

Linking $U(2) \times U(2)$ to $O(4)$ model via decoupling

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The nature of chiral phase transition of massless two flavor QCD depends on the fate of flavor singlet axial symmetry $U_A(1)$ at the critical temperature (T_c). Assuming that a finite $U_A(1)$ breaking remains at T_c , the corresponding three dimensional effective theory is composed of four massless and four massive scalar fields. We study the renormalization group flow of the effective theory in the ϵ -expansion, using a mass dependent renormalization scheme, and determine the region of the attractive basin flowing into the $O(4)$ fixed point with a focus on its dependence on the size of the $U_A(1)$ breaking. The result is discussed from a perspective of the decoupling of massive fields. It is pointed out that, although the effective theory inside the attractive basin eventually reaches the $O(4)$ fixed point, the approaching rate, one of the universal exponents, is different from that of the standard $O(4)$ model. We present the reason for this peculiarity, and propose a novel possibility for chiral phase transition in two-flavor QCD.

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I. INTRODUCTION

Quantum chromodynamics (QCD) is a unique gauge theory in that its nonperturbative phenomena are experimentally observable and thus one can test our understanding on nonperturbative dynamics quantitatively. Understanding the underlying principles of nonperturbative dynamics is not only important in its own right but also interesting because it could provide a solid basis for studying other hypothetical strong coupling gauge theories.

In this paper, we address chiral phase transition of massless two-flavor QCD at vanishing density. This system is obviously different from QCD in real world as it consists of massive flavors,¹ and hence studying this system may be considered to be academic. On the other hand, since this system can be seen as one of extreme cases of real QCD, precise knowledge on this system could provide with foundations for understanding phase diagrams of real QCD as a function of chemical potential, quark masses, or the number of flavors, etc.

The order of chiral phase transition of massless two flavor QCD has been studied in uncountably many works both analytically and numerically, but not settled yet [7]. One of analytical methods is to examine the renormalization group (RG) flow of the corresponding Landau-Ginzburg-Wilson (LGW) theory. In 1983, Pisarski and Wilczek revisited the β functions of linear sigma models (LSMs) calculated in the ϵ expansion and classified by the resulting RG flow the nature of chiral phase transition of QCD with arbitrary number of massless flavors [8]. However, the two flavor case remained uncertain because two distinct effective theories are possible, depending on

the presence of flavor singlet axial [$U_A(1)$] symmetry at the critical temperature (T_c), and they draw different conclusions.

In the case where a large $U_A(1)$ symmetry breaking remains at T_c , $O(4)$ LSM should be analyzed. $O(N)$ LSM has been well studied again both analytically and numerically,² and the existence of the stable infrared fixed point (IRFP), or the Wilson-Fisher fixed point, is established.

On the other hand, when the $U_A(1)$ symmetry is effectively and fully restored at T_c , the symmetry of the system turns to $U_L(2) \times U_R(2)$ [or $O(2) \times O(4)$]. This case has been also studied through various methods and is attracting attention [15–23]. It appears that the nature of the transition in this system is still under debate.

Numerical simulations based on lattice QCD can directly determine the nature of the transition of massless two-flavor QCD without any assumption, in principle. Interestingly, a possibility of first order phase transition is recently reported in one of the lattice calculations [24], while there remain many systematic uncertainties to be checked.

In this work, we will not pursue whether the $U_L(2) \times U_R(2)$ model has an IRFP or not, and would rather focus on the case where the $U_A(1)$ symmetry breaking is small but finite at T_c . Although the size of the symmetry breaking at T_c is determined by nonperturbative dynamics and its precise value is not known yet, it is probable from recent studies that the breaking effect is not large [5,25–28].

This system is interesting from the field theoretical viewpoint. $U_L(2) \times U_R(2)$ LSM contains eight degenerate scalar fields, and by introducing the breaking, half of them gain mass proportional to the breaking. When the size of breaking is infinitely large, the system is simply reduced to

¹For the lattice studies of chiral transition of realistic $2+1$ flavor QCD, see, for example, Refs. [1–6].

²See, for example, Refs. [9–14].

the $O(4)$ LSM and will end up with second order phase transition [8]. Even if the breaking is tiny, we expect that the massive degrees of freedom will decouple from the system and $O(4)$ LSM is eventually realized as the flow goes into the infrared limit. However, we are concerned that the decoupling theorem [29,30] is not obvious in three dimensions because the scalar quartic couplings have a mass dimension.

For example, four-point Green's functions can, in general, have a term like $\bar{g}^2(P^2)/M^2$ due to massive fields with a mass M , where P represents a typical scale of external momenta and $\bar{g}(P^2)$ is an effective quartic coupling connecting light and heavy fields. In three dimensions, $\bar{g}(P^2)$ has a mass dimension, and whether $\bar{g}^2(P^2)/M^2$ vanishes in the $P^2 \rightarrow 0$ limit is determined by P^2 dependence of the running of $\bar{g}^2(P^2)$. Indeed, the presence of nondecoupling effects is reported in Ref. [31], where a theory with a dimensionful scalar cubic coupling is examined in $3+1$ dimensions. It is thus interesting to see in the context of the RG flow how or even whether the decoupling occurs.

We take the ϵ expansion approach to study this system since the ϵ is suitable for investigating the detailed structure of the decoupling on a fundamental level. The calculation is done mainly in a mass-dependent renormalization scheme such that β functions contain information on finite mass of would-be decoupling particles. The consistency with the $\overline{\text{MS}}$ scheme is checked through the calculation of four-point correlation function. As for other parts, we simply follow the standard. With β functions thus obtained, we determine the attractive basin flowing into the $O(4)$ (or Wilson-Fisher) fixed point and see how the area of the basin is affected by the size of the $U_A(1)$ breaking.

We point out that, although the effective theory starting from the inside of the attractive basin eventually reaches the $O(4)$ fixed point, one of the universal exponents turns out to differ from that of the standard $O(4)$ LSM. We present the reason for this peculiarity and propose a novel possibility for chiral phase transition in two-flavor QCD, that is second order phase transition with, say, the $U_A(1)$ broken scaling.

The same system has been studied in the functional renormalization group (FRG) approach in Ref. [32], where the phase transition in the presence of a finite $U_A(1)$ breaking is concluded to be of first order. Since the β functions calculated in the ϵ expansion are embedded in the FRG, the same conclusion is naively expected to be reached. However, our conclusion is different from theirs.

Determining the order of the chiral phase transition of massless two-flavor QCD has some impact on models of dynamical electroweak symmetry breaking with electroweak baryogenesis. For an attempt on the lattice, see Ref. [33,34].

Our analysis is performed at the leading order of the ϵ expansion. Thus, our findings may be significantly affected

by higher orders in the expansion. Furthermore, it is pointed out that the ϵ expansion is sometimes not useful even for qualitative discussions [19]. Nevertheless, we believe that the ϵ expansion suffices for exploring possible scenarios and making a survey of how the decoupling of massive fields occurs along the flow toward the infrared limit.

The rest of the paper is organized as follows. In Sec. II, the effective theory we will discuss is introduced. We briefly summarize the leading ϵ expansion results for the large and vanishing limits of the $U_A(1)$ breaking in Sec. III. The β functions and the RG flow in the presence of a finite $U_A(1)$ breaking are shown in Sec. IV. Based on those results, we determine the attractive basin in Sec. V. The decoupling theorem is addressed in this system in Sec. VI. Summary and outlook are given in Sec. VII. A part of this work has been published in Ref. [35].

