the generators.

In Sec. VII we shall describe the intrinsic geometry of the event horizon in the perturbed spacetime, and to prepare for this discussion we implement a coordinate transformation that keeps the (background) generators at a constant coordinate position. This can be accomplished with $\theta = \vartheta$, $\phi = \varphi + \omega_{\mathrm{H}} v$, in which $\vartheta^A = (\vartheta, \varphi)$ are the new angular coordinates. The transformation can be expressed formally as

$$\theta^A = \vartheta^A + \omega_{\mathrm{H}}^A v, \qquad \omega_{\mathrm{H}}^A \coloneqq \omega_{\mathrm{H}} \phi^A, \qquad (5.2)$$

where $\phi^A = (0, 1)$ is the vector introduced previously to describe rotations around the polar axis. In Sec. VI the transformation will be refined to ensure that the null generators of the event horizon are at a fixed coordinate position also in the perturbed spacetime.

It is easy to show that to first order in ω_{H}^A, the metric becomes

$$g_{vv}^{\text{corot}} = g_{vv} + 2g_{vA}\omega_{\mathrm{H}}^A,$$
$$g_{vA}^{\text{corot}} = g_{vA} + g_{AB}\omega_{\mathrm{H}}^B, \qquad g_{AB}^{\text{corot}} = g_{AB} \qquad (5.3)$$

under a transformation to the corotating frame. Substitution of Eq. (4.6) implies that g_{vv}^{corot} acquires new terms proportional to $\mathcal{E}_A^{\mathsf{q}}\omega_{\mathrm{H}}^A$ and $\mathcal{B}_A^{\mathsf{q}}\omega_{\mathrm{H}}^A$, while g_{vA}^{corot} acquires terms proportional to $\Omega_{AB}\omega_{\rm H}^B$, $\mathcal{E}^{\sf q}_{AB}\omega_{\rm H}^B$, and $\mathcal{B}^{\sf q}_{AB}\omega_{\rm H}^B$. These terms are not independent of the ones that already appear in the metric of Eq. (4.6). To begin, it is easy to see that the definitions of $\omega_{\rm H}^A$ and $\chi^{\sf d}_A$—refer to Eq. (2.15)—imply that

$$\Omega_{AB}\omega_{\rm H}^B = -\frac{1}{4M}\chi^{\sf d}_A. \tag{5.4}$$

Similar identities are obtained by recognizing that a quantity such as $\mathcal{E}^{\sf q}_A\omega_{\rm H}^A = \omega_{\rm H}\mathcal{E}^{\sf q}_\phi$ can be decomposed in scalar harmonics $Y^{\ell{\sf m}}$, while a quantity such as $\mathcal{B}^{\sf q}_{AB}\omega_{\rm H}^B = \omega_{\rm H}\mathcal{B}^{\sf q}_{A\phi}$ can be decomposed in odd-parity vector harmonics $X_A^{\ell{\sf m}}$. In this way we obtain the relations

$$\mathcal{E}^{\sf q}_A\omega_{\rm H}^A = \frac{1}{8M}\chi\partial_\phi\mathcal{E}^{\sf q}, \tag{5.5a}$$

$$\mathcal{B}^{\sf q}_A\omega_{\rm H}^A = -\frac{3}{20M}\mathcal{K}^{\sf d} + \frac{1}{4M}\mathcal{K}^{\sf o}, \tag{5.5b}$$

$$\mathcal{E}^{\sf q}_{AB}\omega_{\rm H}^B = \frac{3}{10M}\mathcal{F}^{\sf d}_A + \frac{1}{6M}\chi\partial_\phi\mathcal{E}^{\sf q}_A - \frac{1}{4M}\mathcal{F}^{\sf o}_A, \tag{5.5c}$$

$$\mathcal{B}^{\sf q}_{AB}\omega_{\rm H}^B = -\frac{3}{10M}\mathcal{K}^{\sf d}_A + \frac{1}{6M}\chi\partial_\phi\mathcal{B}^{\sf q}_A + \frac{1}{4M}\mathcal{K}^{\sf o}_A. \tag{5.5d}$$

Making the substitutions, we find that the metric in the corotating frame takes the same form as in Eq. (4.6), except that the rotational term in $g^{\rm corot}_{vA}$ is now given by

$$\left(\frac{2M^2}{r} - \frac{r^2}{4M}\right)\chi^{\sf d}_A, \tag{5.6}$$

and that a number of radial functions are now shifted relative to their original expression. The transformation produces

$$\hat{e}^{\sf q}_1 \to \hat{e}^{\sf q}_1 + \frac{1}{6}\frac{r}{M}e^{\sf q}_4, \qquad \hat{e}^{\sf q}_4 \to \hat{e}^{\sf q}_4 + \frac{1}{18}\frac{r}{M}e^{\sf q}_7, \qquad \hat{b}^{\sf q}_4 \to \hat{b}^{\sf q}_4 + \frac{1}{18}\frac{r}{M}b^{\sf q}_7, \tag{5.7a}$$

$$k^{\sf d}_1 \to k^{\sf d}_1 - \frac{1}{5}\frac{r}{M}b^{\sf q}_4, \qquad k^{\sf d}_4 \to k^{\sf d}_4 - \frac{1}{10}\frac{r}{M}b^{\sf q}_7, \qquad f^{\sf d}_4 \to f^{\sf d}_4 + \frac{1}{10}\frac{r}{M}e^{\sf q}_7, \tag{5.7b}$$

$$k^{\sf o}_1 \to k^{\sf o}_1 - \frac{1}{3}\frac{r}{M}b^{\sf q}_4, \qquad k^{\sf o}_4 \to k^{\sf o}_4 + \frac{1}{12}\frac{r}{M}b^{\sf q}_7, \qquad f^{\sf o}_4 \to f^{\sf o}_4 + \frac{1}{12}\frac{r}{M}e^{\sf q}_7; \tag{5.7c}$$

the remaining radial functions stay unchanged. Another change in the metric is that the tidal potentials must now be expressed in terms of ϑ^A instead of θ^A; this simply involves substituting ϑ for θ, and $\varphi + \omega_{\rm H}v$ for ϕ. The differentiation with respect to ϕ becomes a differentiation with respect to φ, but since this can still be related to a Lie derivative in the direction of ϕ^A [refer to Eq. (4.7)], there is no pressing need to alter the notation.

VI. HORIZON LOCKING

In Sec. V we introduced a system of angular coordinates ϑ^A such that in the background spacetime of Eq. (3.5)—with the rotational term modified to the expression of Eq. (5.6)—the null generators of the event horizon are described by the parametric equations $v = v$, $r = 2M$, $\vartheta = \text{constant}$, and $\varphi = \text{constant}$. In this section we show that we can choose the six undetermined constants $\gamma^{\sf d}$, $\gamma^{\sf q}$, $\gamma^{\sf o}$, $c^{\sf d}$, $c^{\sf q}$, and $c^{\sf o}$ of Sec. IV in such a way as to leave the parametric description of the horizon unchanged in the perturbed spacetime. In other words, our selection of constants keeps the horizon at $r = 2M$, keeps ϑ^A constant on the null generators, and retains v as a parameter on each generator.

To show that it is indeed possible to keep the coordinate description of the horizon unchanged in the perturbed spacetime, and to obtain the necessary conditions to achieve this, we adapt the discussion of Sec. 3.2 of Ref. [59] to a rotating black hole. The original discussion involved a nonrotating black hole, and it relied on a decomposition of the perturbation in spherical harmonics. Here we allow the background spacetime to describe a slowly rotating black hole, and we abandon the decomposition in spherical harmonics, which is not required for this discussion.

We assume that the background metric $g^{\rm back}_{\alpha\beta}$ of the slowly rotating black hole is expressed in coordinates (v, r, ϑ^A) such that the parametric description of the horizon is given by $v = v$, $r = 2M$, $\vartheta^A = \alpha^A$, in which α^A are constant generator labels. (Notice that our notation here differs from that of Sec. IV. Here the background metric includes the rotation of the black hole. In Sec. IV the rotation was excluded from the background metric.) In general the description of the horizon will be shifted to

$$v = v, \quad r = 2M[1 + B(v, \alpha^A)], \quad \vartheta^A = \alpha^A + \Xi^A(v, \alpha^A) \tag{6.1}$$

in the perturbed spacetime with metric $g_{\alpha\beta} = g^{\rm back}_{\alpha\beta} + p_{\alpha\beta}$. Here $2MB$ and Ξ^A are the components of a Lagrangian displacement vector that takes a horizon point identified by (v, α^A) in the background spacetime to a point also identified by (v, α^A) in the perturbed spacetime. We keep α^A as constant generator labels in the perturbed spacetime, and we keep v as parameter on the generators.

The vector $k^\alpha = \partial x^\alpha/\partial v$ is tangent to the congruence of null generators, and the vectors $e^\alpha_A = \partial x^\alpha/\partial\alpha^A$, orthogonal to k^α, point from one generator to another generator. The condition that k^α be null in the perturbed spacetime gives rise to the differential equation

$$4M\frac{\partial B}{\partial v} - B + p_{vv}(v, 2M, \vartheta^A) = 0 \tag{6.2}$$

for the function $B(v, \alpha^A)$. On the other hand, the condition $k_\alpha e^\alpha_A = 0$ gives rise to

$$(2M)^2\Omega_{AB}\frac{\partial \Xi^B}{\partial v} + 2M\frac{\partial B}{\partial \alpha^A} + 12M^2 B\Omega_{AB}\omega^B_{\rm H} + p_{vA}(v, 2M, \vartheta^A) = 0, \tag{6.3}$$

a differential equation for Ξ^A.

Suppose that we wish to impose $B = 0$, so that the coordinate position of the horizon remains at $r = 2M$ in the perturbed spacetime. Equation (6.2) reveals that p_{vv} must then vanish at $r = 2M$. Conversely, setting $p_{vv} = 0$ implies that $B = B(0, \alpha^A)e^{v/4M}$, which in general is incompatible with the requirement that B remain small to describe a perturbed horizon. The only acceptable solution is $B = 0$, and we conclude that a necessary and sufficient condition for a horizon at $r = 2M$ is the first horizon-locking condition,

$$p_{vv}(v, 2M, \vartheta^A) = 0. \tag{6.4}$$

With $B = 0$ we next find from Eq. (6.3) that $\Xi^A = 0$ implies that p_{vA} must vanish at $r = 2M$. Conversely, setting $p_{vA} = 0$ leads to $\Xi^A = \Xi^A(\alpha^B)$, with the right-hand side independent of v. Such a shift in α^A represents an uninteresting constant relabeling of the null generators, and there is no loss of generality if we simply set $\Xi^A = 0$. A necessary and sufficient condition for the preservation of the generator labels is therefore the second horizon-locking condition,

$$p_{vA}(v, 2M, \vartheta^A) = 0. \tag{6.5}$$

With these conditions we have that the parametric description of the horizon is given by $v = v$, $r = 2M$, and $\vartheta^A = \alpha^A$ also in the perturbed spacetime.

A third horizon-locking condition arises as a consequence of Eq. (6.5) and the vacuum Einstein field equations examined at $r = 2M$. As was shown in Secs. VI A and VI C of Poisson and Vlasov [43], this condition can be formulated in terms of the $h_v^{\ell\mathsf{m}}$ and $h_2^{\ell\mathsf{m}}$ quantities introduced in Eq. (4.5). We have

$$h_2^{\ell\mathsf{m}}(v, r = 2M) = -\frac{8M^2}{(\ell-1)(\ell+2)}\frac{\partial h_v^{\ell\mathsf{m}}}{\partial r}\bigg|_{r=2M}. \tag{6.6}$$

The third horizon-locking condition implicates only the odd-parity sector of the perturbation.

With the radial functions listed in Table III and shifted to the corotating frame by Eq. (5.7), we find that the horizon-locking conditions of Eqs. (6.4), (6.5), and (6.6) can all be enforced if we make the assignments

$$c^{\mathsf{d}} = -\frac{8}{5}, \qquad c^{\mathsf{q}} = -\frac{1}{3}, \qquad c^{\mathsf{o}} = -\frac{8}{3} \tag{6.7}$$

and

$$\gamma^{\mathsf{d}} = -1, \qquad \gamma^{\mathsf{q}} = 2, \qquad \gamma^{\mathsf{o}} = -\frac{4}{3} \tag{6.8}$$

for the undetermined constants of the perturbed metric. We recall that the freedom to choose these six constants represents a refinement of the light-cone gauge that preserves the geometrical meaning of the original coordinates (v, r, θ, ϕ). With the choices of Eqs. (6.7) and (6.8), we have established that a transformation to the corotating frame $(v, r, \vartheta, \varphi)$ produces a metric for which the parametric description of the event horizon is also preserved: it is given by $v = v$, $r = 2M$, $\vartheta =$ constant, and $\varphi =$ constant in both the unperturbed and perturbed spacetimes.

