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The generalized spectral dimension $D_S(T)$ provides a powerful tool for comparing different approaches to quantum gravity. In this work, we apply this formalism to the classical spectral actions obtained within the framework of almost-commutative geometry. Analyzing the propagation of spin-0, spin-1 and spin-2 fields, we show that a nontrivial spectral dimension arises already at the classical level. The effective field theory interpretation of the spectral action yields plateau structures interpolating between a fixed spin-independent $D_S(T) = d_S$ for short and $D_S(T) = 4$ for long diffusion times T . Going beyond effective field theory the spectral dimension is completely dominated by the high-momentum properties of the spectral action, yielding $D_S(T) = 0$ for all spins. Our results support earlier claims that high-energy bosons do not propagate.

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I. INTRODUCTION

The spectral action principle [1,2] provides a framework for unifying gravity and elementary particle physics on the basis of noncommutative geometry [3], see [4–7] for reviews. A key ingredient in the construction is the spectral action [1,2]

$$S_{\chi,\Lambda} = \text{Tr}[\chi(\mathcal{D}^2/\Lambda^2)] \quad (1)$$

where χ is a positive function, \mathcal{D} is a Dirac operator on a noncommutative geometry, Λ is a suitable cutoff scale, and the trace indicates the sum over eigenvalues of \mathcal{D} . Amongst noncommutative geometries almost-commutative ones lead, for suitable choices of the almost-commutative manifold, to a spectral action which can give rise to the standard model of particle physics minimally coupled to gravity [8–12]. These models may also be extended to include physics beyond the standard model [13–16] or supersymmetry [17–20]. Their renormalization has been studied in [21–25] and the phenomenological implications of the resulting effective actions have been carried out, e.g., in [26–28]. A more detailed discussion of the cutoff Λ may be found in [29], and the generalization to noncommutative spaces built from nonassociative algebras has been pursued in [30].

In this work, we follow up on Refs. [31,32] and study the properties of the spectral action beyond the framework of effective field theory. More specifically, we construct the

generalized spectral dimension $D_S(T)$ [33–35] resulting from the spectral action principle and compare our findings with other approaches to quantum gravity.

The basic idea underlying the concept of the generalized spectral dimension $D_S(T)$ is that a test particle diffusing on a given background probes certain features of the background. By now, this “observable” has been computed in many approaches to quantum gravity and a number of quantum gravity inspired models. A common feature shared by many of these studies is a “dynamical dimensional reduction” from a classical spacetime with $D_S(T) = 4$ at macroscopic scales to $D_S(T) = 2$ [36,37]. The widespread use of $D_S(T)$ throughout the quantum gravity community then calls for a detailed understanding which (quantum) features of a model are actually encoded in $D_S(T)$.

In the literature, there are essentially three approaches to compute $D_S(T)$. For the piecewise linear geometries approximating spacetime within Monte Carlo simulations of the gravitational partition sum, one sets up a random walk on the effective quantum geometry. This allows a direct measurement of $D_S(T)$ from the return probability of the random walker, thereby characterizing the fractal features of the geometry. A prototypical example for this setup is provided by the causal dynamical triangulations program [34,38]. The second way towards obtaining a nontrivial $D_S(T)$ starts from a theory where the propagator of the test particle is already modified at the *classical* level. It is this feature that gives rise to the nontrivial spectral dimension in Hořava-Lifshitz gravity [35,39,40], general Lorentz-violating theories [41,42], or fractional quantum field theory [43,44]. Finally, $D_S(T)$ may be modified through nontrivial quantum fluctuations of spacetime

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predicted by a fundamental theory of gravity. This set includes the multifractal spacetimes occurring in the gravitational asymptotic safety program [33,45–48], the microscopic structure of spacetime within loop quantum gravity [49–54], causal set theory [55] or the propagation of particles on κ -Minkowski [56,57] and other noncommutative spaces [58].