II. EFFECTIVE THEORY

We take a linear sigma model (LSM) that has the same global symmetry as that of massless two-flavor QCD around the critical temperature, T_c . Following the standard procedure, we make a working hypothesis that the system undergoes second order phase transition. Then, the order parameters suitably chosen are small and hence is used as an expansion parameter to construct Landau-Ginzburg-Wilson (LGW) field theory. At the critical temperature, the system becomes infrared conformal, and modes with a divergent correlation length arise. Then, the original system defined in four space-time dimensions can be approximately described in three space dimensions. In the following, the calculation is done in $D = 4 - \epsilon$ dimension, and in the end $\epsilon = 1$ is substituted.

The building block of the LSM is a 2×2 complex matrix field

$$\Phi = \sqrt{2}(\phi_0 - i\chi_0)t_0 + \sqrt{2}(\chi_i + i\phi_i)t_i, \quad (1)$$

where $t_0 = 1_{2 \times 2}/2$ and $t_i = \sigma_i/2$ ($i = 1, 2, 3$) is the generator of $SU(2)$ group. ϕ_0 and ϕ_i correspond to σ and π_i in more commonly used name, respectively. Similarly χ_0 and χ_i to η' and δ_i . Thus, χ_0 denotes the isosinglet pseudoscalar, and χ_i the isotriplet scalar. Under chiral and $U_A(1)$ transformations, Φ transform as

$$\Phi \rightarrow e^{2i\theta_A} L^\dagger \Phi R \quad (L \in SU_L(2), R \in SU_R(2), \theta_A \in \text{Re}). \quad (2)$$

$U_V(1)$ symmetry corresponding to the baryon number conservation was omitted. Since Φ can be considered as the order parameter of chiral symmetry, nonzero vacuum expectation value of Φ indicates spontaneous chiral symmetry breaking ($S\chi SB$). Most general renormalizable

Lagrangian conserving chiral and $U_A(1)$ rotations is then given by

$$\begin{aligned} \mathcal{L}_{U(2) \times U(2)} = & \frac{1}{2} \text{tr}[\partial_\mu \Phi^\dagger \partial_\mu \Phi] + \frac{1}{2} m_0^2 \text{tr}[\Phi^\dagger \Phi] \\ & + \frac{\pi^2}{3} g_1 (\text{tr}[\Phi^\dagger \Phi])^2 + \frac{\pi^2}{3} g_2 \text{tr}[(\Phi^\dagger \Phi)^2]. \end{aligned} \quad (3)$$

which is referred to as $U(2) \times U(2)$ LSM. Since we are interested in the system at around T_c , m_0 will be set to zero in the analysis of $U(2) \times U(2)$ LSM.

In order to incorporate the effect of $U_A(1)$ symmetry breaking into the system, the following terms are added

$$\begin{aligned} \mathcal{L}_{\text{total}} = & \mathcal{L}_{U(2) \times U(2)} + \mathcal{L}_{\text{breaking}} \\ = & (1+w) \frac{1}{2} (\partial_\mu \phi_a)^2 + (1-w) \frac{1}{2} (\partial_\mu \chi_a)^2 + \frac{m_\phi^2}{2} \phi_a^2 + \frac{m_\chi^2}{2} \chi_a^2 \\ & + \frac{\pi^2}{3} [\lambda (\phi_a^2)^2 + (\lambda - 2x) (\chi_a^2)^2 + 2(\lambda + g_2 - z) \phi_a^2 \chi_b^2 - 2g_2 (\phi_a \chi_a)^2], \end{aligned} \quad (5)$$

where $\lambda = g_1 + g_2/2 + x + y$, $z = x + 2y$ and a runs 0 to 3. We refer to the theory of Eq. (5) as the $U_A(1)$ broken LSM. The nonzero value of w affects all the terms through the redefinition of the field normalization. In the following, we set the tree level value of w to zero, although w receives radiative corrections at two or higher loops unless both c_A and x are zero. Notice that the c_A term in Eq. (4) separates off the degeneracy between ϕ_a and χ_a as

$$m_\phi^2 = m_0^2 - \frac{c_A}{2}, \quad m_\chi^2 = m_0^2 + \frac{c_A}{2}. \quad (6)$$

In order to reproduce the properties of QCD vacuum, c_A is taken to be positive. Otherwise the parity or isovector symmetry is broken. As usual, $T = T_c$ corresponds to $m_\phi^2 = 0$, which means that only χ 's have a mass of $m_\chi^2 = c_A > 0$. When c_A is infinitely large, χ_a would be decoupled from the system, and the total Lagrangian Eq. (5) becomes $O(4)$ LSM,

$$\mathcal{L}_{O(4)} = \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{\pi^2}{3} \lambda (\phi_a^2)^2. \quad (7)$$

III. RG FLOWS FOR $c_A = 0$ AND ∞

In order to determine the renormalization group (RG) flow of the theory, the β functions in the effective theories are calculated. Loop integrals are regularized by the dimensional regularization with $D = 4 - \epsilon$. In order to see the effects of the massive fields to the β functions, we take a mass dependent renormalization scheme. Here we choose the renormalization conditions that some specific

$$\begin{aligned} \mathcal{L}_{\text{breaking}} = & -\frac{c_A}{4} (\det \Phi + \det \Phi^\dagger) \\ & + \frac{\pi^2}{3} x \text{Tr}[\Phi \Phi^\dagger] (\det \Phi + \det \Phi^\dagger) \\ & + \frac{\pi^2}{3} y (\det \Phi + \det \Phi^\dagger)^2 \\ & + w (\text{tr}[\partial_\mu \Phi^\dagger t_2 \partial_\mu \Phi^* t_2] + \text{H.c.}). \end{aligned} \quad (4)$$

The third term is symmetric under Z_4 , and so is the rest under Z_2 . Rewriting the total Lagrangian in terms of the component fields, we obtain

four-point amputated Green's functions should coincide, at a symmetric, off-shell kinematic point (SYM) $s = t = u = \mu^2$, with their tree level expressions:

$$\Gamma_4(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{\text{SYM}} = -\frac{8}{3} \pi^2 \mu^\epsilon \hat{\lambda}_R \quad (8)$$

$$\begin{aligned} & \Gamma_4(\chi_1(p_1), \chi_1(p_2), \chi_2(p_3), \chi_2(p_4))|_{\text{SYM}} \\ & = -\frac{8}{3} \pi^2 \mu^\epsilon (\hat{\lambda}_R - 2\hat{x}_R) \end{aligned} \quad (9)$$

$$\begin{aligned} & \Gamma_4(\phi_1(p_1), \chi_2(p_2), \phi_1(p_3), \chi_2(p_4))|_{\text{SYM}} \\ & = -\frac{8}{3} \pi^2 \mu^\epsilon (\hat{\lambda}_R + \hat{g}_{2,R} - \hat{z}_R) \end{aligned} \quad (10)$$

$$\Gamma_4(\phi_1(p_1), \chi_2(p_2), \phi_2(p_3), \chi_1(p_4))|_{\text{SYM}} = \frac{4}{3} \pi^2 \mu^\epsilon \hat{g}_{2,R} \quad (11)$$

where $p_{1,2}$ and $p_{3,4}$ are the incoming and outgoing momenta, respectively. $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$ and $u = (p_1 - p_4)^2 = (p_2 - p_3)^2$. The conditions (8)-(11) are for the $U_A(1)$ broken LSM. Those for the $U(2) \times U(2)$ or the $O(4)$ LSM can be obtained by simply omitting irrelevant couplings or conditions. For example, the condition for the $O(4)$ LSM is given by Eq. (8) only. The mass dimension μ^ϵ is factored out from the original quartic couplings as explicitly shown, and the hatted couplings are defined to be dimensionless. Hereafter, the subscript "R"

denoting renormalized one is omitted to avoid notational complexity.