VII. HORIZON GEOMETRY

As discussed at length in Ref. [59], the geometry of the perturbed horizon can be conveniently described by adopting (v, ϑ^A) as intrinsic coordinates. These, we recall, are intimately related to the behavior of the horizon's generators, and as such they provide a preferred coordinate system on the horizon. The line element is given by

$$ds^2 = \gamma_{AB}d\vartheta^A d\vartheta^B, \tag{7.1}$$

where $\gamma_{AB} := g_{\alpha\beta}e^\alpha_A e^\alpha_B = g_{AB}(v, 2M, \vartheta^A)$ is the horizon's intrinsic metric. A considerable virtue of this coordinate system is that the metric is not merely degenerate on the null hypersurface, it is explicitly two-dimensional.

A simple computation involving the corotating metric of Sec. V and the assignments of Eqs. (6.7) and (6.8) reveals that the horizon's intrinsic metric is given by

$$\gamma_{AB} = (2M)^2\Omega_{AB} - \frac{8}{3}M^4\left[\left(1 + \frac{2}{3}\chi\partial_\varphi\right)\times(\mathcal{E}^{\mathsf{q}}_{AB} + \mathcal{B}^{\mathsf{q}}_{AB}) - \mathcal{F}^{\mathsf{o}}_{AB} - \mathcal{K}^{\mathsf{o}}_{AB}\right]. \tag{7.2}$$

The metric involves some of the irreducible potentials introduced in Sec. II, now expressed in terms of ϑ^A through the combination $\vartheta^A + \omega^A_{\rm H}v$, which describes the transformation to the corotating frame. The metric implicates the quadrupolar tidal potentials $\mathcal{E}^{\mathsf{q}}_{AB}$ and $\mathcal{B}^{\mathsf{q}}_{AB}$, and we observe that the rotation induces an azimuthal shift that can be expressed as

$$\left(1+\frac{2}{3}\chi\partial_\varphi\right)\mathcal{E}^{\mathsf{q}}_{AB}(\vartheta,\varphi+\omega_{\rm H}v) = \mathcal{E}^{\mathsf{q}}_{AB}\left(\vartheta,\varphi+\omega_{\rm H}v+\frac{2}{3}\chi\right)+O(\chi^2), \tag{7.3}$$

with a similar equation holding for the terms involving $\mathcal{B}^{\mathsf{q}}_{AB}$. The metric also implicates the octupolar potentials $\mathcal{F}^{\mathsf{o}}_{AB}$ and $\mathcal{K}^{\mathsf{o}}_{AB}$ that result from the coupling of the spin pseudovector χ_a with the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$.

The metric of Eq. (7.2) contains information about the horizon's intrinsic geometry, and it also contains information about the generator-anchored coordinate system that was placed on the event horizon. Purely geometrical information can be extracted by computing the Ricci scalar $\mathcal{R}$ associated with the two-dimensional metric. A simple calculation gives

$$(2M)^2\mathcal{R} = 2-8M^2\left(1+\frac{2}{3}\chi\partial_\varphi\right)\mathcal{E}^{\mathsf{q}}+\frac{40}{3}M^2\mathcal{K}^{\mathsf{o}}, \tag{7.4}$$

and we observe that the dependence on $\mathcal{B}_{ab}$ and $\mathcal{F}_{abc}$ has disappeared—the Ricci scalar involves irreducible potentials of even parity only. In view of Eq. (7.3) and the definitions introduced in Sec. II, an alternative expression for $\mathcal{R}$ is

$$(2M)^2\mathcal{R} = 2-8M^2\mathcal{E}_{ab}\Omega^a_\dagger\Omega^b_\dagger+\frac{40}{3}M^2\mathcal{B}_{\langle ab}\chi_{c\rangle}\Omega^a\Omega^b\Omega^c, \tag{7.5}$$

where

$$\Omega^a := [\sin\vartheta\cos(\varphi+\omega_{\rm H}v), \sin\vartheta\sin(\varphi+\omega_{\rm H}v), \cos\vartheta] \tag{7.6}$$

specifies the direction to a given point on the horizon, and

$$\Omega^a_\dagger := \left[\sin\vartheta\cos\left(\varphi+\omega_{\rm H}v+\frac{2}{3}\chi\right), \sin\vartheta\sin\left(\varphi+\omega_{\rm H}v+\frac{2}{3}\chi\right), \cos\vartheta\right] \tag{7.7}$$

is its shifted version. These results were already displayed and discussed in Sec. I—refer back to Eq. (1.1) and the following discussion.

VIII. PERTURBED SPACETIME IN HARMONIC GAUGE

In the remaining portions of this paper we shall determine the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ of the perturbed black-hole spacetime by immersing the black hole in a post-Newtonian tidal environment, thereby generalizing the work of Taylor and Poisson [55] to include spin. The first step in this exercise is to cast the perturbed black-hole metric in a harmonic coordinate system that will facilitate its matching to the post-Newtonian metric; this shall be our task in this section. We follow and generalize the method outlined in Sec. III of Taylor and Poisson, which consists of (i) transforming the metric perturbation from the light-cone gauge of Sec. IV to a new harmonic gauge, keeping the background coordinates fixed to the (v,r,θ,ϕ) system, and (ii) transforming the background coordinates to a system of Cartesian harmonic coordinates. We adopt the same perspective as in Sec. IV, and choose the background spacetime to have the Schwarzschild metric of Eq. (4.2); the rotational and tidal terms in the metric of the perturbed spacetime are then viewed as a first-order perturbation, and the coupling between them as a second-order perturbation. As before we exclude terms quadratic in χ and quadratic in the tidal moments from the second-order perturbation.

To define the harmonic gauge in the background coordinates (v,r,θ,ϕ), we introduce a collection of four scalar fields $X^{(\mu)}$,

$$X^{(0)} := v-r-2M\ln(r/2M-1), \tag{8.1a}$$

$$X^{(1)} := (r-M)\sin\theta\cos\phi, \tag{8.1b}$$

$$X^{(2)} := (r-M)\sin\theta\sin\phi, \tag{8.1c}$$

$$X^{(3)} := (r-M)\cos\theta, \tag{8.1d}$$

and demand that these be solutions to the scalar wave equation

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta X^{(\mu)} = \frac{1}{\sqrt{-g}}\partial_\alpha(g^{\alpha\beta}\partial_\beta X^{(\mu)}) = 0 \tag{8.2}$$

in the perturbed spacetime with metric $g_{\alpha\beta}$ and covariant-derivative operator ∇_α. A simple computation confirms that each wave equation is satisfied in the background spacetime with metric $g^{\rm back}_{\alpha\beta}$, and $X^{(\mu)}$ are indeed recognized as the Cartesian harmonic coordinates of the Schwarzschild metric. Setting $g_{\alpha\beta} = g^{\rm back}_{\alpha\beta}+p_{\alpha\beta}$, the inverse metric is $g^{\alpha\beta} = g^{\alpha\beta}_{\rm back}-p^{\alpha\beta}+p^{\alpha\mu}p^\beta{}_\mu$ to second order in the perturbation, and the metric determinant is $g = g_{\rm back}(1+p+\frac{1}{2}p^2-\frac{1}{2}p_{\mu\nu}p^{\mu\nu})$; indices on $p_{\alpha\beta}$ are raised with the background metric, and $p := g^{\alpha\beta}_{\rm back}p_{\alpha\beta}$. Making the substitutions in the wave equation, we find that it remains valid to second order in the perturbed spacetime when

$$\nabla^{\rm back}_\alpha(\psi^{\alpha\beta}\partial_\beta X^{(\mu)}) = 0, \tag{8.3}$$

where $\nabla^{\rm back}_\alpha$ is the covariant-derivative operator compatible with the background metric, and

$$\psi^{\alpha\beta} := p^{\alpha\beta} - \frac{1}{2} p g^{\alpha\beta}_{\text{back}} - p^{\alpha\mu} p^{\beta}{}_{\mu} + \frac{1}{2} p p^{\alpha\beta} - \frac{1}{8}(p^2 - 2 p_{\mu\nu} p^{\mu\nu}) g^{\alpha\beta}_{\text{back}}. \tag{8.4}$$

A perturbation $p^{\text{harm}}_{\alpha\beta}$ that satisfies these conditions shall be said to be in the harmonic gauge, irrespective of the choice of background coordinates.

The question that faces us now is the following: Given a perturbation $p^{\text{LC}}_{\alpha\beta}$ presented in the light-cone gauge, how do we obtain a physically equivalent perturbation $p^{\text{harm}}_{\alpha\beta}$ that satisfies the harmonic conditions of Eq. (8.3)? The answer, of course, is to perform a gauge transformation. Given a perturbation $p^{\text{old}}_{\alpha\beta}$ presented in an old gauge, a second-order transformation to a new gauge is described by (see, for example, Ref. [60])

$$p^{\text{new}}_{\alpha\beta} = p^{\text{old}}_{\alpha\beta} - \mathcal{L}_\xi g^{\text{back}}_{\alpha\beta} - \mathcal{L}_\xi p^{\text{old}}_{\alpha\beta} + \frac{1}{2}\mathcal{L}_\xi \mathcal{L}_\xi g^{\text{back}}_{\alpha\beta}, \tag{8.5}$$

in which ξ^α is the generating vector field, and $\mathcal{L}_\xi$ indicates Lie differentiation in the direction of ξ^α. In our current application, $p^{\text{old}}_{\alpha\beta}$ refers to the perturbation in the light-cone gauge, $p^{\text{new}}_{\alpha\beta}$ is the perturbation in harmonic gauge, and inserting Eq. (8.5) within Eq. (8.3) gives rise to a nonlinear, second-order differential equation for the vector ξ^α. Once a solution has been identified, insertion back into Eq. (8.5) produces the desired $p^{\text{harm}}_{\alpha\beta}$.

The complete gauge transformation can be built in stages, first by finding the first-order gauge transformations that separately cast the rotational and tidal perturbations in the harmonic gauge, and next finding the second-order transformation that accounts for the couplings between rotational and tidal terms. It is easy to show that the vector $\xi_\alpha[\chi]$ that transforms the rotational perturbation from the light-cone gauge to the harmonic gauge has

$$\xi_A[\chi] = -\frac{M^2 r}{r - M}\chi^{\mathsf{d}}_A \tag{8.6}$$

as nonvanishing components. The vectors $\xi_\alpha[\mathcal{E}]$ and $\xi_\alpha[\mathcal{B}]$ that transform the tidal perturbations from the light-cone gauge to the harmonic gauge were identified by Taylor and Poisson. They have

$$\xi_v[\mathcal{E}] = -\frac{1}{3} r^3 f \mathcal{E}^{\mathsf{q}}, \qquad \xi_r[\mathcal{E}] = \frac{1}{3} r^3 \mathcal{E}^{\mathsf{q}}, \qquad \xi_A[\mathcal{E}] = -\frac{r^5 f^2}{3(r-M)}\mathcal{E}^{\mathsf{q}}_A \tag{8.7}$$

and

$$\xi_v[\mathcal{B}] = 0, \qquad \xi_r[\mathcal{B}] = 0, \qquad \xi_A[\mathcal{B}] = \frac{1}{3} r^2 (r^2 - 6M^2)\mathcal{B}^{\mathsf{q}}_A. \tag{8.8}$$

The complete, second-order transformation can be expressed as

$$\xi_\alpha = \xi_\alpha[\chi] + \xi_\alpha[\mathcal{E}] + \xi_\alpha[\mathcal{B}] + \xi_\alpha[\chi\mathcal{E}] + \xi_\alpha[\chi\mathcal{B}], \tag{8.9}$$

where $\xi_\alpha[\chi\mathcal{E}]$ accounts for the coupling between the rotational and even-parity tidal terms, while $\xi_\alpha[\chi\mathcal{B}]$ takes care of the coupling between the rotational and odd-parity tidal terms. Each vector can be obtained by inserting Eq. (8.9) within Eq. (8.5), and that within Eq. (8.3), and integrating the coupled differential equations. Because the solution to the first-order problem has been previously identified, the differential equations for $\xi_\alpha[\chi\mathcal{E}]$ and $\xi_\alpha[\chi\mathcal{B}]$ are linear, and contain source terms generated by the known $p^{\text{LC}}_{\alpha\beta}$, $\xi_\alpha[\chi]$, $\xi_\alpha[\mathcal{E}]$, and $\xi_\alpha[\mathcal{B}]$.

