# Noncommutative field with constant background fields and neutral fermions

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Introducing constant background fields into the noncommutative gauge theory, we first obtain a Hermitian fermion Lagrangian which involves a Lorentz violation term, then we generalize it to a new deformed canonical noncommutation relations for fermion field. Massless neutrino oscillation in the deformed canonical noncommutation relations is analyzed. The restriction of the noncommutative coefficients is also discussed. By comparing with the existing experimental data of conventional neutrino oscillations, the order of noncommutative deformed coefficients is given from different ways.

## I. INTRODUCTION

Neutrinos are massless particles in the Standard Model of particle physics. In order to explain neutrino oscillation, the conventional scenario is to assume that neutrino physics is beyond the Standard Model, and then get neutrinos with nonzero mass [1–3], where is a spectrum of three or more neutrino mass eigenstates, and flavor state is a mixing one of mass eigenstates [4]. D. Colladay, V. A. Kostelecký [5], S. Coleman and S. L. Glashow [6] noted that neutrino oscillations may take place for massless neutrinos in which case that Lorentz invariance is violated in the neutrino sector.

Lorentz invariance violation could give a preparative solution to two important experimental problems, namely, the observation of TeV photons and of cosmic ray events above the Greisen-Zatsepin-Kuz'min cutoff [6–11], since the threshold energy at which the cutoff occurs could be altered by modifying special relativity. The high energy tests of Lorentz violating depended on a perturbative framework for neutrinos was discussed in Ref. [7]. The consequences of Lorentz and *CPT* violations to the three-generation neutrino oscillations were abundantly analyzed in the massless neutrino sector [12–14]. The Lorentz-violating extension of the minimal Standard Model [15] including *CPT*-even and *CPT*-odd terms were also studied, the stability and causality are investigated in quantum field theories that incorporate Lorentz and *CPT* violations [16].

The violation of Lorentz invariance may arise from quantum gravity [17–19], random dynamics [20], string theory [21], and field theories with gravity [22]. Another approach is noncommutative field theory [23,24], where any realistic noncommutative theory is found to be physically equivalent to a subset of a general Lorentz-violating Standard-Model extension involving ordinary fields. Spontaneous violation of *CPT*/Lorentz symmetries in string theories is also well known [25], where Lorentz symmetry breakdown is natural when the perturbative string vacuum is unstable.

In order to get the oscillation of massless neutrinos, the introduction of Lorentz- or (and) *CPT*-violating items into the Lagrangian is needed, but for this introduction, people often introduce directly from the Lorentz-violating extension of the Standard Model, instead of introducing directly by noncommutative fields theory. Different from the conventional way, J. M. Carmona *et al.* [26] consider the Hamiltonian of free complex bosonic field theory and investigate the connection between Lorentz invariance violation and noncommutativity of fields by proposing a new Moyal product between functionals, which is consistent with the commutation relations. In that case, they only discuss the effects of neutrinos on noncommutative space.

In this paper, since all noncommutative effects vanish for neutral fermions in noncommutative gauge theories, we introduce neither a new Moyal product nor a Lorentz-violation extension of the Standard Model, but mainly focus on the constant background fields in noncommutative gauge theories, and obtain a Hermitian fermion Lagrangian which involves a Lorentz violation term. The Lorentz violation term deforms the conventional canonical commutation relations. From this we try to generalize our model to new deformed canonical noncommutation relations for the fermion field. On this basis of the new canonical noncommutativity, we obtain the massless neutrino Lagrangian which satisfies the deformed canonical relations. Subsequently, we investigate the restriction of the noncommutative coefficients. Comparing with the existing

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experimental data on conventional neutrino oscillations, finally the order of noncommutative deformation coefficients is given by different ways.

## II. NONCOMMUTATIVE COEFFICIENTS OF NEUTRAL FERMIONS WITH CONSTANT BACKGROUND FIELDS

The idea that the spacetime coordinates do not commute is quite old in mathematics and physics [27]. Noncommutativity [28–32] is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables, such as position and momentum. A. Connes *et al.* [33] build a noncommutative geometry system, and try to apply it to physics or even the Standard Model. N. Seiberg and E. Witten [34] extend earlier ideas about the appearance of noncommutative geometry in string theory with a nonzero $B$ field, and get an equivalence between ordinary gauge fields and noncommutative gauge fields, which is realized by a change of variables. The spacetime noncommutativity undergoes a revival following the realization which occurs naturally in string theory [35].

