Some remarks on relativistic diffusion and the spectral dimension criterion

C. R. Muniz*
Grupo de Física Teórica (GFT), Universidade Estadual do Ceará, Faculdade de Educação, Ciências e Letras de Iguatu, 63508-010 Iguatu, Ceará, Brazil

M. S. Cunha†
Grupo de Física Teórica (GFT), Centro de Ciências e Tecnologia, Universidade Estadual do Ceará, CEP 60740-000 Fortaleza, Ceará, Brazil

R. N. Costa Filho‡
Departamento de Física, Universidade Federal do Ceará, Caixa Postal 6030, Campus do Pici, 60455-760 Fortaleza, Ceará, Brazil

V. B. Bezerra§
Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, CEP 58059-970 João Pessoa-PB, Brazil
(Received 7 October 2014; published 27 January 2015)

The spectral dimension d_s for high energies is calculated using the Relativistic Schrödinger Equation Analytically Continued (RSEAC) instead of the so-called Telegraph's equation (TE), in both ultraviolet (UV) and infrared (IR) regimens. Regarding the TE, the recent literature presents difficulties related to its stochastic derivation and interpretation, advocating the use of the RSEAC to properly describe the relativistic diffusion phenomena. Taking into account that the Lorentz symmetry is broken in UV regime at Lifshitz point, we show that there exists a degeneracy in very high energies, meaning that both the RSEAC and TE correctly describe the diffusion processes at these energy scales, at least under the spectral dimension criterion. In fact, both the equations yield the same result, namely, $d_s = 2$, a dimensional reduction that is compatible with several theories of quantum gravity. This result is reached even when one takes into account a cosmological model, as for example, the de Sitter universe. On the other hand, in the IR regimen, such degeneracy is lifted in favor of the approach via TE, due to the fact that only this equation provides the correct value for d_s, which is equal to the actual number of spacetime dimensions, i.e., $d_s = 4$, while RSEAC yields $d_s = 3$, so that a diffusing particle described by this method experiences a three-dimensional spacetime.

DOI: 10.1103/PhysRevD.91.027501 PACS numbers: 04.50.Kd, 04.60.−m

The classical diffusion equation (CDE) has been studied in a broad context ranging from problems in thermal, electrical, and nuclear engineering, passing by biological ones such as the diffusion of nutrients in the ocean [1], to social and economic issues like the diffusion of new products on the market [2]. In nuclear physics, the neutron flux in a nuclear reactor can be obtained from the CDE solution involving both spatially and temporally variable parameters of diffusion, which also includes terms of absorption and production of neutrons (sinks and sources). This solution is quite well known and provides the balance of the particle average number in an infinitesimal volume of the material, which encloses the reactor core for non-relativistic (slow, thermal) neutrons [3]. However, when one goes to systems with relativistic velocities or high energies, which occur in astrophysical and cosmological scenarios, there is still no agreement about the form of the diffusion equation itself, as well as on the physical interpretation of their solutions based on microscopic stochastic collisions [4].

The first detailed studies on relativistic diffusion processes were carried out independently by Rudberg in 1957 [5], and Schay in 1961 [6]. But it was only in the middle 1980's that the relativistic diffusion drawn more attention, when one has considered the possibility of extending microscopic and stochastic mechanisms to the framework of the special relativity. According to those authors, any relativistic generalization of CDE with constant coefficients should be at least of second order in time. Besides this term, the so-called Telegraph's equation (TE) preserves all the CDE terms. This commonly used relativistic diffusion equation can be formally obtained by simply replacing in CDE the Laplacian operator by the d'Alambertian one, with the temporal variable becoming the proper time. A stochastic, i.e., random walk based derivation of TE shows that it is given by