The rules implicated in the creation of the metric ansatz of Eq. (4.6) can be employed here also, and they imply that the second-order gauge vectors can be decomposed as

$$\xi_v[\chi\mathcal{E}] = r^3 p^{\mathsf{q}}_v \chi \partial_\phi \mathcal{E}^{\mathsf{q}}, \tag{8.10a}$$

$$\xi_r[\chi\mathcal{E}] = r^3 p^{\mathsf{q}}_r \chi \partial_\phi \mathcal{E}^{\mathsf{q}}, \tag{8.10b}$$

$$\xi_A[\chi\mathcal{E}] = r^4 p^{\mathsf{q}} \chi \partial_\phi \mathcal{E}^{\mathsf{q}} + r^4 p^{\mathsf{d}} \mathcal{F}^{\mathsf{d}}_A + r^4 p^{\mathsf{o}} \mathcal{F}^{\mathsf{o}}_A, \tag{8.10c}$$

and

$$\xi_v[\chi\mathcal{B}] = r^3 q^{\mathsf{d}}_v \mathcal{K}^{\mathsf{d}} + r^3 q^{\mathsf{o}}_v \mathcal{K}^{\mathsf{o}}, \tag{8.11a}$$

$$\xi_r[\chi\mathcal{B}] = r^3 q^{\mathsf{d}}_r \mathcal{K}^{\mathsf{d}} + r^3 q^{\mathsf{o}}_r \mathcal{K}^{\mathsf{o}}, \tag{8.11b}$$

$$\xi_A[\chi\mathcal{B}] = r^4 q^{\mathsf{d}} \mathcal{K}^{\mathsf{d}}_A + r^4 q^{\mathsf{o}} \mathcal{K}^{\mathsf{o}}_A + r^4 q^{\mathsf{q}} \chi \partial_\phi \mathcal{B}^{\mathsf{q}}_A, \tag{8.11c}$$

where the various coefficients $p^{\mathsf{q}}_v, p^{\mathsf{q}}_r, \ldots, q^{\mathsf{o}}, q^{\mathsf{q}}$ are functions of r that are determined by integrating the differential equations for $\xi_\alpha[\chi\mathcal{E}]$ and $\xi_\alpha[\chi\mathcal{B}]$. The solutions to these equations are linear superpositions of particular solutions to the sourced equations and general solutions to the homogeneous equations. The homogeneous terms come with a number of integration constants, and these represent the freedom to perform an additional transformation that keeps $p^{\text{harm}}_{\alpha\beta}$ within the harmonic gauge; we have used this freedom to simplify the expressions for the gauge vectors and resulting metric perturbation to the largest extent possible. Our solutions are displayed in Table IV. We recall that the starting point of our gauge transformation is the metric of Eq. (4.6), written in terms of the radial functions of Table III, which involve the six arbitrary constants of the light-cone gauge. Accordingly, the functions listed in Table IV feature a dependence on these constants; this dependence can be eliminated by imposing

TABLE IV. Radial functions appearing in the gauge vectors of Eqs. (8.10) and (8.11); $f \coloneqq 1 - 2M/r$.

$p^{\mathsf{d}} = -2(\gamma^{\mathsf{d}}+1)\frac{M^2}{r^2}\ln r - \frac{(r^2-6Mr+M^2)M}{r^2(r-M)}\gamma^{\mathsf{d}} + \frac{(188r^3-232Mr^2+23M^2r+24M^3)M^2}{30r^3(r-M)^2}$
$p^{\mathsf{q}}_v = -\frac{(r+M)M^3}{3r^4}\gamma^{\mathsf{q}} - \frac{(9r^3-4Mr^2-18M^2r+16M^3)M^2}{18r^4(r-M)}$
$p^{\mathsf{q}}_r = -\frac{M^3}{6r^3}\gamma^{\mathsf{q}} + \frac{(6r^2+Mr-8M^2)M^2}{9r^3(r-2M)}$
$p^{\mathsf{q}} = -\frac{M^3}{3r^3}\gamma^{\mathsf{q}} + \frac{(6r^2-11Mr+6M^2)M^3}{9r^3(r-M)^2}$
$p^{\mathsf{o}} = -\frac{M^2}{2r^2}\gamma^{\mathsf{o}} - \frac{(2r^3-2Mr^2-3M^2r+2M^3)M^2}{3r^3(r-M)^2}$
$q^{\mathsf{d}}_v = -\frac{M^4}{r^4}c^{\mathsf{d}} + \frac{(5r-12M)M}{5r^2}$
$q^{\mathsf{d}}_r = \frac{M^4}{r^3(r-2M)}c^{\mathsf{d}} + \frac{M^2}{5r(r-2M)}$
$q^{\mathsf{d}} = \frac{M^4}{2r^3(r-M)}c^{\mathsf{d}} + \frac{(5r^2-25Mr+17M^2)M}{10r^2(r-M)}$
$q^{\mathsf{q}} = \frac{M^2}{r^2}c^{\mathsf{q}} - \frac{(8r^2-Mr-22M^2)M^2}{18r^3(r-M)}$
$q^{\mathsf{o}}_v = \frac{(5r+2M)M^3}{12r^4}c^{\mathsf{o}} + \frac{2M^2}{3r^2}$
$q^{\mathsf{o}}_r = \frac{M^3}{12r^3}c^{\mathsf{o}} - \frac{(r-M)M^2}{3r^2(r-2M)}$
$q^{\mathsf{o}} = \frac{M^3}{4r^3}c^{\mathsf{o}}$

the choices made in Eqs. (6.7) and (6.8), which anchor the light-cone gauge to the null generators of the perturbed horizon. Another point worthy of notice is that the functions p^{q}_r, q^{d}_r, and q^{o}_r diverge at $r = 2M$: the harmonic gauge is singular at the event horizon, in spite of the fact that the background coordinates are well behaved there.

It is now a simple matter to insert the gauge vector of Eq. (8.9) within Eq. (8.5) to obtain the metric perturbation in the harmonic gauge. The metric of the perturbed spacetime becomes

$$g_{vv} = -f - r^2 e^{\mathsf{q}}_{vv}\mathcal{E}^{\mathsf{q}} + r^2\hat{e}^{\mathsf{q}}_{vv}\chi\partial_\phi\mathcal{E}^{\mathsf{q}} + r^2 k^{\mathsf{d}}_{vv}\mathcal{K}^{\mathsf{d}} + r^2 k^{\mathsf{o}}_{vv}\mathcal{K}^{\mathsf{o}}, \tag{8.12a}$$

$$g_{vr} = 1 + r^2 e^{\mathsf{q}}_{vr}\mathcal{E}^{\mathsf{q}} + r^2\hat{e}^{\mathsf{q}}_{vr}\chi\partial_\phi\mathcal{E}^{\mathsf{q}} + r^2 k^{\mathsf{d}}_{vr}\mathcal{K}^{\mathsf{d}} + r^2 k^{\mathsf{o}}_{vr}\mathcal{K}^{\mathsf{o}}, \tag{8.12b}$$

$$g_{rr} = -2r^2 e^{\mathsf{q}}_{rr}\mathcal{E}^{\mathsf{q}} + r^2\hat{e}^{\mathsf{q}}_{rr}\chi\partial_\phi\mathcal{E}^{\mathsf{q}} + r^2 k^{\mathsf{d}}_{rr}\mathcal{K}^{\mathsf{d}} + r^2 k^{\mathsf{o}}_{rr}\mathcal{K}^{\mathsf{o}}, \tag{8.12c}$$

$$g_{vA} = \frac{2M^2}{r}\chi^{\mathsf{d}}_A + \frac{2}{3}r^3 b^{\mathsf{q}}_v\mathcal{B}^{\mathsf{q}}_A + r^3\hat{e}^{\mathsf{q}}_v\chi\partial_\phi\mathcal{E}^{\mathsf{q}}_A + r^3 f^{\mathsf{d}}_v\mathcal{F}^{\mathsf{d}}_A + r^3 f^{\mathsf{o}}_v\mathcal{F}^{\mathsf{o}}_A + r^3\hat{b}^{\mathsf{q}}_v\chi\partial_\phi\mathcal{B}^{\mathsf{q}}_A, \tag{8.12d}$$

$$g_{rA} = -\frac{(2r-M)M^2}{(r-M)^2}\chi^{\mathsf{d}}_A - r^3 e^{\mathsf{q}}_r\mathcal{E}^{\mathsf{q}}_A - \frac{2}{3}r^3 b^{\mathsf{q}}_r\mathcal{B}^{\mathsf{q}}_A + r^3\hat{e}^{\mathsf{q}}_r\chi\partial_\phi\mathcal{E}^{\mathsf{q}}_A + r^3 f^{\mathsf{d}}_r\mathcal{F}^{\mathsf{d}}_A + r^3 f^{\mathsf{o}}_r\mathcal{F}^{\mathsf{o}}_A + r^3\hat{b}^{\mathsf{q}}_r\chi\partial_\phi\mathcal{B}^{\mathsf{q}}_A + r^3 k^{\mathsf{d}}_r\mathcal{K}^{\mathsf{d}}_A + r^3 k^{\mathsf{o}}_r\mathcal{K}^{\mathsf{o}}_A, \tag{8.12e}$$

$$g_{AB} = r^2\Omega_{AB} - r^4\bar{e}^{\mathsf{q}}\Omega_{AB}\mathcal{E}^{\mathsf{q}} - r^4 e^{\mathsf{q}}\mathcal{E}^{\mathsf{q}}_{AB} + r^4\hat{\bar{e}}^{\mathsf{q}}\Omega_{AB}\chi\partial_\phi\mathcal{E}^{\mathsf{q}} + r^4\hat{e}^{\mathsf{q}}\chi\partial_\phi\mathcal{E}^{\mathsf{q}}_{AB} + r^4 f^{\mathsf{o}}\mathcal{F}^{\mathsf{o}}_{AB} + r^4\hat{b}^{\mathsf{q}}\chi\partial_\phi\mathcal{B}^{\mathsf{q}}_{AB} + r^4\bar{k}^{\mathsf{d}}\Omega_{AB}\mathcal{K}^{\mathsf{d}} + r^4\bar{k}^{\mathsf{o}}\Omega_{AB}\mathcal{K}^{\mathsf{o}} + r^4 k^{\mathsf{o}}\mathcal{K}^{\mathsf{o}}_{AB}, \tag{8.12f}$$

and the radial functions that appear in the metric are listed in Table V. We notice the large number of functions that diverge at $r = 2M$, and observe that while the gauge vector featured the six arbitrary constants of the light-cone gauge, the metric depends only on the two constants c^{d} and γ^{d} that appear in the dipole terms. We also note that the $\frac{2}{3}r^3 b^{\mathsf{q}}_v\mathcal{B}^{\mathsf{q}}_A$ term in g_{vA} corrects a typo contained in Eq. (3.33) of Taylor and Poisson; the correct numerical coefficient is indeed $\frac{2}{3}$ instead of $\frac{1}{3}$.

Now that the perturbed metric has been recast in a harmonic gauge in the background coordinates (v, r, θ, ϕ), the remaining task is to perform a transformation from these coordinates to Cartesian harmonic coordinates $t = X^{(0)}, x = X^{(1)}, y = X^{(2)}, z = X^{(3)}$. We carry this out in two stages. First, we transform the metric from the (v, r, θ, ϕ) coordinates to $(t, \bar{r}, \theta, \phi)$ coordinates, with $v = t + r + 2M\ln(r/2M - 1)$ and $r = \bar{r} + M$. This produces

$$g_{tt} = g_{vv}, \qquad g_{t\bar{r}} = \frac{1}{f}g_{vv} + g_{vr}, \qquad g_{\bar{r}\bar{r}} = \frac{1}{f^2}g_{vv} + \frac{2}{f}g_{vr} + g_{rr} \tag{8.13}$$

and

$$g_{tA} = g_{vA}, \quad g_{\bar{r}A} = \frac{1}{f}g_{vA} + g_{rA}, \quad g_{AB} = g_{AB}. \tag{8.14}$$

In the second stage we introduce Cartesian coordinates x^a that are related in the usual way to the spherical polar coordinates $(\bar{r}, \theta, \phi)$. We express the relationship as $x^a = \bar{r}\Omega^a(\theta^A)$, in which $\Omega^a \coloneqq [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$. The transformation gives

$$g_{tt} = g_{tt}, \qquad g_{ta} = g_{t\bar{r}}\Omega_a + \frac{1}{\bar{r}}g_{tA}\Omega^A_a, \qquad g_{ab} = g_{\bar{r}\bar{r}}\Omega_a\Omega_b + \frac{1}{\bar{r}}g_{\bar{r}A}(\Omega_a\Omega^A_b + \Omega^A_a\Omega_b) + \frac{1}{\bar{r}^2}g_{AB}\Omega^A_a\Omega^B_b, \tag{8.15}$$

TABLE V. Radial functions appearing in the harmonic-gauge metric of Eq. (8.12); $f := 1 - 2M/r$.