In noncommutativity framework, the canonical commutator of the coordinates in the spacetime manifold is

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \qquad [\partial_i, \partial_j] = 0, \qquad (1)$$

with a parameter $\theta$ which is a real and antisymmetric (constant) tensor. In addition, there are operator and Lie algebra forms

$$[x^\mu, x^\nu] = i\hat{\theta}^{\mu\nu}, \qquad [x^\mu, x^\nu] = iC_k^{\mu\nu}x^k. \qquad (2)$$

In the case of Eq. (1), a simple set of derivatives $\partial_i$ can be defined by the relations

$$\partial_i x^j = \delta_i^j, \qquad [\partial_i, \partial_j] = 0, \qquad (3)$$

and the Leibnitz rule.

To construct a noncommutative quantum field theory, one approach is to replace conventional fields with noncommutative fields, and conventional products with Moyal star products [36], namely,

$$f(x) \star g(x) = \exp\left(\frac{1}{2}i\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}\right)f(x)g(x)|_{x=y}. \qquad (4)$$

It is well known that as noncommutative phase space one can occasionally generalize Eq. (3) to

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \qquad [\partial_\mu, \partial_\nu] = -i\Phi_{\mu\nu} \qquad (5)$$

to incorporate an additional background field [25].

Gauge theories, such as quantum electrodynamics (QED), are ordinarily transformed into noncommutative

QED [28] by Moyal star products. The actions of pure noncommutative $U(1)$ Yang-Mills and matter field are, respectively, [37]

$$S_{\rm YM} = \int d^4x \left(-\frac{1}{4q^2}\right)\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu},$$

$$S_{\rm Matter} = \int d^4x \left(\frac{1}{2}i\hat{\bar{\psi}}\gamma^\mu \star \overset{\leftrightarrow}{\hat{D}}_\mu\hat{\psi} - m\hat{\bar{\psi}} \star \hat{\psi}\right)$$

where $q$ is the coupling constant. Then for noncommutative QED, the Hermitian Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}i\hat{\bar{\psi}}\gamma^\mu \star \overset{\leftrightarrow}{\hat{D}}_\mu\hat{\psi} - m\hat{\bar{\psi}} \star \hat{\psi} - \frac{1}{4q^2}\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}, \qquad (6)$$

where $\hat{A}_\mu$ and $\hat{\psi}$ are the fields in noncommutative QED space, and $A_\mu$ together with $\psi$ are for conventional QED, and have

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star,$$

$$\hat{D}_\mu\hat{\psi} = \partial_\mu\hat{\psi} - i\hat{A}_\mu \star \hat{\psi},$$

$$\hat{A} \star \overset{\leftrightarrow}{\hat{D}}_\mu\hat{B} = \hat{A} \star \hat{D}_\mu\hat{B} - \hat{B} \star \hat{D}_\mu\hat{A}. \qquad (7)$$

Noncommutative coefficient $\theta^{\mu\nu}$ are acceptable only when $\theta^{0k} = 0, \theta^{ij} \neq 0 (i, j = 1, 2, 3)$, on account of the difficulties with perturbative unitarity [38]. Nonetheless, presumably certain cases with $\theta^{0j} \neq 0$ can also be allowed, since the presence of observer Lorentz invariance implies that there are no difficulties with perturbative unitarity provided $\theta_{\mu\nu}\theta^{\mu\nu} > 0, \theta_{\mu\nu}\tilde{\theta}^{\mu\nu} = 0$, which can be converted into one with only $\theta^{jk}$ nonzero by a suitable observer Lorentz transformation. It is similar to the condition applying for open bosonic stings where the presence of a nonzero $B^{jk}$ field is equivalent to a constant magnetic field on a $D_p$ brane [34].

N. Seiberg and E. Witten [34] proposed a map from the noncommutative $U(N)$ gauge theory to an ordinary $U(N)$ gauge theory. In 2002, Chaichian, Prešnajder *et al.* [39] presented the no-go theorem; this theorem showed that matter fields in the noncommutative $U(1)$ gauge theory can only have $\pm 1$ or 0 charges and for a generic noncommutative $\Pi_{i=1}^n U(N_i)$ gauge theory, matter fields can be charged under at most two of the $U(N_i)$ gauge group factors. In Refs. [40–43], the authors argued that the Seiberg-Witten map, which relates a noncommutative gauge theory to an ordinary one, paves the way for constructing the noncommutative version of gauge theories based on generic Lie algebras with matter fields in generic representations, thus circumventing the restrictions discussed in Ref. [39]. However, as shown in Ref. [44], the Seiberg-Witten map can only be consistently defined and

used for the gauge theories which respect the no-go theorem.