First we discuss the RG flow for the case with infinitely large c_A . In this case, we deal with $O(4)$ LSM, Eq. (7), which contains only a single coupling $\hat{\lambda}$. From the condition (8), we obtain as the β function [36]

$$\beta_{\hat{\lambda}, c_A=\infty} = \mu \frac{d\hat{\lambda}}{d\mu} = -\epsilon\hat{\lambda} + 2\hat{\lambda}^2. \quad (12)$$

Although the β function is known through higher orders in other scheme [11],³ we showed the one loop result for the later use. $\hat{\lambda}$ reaches the IRFP $\hat{\lambda}_{\text{IR}, c_A=\infty} = \epsilon/2$ as long as the coupling at the initial scale Λ satisfies $\hat{\lambda}(\Lambda) > 0$. The existence of the IRFP meets the working hypothesis, and thus massless two-flavor QCD satisfies the necessary condition for the second order phase transition with the $O(4)$ scaling if c_A is infinitely large [8].

Next, we consider the case with $U_A(1)$ symmetry effectively restored. $U(2) \times U(2)$ LSM in Eq. (3) with $m_0 = 0$ contains two independent couplings, $\hat{\lambda} = \hat{g}_1 + \hat{g}_2/2$ and \hat{g}_2 . With the conditions (8) and (11), their β functions are obtained as [8]

$$\beta_{\hat{\lambda}, c_A=0} = -\epsilon\hat{\lambda} + \frac{8}{3}\hat{\lambda}^2 + \hat{\lambda}\hat{g}_2 + \frac{1}{2}\hat{g}_2^2, \quad (13)$$

$$\beta_{\hat{g}_2, c_A=0} = -\epsilon\hat{g}_2 + 2\hat{\lambda}\hat{g}_2 + \frac{1}{3}\hat{g}_2^2. \quad (14)$$

The one loop β functions (13) and (14) show no IRFP. However, it should be noted that the existence of IRFP and hence the possibility of the continuous transition in $U(2) \times U(2)$ LSM is reported in Refs. [21,22] employing different approaches.

IV. RG FLOW FOR FINITE c_A

We now turn to the $U_A(1)$ broken theory (5) with a finite and positive c_A . The explicit one loop calculation yields

$$\begin{aligned} \beta_{\hat{\lambda}} = & -\epsilon\hat{\lambda} + 2\hat{\lambda}^2 \\ & + \frac{1}{6}f(\hat{\mu})(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2), \end{aligned} \quad (15)$$

$$\begin{aligned} \beta_{\hat{g}_2} = & -\epsilon\hat{g}_2 + \frac{1}{3}\hat{\lambda}\hat{g}_2 + \frac{1}{3}f(\hat{\mu})\hat{g}_2(\hat{\lambda} - 2\hat{x}) \\ & + \frac{1}{3}h(\hat{\mu})\hat{g}_2(4\hat{\lambda} + \hat{g}_2 - 4\hat{z}), \end{aligned} \quad (16)$$

$$\begin{aligned} \beta_{\hat{x}} = & -\epsilon\hat{x} + 4f(\hat{\mu})(\hat{\lambda}\hat{x} - \hat{x}^2) \\ & + \frac{1}{12}(1-f(\hat{\mu}))(8\hat{\lambda}^2 - 6\hat{\lambda}\hat{g}_2 - 3\hat{g}_2^2 + 8\hat{\lambda}\hat{z} + 6\hat{g}_2\hat{z} - 4\hat{z}^2), \end{aligned} \quad (17)$$

$$\begin{aligned} \beta_{\hat{z}} = & -\epsilon\hat{z} + \frac{1}{2}(2\hat{\lambda}^2 - \hat{\lambda}\hat{g}_2 + 2\hat{\lambda}\hat{z}) \\ & - \frac{1}{6}h(\hat{\mu})(4\hat{\lambda}^2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} + 4\hat{z}^2) \\ & + \frac{1}{6}f(\hat{\mu})(-2\hat{\lambda}^2 + 3\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 2\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} \\ & + 12\hat{\lambda}\hat{x} + 6\hat{g}_2\hat{x} - 12\hat{x}\hat{z} + 4\hat{z}^2), \end{aligned} \quad (18)$$

where $\hat{\mu} = \mu/\sqrt{c_A}$ and

$$\begin{aligned} f(\hat{\mu}) &= 1 - \frac{4}{\hat{\mu}\sqrt{4+\hat{\mu}^2}} \arctan \sqrt{\frac{\hat{\mu}^2}{4+\hat{\mu}^2}}, \\ h(\hat{\mu}) &= 1 - \frac{1}{\hat{\mu}^2} \ln[1+\hat{\mu}^2]. \end{aligned} \quad (19)$$

For small $\hat{\mu}$ these functions take the asymptotic forms,

$$f(\hat{\mu}) = \frac{\hat{\mu}^2}{3} + O(\hat{\mu}^4), \quad h(\hat{\mu}) = \frac{\hat{\mu}^2}{2} + O(\hat{\mu}^4), \quad (20)$$

and for large $\hat{\mu}$,

$$\lim_{\hat{\mu} \rightarrow \infty} f(\hat{\mu}) = \lim_{\hat{\mu} \rightarrow \infty} h(\hat{\mu}) = 1. \quad (21)$$

Thus, for infinitely large c_A (or $\hat{\mu} \rightarrow 0$ with μ fixed), $\beta_{\hat{\lambda}}$ [Eq. (15)] reduces to $\beta_{\hat{\lambda}, c_A=\infty}$ [Eq. (12)] as expected. On the other hand, in the $c_A \rightarrow 0$ limit (or $\hat{\mu} \rightarrow \infty$ with μ fixed), the β functions Eqs. (15)–(18) agree with those in Ref. [37], where the calculation is done with $c_A = 0$ in the mass independent scheme. Note that the first term in each of Eqs. (15)–(18) comes from the mass dimension of the original dimensionful quartic couplings. Because of this, the dimensionless couplings behave like $1/\mu$ at the tree level.

With the dimensional regularization, the wave function renormalizations for ϕ and χ do not receive corrections at the one-loop. We take the on-shell scheme in the renormalization of two-point functions. Thus, $\sqrt{c_A}$ is defined to be the pole mass of χ_a and does not depend on the renormalization scale.

Two side remarks related to discrete symmetries are below. Even if we set the mass of χ_a to zero ($c_A = 0$) at tree level, it would potentially receive radiative corrections unless x is also zero and Z_2 symmetry is present. But the associated counterterms allow us to keep the renormalized c_A to zero.

³See also Ref. [10].