$$e^{\mathsf{q}}_{vv} = f^2$$
$$e^{\mathsf{q}}_{vr} = f$$
$$e^{\mathsf{q}}_{rr} = 1$$
$$e^{\mathsf{q}}_{r} = -\frac{(3r^2-6Mr+4M^2)M}{3r(r-M)^2}$$
$$\bar{e}^{\mathsf{q}} = \frac{rf^2}{r-M}$$
$$e^{\mathsf{q}} = \frac{(3r^2-6Mr+2M^2)M}{3r^2(r-M)}$$
$$b^{\mathsf{q}}_{v} = f$$
$$b^{\mathsf{q}}_{r} = 1$$
$$\hat{e}^{\mathsf{q}}_{vv} = \frac{2(2r^2-4Mr+3M^2)M^3}{3r^4(r-M)}$$
$$\hat{e}^{\mathsf{q}}_{vr} = -\frac{(r^2-3Mr+3M^2)(3r^2-4Mr+2M^2)M^2}{3r^3(r-M)^2(r-2M)}$$
$$\hat{e}^{\mathsf{q}}_{rr} = \frac{(r^6-8Mr^5+36M^2r^4-82M^3r^3+96M^4r^2-56M^5r+12M^6)M^2}{3r^2(r-M)^4(r-2M)^2}$$
$$\hat{e}^{\mathsf{q}}_{v} = \frac{2(7r^2-15Mr+6M^2)M^3}{9r^4(r-M)}$$
$$\hat{e}^{\mathsf{q}}_{r} = -\frac{(15r^5-43Mr^4+15M^2r^3+48M^3r^2-40M^4r+12M^5)M^2}{9r^3(r-M)^3(r-2M)}$$
$$\hat{\bar{e}}^{\mathsf{q}} = \frac{(5r^3-2Mr^2-10M^2r+12M^3)}{6r^3(r-M)^2}$$
$$\hat{e}^{\mathsf{q}} = -\frac{5(3r^2-6Mr+4M^2)}{18r^2(r-M)^2}$$
$$\hat{b}^{\mathsf{q}}_{v} = \frac{2(r^2+2M^2)M^3}{9r^4(r-M)}$$
$$\hat{b}^{\mathsf{q}}_{r} = \frac{2(3r^2-Mr+M^2)M^2}{9r^3(r-M)}$$
$$\hat{b}^{\mathsf{q}} = \frac{5M^2}{18r^2}$$
$$k^{\mathsf{d}}_{vv} = \frac{M^4}{r^4}c^{\mathsf{d}} + \frac{2(5r^2-12Mr+5M^2)M}{5r^3}$$
$$k^{\mathsf{d}}_{vr} = -\frac{M^4}{r^3(r-2M)}c^{\mathsf{d}} - \frac{2(5r^4-22Mr^3+33M^2r^2-20M^3r+5M^4)M}{5r^2(r-M)^2(r-2M)}$$
$$k^{\mathsf{d}}_{rr} = \frac{2M^4}{r^2(r-2M)^2}c^{\mathsf{d}} + \frac{2(3r^3-12Mr^2+15M^2r-4M^3)M^2}{5r(r-M)^2(r-2M)^2}$$
$$k^{\mathsf{d}}_{r} = -\frac{M^5}{2r^2(r-M)^2(r-2M)}c^{\mathsf{d}} - \frac{(5r^3-18Mr^2+24M^2r-14M^3)M}{10r(r-M)^2(r-2M)}$$
$$\bar{k}^{\mathsf{d}} = \frac{M^4}{r^3(r-M)}c^{\mathsf{d}} - \frac{(5r^2-7Mr+5M^2)M}{5r^2(r-M)}$$
$$f^{\mathsf{d}}_{v} = -\frac{M}{r}\gamma^{\mathsf{d}} - \frac{2(5r^2-8Mr+4M^2)}{5r^3(r-M)}$$
$$f^{\mathsf{d}}_{r} = \frac{(r+M)(r^2-Mr+2M^2)M}{r^2(r-M)^2}\gamma^{\mathsf{d}} + \frac{(10r^3-15Mr^2+23M^2r-16M^3)M^2}{5r^2(r-M)^3}$$
$$k^{\mathsf{o}}_{vv} = -\frac{2(3r^2-5Mr-M^2)M^2}{3r^4}$$
$$k^{\mathsf{o}}_{vr} = \frac{2(3r^4-12Mr^3+14M^2r^2-3M^3r-M^4)M^2}{3r^3(r-M)^2(r-2M)}$$
$$k^{\mathsf{o}}_{rr} = -\frac{2(3r^4-12Mr^3+13M^2r^2-2M^4)M^2}{3r^2(r-M)^2(r-2M)^2}$$
$$k^{\mathsf{o}}_{r} = \frac{(r-M)M^2}{r^2(r-2M)}$$
$$\bar{k}^{\mathsf{o}} = -\frac{2(r+M)M^2}{3r^3}$$
$$k^{\mathsf{o}} = \frac{4M^2}{3r^2}$$
$$f^{\mathsf{o}}_{v} = \frac{2(r-3M)M^3}{3r^3(r-M)}$$
$$f^{\mathsf{o}}_{r} = -\frac{(5r^2-11Mr+2M^2)M^3}{3r^2(r-M)^3}$$
$$f^{\mathsf{o}} = -\frac{2M^4}{3r^2(r-M)^2}$$

where $\Omega^A_a := \delta_{ab}\Omega^{AB}\partial_B\Omega^b = \bar{r}\partial\theta^A/\partial x^a$. The manipulations that lead to Eq. (8.15) from Eq. (8.12) are straightforward, but they produce very long expressions for the metric components.

Fortunately, only the components g_{tt} and g_{ta} are required in the matching of the black-hole metric with the post-Newtonian metric to be introduced in the following section. And because the matching will be carried out at the first-and-a-half post-Newtonian (1.5PN) approximation, the expressions for g_{tt} and g_{ta} can be simplified by discarding all terms that occur at higher post-Newtonian orders. The rules of truncation are simple, and to formulate them we recall from Sec. I that $\mathcal{E}_{ab}$ scales as m_2/b^3, $\mathcal{B}_{ab}$ scales as m_2u/b^3, and $\chi \ll 1$ is a scale-free quantity; here m_2 is a mass scale for the external matter, b is a length scale for its distance to the black hole, and u is a velocity scale. These scalings imply that $\mathcal{E}_{ab}$ is of Newtonian order (0PN), while $\mathcal{B}_{ab}$ is of 0.5PN order because of the additional factor of u; the tidal moments, however, also contain higher-order post-Newtonian corrections.

Let us examine g_{tt}. A term proportional to $M/\bar{r}$ in g_{tt} is a Newtonian term, and each additional factor of $M/\bar{r}$ increases the post-Newtonian order by one unit. The leading tidal term proportional to $\bar{r}^2\mathcal{E}_{ab}$ is a Newtonian term, and again, each additional factor of $M/\bar{r}$ gives rise to a higher post-Newtonian correction. Turning next to the couplings between rotational and tidal terms, we see that $\bar{r}^2\chi\partial_\phi\mathcal{E}^{\mathsf{q}}$ would be of Newtonian order, but that the radial function $\hat{e}^{\mathsf{q}}_{vv}$, which leads off at order $(M/\bar{r})^3$, promotes it to a 3PN term that can be neglected. Similarly, $\bar{r}^2\mathcal{K}^{\mathsf{d}}$ and $r^2\mathcal{K}^{\mathsf{o}}$ lead off at 0.5PN order, and here the radial functions promote the dipole term to 1.5PN order (this we keep), while the octupole term is promoted to 2.5PN order (and can be neglected).

For g_{ta}, tradition dictates that the post-Newtonian counter must be increased by a half unit. The leading term is proportional to $\bar{r}^2\mathcal{B}_{ab}$, which is declared to be of 1PN order; each additional factor of $M/\bar{r}$ increases the post-Newtonian order by one unit. The rotational term occurs at 1.5PN order, and the post-Newtonian order of the coupling terms can be determined by adapting the rules spelled out previously for g_{tt}; the conclusion is that the only contribution at 1.5PN order comes from the dipole term proportional to $\bar{r}^2\mathcal{F}^{\mathsf{d}}_a$.

From these considerations we arrive at

$$g_{tt} = -1 + \frac{2M}{\bar{r}} - \frac{2M^2}{\bar{r}^2} - \bar{r}^2\left(1 - \frac{2M}{\bar{r}}\right)\mathcal{E}^{\mathsf{q}} + 2M\bar{r}\mathcal{K}^{\mathsf{d}}, \tag{8.16a}$$

$$g_{ta} = \frac{2M^2}{\bar{r}^2}\chi^{\mathsf{d}}_a + \frac{2}{3}\bar{r}^2\mathcal{B}^{\mathsf{q}}_a - \gamma^{\mathsf{d}}M\bar{r}\mathcal{F}^{\mathsf{d}}_a \tag{8.16b}$$

for the relevant components of the metric truncated through 1.5PN order. Inserting the definitions for the tidal potentials, restoring the factors of G and c that were previously set equal to unity, and replacing the dimensionless spin χ^a with $S^a = (GM^2/c)\chi^a$, we obtain the explicit expression

$$g_{tt} = -1 + \frac{2GM}{c^2\bar{r}} - 2\left(\frac{GM}{c^2\bar{r}}\right)^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2\bar{r}}\right)\mathcal{E}_{ab}x^a x^b + \frac{2}{c^4}\mathcal{B}_{ap}\hat{S}^p x^a, \tag{8.17a}$$

$$g_{ta} = \frac{2G}{c^3}\frac{(\boldsymbol{x}\times\boldsymbol{S})_a}{\bar{r}^3} + \frac{2}{3c^3}\epsilon_{abp}\mathcal{B}^p{}_c x^b x^c - \frac{\gamma^{\mathsf{d}}}{c^3}\epsilon_{abp}\mathcal{E}^p{}_q\hat{S}^q x^b \tag{8.17b}$$

for the black-hole metric, where $\hat{S}^a \coloneqq S^a/M$ and $(\boldsymbol{x}\times\boldsymbol{S})_a \coloneqq \epsilon_{abc}x^b S^c$. This is our final result in this section: We have the metric of a slowly rotating black hole perturbed by a tidal environment, expressed in Cartesian harmonic coordinates, and truncated through 1.5PN order. It should be noted that the expression of the metric in terms of S^a instead of χ^a eliminates some factors of c^{-1} that were present implicitly in Eq. (8.16). As a result, the post-Newtonian order of each term involving S^a has been altered by a half unit, and we now find that 1.5PN terms no longer appear explicitly in the metric of Eq. (8.17). It is interesting to note that at this post-Newtonian order, the metric features a dependence on the light-cone gauge constant γ^{d}; the dipole term in g_{ta} therefore reflects a choice of gauge, and we recall that the meaning of this gauge freedom was explained near the end of Sec. IV—refer back to the discussion surrounding Eq. (4.9).

IX. TIDAL MOMENTS FOR A SLOWLY ROTATING BLACK HOLE

In this section the black hole is imagined to be a member of a post-Newtonian system that could contain one or more companion bodies, or a continuous distribution of matter that is taken to be well separated from the black hole; the immediate vicinity of the black hole is assumed to be empty of matter. The tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ shall be determined by matching the black-hole metric of Eq. (8.17) to the post-Newtonian metric that describes the weak mutual gravity between the black hole and the external matter. The black-hole metric is valid in a neighborhood of the black hole, where $\bar{r} < r_{\max}$ with $r_{\max} \ll b$, and the post-Newtonian metric is valid in a region that excludes the black hole, where $\bar{r} > r_{\min}$ with $r_{\min} \gg GM/c^2$. When $GM/c^2 \ll b$ it is possible to identify an overlap region described by $r_{\min} < \bar{r} < r_{\max}$, in which both descriptions are valid (refer back to Fig. 2); the matching of the black-hole and post-Newtonian metrics can be carried out in this region, and among other important pieces of information, the procedure returns expressions for the tidal moments.

The matching procedure is described in great detail in Taylor and Poisson [55], and their work can be directly imported here with very few alterations to account for the black-hole spin. We provide here a concise discussion that highlights the main differences with the previous work.

The starting point (Sec. IV of Taylor and Poisson) is the post-Newtonian metric

$$g_{tt} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}(\Psi - U^2) + O(c^{-5}), \tag{9.1a}$$

$$g_{ta} = -\frac{4}{c^3}U_a + O(c^{-5}) \tag{9.1b}$$

that describes the mutual gravity between the black hole and the external matter. The metric is presented in a harmonic coordinate system (t, x^a) attached to the system's barycenter, and it involves a Newtonian potential U, a vector potential U_a, and a post-Newtonian potential $\Psi = \psi + \frac{1}{2}\partial_{tt}X$ conveniently decomposed in terms of another potential ψ and a superpotential X. Because the region of space surrounding the black hole is empty of matter, the potentials there satisfy the vacuum field equations $\nabla^2 U = 0$, $\nabla^2 U_a = 0$, $\nabla^2\psi = 0$, and $\nabla^2 X = 2U$, where ∇^2 is the familiar Laplacian operator of three-dimensional flat space; the harmonic coordinate condition implies that the Newtonian and vector potentials are tied by the gauge condition $\partial_t U + \partial_a U^a = 0$.