To the first order, the Seiberg-Witten map reads [34,45–47]

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\gamma\rho}A_\gamma(\partial_\rho A_\mu + F_{\rho\mu})$$

$$\hat{\psi} = \psi - \frac{1}{2}\theta^{\gamma\rho}\partial_\rho\psi. \qquad (8)$$

Substitute Eq. (4) and the Seiberg-Witten map equation (8) into the Lagrangian (6), then in the new Lagrangian the gauge invariant is given manifestly, and it consists of ordinary QED plus nonrenormalizable Lorentz-violating corrections. It should be noted that noncommutative effects vanish for neutral fermions. One will get the same results if one use the Moyal star products [Eq. (4)] in the case of a canonical commutator of the coordinates, Eq. (1), because of such a property of the Moyal star products

$$\int d^4x f(x) \star g(x) = \int d^4x g(x) \star f(x) = \int d^4x f(x)g(x). \qquad (9)$$

In the no-go theorem, neutral fermions could be in "adjoint" representation of noncommutative $U(1)$ gauge theory. The neutral fermions field, although is not carrying any $U(1)$ charge, similarly to noncommutative photons, carries the corresponding dipole moment [39]. Considering the above situation, in order to get other noncommutative effects of neutral fermions, it seems to be a reasonable consideration by generalizing its Lagrangian to noncommutative phase space [depending on Eq. (5)], which is equivalent to attach a constant background field. But we will encounter some complex issues.

In this paper, in order to avoid these difficulties above, we mainly pay attention to these cases with constant electromagnetic background fields in noncommutative fields, and try to generalize this model to new canonical noncommutation relations. In the new noncommutativity, we assume that all fermions be applied to these new relations; then we investigate the neutrino Lagrangian which involves a Lorentz violation term.

One can get the relation (6) when one considers noncommutative QED under the canonical commutation relation, Eq. (1), and then substitutes the Seiberg-Witten map [Eq. (8)] into the Lagrangian equation (6). The relation at the leading order in noncommutative coefficients is given as

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}\gamma^\mu\overleftrightarrow{D}_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Theta(\theta^2)$$

$$+ \frac{1}{8}\theta^{\alpha\beta}q[2mF_{\alpha\beta}\bar{\psi}\psi + F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - 4F_{\alpha\mu}F_{\beta\nu}F^{\mu\nu}]$$

$$+ \frac{1}{8}i\theta^{\alpha\beta}q[2F_{\alpha\mu}\bar{\psi}\gamma^\mu\overleftrightarrow{D}_\beta\psi - F_{\alpha\beta}\bar{\psi}\gamma^\mu\overleftrightarrow{D}_\mu\psi]. \qquad (10)$$

where we have redefined $A_\mu \to qA_\mu$. If we make the replacement

$$F_{\alpha\beta} \to F_{\alpha\beta} + f_{\alpha\beta}, \qquad (11)$$

where $f_{\alpha\beta}$ is a constant background field and $F_{\alpha\beta}$ is understood to be a small fluctuation, it yields the Hermitian Lagrangian describing the leading-order effects of noncommutativity in constant background fields

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}\gamma^\mu\overleftrightarrow{D}^\nu(g_{\mu\nu} + \eta_{\mu\nu})\psi - m\bar{\psi}\psi$$

$$- \frac{1}{4}(g_{\alpha\mu}g_{\beta\nu} + \kappa_{\alpha\beta\mu\nu})F^{\alpha\beta}F^{\mu\nu} + \Theta(\eta^2, \kappa^2), \quad (12)$$

where $g_{\mu\nu}$ is the metric tensor, dimensionless coefficients $\eta_{\mu\nu}$ and $\kappa_{\alpha\beta\mu\nu}$ are constant tensors associated with both the constant electromagnetic fields and the coefficients of the canonical commutator,

$$\eta_{\mu\nu} = -1/2qf_\mu{}^\sigma\theta_{\sigma\nu}, \qquad (13)$$

where $q$ is replaced with a scaled effective value $q \to (1 + 1/4qf^{\mu\sigma}\theta_{\mu\sigma})q$. If one removes the mass $m$, and lets electromagnetic background fluctuation $F_{\mu\nu} \to 0$, then an effective Lagrangian is given

$$\mathcal{L} = \frac{1}{2}ig_{\mu\nu}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi + \frac{1}{2}i\gamma^\mu\eta_{\mu\nu}\bar{\psi}\overleftrightarrow{\partial}^\nu\psi. \qquad (14)$$

Next, we generalize Eq. (14) to coupled field $\psi_{i,j}$,

$$\mathcal{L} = \frac{1}{2}i\bar{\psi}_i\gamma^\mu\overleftrightarrow{\partial}_\mu\psi_i + \frac{1}{2}i\eta_{\mu\nu(ij)}\bar{\psi}_i\gamma^\mu\overleftrightarrow{\partial}^\nu\psi_j, \qquad (15)$$

where $\eta_{\mu\nu(ij)}$ violate Lorentz invariance. Then we can obtain the conjugate momentum and the canonical commutation relations for the fermion field,

$$\pi_i = i(\delta_{ij} + \eta_{00(ij)})\psi_j^\dagger = \Lambda_{ij}^{-1}\psi_j^\dagger$$

$$\{\psi_i(x), \psi_j^\dagger(y)\} = \Lambda_{ij}\delta(x - y), \qquad (16)$$

with the coefficient of the constant background field $\eta_{\mu\nu(ij)}$ diagonal in the spacetime indices, where the Lorentz violation term deforms the conventional canonical commutation relations.