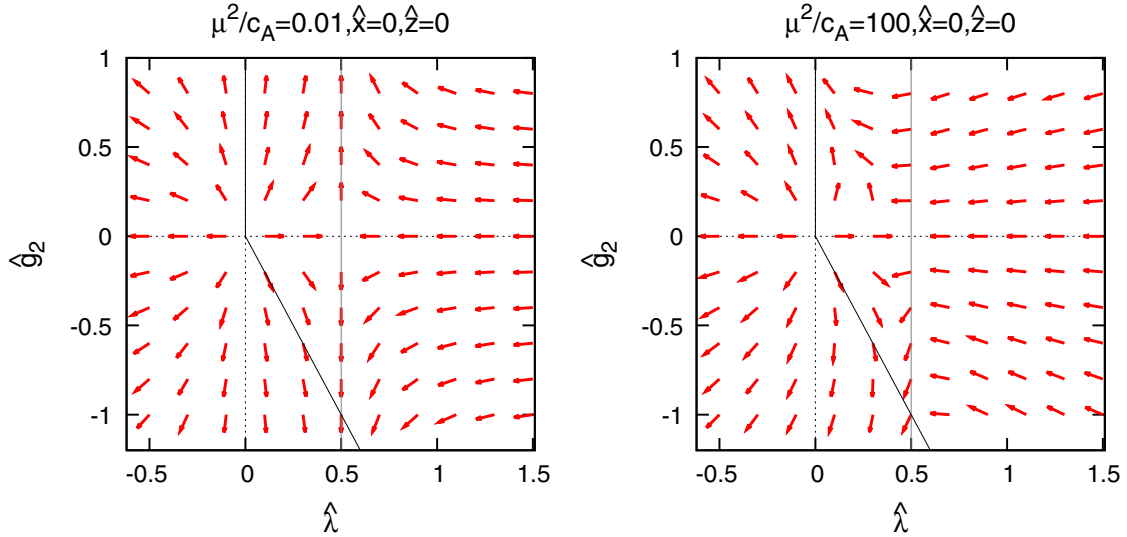


FIG. 1 (color online). The RG flow of the couplings in the $U_A(1)$ broken LSM (5) projected on to the $\hat{\lambda}$ - \hat{g}_2 plane. μ^2/c_A is 0.01 (left) and 100 (right). The length of arrow does not represent the velocity of the flow. The solid lines show the stability bound obtained at the tree level analysis of the effective potential for the $U(2) \times U(2)$ LSM [38]. The dashed and dotted lines are just to guide the eyes.

Another remark is that $\hat{y} = 0$ at a certain scale can be kept at the different scale only if Z_2 symmetry is preserved, i.e., both c_A and \hat{x} are zero. We can explicitly check this in the β functions (15)–(18). These features are not affected by higher orders of the perturbation series.

The β functions in (15)–(18) indicate no stable IRFP. Figure 1 shows an example of the RG flow in the $U_A(1)$ broken LSM with $\epsilon = 1$, where the flow is projected on to the $\hat{\lambda}$ - \hat{g}_2 plane for clarity. In this example, \hat{x} and \hat{z} are set to zero everywhere. The direction of the flow at each point is indicated by the arrow. It turns out that at a region far from the line along $\hat{\lambda} = 1/2$ the flow depends on μ^2/c_A only weakly while it is drastically changed in the vicinity of the line for $\hat{g}_2 > 0$.

To see other aspects of the RG flow, the flow is calculated for two initial conditions, $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$ and $(0.75, 0.25, 0, 0)$ with varying c_A/Λ^2 . Figure 2 shows the result projected onto the $\hat{\lambda}$ - \hat{g}_2 plane, where the flows are classified into two types: one approaching $\hat{\lambda} = 1/2$ (solid curves) and the other going $\hat{\lambda} = -\infty$ (dashed curves). In the latter case (dashed curves), \hat{g}_2 also diverges, i.e., not approaching some finite value, and then one usually expects first order phase transition.

In the former case (solid curves), the flow never reaches an IRFP because it does not exist, at least, at this order, but projecting it onto the $\hat{\lambda}$ -axis, it appears to reach the IRFP, $\hat{\lambda} = \epsilon/2$. In the infrared limit, μ^2/c_A becomes arbitrary small as long as c_A is finite. Then χ would be effectively

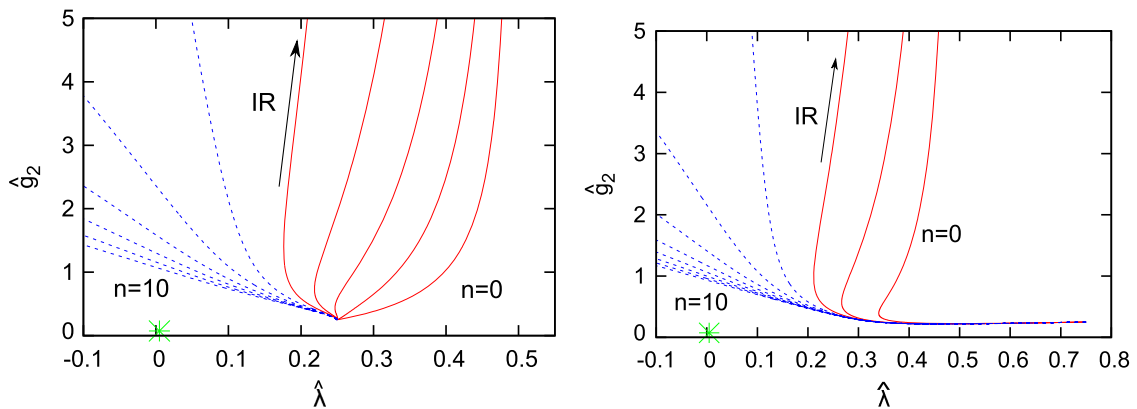


FIG. 2 (color online). The RG flow of the couplings in the $U_A(1)$ broken LSM (5) on the $\hat{\lambda}$ - \hat{g}_2 plane. Two initial conditions are chosen to be $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$ and $c_A/\Lambda^2 = (\frac{1}{2n+1})^2$ (left), and $(0.75, 0.25, 0, 0)$ and $c_A/\Lambda^2 = (\frac{1}{10(2n+1)})^2$ (right), as an example, where $n = 0, \dots, 10$. The IRFP of $U(2) \times U(2)$ LSM reported in Ref. [21] is plotted at $(\hat{\lambda}, \hat{g}_2) \sim (0.0048, 0.073)$ (cross) as a reference.

seen as a very massive field and decoupled from the system. Actually, $\hat{\lambda} = \epsilon/2$ is the IRFP of $O(4)$ LSM (7), which seems to support our interpretation that the $U_A(1)$ broken theory (5) is reduced to the $O(4)$ LSM in the IR limit via the decoupling of χ . This point is further discussed in the Sec. VI.

When approaching the $O(4)$ fixed point, $\hat{g}_2(\mu)$ and $\hat{z}(\mu)$ diverge as we will see below, but the terms including those couplings in $\beta_{\hat{\lambda}}$ asymptotically vanish due to the suppression of $f(\hat{\mu})$ [see Eq. (20)]. It means that although the couplings connecting ϕ and χ diverge the perturbative expansion of $\beta_{\hat{\lambda}}$ is still sensible as long as this suppression works.

It is interesting to note that the approaching rate to $\hat{\lambda} = \epsilon/2$ differs from that in the ordinary $O(4)$ LSM model. In order to see this, we substitute $\lambda = \epsilon/2$ into $\beta_{\hat{g}_2}$, $\beta_{\hat{z}}$ and $\beta_{\hat{x}}$, and pick up the dominant terms in the $\mu \rightarrow 0$ limit to obtain

$$\beta_{\hat{g}_2} \approx -\frac{5}{6}\epsilon\hat{g}_2, \quad (22)$$

$$\beta_{\hat{x}} \approx -\epsilon\hat{x} + \frac{1}{12}(-3\hat{g}_2^2 + 6\hat{g}_2\hat{z} - 4\hat{z}^2), \quad (23)$$

$$\beta_{\hat{z}} \approx -\frac{1}{2}\epsilon\hat{z} - \frac{1}{4}\epsilon\hat{g}_2, \quad (24)$$

where we have assumed that in the $\mu \rightarrow 0$ limit the terms proportional to $f(\hat{\mu})$ and $h(\hat{\mu})$ are smaller than the other terms. Equation (22) is easily solved, and the others too by expressing the couplings as $\hat{z}(\mu) \sim \mu^a$ and $\hat{x}(\mu) \sim \mu^b$ with unknown constants a and b . Then, the asymptotic behaviors of $\hat{g}_2(\mu)$, $\hat{x}(\mu)$, and $\hat{z}(\mu)$ in the vicinity of $\hat{\lambda} = \epsilon/2$ are found to be related to each other as

$$\hat{g}_{2,\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{g}_2(\mu) = c \left(\frac{\mu}{\sqrt{c_A}} \right)^{-5\epsilon/6}, \quad (25)$$