The potentials must reflect the presence of a black hole near the world line described by $\boldsymbol{z}(t)$, and they must reflect the presence of the external matter. The matching procedure compels us to model the black hole as an object possessing both mass and spin, but no higher-order multipole moments. The potentials are given by

$$U(t,\boldsymbol{x}) = \frac{GM}{R} + U_{\rm ext}(t,\boldsymbol{x}), \tag{9.2a}$$

$$U^a(t,\boldsymbol{x}) = -\frac{G(\boldsymbol{R}\times\boldsymbol{S})^a}{2R^3} + \frac{GMu^a}{R} + U^a_{\rm ext}(t,\boldsymbol{x}), \tag{9.2b}$$

$$\psi(t,\boldsymbol{x}) = \frac{GM\mu}{R} + \frac{G\boldsymbol{p}\cdot\boldsymbol{R}}{R^3} + \psi_{\rm ext}(t,\boldsymbol{x}), \tag{9.2c}$$

$$X(t,\boldsymbol{x}) = GMR + X_{\rm ext}(t,\boldsymbol{x}), \tag{9.2d}$$

where the black-hole terms are clearly distinguished from the external terms. With this we find that

$$\Psi(t,\boldsymbol{x}) = \frac{GM}{R}\left(\mu + \frac{1}{2}u^2\right) - \frac{GM}{2R^3}(\boldsymbol{u}\cdot\boldsymbol{R})^2 - \frac{GM}{2R}\boldsymbol{a}\cdot\boldsymbol{R} + \frac{G\boldsymbol{p}\cdot\boldsymbol{R}}{R^3} + \Psi_{\rm ext}(t,\boldsymbol{x}), \tag{9.3}$$

where $\Psi_{\rm ext} \coloneqq \psi_{\rm ext} + \frac{1}{2}\partial_{tt}X_{\rm ext}$. We have introduced $\boldsymbol{R} \coloneqq \boldsymbol{x} - \boldsymbol{z}(t)$, $R \coloneqq |\boldsymbol{R}|$, $\boldsymbol{u}(t) \coloneqq d\boldsymbol{z}/dt$, $\boldsymbol{a}(t) \coloneqq d\boldsymbol{u}/dt$, and

$u^2 := \boldsymbol{u}\cdot\boldsymbol{u}$. The mass monopole moment M is identified with the black-hole mass, and the current dipole moment $\boldsymbol{S}$ is identified with its spin; $\mu(t)$ is a post-Newtonian correction to the monopole moment, and $\boldsymbol{p}(t)$ is a post-Newtonian dipole moment that must be included to ensure a successful matching to the black-hole metric. It is easy to verify that the potentials of Eq. (9.2) satisfy the vacuum field equations and the harmonic gauge condition (which requires the presence of u^a in the vector potential).

The post-Newtonian multipoles $\mu(t)$ and $\boldsymbol{p}(t)$ are not determined by the field equations, because the validity of the post-Newtonian metric does not extend beyond $R = r_{\rm min}$. If the black hole were replaced by a material body with weak internal gravity, integrating the field equations with matter (refer to Secs. 9.2.2 and 9.5.8 of Poisson and Will [61]) would return the same potentials with $\mu = \frac{3}{2}u^2 - U_{\rm ext}(t,\boldsymbol{z})$ and $\boldsymbol{p} = \frac{1}{2}(3+\lambda)\boldsymbol{u}\times\boldsymbol{S}$, where λ is a dimensionless quantity that parametrizes the choice of representative world line to describe the body's center-of-mass; the choice $\lambda = 1$ can be described as "imposing the covariant spin supplementary condition." In the sequel we shall verify that matching the black-hole and post-Newtonian metrics produces the same expressions for μ and $\boldsymbol{p}$; we shall also see that the choice $\lambda = 1$ is natural and gives rise to important simplifications.

To carry out the matching it is necessary to transform the post-Newtonian metric from the barycentric frame (t, x^a) to another harmonic frame $(\bar{t}, \bar{x}^a)$ that is at all times centered on the black hole; we shall refer to this new frame as the black-hole frame. The transformation was first described in the context of weakly gravitating bodies by Kopeikin [62], Brumberg and Kopeikin [63], and Damour, Soffel, and Xu [64], then extended to compact bodies by Racine and Flanagan [65]; its details are reviewed in Sec. V of Taylor and Poisson. The end result is the following expressions for the transformed potentials,

$$\bar{U} = \frac{GM}{\bar{r}} + {}_0U + {}_1U_a\bar{x}^a + {}_2U_{ab}\bar{x}^a\bar{x}^b + O(\bar{r}^3), \quad (9.4a)$$

$$\bar{U}^a = -\frac{G(\bar{\boldsymbol{x}}\times\boldsymbol{S})^a}{2\bar{r}^3} + {}_0U^a + {}_1U^a{}_b\bar{x}^b + {}_2U^a{}_{bc}\bar{x}^b\bar{x}^c + O(\bar{r}^3), \quad (9.4b)$$

$$\bar{\Psi} = \frac{G\boldsymbol{q}\cdot\bar{\boldsymbol{x}}}{\bar{r}^3} + \frac{GM}{\bar{r}}(\mu + \dot{A} - 2v^2) + {}_0\Psi + {}_1\Psi_a\bar{x}^a + {}_2\Psi_{ab}\bar{x}^a\bar{x}^b + O(\bar{r}^3), \quad (9.4c)$$

where $\bar{r}^2 := \delta_{ab}\bar{x}^a\bar{x}^b$, $\boldsymbol{q} := \boldsymbol{p} - 2\boldsymbol{u}\times\boldsymbol{S} - M(\boldsymbol{H} - A\boldsymbol{u})$, and all other symbols are defined in Taylor and Poisson. It is important to note that spin terms appear in the singular pieces of the potentials; for example, the spin vector $\boldsymbol{S}$ now appears within $\bar{U}^a$ [compare with Eq. (5.34) of Taylor and Poisson] and it appears also, along with the dipole moment $\boldsymbol{p}$, in the expression for $\boldsymbol{q}$ [compare with Eq. (5.35) of Taylor and Poisson]. It is equally important to note that the spin terms do not appear in the regular pieces, which are listed explicitly in Eqs. (5.36)–(5.44) of Taylor and Poisson.

The transformed potentials are next inserted into the transformed metric, which is now ready to be compared with the black-hole metric of Eq. (8.17). The metrics must agree, and the comparison gives rises to matching conditions that determine the unknown functions that enter the transformation between the (t, x^a) and $(\bar{t}, \bar{x}^a)$ coordinates, the unknown functions μ and $\boldsymbol{p}$ that enter the post-Newtonian metric, and finally, the tidal moments $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$. (We place an overbar on the tidal moments to indicate that these are defined in the black-hole frame; this notation differs from the one adopted in earlier sections of this paper. Below we shall also introduce tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ that are defined in the barycentric frame of the post-Newtonian spacetime.) The details of the matching are described in Sec. VI of Taylor and Poisson, and very few alterations are required to account for the black-hole spin.

The matching conditions that do acquire spin terms are

$${}_1U_a + \frac{1}{c^2}({}_1\Psi_a - \bar{\mathcal{B}}_{ab}\hat{S}^b) = O(c^{-4}), \quad (9.5a)$$

$$\boldsymbol{p} - 2\boldsymbol{u}\times\boldsymbol{S} - M(\boldsymbol{H} - A\boldsymbol{u}) = O(c^{-2}), \quad (9.5b)$$

$${}_1U_{ab} - \frac{1}{4}\gamma^{\rm d}\epsilon_{abp}\bar{\mathcal{E}}^p{}_q\hat{S}^q = O(c^{-2}), \quad (9.5c)$$

which replace Eqs. (6.7), (6.8), and (6.13) of Taylor and Poisson, respectively. The second equation is the only matching condition that permits the determination of $\boldsymbol{H}$, which appears in the coordinate transformation, and $\boldsymbol{p}$, which appears in the post-Newtonian metric. In the absence of spin $\boldsymbol{p}$ would vanish identically, and the matching condition would produce $\boldsymbol{H} = A\boldsymbol{u} + O(c^{-2})$. In the presence of spin it is natural to keep this relation unaltered, and to separately impose

$$\boldsymbol{p} = 2\boldsymbol{u}\times\boldsymbol{S}. \quad (9.6)$$

While this choice is natural, other possibilities cannot be excluded. For example, $\boldsymbol{p}$ could be generalized to $\frac{1}{2}(3+\lambda)\boldsymbol{u}\times\boldsymbol{S}$ at the cost of altering the expression for $\boldsymbol{H}$; as was mentioned previously, this would generate the freedom to shift the representative world line taken to represent the black hole's center-of-mass in the post-Newtonian metric. For convenience and simplicity we prefer to leave $\boldsymbol{H}$ unaffected by spin terms, and to stick with $\boldsymbol{p}$ as determined by Eq. (9.6).

The spin-induced changes in the matching conditions impact only some of the results obtained by Taylor and Poisson. Among the results that are not changed are the expression for μ given by their Eq. (6.24)—the expected

$\mu = \frac{3}{2}v^2 - U_{\rm ext}$—and the expressions for the tidal moments given by their Eqs. (6.36)–(6.37); we have

$$\mathcal{E}_{ab} = -\partial_{ab}U_{\rm ext} + \frac{1}{c^2}[-\partial_{\langle ab\rangle}\Psi_{\rm ext} + 4u^c(\partial_{ab}U_c^{\rm ext} - \partial_{c\langle a}U_{b\rangle}^{\rm ext}) - 4\partial_{t\langle a}U_{b\rangle}^{\rm ext} - 2(u^2 - U_{\rm ext})\partial_{ab}U_{\rm ext} + 3u^c u_{\langle a}\partial_{b\rangle c}U_{\rm ext} + 2u_{\langle a}\partial_{b\rangle t}U_{\rm ext} + 3\partial_{\langle a}U_{\rm ext}\partial_{b\rangle}U_{\rm ext}] + O(c^{-4}) \tag{9.7}$$

and

$$\mathcal{B}_{ab} = 2\epsilon_{pq(a}\partial^p{}_{b)}(U^q_{\rm ext} - u^q U_{\rm ext}) + O(c^{-2}), \tag{9.8}$$

where the potentials are evaluated at $\boldsymbol{x} = \boldsymbol{z}(t)$ after differentiation, and indices within angular brackets are symmetrized and made tracefree; for example $u_{\langle a}\partial_{b\rangle} = \frac{1}{2}(u_a\partial_b + u_b\partial_a) - \frac{1}{3}\delta_{ab}u^c\partial_c$.

Among the results that do change to account for spin are the black hole's equations of motion, which become

$$a_a = \partial_a U_{\rm ext} + \frac{1}{c^2}[\partial_a\Psi_{\rm ext} - 4(\partial_a U_b^{\rm ext} - \partial_b U_a^{\rm ext})u^b + 4\partial_t U_a^{\rm ext} + (u^2 - 4U_{\rm ext})\partial_a U_{\rm ext} - u_a(4u^b\partial_b U_{\rm ext} + 3\partial_t U_{\rm ext}) - \mathcal{B}_{ab}\hat{S}^b] + O(c^{-4}) \tag{9.9}$$

and which differ from Eq. (6.30) of Taylor and Poisson by the presence of the Mathisson-Papapetrou spin force [66–68] proportional to $\mathcal{B}_{ab}\hat{S}^b$.

The tidal moments of Eqs. (9.7) and (9.8) are defined in the barycentric frame (t, x^a) of the post-Newtonian metric. The transformation to the black-hole frame $(\bar{t}, \bar{x}^a)$ is described by Eqs. (6.33)–(6.35) of Taylor and Poisson. We have

$$\bar{\mathcal{E}}_{ab}(\bar{t}) = \mathcal{N}_a{}^c(t)\mathcal{N}_b{}^d(t)\mathcal{E}_{cd}(t),$$
$$\bar{\mathcal{B}}_{ab}(\bar{t}) = \mathcal{N}_a{}^c(t)\mathcal{N}_b{}^d(t)\mathcal{B}_{cd}(t), \tag{9.10}$$

where

$$\mathcal{N}_{ab}(t) \coloneqq \delta_{ab} - \frac{1}{c^2}\epsilon_{abc}R^c(t) + O(c^{-4}) \tag{9.11}$$

describes a post-Newtonian precession of the black-hole frame relative to the barycentric frame. The relation between the time coordinates is given by $\bar{t} = t - c^{-2}A(t) + O(c^{-4})$, with $A(t)$ determined by

$$\frac{dA}{dt} = \frac{1}{2}u^2 + U_{\rm ext}(t, z); \tag{9.12}$$

this equation is unchanged with respect to Eq. (6.35) of Taylor and Poisson. The precession vector $\boldsymbol{R}(t)$ is determined by

$$\epsilon_{abc}\frac{dR^c}{dt} = -4\partial_{[a}U_{b]}^{\rm ext} - 3u_{[a}\partial_{b]}U_{\rm ext} - \gamma^{\sf d}\epsilon_{abc}\mathcal{E}^c{}_p\hat{S}^p. \tag{9.13}$$

Notice that this equation acquires a spin term that was not present in Eq. (6.35) of Taylor and Poisson. We observe that the precession of the black-hole frame features a dependence on the gauge constant $\gamma^{\sf d}$, which was already associated with precession effects in the discussion surrounding Eq. (4.9).