*celio.muniz@uece.br
†marcony.cunha@uece.br
‡rai@fisica.ufc.br
§valdir@fisica.ufpb.br

$$\beta \frac{\partial^2 \rho(\mathbf{x}, \mathbf{x}'; \tau, \tau')}{\partial \tau^2} + \frac{\partial \rho(\mathbf{x}, \mathbf{x}'; \tau, \tau')}{\partial \tau} = D \Delta \rho(\mathbf{x}, \mathbf{x}'; \tau, \tau'), \tag{1}$$

where D is the diffusion coefficient dependent on the propagation speed, which cannot be arbitrarily large as in CDE; Δ is the Laplacian operator, which in curved spaces is given by $\Delta \equiv |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij} \partial_j)$ and corresponds to the Laplace-Beltrami operator, and $\beta > 0$ is the relaxation time parameter, that also measures the correlation in microscopic motion, namely, how each particle continues to move itself in the same direction as previously. Such quantity introduces memory leading to a non-Markovian process [7] arising from the random structure of TE.

A recently published paper [8] points out difficulties related to the stochastic derivation and interpretation of TE, advocating the use of another equation to properly describe relativistic diffusion phenomena. This equation is built from the relativistic kinetic energy operator for a free massive particle described in quantum mechanics, given by $K = \sqrt{p^2 + m^2} - m$, identifying p^2 with $-\Delta$ and K with $i\partial/\partial t$ ($c = \hbar = 1$), and after this, doing a Wick rotation, $t \to -i\tau$. The obtained equation is called Relativistic Schrödinger Equation Analytically Continuated (RSEAC), and it seems to describe diffusion processes in high energies without ambiguities, better than the TE, since the stochastic process associated to the former does not present sharp (singular) propagating wavefronts which arises in the latter. Besides this, the RSEAC also describes a non-Markovian diffusion process.

In very high energies (possibly at Planck scale or beyond), some fundamental theories require the Lorentz symmetry breaking, and a natural question that arises is about the correct diffusion equation at these scales, since that we would have a type of nonrelativistic behavior again. The Hořava-Lifshitz theory, for example, is a recent attempt to quantize gravity [9–11], whose renormalizability via power-counting is warranted through anisotropy between space and time directions existing at ultraviolet (UV) scales. Such anisotropy is implemented by means of scaling differently time and space, in the form $t \to a^z t$ and $x^i \to a x^i$, where a is a scale factor and z is a dynamical critical exponent that goes to the unity at large distances. As expected,this theory restores general relativity at the IR region. In a d-dimensional space, the renormalizability of the theory is warranted for $z = d$, at least.

In this paper, we will compare the two relativistic diffusion mechanisms described by RSEAC and TE, by calculating the spectral dimension associated to them, in both UV and IR regimens. This quantity can be interpreted as an effective spacetime dimension experienced by a diffusing particle.

In a seminal papers, Hořava [11] used the TE to calculate the spectral dimension of the Universe, finding the following result

$$d_s = 1 + \frac{d}{z}, \tag{2}$$

so that at UV scale, $d_s = 2$, and at IR one, $d_s = 4$. This expression does not depend on any temporal parameter, although it is possible to obtain a continuous interpolation between these two limits through the diffusion time [12]. Then, we will employ the Hořava's procedure to obtain the spectral dimension in these limits, now considering in this analysis the RSEAC. The calculations will also be performed by taking into account the cosmological evolution for a flat universe at UV scale, using the model which is possibly associated with its inflationary phase—the de Sitter one, which is also compatible with the Hořava-Lifshitz gravity as shown in masato.