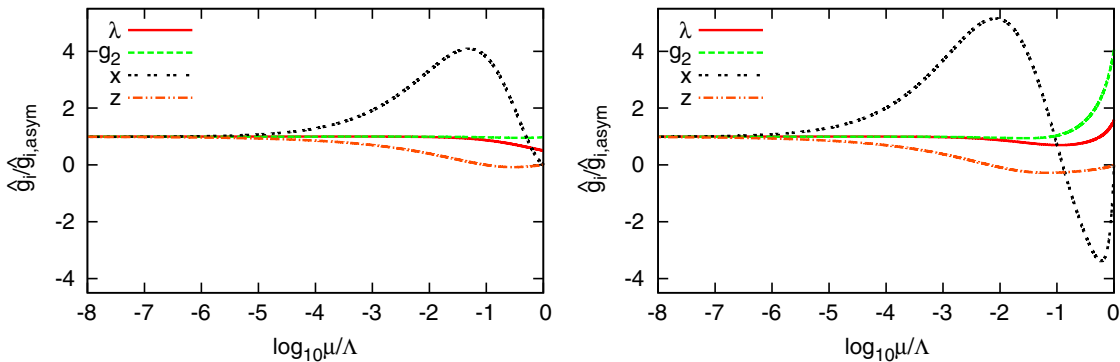


FIG. 3 (color online). The μ dependence of the couplings is shown for two different initial conditions. Each coupling is normalized by its asymptotic behavior shown in Eqs. (25)–(27) and (29). The initial conditions are $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$ and $c_A/\Lambda^2 = 1$ (left), and $(0.75, 0.25, 0, 0)$ and $c_A/\Lambda^2 = 0.01$ (right). The constant c in Eq. (25) is 0.2613774 and 0.4201792, respectively.

$$\hat{x}_{\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{x}(\mu) = \frac{3}{32} \hat{g}_{2,\text{asym}}^2(\mu), \quad (26)$$

$$\hat{z}_{\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{z}(\mu) = \frac{3}{4} \hat{g}_{2,\text{asym}}(\mu), \quad (27)$$

where the constant c depends on the initial condition. This behavior is consistent with the assumption above and confirmed in the numerical calculation as shown in Fig. 3.

Substituting $\hat{\lambda} = 1/2 + \alpha$ and the asymptotic behavior Eqs. (25)–(27) into Eq. (15), we obtain

$$\mu \frac{d\alpha}{d\mu} \approx \alpha + \frac{c^2}{24} \hat{\mu}^{2-\frac{5\epsilon}{3}}, \quad (28)$$

Then, as $\mu \rightarrow 0$, $\hat{\lambda}$ behaves like

$$\hat{\lambda}_{\text{asym}} = \frac{\epsilon}{2} - \frac{c^2}{8(5\epsilon - 3)} \hat{\mu}^{2-\frac{5\epsilon}{3}}. \quad (29)$$

The approaching rate in this case turns out to be $\sim \mu^{1/3}$ for $\epsilon = 1$ while in ordinary $O(4)$ LSM (7) it is linear in μ . It is also interesting to note that $\hat{\lambda}$ always approaches $1/2$ from below as demonstrated in Fig. 3. This is not the case in the ordinary $O(4)$ LSM. The origin of the discrepancy in the approaching rate is addressed in Sec. VI.

V. ATTRACTIVE BASIN

Next, we present the attractive basin flowing into the $O(4)$ fixed point. We survey the initial coupling space on the $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda))$ plane with two values of $c_A/\Lambda^2 = 1$ and 0.01 , shown in Figs. 4 and 5, respectively. The attractive basin is represented by the hatched area. $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$ are also varied as shown in the figures. It is seen that the attractive basin shrinks especially in the \hat{g}_2 direction as c_A/Λ^2 decreases and is not very sensitive to $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$, unless $\hat{x}(\Lambda) > 0$ and $\hat{z}(\Lambda) < 0$, in the region we studied.

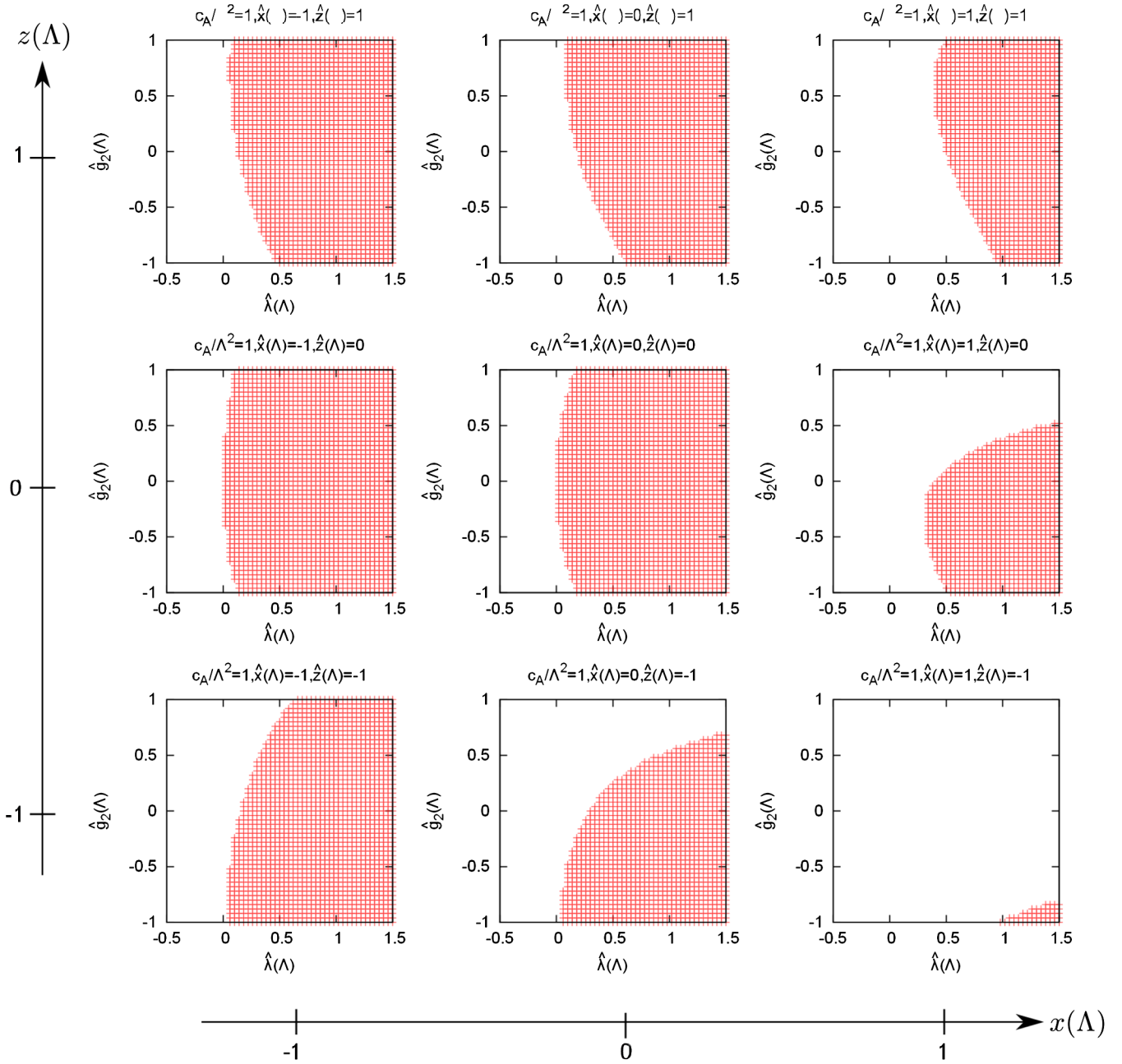


FIG. 4 (color online). The attractive basin in the $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda))$ plane (hatched area) is shown, where $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$ are varied from -1 to 1 as indicated. $c_A/\Lambda^2 = 1$.