Next, let us put aside the background field, and assume that Eq. (16) applies to all fermion fields. In fact, for the fermion field, Eq. (16) inspired us to build new noncommutativity relations. We may assume a new noncommutativity by deforming the commutator of fields in analogy with the deformation commutator, such as Eq. (16). If it preserves the locality in the new set of canonical commutation relations, they become

$$\{\pi_\alpha(x), \pi_\beta(y)\} = 0,$$
$$\{\psi_i(x), \psi_j^\dagger(y)\} = \Lambda_{ij}\delta(x - y). \qquad (17)$$

Now we can obtain an effective massless neutrino Lagrangian which satisfies this deformation noncommutative condition, Eqs. (16) and (17),

$$\mathcal{L} = \frac{1}{2}i\bar\nu_\alpha\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\nu_\alpha + \frac{1}{2}i\eta_{\mu\nu(\alpha\beta)}\bar\nu_\alpha\gamma^\mu\overset{\leftrightarrow}{\partial^\nu}\nu_\beta \qquad (18)$$

where $\alpha, \beta$ are flavor indices, $\mu, \nu$ are spinor indices, and the constant tensor $\eta_{\mu\nu(\alpha\beta)}$ is related to the flavor and spinor indices. There it contains renormalizable Lorentz-violating corrections. On the basis, the Lagrangian may provide a general rotation-invariant model of three active massless neutrinos in noncommutative field theory.

The coefficients $\eta$ on deformation noncommutation relations violates Lorentz invariance but not *CPT* symmetry which is in agreement with Refs. [48,49]. A general form for the quadratic sector of a renormalizable Lorentz and *CPT*-violating Lagrangian that describes a single massive spin-$\frac{1}{2}$ is given in Ref. [16]

$$\mathcal{L} = \frac{1}{2}i\bar\psi\Gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\psi - \bar\psi M\psi, \qquad (19)$$

where

$$\Gamma^\mu := \gamma^\mu + c^{\nu\mu}\gamma_\nu + d^{\nu\mu}\gamma_5\gamma_\nu + e^\mu + if^\mu\gamma_5 + \frac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu},$$

$$M := m + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \frac{1}{2}\rho^{\mu\nu}\sigma_{\mu\nu}. \qquad (20)$$

Coefficients for Lorentz violation are all real, $c_{\mu\nu}$ and $d_{\mu\nu}$ are traceless, $g_{\lambda\mu\nu}$ antisymmetric in its first two indices, and $\rho_{\mu\nu}$ antisymmetric. Just as it is pointed out in Ref. [16] that all the parameters violate Lorentz invariance, $a_\mu, b_\mu, e_\mu, f_\mu, g_{\lambda\mu\nu}$ also break *CPT*. Under the above assumptions, these parameters about Lorentz and *CPT* violation are given when they are applied to massless neutrinos [50]. On this base, Ref. [50] discussed the general equations of motion for the free propagation of neutrinos, which can be written as a first-order differential operator acting on the object $\nu_\beta$.

Now we go back to Eq. (18), where the constant tensor $\eta_{\mu\nu}$ denotes the noncommutative effects, and it violates Lorentz invariance but keeps *CPT* symmetry. For the convenience of calculation, the Lagrangian can be transformed into another form since divergence terms have no contribution to the action

$$\mathcal{L} = \frac{1}{2}i\bar\nu_\alpha\gamma^\mu\partial_\mu\nu_\alpha + \frac{1}{2}i\eta_{\mu\nu(\alpha\beta)}\bar\nu_\alpha\gamma^\mu\partial^\nu\nu_\beta, \qquad (21)$$

and the Euler Lagrange equation of motion satisfies

$$i\gamma^0\partial_0\nu_\alpha + i\gamma^i\partial_i\nu_\alpha + i\eta_{\mu\nu(\alpha\beta)}\gamma^\mu\partial^\nu\nu_\beta = 0. \qquad (22)$$

Comparing Eq. (22) with the conventional equations of motion $(i\delta_{\alpha\beta}\partial_0 - \mathcal{H}_{\alpha\beta})\nu_\alpha = 0$, the Hamiltonian for neutrinos is given as

$$\mathcal{H}_{\alpha\beta} = -i\gamma^0\gamma^i\partial_i - i\eta_{\mu\nu(\alpha\beta)}\gamma^0\gamma^\mu\partial^\nu. \qquad (23)$$