The conclusion of this section is that while the tidal moments of Eqs. (9.7) and (9.8) do not depend explicitly on the black-hole spin $\boldsymbol{S}$, the acceleration vector $\boldsymbol{a}$ and the precession vector $\boldsymbol{R}$ both do. This implies that $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$ possess an implicit dependence upon $\boldsymbol{S}$ provided by the motion and precession of the black-hole frame.

X. TIDAL MOMENTS FOR A TWO-BODY SYSTEM

In this section we specialize the results of the preceding section to a post-Newtonian system consisting of two bodies, the black hole and a companion. We adapt the notation to this specific situation: the black hole's mass, spin, position, velocity, and acceleration will now be denoted m_1, $\boldsymbol{S}_1$, z_1, $\boldsymbol{u}_1$, and $\boldsymbol{a}_1$, respectively, and the quantities associated with the companion will be m_2, $\boldsymbol{S}_2$, z_2, $\boldsymbol{u}_2$, and $\boldsymbol{a}_2$. We recall that $\hat{\boldsymbol{S}}_1 \coloneqq \boldsymbol{S}_1/m_1$, $\hat{\boldsymbol{S}}_2 \coloneqq \boldsymbol{S}_2/m_2$, and we also introduce $\boldsymbol{b} \coloneqq z_1 - z_2$ and $\boldsymbol{u} \coloneqq \boldsymbol{u}_1 - \boldsymbol{u}_2$. Finally, we let $b \coloneqq |\boldsymbol{b}|$ be the distance between the black hole and its companion, and $\boldsymbol{n} \coloneqq \boldsymbol{b}/b$ be a unit vector pointing from the companion to the black hole. The positions, velocities, and accelerations all refer to the barycentric frame (t, x^a) of the post-Newtonian metric.

The black hole's acceleration $\boldsymbol{a}_1$, given by Eq. (9.9), and the barycentric tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, given by Eqs. (9.7) and (9.8), can be calculated from the external potentials $U_{\rm ext}$, $U^a_{\rm ext}$, and $\Psi_{\rm ext}$, which are now taken to be those generated by the companion. Whether this is a black hole, a material compact body, or a weakly self-gravitating body, the external potentials will take the same form as the black-hole terms in Eqs. (9.2) and (9.3). We therefore have

$$U_{\rm ext}(t, \boldsymbol{x}) = \frac{Gm_2}{R_2}, \tag{10.1a}$$

$$U^a_{\rm ext}(t, \boldsymbol{x}) = -\frac{G(\boldsymbol{R}_2 \times \boldsymbol{S}_2)^a}{2R_2^3} + \frac{Gm_2 u_2^a}{R_2}, \tag{10.1b}$$

$$\Psi_{\rm ext}(t, \boldsymbol{x}) = \frac{Gm_2}{R_2}\left(2u_2^2 - \frac{Gm_1}{b}\right) - \frac{Gm_2}{2R_2^3}(\boldsymbol{u}_2\cdot\boldsymbol{R}_2)^2 - \frac{Gm_2}{2R_2}\boldsymbol{a}_2\cdot\boldsymbol{R}_2 + \frac{2G(\boldsymbol{u}_2 \times \boldsymbol{S}_2)\cdot\boldsymbol{R}_2}{R_2^3}, \tag{10.1c}$$

where $\boldsymbol{R}_2 := \boldsymbol{x} - \boldsymbol{z}_2$ and $R_2 := |\boldsymbol{R}_2|$. In $\Psi_{\rm ext}$ we have inserted the appropriate expressions for μ and $\boldsymbol{p}$, and in future manipulations we shall also substitute the appropriate expression $\boldsymbol{a}_2 = Gm_1\boldsymbol{n}/b^2 + O(c^{-2})$ for the acceleration of the companion body.

We first involve the external potentials in a calculation of the barycentric tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$. The contributions that are independent of spin were previously computed by Taylor and Poisson [55]—see their Eqs. (7.10) and (7.11)—and we shall focus here on the contributions arising from the spin terms in the external potentials. Going through these calculations it is convenient to go back and forth between the spin vector and tensor of each body, using the relations

$$S^{ab} = \epsilon^{abc} S_c, \qquad S^a = \frac{1}{2}\epsilon^{abc} S_{bc}. \tag{10.2}$$

The tensorial notation, in particular, is helpful to simplify the spin part of the external potentials, which can be expressed as

$$U^a_{\rm ext}[\text{spin}] = \frac{1}{2} G S_2^{ab} \partial_b \frac{1}{R_2},$$
$$\Psi_{\rm ext}[\text{spin}] = -2 G S_2^{ab} u_{2b} \partial_a \frac{1}{R_2}. \tag{10.3}$$

This form allows the various derivatives of the potentials to be related to derivatives of R_2^{-1}, and therefore to be expressed in terms of symmetric tracefree tensors after evaluation at $\boldsymbol{x} = \boldsymbol{z}_1$. For example, $\partial_{ab} R_2^{-1}$ becomes $3n_{\langle ab\rangle}/b^3$, and $\partial_{abc} R_2^{-1}$ becomes $-15n_{\langle abc\rangle}/b^4$. After straightforward manipulations we find that the spin contributions to the barycentric tidal moments are given by

$$\mathcal{E}_{ab}[\text{spin}] = \frac{15G}{c^2 b^4}(S_{2a}{}^c n_{\langle bcd\rangle} + S_{2b}{}^c n_{\langle acd\rangle} - 2S_{2d}{}^c n_{\langle abc\rangle}) u^d \tag{10.4}$$

and

$$\mathcal{B}_{ab}[\text{spin}] = -\frac{15G}{b^4} S_2^c n_{\langle abc\rangle}. \tag{10.5}$$

These contributions to the tidal moments can be added to those listed in Eqs. (7.10) and (7.11) of Taylor and Poisson.

We next involve the external potentials in a calculation of the black hole's acceleration $\boldsymbol{a}_1$. We focus our attention on the spin terms coming both from the external potentials (terms proportional to $\boldsymbol{S}_2$) and the Mathisson-Papapetrou spin force (terms proportional to $\boldsymbol{S}_1$ and to the product of each spin), and show that they reproduce the well-known expressions for the leading-order spin-orbit and spin-spin forces. It is known that the other terms in $\boldsymbol{a}_1$ reproduce the standard Newtonian and post-Newtonian terms—see Eq. (9.245) of Poisson and Will [61]—in the equations of motion.

The spin terms arising from the external potentials in Eq. (9.9) are calculated using the methods outlined previously for the computation of the tidal moments. The spin terms arising from the Mathisson-Papapetrou spin force are obtained from

$$\mathcal{B}_{ab}\hat{S}_1^b = (2\hat{S}_1^{bc}\partial_{ab} + \hat{S}_{1a}{}^b \partial_{bc})(U^c_{\rm ext} - u_1^c U_{\rm ext}), \tag{10.6}$$

which follows from Eq. (9.8) after involving Eq. (10.2) and rearranging the products of permutation symbols. Substituting the appropriate expressions for the external potentials gives

$$\mathcal{B}_{ab}\hat{S}_1^b = -\frac{3Gm_2}{b^3}(\hat{S}_{1a}{}^b n_{\langle bc\rangle} + 2\hat{S}_1^{bc} n_{\langle ab\rangle}) u^c - \frac{15Gm_2}{b^4}\hat{S}^b_{1p}\hat{S}_2^{pc} n_{\langle abc\rangle}. \tag{10.7}$$

With all this we find that the spin terms in the black hole's acceleration are given by

$$a_{1a}[\text{spin}] = \frac{3Gm_2}{c^2 b^3}[2n_{\langle ac\rangle}(\hat{S}^c_{1b} + \hat{S}^c_{2b}) + n_{\langle bc\rangle}(\hat{S}_1^{ac} + 2\hat{S}_2^{ac})]u^b + \frac{15Gm_2}{c^2 b^4}\hat{S}^b_{1p}\hat{S}_2^{pc} n_{\langle abc\rangle}. \tag{10.8}$$

The first group of terms, linear in the spins, is the well-known expression for the leading-order spin-orbit acceleration of a post-Newtonian body—see Eq. (9.245) of Poisson and Will, evaluated with $\lambda = 1$—and the last term, bilinear in the spins, is the leading-order spin-spin acceleration—see Eq. (9.190) of Poisson and Will. As promised, we have verified that the acceleration of Eq. (9.9) reproduces the standard results of post-Newtonian theory. The spin acceleration $\boldsymbol{a}_2[\text{spin}]$ of the companion can be obtained directly from Eq. (10.8) by switching the body labels and letting $\boldsymbol{n} \to -\boldsymbol{n}$, $\boldsymbol{u} \to -\boldsymbol{u}$.

To conclude this section we calculate the spin contributions to dA/dt and dR^a/dt, which are involved in the transformation from the barycentric frame (t, x^a) to the black-hole frame $(\bar{t}, \bar{x}^a)$. The calculation proceeds from Eqs. (9.12) and (9.13), it involves the same external potentials that were computed previously, and it involves the Newtonian piece of $\mathcal{E}_{ab}$, given by Eq. (7.10) of Taylor and Poisson. After straightforward manipulations we obtain

$$\frac{dA}{dt}[\text{spin}] = 0,$$
$$\frac{dR_a}{dt}[\text{spin}] = -\frac{3Gm_2}{b^3}(-\gamma^{\rm d}\hat{S}_1^b + \hat{S}_2^b) n_{\langle ab\rangle}. \tag{10.9}$$

These contributions are to be added to the nonspinning contributions listed in Eq. (7.12) of Taylor and Poisson.

XI. TIDAL MOMENTS FOR A CIRCULAR BINARY

In this section we continue our computation of the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ when the black hole is a member of a two-body system, and specialize the situation examined in the previous section to circular motion. We now assume that $b =$ constant, and to be consistent we take the spin vectors $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ to be either aligned or antialigned with the orbital angular-momentum vector. We describe the motion in terms of the orbital phase angle $\phi \coloneqq \omega t$, with ω denoting the orbital angular velocity in the barycentric frame (t, x^a) of the post-Newtonian metric. We let the orbital motion take place in the x-y plane of the coordinate system, and in these coordinates we denote by $\boldsymbol{n} = [\cos\phi, \sin\phi, 0]$ the unit vector that points from the companion to the black hole. We also introduce the additional basis vectors $\boldsymbol{\phi} \coloneqq [-\sin\phi, \cos\phi, 0]$ and $\boldsymbol{\ell} \coloneqq [0, 0, 1]$, with $\boldsymbol{\ell}$ pointing in the direction of the orbital angular-momentum vector. In terms of the vectorial basis we have $\boldsymbol{b} = b\boldsymbol{n}$, $\boldsymbol{u} = \omega b \boldsymbol{\phi}$, $\boldsymbol{S}_1 = S_1 \boldsymbol{\ell}$ and $\boldsymbol{S}_2 = S_2 \boldsymbol{\ell}$, with $S_{1,2}$ positive (negative) when $\boldsymbol{S}_{1,2}$ is aligned (antialigned) with $\boldsymbol{\ell}$. We resume the use of the dimensionless spin variables $\chi_{1,2}$ defined by

$$S_1 = \frac{Gm_1^2}{c}\chi_1, \qquad S_2 = \frac{Gm_2^2}{c}\chi_2. \tag{11.1}$$

We shall work with the mass combinations $m \coloneqq m_1 + m_2$ (total mass) and $\eta \coloneqq m_1 m_2/m^2$ (symmetric mass ratio).

A concrete expression for the relative acceleration $\boldsymbol{a} \coloneqq \boldsymbol{a}_1 - \boldsymbol{a}_2$ of the two-body system can be obtained from Eqs. (9.142) and (9.252) of Poisson and Will [61] (the spin-orbit terms must be evaluated with $\lambda = 1$). After specialization to circular motion, which implies $\boldsymbol{a} = -b\omega^2 \boldsymbol{n}$, and substitution of Eq. (11.1), the equations of motion produce the following expression for the orbital velocity:

$$u^2 = (\omega b)^2 = \frac{Gm}{b}\left[1 - (3-\eta)\frac{Gm}{c^2 b} - \tilde{\chi}\left(\frac{Gm}{c^2 b}\right)^{3/2} + O(c^{-4})\right], \tag{11.2}$$

where

$$\tilde{\chi} \coloneqq \frac{m_1(2m_1 + 3m_2)}{m^2}\chi_1 + \frac{m_2(2m_2 + 3m_1)}{m^2}\chi_2. \tag{11.3}$$

This expression incorporates the Newtonian, post-Newtonian, and spin-orbit terms in the relative acceleration, but it neglects the spin-spin terms, which contribute at order c^{-4} by virtue of the scalings displayed in Eq. (11.1). The expression agrees with Eq. (4.5) of Ref. [69].