Here let us assume that Λ is the cutoff scale below which the $U_A(1)$ broken LSM well describes massless two-flavor QCD and that the size of c_A is much smaller than Λ . Then, in order for the $U_A(1)$ broken LSM to undergo second order phase transition via the $O(4)$ fixed point, the initial condition, especially $\hat{g}_2(\Lambda)$, has to be suitably tuned.

VI. DECOUPLING

In this section, the decoupling theorem [29,30] is revisited in this system. The theorem states that with a few exceptions [31,39] the existence of heavy particles is

unknowable in low energy experiments as long as the momentum scale is much smaller than the heavy particles' mass. If the theorem holds in the present case, any n -point Green's functions consisting only of ϕ_a in the $U_A(1)$ broken LSM should agree with those in the ordinary $O(4)$ LSM in the infrared limit. Thus, even if $\hat{\lambda}$ approaches the IRFP of the $O(4)$ LSM and the $U_A(1)$ broken LSM appears to reduce to the $O(4)$ LSM, the observed discrepancy in the approaching rate indicates that the decoupling theorem does not hold in the $U_A(1)$ broken LSM.

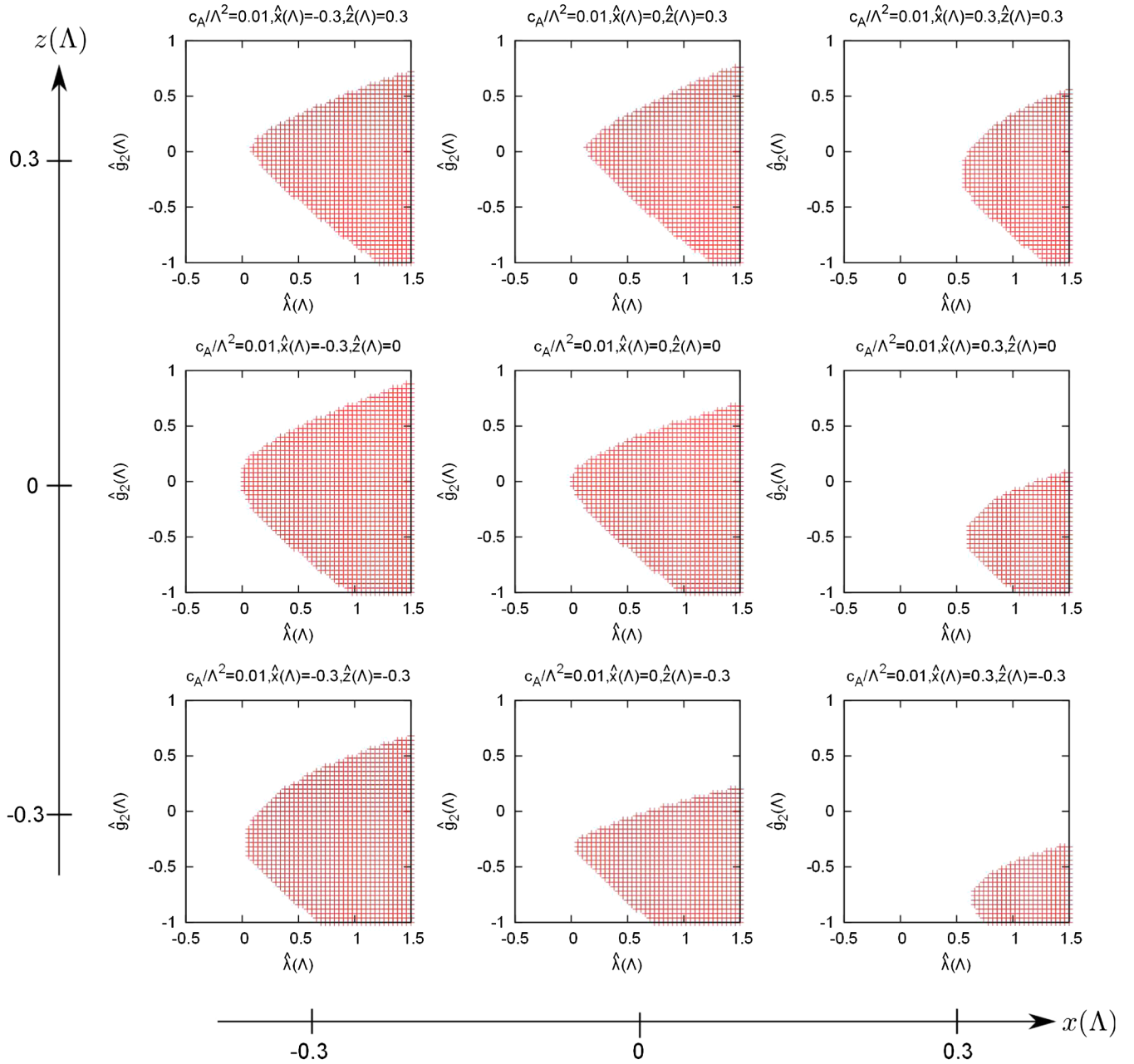


FIG. 5 (color online). The same plot as Fig. 4 but for $c_A/\Lambda^2 = 0.01$. $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$ are varied from -0.3 to 0.3 .

To see this more explicitly, we calculate the four-point Green's function of ϕ_a in the ordinary $O(4)$ and the $U_A(1)$ broken LSM. In each LSM, the calculation is done with two renormalization schemes, one being the symmetric scheme defined in (8)–(11) and another being the $\overline{\text{MS}}$ scheme, to examine the scheme dependence. The external momenta are set to $s = t = u = P^2$. Since we consider the case where P^2 is extremely small, the RG improvement is carried out.