In this paper, there are four parameters: $\theta'^{\mu\nu}$ in the canonical commutator of the coordinates in the spacetime manifold (1), $\theta^{\mu\nu}$ in the noncommutative field theory (6), $\eta^{\mu\nu}$ in the new deformed canonical commutation relations (18), and $\eta^{\mu\nu}$ in the Lorentz- and *CPT*-violating extension of the Standard Model (19), respectively. As in the above analysis, in the canonical commutator of the coordinates, $\theta'^{\mu\nu}$ is a real and antisymmetric (constant) tensor; in the noncommutative field theory it meets this relationship $\theta^{\mu\nu}\theta_{\mu\nu} > 0$ and $\theta^{\mu\nu}\tilde\theta_{\mu\nu} = 0$. The new neutrino Lagrangian and the new commutation relation (18) requiring Hermiticity implies the noncommutative coefficients $\eta^{\mu\nu}$ are Hermitian in generation space. Some features of this model are similar to the conventional massive-neutrino case; however, there is unusual energy dependence. Moreover, for the single fermion extension model (19), the dimensionless coefficients $\eta^{00} > 0$ and $\eta^{00} < 0$ (which resembles the noncommutative coefficient $\eta_{\mu\nu}$) imply that instabilities and microcausality violation arise at the Planck scale [16]. However, in Eq. (18), we do not need to consider this condition. The restriction of rotation invariance provides an additional special limit of the theory equation (18), which can significantly reduce the complexity of calculations. In addition, restricting Hamiltonian to rotation invariant leaves two coefficients $\eta_{\alpha\beta}^{00}$ and $\eta_{\alpha\beta}^{ij} = \frac{1}{3}\eta_{\alpha\beta}^{kk}\delta^{ij}$. It may be assumed $\eta_{\alpha\beta}^{00} - \eta_{\alpha\beta}^{jj} = 0$ since the trace component $g_{\mu\nu}\eta^{\mu\nu}$ is Lorentz invariant and can be absorbed into the usual kinetic term (it is unobservable), so only one of two matrices with noncommutative coefficients is independent, and leaves only the matrices $\eta_{00(\alpha\beta)}$.

## III. THE ORDER OF THE NONCOMMUTATIVE COEFFICIENT

From what has been discussed above, the Hamiltonian of massless neutrino in deformation canonical commutation relations space is given, and the restriction to the noncommutative coefficients $\eta$ is also discussed. In the following sections, we will discuss the order of the noncommutative coefficient by means of different methods (more detail can be found in Refs. [50,51]).

With the relation (18), the Hamiltonian matrix for neutrinos can be given, substituting it into the Euler-Lagrange equation of motion, and the general dynamical equation for neutrinos could be obtained as

$$h_{\alpha\beta} = |\vec{p}|\delta_{\alpha\beta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \frac{1}{|\vec{p}|}\begin{pmatrix} -[\eta^{\mu\nu}p_\mu p_\nu]_{\alpha\beta} & 0 \\ 0 & -[\eta^{\mu\nu}p_\mu p_\nu]^*_{\alpha\beta} \end{pmatrix}. \quad (24)$$

The four-dimension momentum may be taken as $p_\mu = (|\vec{p}|, -\vec{p})$ at leading order.

For the convenience of the following discussions, we first consider the conventional neutrino oscillation. In the mechanism of the conventional oscillation of the massive-neutrino, the simplest form of probability for oscillation between two species of particles with a mixing angle $i, j$ and energy levels $E_i, E_j$ is given as

$$P_{i\to i} = 1 - \sin^2(2\theta)\sin^2\left(\frac{1}{4E}\Delta m^2 L\right),$$

$$P_{i\to j} = \sin^2(2\theta_{ij})\sin^2\left(\frac{1}{2}\Delta E_{ij}t\right)$$

$$= \sin^2(2\theta_{ij})\sin^2\left[\frac{1}{2}\left(\sqrt{p^2 + m_i} - \sqrt{p^2 + m_j}\right)t\right]$$

$$\approx \sin^2(2\theta_{ij})\sin^2\left(\frac{1}{4E}\Delta m_{ij}^2 L\right). \quad (25)$$

Then we turn back to the deformation canonical non-commutation relations (16) and the Hamiltonian density (23); the dynamical equation can be written as

$$\dot{\nu}_\alpha = -\Lambda_{\alpha\beta}(\alpha \cdot \nabla \nu_\beta), \quad (26)$$

where $\alpha = \gamma^0\vec{\gamma}$. Turning it into momentum space, the form above can be written as

$$E\nu_\alpha = \Lambda_{\alpha\beta}(\alpha \cdot \vec{p}\nu_\beta). \quad (27)$$