For the classical spectral action, the nontrivial spectral dimension originates from the noncanonical momentum dependence of the theory’s propagators. The essentially new feature, making this model worthwhile to study, is that the (inverse) full propagators of the model are nonanalytic functions of the momentum. This will lead to a rather surprising behavior of the spectral dimension computed in this framework.

The rest of this work is devoted towards studying these novel features. In Secs. II and III we review the relevant aspects of the spectral action and the generalized spectral dimension, respectively. The spectral dimension resulting from the spectral action is computed in Sec. IV and we conclude with a brief summary and outlook in Sec. V.

II. THE SPECTRAL ACTION AND ITS BOSONIC PROPAGATORS

We start by reviewing the spectral action and its connection to the heat kernel mainly following [7,31]. The basic ingredient for constructing the spectral action is an almost-commutative manifold $M \times F$. Here M is a Riemannian spin manifold which plays the role of the Euclidean spacetime and F a finite, generally noncommutative, space encoding the internal degrees of freedom. The geometry of M has an operator algebraic description in terms of the canonical triple

$$M := (C^\infty(M), L^2(M, S), \mathcal{D}) \quad (2)$$

where $C^\infty(M)$ is the set of smooth functions on M , $L^2(M, S)$ is the Hilbert space of square-integrable spinors on M , and \mathcal{D} is the (Euclidean) Dirac operator acting on this Hilbert space. Similarly, the geometry of F can be captured by a triple

$$F := (\mathcal{A}_F, \mathcal{H}_F, D_F), \quad (3)$$

where \mathcal{H}_F is a finite-dimensional Hilbert space of complex dimension N , \mathcal{A}_F is an algebra of $N \times N$ matrices acting on \mathcal{H}_F , and D_F is a Hermitian operator, given by a Hermitian $N \times N$ matrix.

In order to illustrate the propagation of particles resulting from this setup, it suffices to chose the simplest internal space, taking F as a single point. In this case the triple $F = (\mathbb{C}, \mathbb{C}, 0)$ and the Dirac operator on the product space reduce to the one on M . Concretely, we will consider

$$\mathcal{D} = \mathcal{D} + \gamma_5 \phi, \quad (4)$$

where the covariant derivative

$$\mathcal{D} = i\gamma^\mu (\nabla_\mu^{\text{LC}} + iA_\mu), \quad (5)$$

contains the Levi-Cevita spin connection and the gauge potential A_μ . Consequently, the resulting spectral action comprises a spin-2 field, the graviton, a massless $U(1)$ gauge field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and a scalar ϕ . In the sequel, we will study the transverse traceless fluctuations $h_{\mu\nu}$ with $\partial^\mu h_{\mu\nu} = 0$, $\delta^{\mu\nu} h_{\mu\nu} = 0$ only and impose the Landau gauge for the spin-1 field $\partial^\mu A_\mu = 0$, limiting ourselves to the propagation of the physical degrees of freedom.

The operator \mathcal{D}^2 appearing in the spectral action (1) can then be cast into the standard form of a Laplace-type operator

$$\mathcal{D}^2 = -(\nabla^2 + E) \quad (6)$$

with the endomorphism E given by

$$E = -i\gamma^\mu \gamma_5 \nabla_\mu \phi - \phi^2 - \frac{1}{4}R + \frac{i}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}. \quad (7)$$

Moreover, the curvature of $\nabla_\mu = \nabla_\mu^{\text{LC}} + iA_\mu$ is given by

$$\Omega_{\mu\nu} := [\nabla_\mu, \nabla_\nu] = -\frac{1}{4}\gamma^\rho \gamma^\sigma R_{\rho\sigma\mu\nu} + iF_{\mu\nu}. \quad (8)$$

At this stage, it is illustrative to follow [31] and consider the special case where the generating function in Eq. (1) is given by $\chi(z) = e^{-z}$. In this case, the spectral action (1) coincides with the heat trace

$$S_{\chi, \Lambda} = \text{Tr}(e^{-t\mathcal{D}^2}) = K(\mathcal{D}^2, t) \quad \text{with } t := \Lambda^{-2}, \quad (9)$$

which is a well-studied object, see, e.g., [32,59–62].