A. Ordinary $O(4)$ LSM

First, we present the four-point function, $G_{O(4)}^{(4)}(\{p_i\}, \hat{\lambda}; \mu)$, in the ordinary $O(4)$ LSM, (7). Calculating it to one loop,

and performing the RG improvement, which is described in the next subsection in detail, one obtains

$$G_{O(4)}^{(4)}(\{p_i\}, \hat{\lambda}; \mu) = \left(\Pi_1^4 \frac{-1}{p_i^2} \right)^4 P^\epsilon \mathcal{G}_{O(4)}^{(4)}(\bar{\lambda}), \quad (30)$$

where

$$\mathcal{G}_{O(4)}^{(4), \text{sym}}(\bar{\lambda}) = -\frac{8}{3} \pi^2 \bar{\lambda}, \quad \mathcal{G}_{O(4)}^{(4), \overline{\text{MS}}}(\bar{\lambda}) = -\frac{8}{3} \pi^2 (\bar{\lambda} - 2\bar{\lambda}^2), \quad (31)$$

for symmetric and $\overline{\text{MS}}$ scheme, respectively, and $\bar{\lambda}(P)$ satisfies

$$\frac{d\bar{\lambda}(P)}{d\ln[P/\mu]} = -\epsilon\bar{\lambda} + 2\bar{\lambda}^2, \quad (32)$$

independently of the scheme at this order. Then, the asymptotic behavior of the coupling in $P \rightarrow 0$ is given by

$$\bar{\lambda}(P \rightarrow 0) \rightarrow \frac{\epsilon}{2} + c' \left(\frac{P}{\mu}\right)^\epsilon \quad (33)$$

with unknown constant c' , and hence those of the four-point function

$$\mathcal{G}_{O(4)}^{(4)\text{sym}}(P \rightarrow 0) \rightarrow -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} + c' \left(\frac{P}{\mu}\right)^\epsilon \right\}, \quad (34)$$

$$\mathcal{G}_{O(4)}^{(4)\overline{\text{MS}}}(P \rightarrow 0) \rightarrow -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} - \frac{\epsilon^2}{2} + c' \left(\frac{P}{\mu}\right)^\epsilon \right\}, \quad (35)$$

are obtained.⁴ Therefore, at the one loop, the approaching rate of the four-point function of ϕ_a to its asymptotic value is P^ϵ and independent of renormalization scheme.

B. $U_A(1)$ broken LSM with symmetric scheme

Next, we calculate the four-point function in the $U_A(1)$ broken LSM, renormalized with the conditions (8)–(11). In the following, the couplings are, for convenience, rewritten as

$$\begin{aligned} \lambda_1 &= \frac{\pi^2}{3}\lambda, & \lambda_2 &= \frac{\pi^2}{3}(\lambda - 2x), \\ \lambda_3 &= \frac{2}{3}\pi^2(\lambda + g_2 - z), & \lambda_4 &= -\frac{2}{3}\pi^2g_2, \end{aligned}$$

and $\rho = c_A/\mu^2$ is introduced. To one loop, the four-point function is given by

$$\begin{aligned} G_1^{(4)\text{sym}}(\{p_i\}, \{\hat{\lambda}_i\}, \rho; \mu) &= \langle 0 | \phi_1(p_1) \phi_1(p_2) \phi_2(p_3) \phi_2(p_4) | 0 \rangle \\ &= \left(\prod_{i=1}^4 \frac{-1}{p_i^2} \right) \mu^\epsilon g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho), \end{aligned} \quad (36)$$

where the dimensionless function $g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho)$ is

$$\begin{aligned} g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho) &= -8\hat{\lambda}_1 - \frac{1}{\pi^2} \int_0^1 d\xi \{ 2^4 \hat{\lambda}_1^2 \ln[P^2/\mu^2] + 2^2 \hat{\lambda}_1^2 (\ln[P^2/\mu^2] + \ln[P^2/\mu^2]) \\ &\quad + (\hat{\lambda}_3 \hat{\lambda}_4 + 2\hat{\lambda}_3^2) \ln[\{\rho + \xi(1-\xi)P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}] \\ &\quad + 2^{-2} \hat{\lambda}_4^2 (\ln[\{\rho + \xi(1-\xi)P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}] + \ln[\{\rho + \xi(1-\xi)P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}]) \}. \end{aligned} \quad (37)$$

From the RG equation,

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial \hat{\lambda}_i} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma_\phi \right] G^{(4)\text{sym}}(\{p_i\}, \{\hat{\lambda}_i\}, \rho; \mu) = 0, \quad (38)$$

that for $g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho)$ is obtained as

$$\left[\frac{\partial}{\partial \ln[P/\mu]} - \sum_i \beta_i(\{\hat{\lambda}_i\}, \rho) \frac{\partial}{\partial \hat{\lambda}_i} - \beta_\rho(\{\hat{\lambda}_i\}, \rho) \frac{\partial}{\partial \rho} - 4\gamma_\phi(\{\hat{\lambda}_i\}, \rho) - \epsilon \right] g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho) = 0, \quad (39)$$

where the derivative with regard to μ is altered to that of P/μ . Using the fact that $\gamma_\phi = 0$ at the one loop, the solution is given by

$$\begin{aligned} g_1^{(4)\text{sym}}(P/\mu, \{\hat{\lambda}_i\}, \rho) &= \mathcal{G}_1^{(4)\text{sym}}(\{\bar{\lambda}_i(P)\}, \bar{\rho}(P)) \exp \left[\epsilon \int_0^{\ln[P/\mu]} d\ln[P'/\mu] \right] \\ &= \left(\frac{P}{\mu}\right)^\epsilon \mathcal{G}_1^{(4)\text{sym}}(\{\bar{\lambda}_i(P)\}, \bar{\rho}(P)), \end{aligned} \quad (40)$$

where $\bar{\lambda}_i$ and $\bar{\rho}$ satisfy

$$\frac{d}{d\ln[P/\mu]} \bar{\lambda}_i(P) = \beta_i(\{\bar{\lambda}_i\}, \bar{\rho}), \quad \frac{d}{d\ln[P/\mu]} \bar{\rho}(P) = -2\bar{\rho}(P), \quad (41)$$

⁴The $O(\epsilon^2)$ term in (35) is subject to the next to leading order.

and the boundary conditions are set by

$$\bar{\lambda}_i(P=\mu) = \hat{\lambda}_i(\mu), \quad \bar{\rho}(P=\mu) = \rho = c_A/\mu^2. \quad (42)$$

Then, we obtain, as the RG improved one,

$$\mathcal{G}_1^{(4),\text{sym}}(\bar{\lambda}_i, \bar{\rho}) = -\frac{8}{3}\pi^2 \bar{\lambda}(P). \quad (43)$$

From the asymptotic behavior of $\bar{\lambda}(P \rightarrow 0)$, the asymptotic behavior of the four-point function in $P \rightarrow 0$ is found to be

$$\mathcal{G}_1^{(4),\text{sym}}(\{\bar{\lambda}_i\}, \bar{\rho}) \rightarrow -\frac{8}{3}\pi^2 \left\{ \frac{1}{2} - k \left(\frac{P}{\mu} \right)^{2-5\epsilon/3} \right\}, \quad (44)$$

with a constant k . Thus, in this scheme the asymptotic behavior of the four-point function is that of $\bar{\lambda}(P)$ as it should be.

C. $U_A(1)$ broken LSM with $\overline{\text{MS}}$ scheme

To check the scheme dependence of the infrared behavior of the four-point function, the calculation is repeated in $\overline{\text{MS}}$ scheme. β functions in this scheme is easily obtained from (15)–(18) by putting $f(\hat{\mu}) = 1$ and $h(\hat{\mu}) = 1$. Thus, β functions do not contain any information on the decoupling by definition. In this subsection, the couplings are defined in the $\overline{\text{MS}}$ scheme except for ρ , unless otherwise stated. Following the same procedure in Sec. VI B, we obtain, as the RG improved one,

$$\begin{aligned} \mathcal{G}_1^{(4)\overline{\text{MS}}}(\{\bar{\lambda}_i\}, \bar{\rho}) &= -\frac{8}{3}\pi^2 \left\{ \bar{\lambda} - 2\bar{\lambda}^2 + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right. \\ &\quad \left. \times \frac{1}{2} \int_0^1 dx \ln[\bar{\rho} + x(1-x)] \right\}. \end{aligned} \quad (45)$$

In contrast to the symmetric scheme, the χ mass ($\bar{\rho}$) dependence appears here.