For simplicity, we choose the $\Lambda$ matrix and unitary matrix $\Gamma$ in two dimensions

$$\Lambda = \begin{pmatrix} 1 & \lambda \\ \lambda^* & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \tau & 1/\sqrt{2} \\ -\tau & 1/\sqrt{2} \end{pmatrix}, \quad (28)$$

where $\Gamma$ is diagonalized, and $\Lambda$, $\lambda^*/\tau - \tau\lambda = 0$, and $\lambda$ are associated with $\eta_{\alpha\beta}$. Then the energy spectrum for the neutrino is

$$E_{i,j}^\pm = \pm(1 + \varepsilon^{ij}|\lambda|)|\vec{p}|, \quad (29)$$

with $\varepsilon^{ij}$ the two-dimensional Levi-Civita symbol.

Taking $\nu_1$ and $\nu_2$ as the eigenstates of energy values in Eq. (29), the energy eigenstates can be determined through the diagonalized matrix $\Gamma$

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \Gamma \cdot \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}, \quad (30)$$

where $\nu_{1,2}$ and $\nu_{\alpha,\beta}$ represent different energy eigenstates and the flavor eigenstates. The time evolution can be determined by the energy spectrum as follows:

$$\nu_{i,j}(t) = e^{-iE_{i,j}^+ t + i\vec{p}\cdot\vec{x}}\nu_{1,2}(0)$$

$$= \kappa \cdot (\varepsilon^{ij}\tau \cdot \nu_\alpha + \nu_\beta), \quad (31)$$

with coefficients $\kappa$ relevant to unitary matrix coefficients $\tau$. The above equation can be parametrized as

$$\nu_1 = \cos\theta_{12}\nu_\alpha + \sin\theta_{12}\nu_\beta,$$

$$\nu_2 = -\sin\theta_{12}\nu_\alpha + \cos\theta_{12}\nu_\beta. \quad (32)$$

Depending on relations (30), the above relations can be inverted, then the evolution of the flavor state $\nu_\alpha$ in time $t$ can be given as

$$\nu_\alpha(t) = [(\cos^2\theta_{12}e^{-iE_1^+ t} + \sin^2\theta_{12}e^{-iE_2^+ t})\nu_\alpha(0)$$

$$+ \frac{1}{2}\sin 2\theta_{12}(e^{-iE_1^+ t} - e^{-iE_2^+ t})\nu_\beta(0)]e^{i\vec{p}\cdot\vec{x}}. \quad (33)$$

Then the probability of finding the flavor state $\nu_\beta$ at time $t$ can be given by

$$P_{\nu_\alpha\to\nu_\beta} = \sin^2(2\theta_{12})\sin^2(|\vec{p}||\lambda|t)$$

$$\approx \sin^2(2\theta_{12})\sin^2(E|\lambda|L). \quad (34)$$

The above probability relations can be inverted by taking the fact that for $|\lambda| \ll 1, E \sim |\vec{p}|$, and velocities close to $c, t \to L$ (the path length traversed by the neutrino).

The analysis above for two neutrino flavors can be extended to incorporate three flavors when generalizing the deformation parameters and the mixing angles. Then the probability for oscillation between neutrino flavors can be given as follows:

$$P_{\nu_\alpha\to\nu_\beta} \approx \sin^2(2\theta_{ij})\sin^2(E|\lambda_{ij}|L), \quad (35)$$

where $i, j = 1, 2, 3$. Comparing with the conventional massive-neutrinos relations (25) and using the above results from the solar neutrino and atmospheric neutrino experiments with $\Delta m_{ij}^2$ and $E$, one can get relation $\Delta m_{ij}^2/4E^2 = |\lambda_{ij}|$. The solar and atmospheric neutrino experimental data [52,53] were given as $\Delta m_{12}^2 \sim 7.67 \times 10^{-5}$ eV$^2$, $E \sim 1$ MeV and $\Delta m_{23}^2 \sim 2.32 \times 10^{-3}$ eV$^2$, $E \sim 1.8$ GeV; then the order of the deformation parameters is given as follows:

$$|\lambda_{12}| \sim 10^{-17}, \qquad |\lambda_{23}| \sim 10^{-22}. \qquad (36)$$