As we are interested in the propagation of fields we will extract the (inverse) propagators of the matter fields by expanding (9) up to second order in ϕ , A_μ . The inverse propagator in the gravity sector is obtained by expanding $g_{\mu\nu}$ around flat (Euclidean) space

$$g_{\mu\nu} = \delta_{\mu\nu} + \Lambda^{-1}h_{\mu\nu}. \quad (10)$$

The inclusion of Λ ensures that $h_{\mu\nu}$ has the same mass dimension as the matter fields and gives rise to the canonical form of the graviton propagator. Comparing (10) with the stand expansion of $g_{\mu\nu}$ used in perturbation theory, $g_{\mu\nu} = \delta_{\mu\nu} + \sqrt{16\pi G_N}h_{\mu\nu}$ with G_N being Newton’s constant, identifies the natural scale for Λ as the Planck mass $m_{\text{Pl}} = (8\pi G_N)^{-1/2}$ (also see [63] for a related discussion). A lengthy but in principle straightforward calculation [31,32,59–61] then yields the expression for the inverse propagators of the physical fields including the full-momentum dependence

$$K^{(2)}(\mathcal{D}^2, t) = \int d^4x [\phi F_0(-t\partial^2)\phi + A_\mu F_1(-t\partial^2)A_\mu + \Lambda^{-2} h_{\mu\nu} F_2(-t\partial^2)h_{\mu\nu}]. \quad (11)$$

Here $K^{(2)}$ indicates that we retained the second order of the fields only and we explicitly exhibit the factor Λ^{-2} coming from the expansion (10). The structure functions F_s coincide with the standard heat kernel result for spin- s fields and are given by

$$F_0(z) = \frac{t^{-1}}{(4\pi)^2} (-4 + 2zh(z)), \quad (12)$$

$$F_1(z) = \frac{t^{-1}}{(4\pi)^2} (-4 + 4h(z) + 2zh(z)), \quad (13)$$

$$F_2(z) = \frac{t^{-2}}{(4\pi)^2} \left(-2 + h(z) + \frac{1}{4}zh(z) \right), \quad (14)$$

with

$$h(z) = \int_0^1 dx e^{-\alpha(1-\alpha)z}. \quad (15)$$

Note that these functions are nonanalytic in

$$z = \frac{p^2}{\Lambda^2} = tp^2. \quad (16)$$

The inverse of these structure functions provide the classical propagators of the theory. For illustration, we show the ‘‘reduced’’ structure functions

$$G_s(z) = (4\pi)^2 t^{\alpha_s} F_s(z) \quad (17)$$

with $F_s(z)$ defined in Eqs. (12)–(14) and $\alpha_s = (1, 1, 2)$ in Fig. 1.

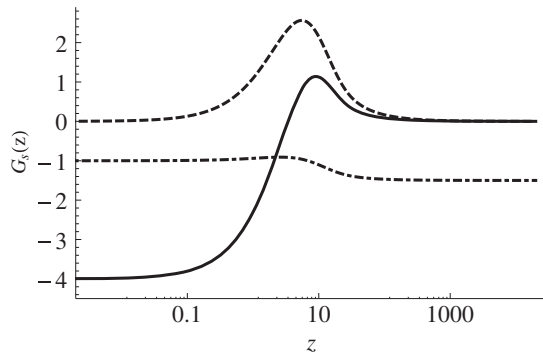


FIG. 1. Illustration of the momentum dependence of the structure functions $G_s(z)$ (17): Spin 0: solid thick line, spin 1: dashed thick line, and spin 2: dash-dotted thick line.