Let us calculate the spectral dimension of the spacetime using a diffusion law different from that one employed in Hořava's paper [11] and generalized in [13]. The principle behind this is that at very short distances the spacetime behaves as a microscopically chaotic and discrete system, with its evolution obeying purely stochastic laws. Then, instead of TE we assume that, in the continuum limit, the mathematical law governing this kind of process is the RSEAC, which is given by [8]

$$\frac{\partial \rho(\mathbf{x}, \mathbf{x}'; \sigma)}{\partial \sigma} = (m - \sqrt{m^2 - \Delta}) \rho(\mathbf{x}, \mathbf{x}'; \sigma), \tag{3}$$

where σ is an external temporal parameter and m is related to the diffusion coefficient. When $m = 0$, we have the fractional Schrödinger equation analytically continued [14]. The isotropic point-source solution of Eq. (3) is given by

$$\begin{aligned} P(\sigma) &= \rho(\mathbf{x}, \mathbf{x}'; \sigma)|_{\mathbf{x}=\mathbf{x}'} \\ &= C_d \int_0^\infty k^{d-1} \exp\left[\sigma(m - \sqrt{m^2 + k^2})\right] dk, \end{aligned} \tag{4}$$

which is integrated in the momenta space and C_d is a constant which depends on the spatial topological dimension, d. The Lorentz symmetry breaking predicted in the Hořava-Lifshitz theory will be considered by introducing the dynamical critical exponent, z, in Eq. (4) in such a way that $k^2 \to k^{2z}$.

Next, we calculate the spectral dimension, d_s, by means of its definition

$$d_s = -2 \frac{d \log P(\sigma)}{d \log \sigma}. \tag{5}$$

For our purpose, it is convenient to rewrite the spectral dimension (5) as

$$d_s = -\frac{2\sigma}{P(\sigma)}\frac{dP(\sigma)}{d\sigma}. \tag{6}$$

Then, substituting Eq. (4) into Eq. (6), we get

$$d_s = -\frac{2\sigma \int_0^\infty k^{d-1} g(k) \exp[\sigma g(k)] dk}{\int_0^\infty k^{d-1} \exp[\sigma g(k)] dk}, \tag{7}$$

where $g(k) = (m - \sqrt{k^{2z} + m^2})$.

A numerical analysis of Eq. (7) shows that, in UV limit, when $z = d$ and $\sigma \to 0$, we have $d_s = 2$ for any value of m. This is in total agreement with the result found in [11] as well as with several other quantum gravity theories ([15] and references therein).

In what follows, we modify the diffusion equation by introducing the Laplace-Beltrami operator, in order to consider the Friedman-Robertson-Walker (FRW) metric, which can be written, in isotropic coordinates [16], as

$$ds^2 = -dt^2 + \frac{a^2(t)}{(1 + \kappa r^2/4)^2}\delta_{ij} dx^i dx^j, \tag{8}$$

Taking into account the spacetime anisotropy via $k^2 \to k^{2z}$, in the background spacetime given by Eq. (8), we can write the solution of Eq. (3) as

$$P(\sigma) = C_d \int_0^\infty k^{d-1} \exp\left[\sigma(m - \sqrt{m^2 + a^{-2}(\sigma) k^{2z}})\right] dk, \tag{9}$$

in which we considered $t \equiv \sigma$ and $\kappa = 0$. Now let us consider a FRW cosmological model compatible with the Hořava-Lifshitz gravity—the de Sitter model [17], which probably prevailed in the primordial universe, during the inflationary phase, which means, at very high energy scales. In this model the scale factor varies as

$$a(\sigma) = a_0 \exp(H_0 \sigma). \tag{10}$$

Plugging the above equation into Eq. (9) and this into Eq. (6), the numerical analysis still yields $d_s = 2$, where $\sigma \to 0$, $z = d$ (UV regimen), for every m, a_0, and H_0 (all greater than zero). In particular, if we make the same analysis for TE, from this cosmological point of view, we obtain an identical result.

On the other hand, without taking into account any cosmological model, in the IR limit ($d = 3$, $z = 1$, and $\sigma \to \infty$) we obtain $d_s = 3$. This is an unexpected result, since that the correct spectral dimension value should be $d_s = 4$, which is the actual number of spacetime topological dimensions, as it was found in the Hořava's paper [11]. Thus, under this point of view, the RSEAC fails at the IR scale, since the diffusing particle that it describes continues experiencing a spacetime with reduced dimension at low energies, which means that this method is not appropriate to describe diffusion processes in the IR region.