Works above only involve the two-generation special case; next, we turn to another useful parametrization of the noncommutative coefficient by CKM-like mixing angles and phases. Restricting $h_{\alpha\beta}$ to rotation-invariant models leaves only coefficients $\eta_{\alpha\beta}^{00}$ and $\eta_{\alpha\beta}^{ij} = \frac{1}{3}\eta_{\alpha\beta}^{kk}\delta^{ij}$. Since the trace component $g_{\mu\nu}\eta^{\mu\nu}$ is Lorentz invariant and can be absorbed into the usual kinetic term (it is unobservable), so it may be assumed as zero for convenience; only one of these matrices is independent. Taking into account the frame dependent of the latter theory, one assumes rotation invariance in the Sun-centered frame $(S, x, y, z)$ for definiteness; then the effective Hamiltonian (24) reduces to the block-diagonal form

$$(h_{\rm sc})_{\alpha\beta} = {\rm diag}\left\{ \left( -\frac{4}{3}\eta^{SS}E \right)_{\alpha\beta}, \ \left( -\frac{4}{3}\eta^{SS}E \right)^{*}_{\alpha\beta} \right\}, \qquad (37)$$

where the irrelevant kinetic term is dropped.

The features of the Hamiltonian above are similar to the conventional massive-neutrino case. It provides a general rotationally invariant model of three active neutrinos. Similar to the analysis of conventional massive-neutrino mixing procedures, the effective Hamiltonian can be diagonalized with a unitary matrix $U_{\rm sc}$,

$$h_{\rm sc} = U_{\rm sc}^{\dagger} E_{\rm sc} U_{\rm sc}, \qquad (38)$$

where $E_{\rm sc}$ is a diagonal matrix.

The deformation noncommutative coefficient matrix can be parametrized with three eigenvalues and a constant unitary matrix $A$. For each coefficient $\eta^{\mu\nu}$ has

$$\eta^{\mu\nu} = (A^{\mu\nu})^{\dagger}{\rm diag}\{\eta_{(1)}^{\mu\nu}, \eta_{(2)}^{\mu\nu}, \eta_{(3)}^{\mu\nu}\}A^{\mu\nu}. \qquad (39)$$

The unitary diagonalizing matrices $A^{\mu\nu}$ are given so that the Hamiltonian above takes the block diagonal form as

$$U_{\rm sc} = \begin{pmatrix} A & 0 \\ 0 & A^{*} \end{pmatrix}. \qquad (40)$$

The above decomposition is frame dependent as we have discussed; there is one restriction to this decomposition to the standard Sun-centered celestial equatorial frame as mentioned. In order to get a CKM-like decomposition of the $U$ matrices we take mixing angles and phases with $\eta^{\mu\nu}$ by $\beta_{(12)}^{\mu\nu}, \beta_{(13)}^{\mu\nu}, \beta_{(23)}^{\mu\nu}$ and $\delta^{\mu\nu}, \gamma_1^{\mu\nu}, \gamma_2^{\mu\nu}$. Then the $U$ matrices can be written explicitly. Here, we ignore the tedious steps,

$$U^{\mu\nu} = ({\rm CKM-like})\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma_1^{\mu\nu}} & 0 \\ 0 & 0 & e^{i\gamma_2^{\mu\nu}} \end{pmatrix}. \qquad (41)$$

In addition, considering that the $\gamma$ matrix of phases can be absorbed into the amplitudes in the conventional massive-neutrino analysis, so these $\gamma$ phases can be neglected.

We replace $\eta^{\mu\nu}$ with $\eta^{SS}$ since we have assumed rotation invariance in the Sun-centered frame. To mimic the usual massive-neutrino solution, there we consider only taking vanishing phases, $\theta_{13}^{SS}$ and $\theta_{23}^{SS} = \pi/4$. This leaves three degrees of freedom, two eigenvalue differences, and one mixing angle $\theta_{12}^{SS}$.

Using the time evolution operator $S_{\alpha\beta} = U_{\rm sc}^{\dagger}e^{-iE_{\rm sc}t}U_{\rm sc}$, then the probabilities for massless neutrino of $\nu_\beta$ oscillating into neutrino of $\nu_\alpha$ in time $t$ can be written as $P_{\nu_\beta \to \nu_\alpha} = |S_{\nu_\alpha\nu_\beta}(t)|^2$. The probabilities are given as

$$P_{\nu_e \to \nu_\mu} = P_{\nu_e \to \nu_\tau} = \frac{1}{2}\sin 2\theta \sin^2(\lambda \cdot EL/2)$$
$$P_{\nu_e \to \nu_e} = 1 - \sin 2\theta \sin^2(\lambda \cdot EL/2), \qquad (42)$$

where $\lambda = \frac{4}{3}[\eta_{(2)}^{SS} - \eta_{(1)}^{SS}]$. Considering the LSND and KamLAND results $P_{\bar{\nu}_e \to \bar{\nu}_e} = 61\%$, $1$ Mev $\leq E \leq 10$ Mev and $L = 138 \sim 214$ km, and $P_{\bar{\nu}_e \to \bar{\nu}_e} = 26\%$, $E \sim 45$ Mev and $L = 30$ m, replacing $\Delta m^2/2E$ with $\lambda E$ as the previous calculation methods, we get

$$\lambda \sim 10^{-17}. \qquad (43)$$

In addition, considering from this relation (24), one just studies the left-handed neutrino and ignores the right-handed antineutrinos because the two terms have the similar form; then one has the simplified Hamiltonian as

$$h = |\vec{p}|\delta_{\alpha\beta} + \eta_{\mu\nu(\alpha\beta)}\frac{p^\mu p^\nu}{|\vec{p}|}, \qquad (44)$$

where the minus sign has been incorporated into the deformation noncommutative coefficients.