It is a simple matter to specialize the expressions of Eqs. (10.4) and (10.5) to circular motion. Making use of Eqs. (10.2) and (11.2), we obtain

$$\mathcal{E}_{ab}[\text{spin}] = \frac{6Gm_2}{b^3}(u/c)^3 \frac{m_2}{m}\chi_2(\phi_{\langle ab\rangle} + 2n_{\langle ab\rangle}) + O(c^{-5}), \tag{11.4a}$$

$$\mathcal{B}_{ab}[\text{spin}] = \frac{6Gm_2}{b^3}(u^2/c)\frac{m_2}{m}\chi_2 \ell_{(a}n_{b)} + O(c^{-3}). \tag{11.4b}$$

Combining this with Eqs. (7.19) and (7.20) of Taylor and Poisson [55], we arrive at

$$\mathcal{E}_{ab} = -\frac{3Gm_2}{b^3}\left\{\left[1 - \frac{m_1 + 2m_2}{2m}(u/c)^2 - \frac{4m_2}{m}\chi_2(u/c)^3\right]n_{\langle ab\rangle} + \left[(u/c)^2 - \frac{2m_2}{m}\chi_2(u/c)^3\right]\phi_{\langle ab\rangle}\right\} + O(c^{-4}), \tag{11.5a}$$

$$\mathcal{B}_{ab} = -\frac{6Gm_2}{b^3}u\left[1 - \frac{m_2}{m}\chi_2(u/c)\right]\ell_{(a}n_{b)} + O(c^{-2}), \tag{11.5b}$$

complete expressions for the barycentric tidal moments, expanded through 1.5PN order.

The transformation of the tidal moments from the barycentric frame to the black-hole frame involves the rotation tensor $\mathcal{N}_{ab}(t)$ of Eq. (9.11), which is determined by $A(t)$ and $\boldsymbol{R}(t)$ as obtained by integrating Eqs. (9.12) and (9.13). The spin contributions to dA/dt and dR^a/dt were displayed in Eq. (10.9), and again it is a simple matter to specialize these results to circular motion. After inclusion of Eqs. (7.21) and (7.22) of Taylor and Poisson, we find that

$$\frac{dA}{dt} = \frac{(2m_1 + 3m_2)m_2}{2m^2}u^2 + O(c^{-2}), \tag{11.6a}$$

$$\frac{dR^a}{dt} = -\frac{Gm_2}{b^2}u\left[\frac{4m_1 + 3m_2}{2m} + \left(\gamma^{\mathsf{d}}\frac{m_1}{m}\chi_1 - \frac{m_2}{m}\chi_2\right)(u/c)\right]\ell^a + O(c^{-2}). \tag{11.6b}$$

These equations imply that

$$t = \left[1 + \frac{(2m_1 + 3m_2)m_2}{2m^2}(u/c)^2 + O(c^{-4})\right]\bar{t} \tag{11.7}$$

and $c^{-2}\boldsymbol{R}(t) = -(\Omega t)\boldsymbol{\ell}$, where

$$\Omega := \sqrt{\frac{Gm}{b^3}}\left[\frac{(4m_1+3m_2)m_2}{2m^2}(u/c)^2 + \frac{m_2}{m}\left(\gamma^{\mathsf{d}}\frac{m_1}{m}\chi_1 - \frac{m_2}{m}\chi_2\right)(u/c)^3 + O(c^{-4})\right] \tag{11.8}$$

is the precessional angular frequency of the black-hole frame $(\bar{t}, \bar{x}^a)$ relative to the barycentric frame (t, x^a).

With these results it is easy to show that $\mathcal{N}_{ab}(t)$ has the following effect on the basis vectors associated with the circular motion:

$$\mathcal{N}_a{}^c n_c = \bar{n}_a := n_a(\phi - \Omega t),$$
$$\mathcal{N}_a{}^c \phi_c = \bar{\phi}_a := \phi_a(\phi - \Omega t),$$
$$\mathcal{N}_a{}^c \ell_c = \bar{\ell}_a := \ell_a. \tag{11.9}$$

We see that the dependence of each vector on the orbital phase $\phi = \omega t$ must be replaced by a dependence on $\phi - \Omega t = (\omega - \Omega)t$. If we also incorporate the transformation of the time coordinate, we find that

$$\bar{\boldsymbol{n}} = [\cos\bar{\phi}, \sin\bar{\phi}, 0],$$
$$\bar{\boldsymbol{\phi}} = [-\sin\bar{\phi}, \cos\bar{\phi}, 0],$$
$$\bar{\boldsymbol{\ell}} = [0,0,1], \tag{11.10}$$

where $\bar{\phi} = \bar{\omega}\,\bar{t}$ is now the phase of the tidal field, with $\bar{\omega} := (\omega - \Omega)(t/\bar{t})$ describing the angular frequency of the tidal moments as measured in the black-hole frame. Involving Eqs. (11.2), (11.7), and (11.8), we see that a concrete expression for the tidal angular frequency is

$$\bar{\omega} = \sqrt{\frac{Gm}{b^3}}\left[1 - \frac{1}{2}(3+\eta)(u/c)^2 - \frac{1}{2}\bar{\chi}(u/c)^3 + O(c^{-4})\right], \tag{11.11}$$

where

$$\bar{\chi} := \frac{m_1}{m^2}[2m_1 + (3+2\gamma^{\mathsf{d}})m_2]\chi_1 + 3\eta\chi_2. \tag{11.12}$$

In view of the discussion of Sec. VI, which provides a precise definition of the black-hole frame in terms of the null generators of the deformed event horizon, it is appropriate to substitute $\gamma^{\mathsf{d}} = -1$ in this expression.

With all this we find that the tidal moments in the black-hole frame take the same form as in Eq. (11.5), but with each vector transformed in the way just described. Concretely, we have

$$\bar{\mathcal{E}}_{ab} = -\frac{3Gm_2}{b^3}\left\{\left[1 - \frac{m_1+2m_2}{2m}(u/c)^2 - \frac{4m_2}{m}\chi_2(u/c)^3\right]\bar{n}_{\langle ab\rangle} + \left[(u/c)^2 - \frac{2m_2}{m}\chi_2(u/c)^3\right]\bar{\phi}_{\langle ab\rangle}\right\} + O(c^{-4}), \tag{11.13a}$$

$$\bar{\mathcal{B}}_{ab} = -\frac{6Gm_2}{b^3}u\left[1 - \frac{m_2}{m}\chi_2(u/c)\right]\bar{\ell}_{(a}\bar{n}_{b)} + O(c^{-2}), \tag{11.13b}$$

with each basis vector $\bar{\boldsymbol{n}}$, $\bar{\boldsymbol{\phi}}$, and $\bar{\boldsymbol{\ell}}$ expressed in terms of $\bar{\phi} = \bar{\omega}\,\bar{t}$.

The individual components of $\bar{\mathcal{E}}_{ab}$ are best displayed in terms of the combinations $\bar{\mathcal{E}}_0 := \frac{1}{2}(\bar{\mathcal{E}}_{11} + \bar{\mathcal{E}}_{22})$, $\bar{\mathcal{E}}_{1c} := \bar{\mathcal{E}}_{13}$, $\bar{\mathcal{E}}_{1s} := \bar{\mathcal{E}}_{23}$, $\bar{\mathcal{E}}_{2c} := \frac{1}{2}(\bar{\mathcal{E}}_{11} - \bar{\mathcal{E}}_{22})$, and $\bar{\mathcal{E}}_{2s} := \bar{\mathcal{E}}_{12}$. We have that the nonvanishing components are

$$\bar{\mathcal{E}}_0 = -\frac{Gm_2}{2b^3}\left[1 + \frac{m_1}{2m}(u/c)^2 - 6\frac{m_2}{m}\chi_2(u/c)^3 + O(c^{-4})\right], \tag{11.14a}$$

$$\bar{\mathcal{E}}_{2c} = -\frac{3Gm_2}{2b^3}\left[1 - \frac{3m_1+4m_2}{2m}(u/c)^2 - 2\frac{m_2}{m}\chi_2(u/c)^3 + O(c^{-4})\right]\cos 2\bar{\phi}, \tag{11.14b}$$

$$\bar{\mathcal{E}}_{2s} = -\frac{3Gm_2}{2b^3}\left[1 - \frac{3m_1+4m_2}{2m}(u/c)^2 - 2\frac{m_2}{m}\chi_2(u/c)^3 + O(c^{-4})\right]\sin 2\bar{\phi}. \tag{11.14c}$$

With similar definitions holding for $\bar{\mathcal{B}}_{ab}$, we have that its nonvanishing components are

$$\bar{\mathcal{B}}_{1c} = -\frac{3Gm_2}{b^3}u\left[1 - \frac{m_2}{m}\chi_2(u/c) + O(c^{-2})\right]\cos\bar{\phi}, \tag{11.15a}$$

$$\bar{\mathcal{B}}_{1s} = -\frac{3Gm_2}{r^3}u\left[1 - \frac{m_2}{m}\chi_2(u/c) + O(c^{-2})\right]\sin\bar{\phi}. \tag{11.15b}$$

These results were already displayed in Sec. I—refer back to Eq. (1.9) and the following equations. In the expressions listed in Sec. I, the overbars were omitted on the tidal moments, and the tidal phase was expressed more generally as $\bar{\phi} = \bar{\omega}(\bar{t} - \bar{t}_0)$ and then rewritten as $\bar{\phi} = \bar{\omega}(v - v_0)$ in terms of the advanced-time coordinate v. The expressions also incorporated the phase shift $\delta\phi = \frac{8}{3}(m_1/m)(u/c)^3$ to be calculated in Sec. XII.

XII. INCLUSION OF TIME-DERIVATIVE TERMS

A complete determination of the tidal moments at 1.5PN order should incorporate terms in the metric that involve $\dot{\mathcal{E}}_{ab} := d\mathcal{E}_{ab}/dv$. These terms were neglected in Eq. (4.6), but they were present in the metrics constructed by Poisson [42] and Poisson and Vlasov [43]. The relevant pieces of the metric perturbation, which include both the $\mathcal{E}_{ab}$ and $\dot{\mathcal{E}}_{ab}$ terms, are given by

$$p_{vv} = -r^2 e^{\mathrm{q}}_1 \mathcal{E}^{\mathrm{q}} + \frac{1}{3} r^3 e^{\mathrm{q}}_2 \dot{\mathcal{E}}^{\mathrm{q}}, \tag{12.1a}$$

$$p_{vA} = -\frac{2}{3} r^3 e^{\mathrm{q}}_4 \mathcal{E}^{\mathrm{q}}_A + \frac{1}{3} r^4 e^{\mathrm{q}}_5 \dot{\mathcal{E}}^{\mathrm{q}}_A, \tag{12.1b}$$

$$p_{AB} = -\frac{1}{3} r^4 e^{\mathrm{q}}_7 \mathcal{E}^{\mathrm{q}}_{AB} + \frac{5}{18} r^5 e^{\mathrm{q}}_8 \dot{\mathcal{E}}^{\mathrm{q}}_{AB}, \tag{12.1c}$$

where $\dot{\mathcal{E}}^{\mathrm{q}}$, $\dot{\mathcal{E}}^{\mathrm{q}}_A$, and $\dot{\mathcal{E}}^{\mathrm{q}}_{AB}$ are tidal potentials constructed from $\dot{\mathcal{E}}_{ab}$; the radial functions e^{q}_1, e^{q}_4, and e^{q}_7 were listed in Table III, while e^{q}_2, e^{q}_5, and e^{q}_8 are displayed in Table VI. It should be noted that the residual freedom of the light-cone gauge and the freedom to redefine the tidal moments according to $\mathcal{E}_{ab} \to \mathcal{E}_{ab} + pM\dot{\mathcal{E}}_{ab}$, where p is an arbitrary number, were exploited to ensure that e^{q}_2, e^{q}_5, and e^{q}_8 all go to zero when $r = 2M$; this implies that the horizon metric of Eq. (7.2) does not acquire a contribution from $\dot{\mathcal{E}}_{ab}$. The $\dot{\mathcal{E}}_{ab}$ terms of Eq. (12.1) can be added to the metric of Eq. (4.6) to obtain a more complete description of the geometry of a tidally deformed, slowly rotating black hole.

That the $\dot{\mathcal{E}}_{ab}$ terms should be incorporated in a 1.5PN calculation of the tidal moments can be seen from the following argument. Equation (12.1) reveals that the leading tidal term in g_{vv} is proportional to $r^2\mathcal{E}_{ab}$, while the time-derivative term is proportional to $r^3\dot{\mathcal{E}}_{ab}$. If we take r to be of the same order of magnitude as M and denote by τ the time scale associated with changes in the tidal environment, we find that the time-derivative term is of order M/τ relative to the leading term. With τ of order $\sqrt{b^3/M}$, where b is the distance scale to the external matter, we find that $M/\tau \sim (M/b)^{3/2} \sim u^3$, where $u \sim b/\tau$ is the velocity scale. Restoring factors of c, we have found that the time-derivative term is smaller than the leading term by a factor of order $(u/c)^3$, and that it therefore represents a 1.5PN correction to the metric.