In summary, we have considered that the RSEAC could correctly describe relativistic processes of diffusion, as it is claimed in the recent literature [8]. In fact, we have showed that this is not true for all energy scales, by calculating the spectral dimension of the Universe using this equation, instead of that one usually employed in relativistic diffusion models, namely, the Telegraph's equation (TE). By means of a numerical analysis, we found that $d_s = 2$ at UV scale, i.e., when the diffusion time (σ) tends to zero and the anisotropic parameter (z) is numerically equal to the space topological dimension. With this, we can assert that there exists a degeneracy in very high energies, since that the two relativistic diffusion models provide the same spectral dimension, at least in those theories in which the Lorentz symmetry is broken at UV scale, as it is in the Hořava-Lifshitz gravity.

The calculations were also performed taking into account the de Sitter cosmological model, by considering the corresponding FRW metric in RSEAC, and they have given the same result. The mentioned degeneracy was reinforced when we did the same procedure, *mutatis mutandis*, by introducing de Sitter model in TE with anisotropic scaling, again obtaining $d_s = 2$. Then Hořava's work [11] was extended by adding this cosmological feature.

At IR scale, namely, when $\sigma \to \infty$, $d = 3$, and $z = 1$, where we no longer consider cosmological models, our approach gave the result $d_s = 3$ for the RSEAC, which is totally unsatisfactory, since the expected value should be equal to the actual number of topological dimensions of the physical spacetime. This shows up that the dimensional reduction remains, even in low energies. In fact, the expected value ($d_s = 4$) happens when one uses the TE in the context of different quantum gravity approaches [15].

From the obtained results, we conclude that the claim that the method based on the RSEAC provides a suitable description of diffusion phenomena is not valid in the general sense. The RSEAC is free from singularities with respect to wavefronts that it describes and the TE is not. On the other hand, the former fails in the IR region while the latter gives a correct result regarding the spectral dimension. The inconsistences of both equations certainly indicate that a proper generalized method without the specific difficulties of the RSEAC and TE, which must contain the positive points of both methods, should be constructed in order to get a correct description of diffusion phenomena at different energy scales.

V. B. Bezerra and R. N. Costa Filho would like to thank CNPq for partial financial support.

[1] A. Okubo and S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, 2nd ed. (Springer, New York, 2010).
[2] V. Mahajan and R. A. Peterson, *Models for Innovation Diffusion* (Sage Publications, New York, 1985).
[3] W. M. Stacey, *Nuclear Reactor Physics*, 2nd ed. (Wiley-VCH, New York, 2007).
[4] J. Herrmann, Phys. Rev. E **80**, 051110 (2009).
[5] H. Rudberg, *On the Theory of Relativistic Diffusion* (Almqvist and Wiksells, Stockholm, 1957).
[6] G. Schay, Ph.D. thesis, Princeton University, 1961.
[7] J. Dunkel, P. Talkner, and P. Hänggi, Phys. Rev. D **75**, 043001 (2007).
[8] B. Baeumer, M. M. Meerschaert, and M. Naber, Phys. Rev. E **82**, 011132 (2010).
[9] P. Hořava, Phys. Rev. D **79**, 084008 (2009).
[10] P. Hořava, J. High Energy Phys. 03 (2009) 020.
[11] P. Hořava, Phys. Rev. Lett. **102**, 161301 (2009).
[12] F. A. Brito and E. Passos, Europhys. Lett. **99**, 60003 (2012).
[13] T. P. Sotiriou, M. Visser, and S. Weinfurtner, Phys. Rev. D **84**, 104018 (2011).
[14] N. Laskin, Phys. Rev. E **66**, 056108 (2002).
[15] S. Carlip, AIP Conf. Proc. **1196**, 72 (2009).
[16] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
[17] M. Minamitsuji, Phys. Lett. B **684**, 194 (2010).