The structure functions (12)–(14) possess an early-time expansion for momenta $p^2 \ll \Lambda^2$, i.e., $z < 1$. Using that

$$h(z) = 1 - \frac{z}{6} + \frac{z^2}{60} - \frac{z^3}{840} + \mathcal{O}(z^4) \quad (18)$$

one finds

$$(4\pi)^2 t F_0(z) = -4 + 2z - \frac{z^2}{3} + \frac{z^3}{30} + \dots,$$

$$(4\pi)^2 t F_1(z) = \frac{4}{3}z - \frac{4}{15}z^2 + \frac{1}{35}z^3 + \dots,$$

$$(4\pi t)^2 F_2(z) = -1 + \frac{z}{12} - \frac{z^2}{40} + \frac{z^3}{336} + \dots \quad (19)$$

The early-time expansion of the structure functions [truncated at $\mathcal{O}(z^3)$ and $\mathcal{O}(z)$, respectively] is shown in Fig. 2. Comparing the result to the functions $F_s(z)$ including the full z -dependence it is easily seen that the truncation drastically modifies the behavior of the (inverse) propagators for large momenta. While the truncated $F_s(z)$ diverge the full structure functions remain finite. As we will see in Sec. IV, this feature will have a drastic effect on the spectral dimension of the theory.

At this stage, the following remarks are in order. The constant term appearing in the expansion of F_0 plays the role of a mass term for ϕ . The sign thereby indicates that the squared mass is negative. This is a remnant of the fact that the scalar ϕ acquires a nontrivial vacuum expectation value via the Higgs mechanism [1, 11, 24]. Since $K^{(2)}$, by construction, contains the terms quadratic in ϕ only the scalar potential is not included in (11) so that the stabilization of ϕ cannot be demonstrated in this approximation. The constant term in F_2 indicates the presence of a positive cosmological constant, acting like a mass term for the graviton, while the structure of F_1 indicates that the gauge field remains massless.

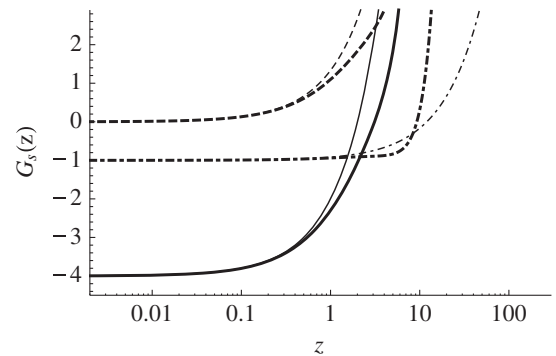


FIG. 2. Propagators obtained from truncating $G_s(z)$ at z^3 , following the spirit of effective field theory (spin 0: solid thick line, spin 1: dashed thick line, and spin 2: dash-dotted). The expansions up to linear order in z are shown as corresponding thin lines.

In order to understand the behavior of the theory at high energies, it is also useful to carry out the late-time expansion of the structure functions, capturing the behavior for $z \gg 1$. In this case

$$h(z) = \frac{2}{z} + \frac{4}{z^2} + \frac{24}{z^3} + \frac{240}{z^4} + \dots \quad (20)$$

which yields

$$\begin{aligned} (4\pi)^2 t F_0(z) &= \frac{8}{z} + \frac{48}{z^2} + \frac{480}{z^3} + \dots, \\ (4\pi)^2 t F_1(z) &= \frac{16}{z} + \frac{64}{z^2} + \frac{576}{z^3} + \dots, \\ (4\pi t)^2 F_2(z) &= -\frac{3}{2} + \frac{3}{z} + \frac{10}{z^2} + \frac{84}{z^3} + \dots \end{aligned} \quad (21)$$