Since we are interested in the P dependence of $\mathcal{G}_1^{(4)\overline{\text{MS}}}$, we differentiate it with regard to $\ln(P/\mu)$. Neglecting higher order terms, it yields

$$\begin{aligned} \frac{d\mathcal{G}_1^{(4)\overline{\text{MS}}}(\{\bar{\lambda}_i\}, \bar{\rho})}{d\ln[P/\mu]} &= -\frac{8}{3}\pi^2 \left\{ \frac{d}{d\ln[P/\mu]} \bar{\lambda} \right. \\ &\quad + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \\ &\quad \left. \times \frac{1}{2} \frac{d\bar{\rho}}{d\ln[P/\mu]} \frac{\partial}{\partial \bar{\rho}} \int_0^1 dx \ln[\bar{\rho} + x(1-x)] \right\}. \end{aligned} \quad (46)$$

Now, using the following,

$$\frac{\partial}{\partial \bar{\rho}} \int_0^1 dx \ln[\bar{\rho} + x(1-x)] = \frac{1}{\bar{\rho}} (1 - f(1/\bar{\rho})). \quad (47)$$

$$\frac{d}{d\ln[P/\mu]} \bar{\lambda} = -\epsilon \bar{\lambda} + \frac{8}{3}\bar{\lambda}^2 + \bar{\lambda}\bar{g}_2 + \frac{1}{2}\bar{g}_2^2 - \frac{4}{3}\bar{\lambda}\bar{z} - \bar{g}_2\bar{z} + \frac{2}{3}\bar{z}^2, \quad (48)$$

we obtain

$$\begin{aligned} \frac{d\mathcal{G}_1^{(4)\overline{\text{MS}}}(\{\bar{\lambda}_i\}, \bar{\rho})}{d\ln[P/\mu]} &= -\frac{8}{3}\pi^2 \left\{ -\epsilon \bar{\lambda} + 2\bar{\lambda}^2 + \frac{1}{6}f(1/\bar{\rho}) \right. \\ &\quad \left. \times (4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right\} \\ &= \frac{d\mathcal{G}_1^{(4)\text{sym}}(\{\bar{\lambda}_i\}, \bar{\rho})}{d\ln[P/\mu]}. \end{aligned} \quad (49)$$

The last line holds because ϵ is counted as the same order as the couplings. Thus, it is confirmed that the P dependence of the four-point function agrees between two schemes.

D. Reason for the different approaching rate

Here let us explore reasons for the different approaching rate. The reason seems to be simply originating from the fact that the quartic couplings describing interactions between the light ϕ_a and heavy χ_b fields have a mass dimension in three dimensional theory.

The contribution of massive fields (χ_b) with a mass M to a renormalized Green's function of light fields (ϕ) at external momentum P will take the form of $\hat{g}^2(P)P^2/M^2$ when $P^2/M^2 \ll 1$, where \hat{g} represents a generic dimensionless quartic coupling and is related to the coupling in Lagrangian as $g = \mu^\epsilon \hat{g}$. This is indeed seen in Eq. (45), if one expands the logarithmic term assuming $1/\bar{\rho}(P) = P^2/c_A \ll 1$.

If $D = 4$ (or $\epsilon = 0$), $\hat{g}^2(P)P^2/M^2$ will vanish as $P^2 \rightarrow 0$ because $\hat{g}^2(P)$ depends on P , at most, logarithmically, but when $D = 3$ (or $\epsilon = 1$), it does not in general because the factor P^2 can be compensated by $\hat{g}^2(P)$, which behaves $\sim 1/P^2$ at the tree level. Thus, in general, the decoupling theorem does not hold when a coupling has a mass dimension. The same conclusion is reported in Ref. [31], where nondecoupling effects of the scalar cubic interaction in $3+1$ dimensions is studied.

Another and more important reason is below. Usually, the approaching rate is argued in terms of more familiar quantity, ω , defined by

$$\omega = \left. \frac{d\beta_{\hat{\lambda}}}{d\hat{\lambda}} \right|_{\hat{\lambda}=\hat{\lambda}_{\text{IRFP}}}, \quad (50)$$

which is one of the universal exponents. The above results yield

$$\omega_{O(4)} = \epsilon \quad \text{and} \quad \omega_{U_A(1)\text{broken}} = 2 - 5\epsilon/3, \quad (51)$$

for the $O(4)$ and the $U_A(1)$ broken LSM, respectively.

According to the general argument of renormalization group, ω is determined by the RG dimension of the leading irrelevant operator in a model under consideration. While $(\phi_a^2)^2$ is the one in the $O(4)$ LSM, it is not evident in the $U_A(1)$ broken LSM but should not be the same as the $O(4)$ LSM because $\omega_{O(4)} \neq \omega_{U_A(1)\text{broken}}$.

One possible candidate is $(\phi_a \chi_a)^2$, which should become eventually irrelevant since its effects to the low energy behavior are expected to vanish as χ_a decouples from the system. Since the coefficient of $(\phi_a \chi_a)^2$ term is \hat{g}_2 , we calculate ω with $\hat{g}_2 = 0$ as a trial and obtain $\omega_{\hat{g}_2=0} = \omega_{O(4)} = \epsilon$. Then, it is concluded from this observation that the operator $(\phi_a \chi_a)^2$ effectively plays a role of the leading irrelevant operator in the $U_A(1)$ broken LSM. Therefore, the $U_A(1)$ broken LSM is the system which is invariant under $O(4)$ rotation for ϕ_a in the IR limit, but does not obey the $O(4)$ scaling.

It is important to notice that our study suggests a novel possibility for the nature of chiral phase transition of two-flavor QCD. Currently, three possibilities remains: (i) first order, (ii) second order with the $O(4)$ scaling, (iii) second order with the $U(2) \times U(2)$ scaling. We suggest a new one: (iv) second order with, say, the $U_A(1)$ broken scaling.

VII. SUMMARY AND OUTLOOK

The nature of the chiral phase transition of massless two-flavor QCD depends on the fate of $U_A(1)$ symmetry at the critical temperature. Two extreme cases with infinitely large and vanishing $U_A(1)$ breaking have been well studied

relying on effective theories and seem to have their respective IRFP although the latter is not settled yet. We have studied the case with a finite $U_A(1)$ breaking.

The RG flow of $U(2) \times U(2)$ LSM with a finite $U_A(1)$ breaking is investigated in the ϵ expansion. It turns out that if the couplings start from a certain region, i.e., attractive basin, one of the couplings flows into the same fixed point as the one in $O(4)$ LSM although the approaching rate is different from the $O(4)$ case. The interpretation of this is that the $U_A(1)$ broken LSM approaches the $O(4)$ LSM in the IR limit via the decoupling of the massive fields.

The attractive basin flowing into the $O(4)$ fixed point shrinks as c_A decreases. Thus, for smaller c_A , the phase transition of massless two flavor QCD favors the first order phase transition more than the second.

The observed discrepancy in the approaching rate is caused by the nondecoupling effect. In other words, the decoupling rate of the massive fields is slower than the approaching rate in the standard $O(4)$ LSM, and it effectively changes the RG dimension of the leading irrelevant operator through $(\phi_a \chi_a)^2$. In order to establish the nondecoupling, it is clearly interesting to calculate the other critical exponents and compare with those of the $O(4)$ LSM.

The existence of an IRFP just satisfies a necessary condition for second order phase transition. The phase transition can be more clearly investigated by calculating the effective potential. Such a study is ongoing [40].

The analysis here consists of simple one-loop calculations, and hence the results are neither quantitative nor conclusive. Nevertheless, we believe that this simple analysis is still useful to explore possible scenarios and offers a good starting point for further study.

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