There were, nevertheless, good reasons to neglect the $\dot{\mathcal{E}}_{ab}$ terms until now. These were not included in the metric of Eq. (4.6) because unless χ is extremely small, they are negligible compared to the $\chi\mathcal{E}_{ab}$ terms when r is comparable to M, which was the prevailing context until the very end of Sec. VIII. But because they create 1.5PN corrections to the tidal moments, we now incorporate the $\dot{\mathcal{E}}_{ab}$ in a more complete description of the metric, and an improved determination of the tidal moments.

The metric perturbation of Eq. (12.1) can be recast in the harmonic gauge by following the methods described in

TABLE VI. Radial functions associated with the $\dot{\mathcal{E}}_{ab}$ terms in the metric; $f := 1 - 2M/r$.

$$e^{\mathrm{q}}_2 = f\left[1 + \frac{M}{2r}\left(5 + 12\ln\frac{r}{2M}\right) - \frac{M^2}{r^2}\left(27 + 12\ln\frac{r}{2M}\right) + \frac{14M^3}{r^3} + \frac{12M^4}{r^4}\right]$$

$$e^{\mathrm{q}}_5 = f\left[1 + \frac{M}{3r}\left(13 + 12\ln\frac{r}{2M}\right) - \frac{10M^2}{r^2} - \frac{12M^3}{r^3} - \frac{8M^4}{r^4}\right]$$

$$e^{\mathrm{q}}_8 = 1 + \frac{4M}{5r}\left(4 + 3\ln\frac{r}{2M}\right) - \frac{36M^2}{5r^2} - \frac{8M^3}{5r^3}\left(7 + 3\ln\frac{r}{2M}\right) + \frac{48M^4}{5r^4}$$

$$z_v = \frac{2(r-2M)M(35r^3-12Mr^2+24M^2r-8M^3)}{105r^5}\ln\frac{r}{2M} + \frac{(r-2M)(150r^4+415Mr^3+342M^2r^2-3540M^3r-288M^4)}{630r^5}$$

$$z_r = -\frac{2M(35r^3-12Mr^2+24M^2r-8M^3)}{105r^4}\ln\frac{r}{2M} - \frac{405r^5+34250Mr^4-103036M^2r^3+63232M^3r^2+6584M^4r+1152M^5}{1260r^4(r-2M)}$$

$$z = \frac{2M(r-2M)^2}{3r^2(r-M)}\ln\frac{r}{2M} + \frac{5r^5+317Mr^4-1036M^2r^3+1056M^3r^2-272M^4r-48M^5}{18r^4(r-M)}$$

$$w_{vv} = \frac{2M(r-2M)^2}{r^3}\ln\frac{r}{2M} + \frac{9r^5-12Mr^4-582M^2r^3+1662M^3r^2-1072M^4r+28M^5}{9r^5}$$

$$w_{vr} = -\frac{2(r-2M)M(35r^2-8Mr+8M^2)}{35r^4}\ln\frac{r}{2M} - \frac{405r^6-630Mr^5-18870M^2r^4+51654M^3r^3-28304M^4r^2-2812M^5r-192M^6}{315r^5(r-2M)}$$

$$w_{rr} = \frac{M(140r^2-32Mr+32M^2)}{35r^3}\ln\frac{r}{2M} + \frac{2(405r^6+24885Mr^5-120930M^2r^4+171634M^3r^3-64144M^4r^2-2812M^5r-192M^6)}{315r^4(r-2M)^2}$$

$$w_v = \frac{16M^2(r-2M)(3r^2-6Mr+2M^2)}{105r^5}\ln\frac{r}{2M} + \frac{(r-2M)(60r^5-125Mr^4-1432M^2r^3+3672M^3r^2-2832M^4r+552M^5)}{315r^5(r-M)}$$

$$w_r = \frac{2M^2(81r^4-114Mr^3+4M^2r^2+80M^3r-16M^4)}{105r^4(r-M)^2}\ln\frac{r}{2M} - \frac{60r^7-6290Mr^6+27833M^2r^5-42192M^3r^4+21018M^4r^3+5192M^5r^2-6340M^6r+1104M^7}{315r^4(r-M)^2(r-2M)}$$

$$\bar{w} = \frac{2M(r-2M)^2}{r^2(r-M)}\ln\frac{r}{2M} + \frac{9r^5+960Mr^4-3498M^2r^3+4066M^3r^2-1532M^4r+28M^5}{9r^4(r-M)}$$

$$w = \frac{2M^2(3r^2-6Mr+2M^2)}{r^3(r-M)}\ln\frac{r}{2M} - \frac{M(153r^3-492Mr^2+538M^2r-188M^3)}{9r^3(r-M)}$$

Sec. VIII. The calculations are simpler here, because the gauge transformation is to be carried out to first order only in the perturbation. A complication arises, however, because the time dependence of the tidal moments $\mathcal{E}_{ab}(v)$ must now be taken into account. Recalling Eq. (8.7), the generating vector field is expressed as

$$\xi_v = -\frac{1}{3}r^3 f\mathcal{E}^{\mathsf{q}} + r^4 z_v \dot{\mathcal{E}}^{\mathsf{q}},$$
$$\xi_r = \frac{1}{3}r^3\mathcal{E}^{\mathsf{q}} + r^4 z_r \dot{\mathcal{E}}^{\mathsf{q}},$$
$$\xi_A = -\frac{r^5 f^2}{3(r-M)}\mathcal{E}^{\mathsf{q}}_A + r^4 z \dot{\mathcal{E}}^{\mathsf{q}}_A, \qquad (12.2)$$

where z_v, z_r, and z are radial functions determined by integrating the differential equations issued from the harmonic conditions of Eq. (8.3). The solutions are listed in Table VI, for some suitable choice of integration constants. After inserting the gauge vector in Eq. (8.5) and truncating to first order, we find that the harmonic-gauge form of the perturbation is given by

$$p_{vv} = -r^2 e^{\mathsf{q}}_{vv}\mathcal{E}^{\mathsf{q}} + r^3 w_{vv}\dot{\mathcal{E}}^{\mathsf{q}}, \qquad (12.3a)$$

$$p_{vr} = r^2 e^{\mathsf{q}}_{vr}\mathcal{E}^{\mathsf{q}} + r^3 w_{vr}\dot{\mathcal{E}}^{\mathsf{q}}, \qquad (12.3b)$$

$$p_{rr} = -2r^2 e^{\mathsf{q}}_{rr}\mathcal{E}^{\mathsf{q}} + r^3 w_{rr}\dot{\mathcal{E}}^{\mathsf{q}}, \qquad (12.3c)$$

$$p_{vA} = r^4 w_v \dot{\mathcal{E}}^{\mathsf{q}}, \qquad (12.3d)$$

$$p_{rA} = -r^3 e^{\mathsf{q}}_r \mathcal{E}^{\mathsf{q}}_A + r^3 w_r \dot{\mathcal{E}}^{\mathsf{q}}_A, \qquad (12.3e)$$

$$p_{AB} = -r^4 \bar{e}^{\mathsf{q}}\Omega_{AB}\mathcal{E}^{\mathsf{q}} - r^4 e^{\mathsf{q}}\mathcal{E}^{\mathsf{q}}_{AB} + r^5\bar{w}\Omega_{AB}\dot{\mathcal{E}}^{\mathsf{q}} + r^5 w\dot{\mathcal{E}}^{\mathsf{q}}_{AB}, \qquad (12.3f)$$

where the various e^{q}-functions are listed in Table V, while the w-functions are listed in Table VI.

We next follow the procedure described near the end of Sec. VIII: We carry out a transformation to Cartesian harmonic coordinates, and express the metric perturbation as a post-Newtonian expansion truncated to 1.5PN order. There is no need to go through the details again here, but one aspect to be careful about is that the transformation to harmonic coordinates involves expressing $\mathcal{E}_{ab}$ as a function of $v = t + r\Delta$, where $\Delta := 1 + (2M/r)\ln(r/2M - 1)$. Because the time dependence of the tidal moments is slow, we can write this as

$$\mathcal{E}_{ab}(t + r\Delta) = \mathcal{E}_{ab}(t) + r\Delta\dot{\mathcal{E}}_{ab}(t) + \cdots \qquad (12.4)$$

and express p_{vv} (say) as

$$p_{vv} = -r^2 e^{\mathsf{q}}_{vv}\mathcal{E}^{\mathsf{q}}(t) + r^3(w_{vv} - \Delta e^{\mathsf{q}}_{vv})\dot{\mathcal{E}}^{\mathsf{q}}(t). \qquad (12.5)$$

Completing the calculations, and inserting the rotation-tidal couplings that were considered previously, we find that the metric of Eq. (8.17) becomes

$$g_{tt} = -1 + \frac{2GM}{c^2\bar{r}} - 2\left(\frac{GM}{c^2\bar{r}}\right)^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2\bar{r}}\right)\mathcal{E}_{ab}x^a x^b + \frac{8}{3}\frac{GM}{c^5}\dot{\mathcal{E}}_{ab}x^a x^b + \frac{2}{c^4}\mathcal{B}_{ap}\hat{S}^p x^a, \qquad (12.6a)$$

$$g_{ta} = \frac{2G}{c^3}\frac{(\boldsymbol{x}\times\boldsymbol{S})_a}{\bar{r}^3} + \frac{2}{3c^3}\epsilon_{abp}\mathcal{B}^p{}_c x^b x^c - \frac{\gamma^{\mathsf{d}}}{c^3}\epsilon_{abp}\mathcal{E}^p{}_q\hat{S}^q x^b \qquad (12.6b)$$

when time-derivative terms are properly incorporated. It should be noted that the tidal moments are now expressed as functions of harmonic time t, and that g_{ta} is actually unchanged relative to Eq. (8.17).

The metric of Eq. (12.6) can be rewritten in a form identical to Eq. (8.17) by simply introducing the new tidal moments

$$\mathcal{E}^{\sharp}_{ab} := \mathcal{E}_{ab} - \frac{8}{3}\frac{GM}{c^3}\dot{\mathcal{E}}_{ab}. \qquad (12.7)$$

With this redefinition, the calculations presented in Secs. IX, X, XI proceed completely unchanged. The tidal moments determined in these sections are then $\mathcal{E}^{\sharp}_{ab}$, and the true moments $\mathcal{E}_{ab}$ are recovered by inverting Eq. (12.7),

$$\mathcal{E}_{ab} = \mathcal{E}^{\sharp}_{ab} + \frac{8}{3}\frac{GM}{c^3}\dot{\mathcal{E}}^{\sharp}_{ab} + O(c^{-6}). \qquad (12.8)$$

In this way Eq. (9.7) becomes

$$\mathcal{E}_{ab} = \mathcal{E}^{\sharp}_{ab} - \frac{8}{3}\frac{GM}{c^3}(\partial_{tab}U_{\rm ext} + u^c\partial_{abc}U_{\rm ext}) + O(c^{-4}), \qquad (12.9)$$

where $\mathcal{E}^{\sharp}_{ab}$ is the expression previously displayed in Eq. (9.7). Specializing to a two-body system, and implementing the change of notation described at the beginning of Sec. X, this is

$$\mathcal{E}_{ab} = \mathcal{E}^{\sharp}_{ab} + 40\frac{G^2 m_1 m_2}{c^3}\frac{n_{\langle abc\rangle}u^c}{b^4} + O(c^{-4}), \qquad (12.10)$$

where $\mathcal{E}^{\sharp}_{ab}$ is now the expression shown in Eq. (7.10) of Taylor and Poisson [55], added to the spin terms displayed in Eq. (10.4). Specializing further to a two-body system in circular motion, the true tidal moments are given by

$$\mathcal{E}_{ab} = \mathcal{E}^{\sharp}_{ab} - 16\frac{Gm_2}{b^3}\frac{m_1}{m}(u/c)^3 n_{(a}\phi_{b)}, \qquad (12.11)$$

where $\mathcal{E}^{\sharp}_{ab}$ now stands for the expression of Eq. (12.8).

For a binary system in circular motion, Eq. (12.8) implies that $\mathcal{E}_{ab}$ and $\mathcal{E}^{\sharp}_{ab}$ differ only by a constant shift in the phase. The equation can indeed be rewritten as

$$\mathcal{E}_{ab} = \mathcal{E}^{\sharp}_{ab} + \frac{8}{3}\frac{GM\omega}{c^3}\partial_\phi \mathcal{E}^{\sharp}_{ab} + O(c^{-6}), \tag{12.12}$$

and this can in turn be expressed as

$$\mathcal{E}_{ab}(\phi) = \mathcal{E}^{\sharp}_{ab}(\phi + \delta\phi), \qquad \delta\phi \coloneqq \frac{8}{3}\frac{m_1}{m}(u/c)^3 + O(c^{-5}). \tag{12.13}$$

It is easy to show that this equation is compatible with Eq. (12.11). This constant phase shift is largely uninteresting, but the more general situation described by Eq. (12.10) can indeed describe interesting, time-dependent phasing effects.

ACKNOWLEDGMENTS

I am grateful to the Institut d'Astrophysique de Paris, where part of this work was completed, for its warm hospitality. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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