To reduce the deformation noncommutative coefficients, we assume the neutrino Hamiltonian matrix can be simplified and then diagonalized by unitary matrix $\Gamma$,

$$\Gamma^{\dagger}\mathcal{H}\Gamma = \Gamma \begin{pmatrix} E & \eta_{e\mu}^{00}E & 0 \\ \eta_{e\mu}^{00}E & E + \eta_{\mu\mu}^{00}E & \eta_{\mu\tau}^{00}E \\ 0 & \eta_{\mu\tau}^{00}E & E \end{pmatrix}\Gamma^{\dagger}$$
$$= E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + A - B & 0 \\ 0 & 0 & 1 + A + B \end{pmatrix}, \qquad (45)$$

where $\eta_{e\mu}^{00}$, $\eta_{\mu\tau}^{00}$, and $\eta_{\mu\mu}^{00}$ are three nonzero noncommutative parameters in the specific model, the other noncommutative parameters are zero, and

$$A = \eta_{\mu\mu}^{00}/2, \qquad B = \sqrt{(\eta_{e\mu}^{00})^2 + (\eta_{\mu\tau}^{00})^2 + (\eta_{\mu\mu}^{00})^2/4}. \quad (46)$$

Then the eigenstates can be the linear combination of flavor eigenstates since the unitary matrix $\Gamma$.

Using the same method mentioned above, with the unitary matrix $\Gamma$ and the diagonal eigenenergy matrix, finally the oscillation probabilities are given as

$$P_{\nu_e \to \nu_\mu} = \frac{(\eta_{e\mu}^{00})^2}{B^2} \cdot \sin^2(\text{BEL}),$$

$$P_{\nu_\mu \to \nu_\tau} = \frac{(\eta_{\mu\tau}^{00})^2}{B^2} \cdot \sin^2(\text{BEL}),$$

$$P_{\nu_\mu \to \nu_\mu} = 1 - \frac{(\eta_{e\mu}^{00})^2 + (\eta_{\mu\tau}^{00})^2}{B^2} \cdot \sin^2(\text{BEL}). \quad (47)$$

Comparing the above equation with the analysis of the oscillation on noncommutative space such as that mentioned in the previous content, then the order of noncommutative parameters can be identified

$$|\eta_{e\mu}^{00}| \sim 10^{-17}, \qquad |\eta_{\mu\tau}^{00}| \sim 10^{-22}. \quad (48)$$

Depending on the data above, the remaining coefficient $\eta_{\mu\mu}^{00}$ can be chosen to match other experimental data, and the order of the remaining noncommutative parameters can be identified. Comparing in three relations, (36), (43), and (48), it is easy to find that one can get the same order of noncommutative coefficients by different methods which is expected in advance. Just as shown in this paper, the Lagrangian which involves the new deformation canonical noncommutation relations may give a model of three active massless neutrinos.

## IV. SUMMARY

In order to get the oscillation of massless neutrinos, the conventional scenario is depending on introducing the Lorentz-violation extension of the Standard Model. Different from the previous treatment, in this paper we try to get oscillation of the massless neutrinos mainly by introducing new canonical anticommutation relations. Considering that all noncommutative effects vanish for neutral fermions in noncommutative gauge theory, and the generalization of Lagrangian to noncommutative phase space will encounter some complex issues (such as the Moyal star products can not apply to this case), we directly introduce constant background fields into the noncommutative gauge theory and obtain a Hermitian fermion Lagrangian which involves a Lorentz violation term. The Lorentz violation term deforms the conventional canonical commutation relations. Then we try to generalize this model to a new deformed canonical noncommutation relations for the fermion field. On this basis of the new deformed canonical noncommutation relations, we investigate the massless neutrino Lagrangian which satisfies the deformed canonical relations. The coefficient $\eta$ in the massless neutrino Lagrangian violates Lorentz invariance but keeps $CPT$ symmetry; the restriction of the noncommutative coefficients is also discussed. By comparing with the existing experimental data of conventional neutrino oscillations, the order of noncommutative deformed coefficients is given. From the discussion above, we conclude that the Lagrangian which satisfies deformed canonical noncommutation relations may give a model of three active massless neutrinos.

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