A typical viewpoint adopted in the spectral action approach to particle physics consider the actions generated by (1) as effective actions which should be truncated at a certain power of the cutoff Λ^{-2} . In this case the early-time expansion (19) allows us to construct the effective action resulting from an arbitrary function χ . This uses the fact that the Trace (1) can be related to the heat kernel (9) using

$$\begin{aligned} S_{\chi,\Lambda} &= \text{Tr}[\chi(\mathcal{D}^2/\Lambda^2)] \\ &= \int_0^\infty dy \tilde{\chi}(y) \text{Tr}[e^{-y\mathcal{D}^2/\Lambda^2}], \end{aligned} \quad (22)$$

where $\tilde{\chi}(y)$ is the inverse Laplace transform of $\chi(z)$. Evaluating the operator trace in (22) based on the early-time expansion then yields the systematic expansion of $S_{\chi,\Lambda}$ in (inverse) powers of the cutoff. The χ -dependent coefficients in this expansion are given by

$$Q_n[\chi] \equiv \int_0^\infty dy y^{-n} \tilde{\chi}(y) \quad (23)$$

and can be computed by standard Mellin-transform techniques [66]. Thus the $Q_n \equiv Q_n[\chi]$ are real numbers which are normalized such that $Q_n = 1$ for $\chi = \exp(-tz)$.

The part of the spectral action containing the terms quadratic in the fields is then given by

$$\begin{aligned} S_{\chi,\Lambda}^{(2)} &= \frac{\Lambda^2}{(4\pi)^2} \int d^4x [\phi \mathcal{F}_{0,\chi}(-\partial^2/\Lambda^2) \phi \\ &\quad + A_\mu \mathcal{F}_{1,\chi}(-\partial^2/\Lambda^2) A_\mu + h_{\mu\nu} \mathcal{F}_{2,\chi}(-\partial^2/\Lambda^2) h_{\mu\nu}], \end{aligned} \quad (24)$$

with

$$\begin{aligned} \mathcal{F}_{0,\chi}(z) &= -4Q_1 + 2Q_0 z - \frac{Q_{-1}}{3} z^2 + \frac{Q_{-2}}{30} z^3 + \dots, \\ \mathcal{F}_{1,\chi}(z) &= \frac{4Q_0}{3} z - \frac{4Q_{-1}}{15} z^2 + \frac{Q_{-2}}{35} z^3 + \dots, \\ \mathcal{F}_{2,\chi}(z) &= -Q_2 + \frac{Q_1}{12} z - \frac{Q_0}{40} z^2 + \frac{Q_{-1}}{336} z^3 + \dots. \end{aligned} \quad (25)$$

The momenta (23) can be adjusted by choosing a suitable function χ . Note, however, that the Q_n 's appearing in the matter and gravitational sector of (24) *cannot be adjusted* independently, however, since they are generated by the same function χ .

We close this section by discussing possible truncations of the expansion (11):

Truncating the moments Q_n . From the mathematical viewpoint it is tempting to choose a generating function χ whose moments $Q_n[\chi]$ vanish for all values $n \geq n_{\max}$. This leads to the rather peculiar property that the highest powers of $-\partial^2$ appearing in the matter and gravitational sector *come with opposite signs*. In other words adjusting the Q_n in such a way that the propagators in the matter sector are stable at high momenta implies an instability of the gravitational propagator and vice versa. Thus ‘‘truncating’’ the theory by adjusting the momenta Q_n gives rise to a dynamical instability of the theory.

The effective field theory viewpoint. A similar (though not equivalent) strategy interprets the expansion (11) as an effective field theory, which should be truncated at a given power of the cutoff Λ . Retaining the relevant and marginal operators then provides a good description of the physics as long as $p^2/\Lambda^2 \ll 1$. While it is possible to systematically compute quantum corrections to an effective action, this expansion breaks down if the momenta are of the order of the Planck scale. A detailed analysis then reveals that the Q_n 's can be adjusted in such a way that all propagators of the theory are stable. Thus we will focus on this case in the sequel.

III. THE GENERALIZED SPECTRAL DIMENSION

The motivation for studying the (generalized) spectral dimension associated with a particle physics model comes from the idea that a test particle diffusing on a given fixed background feels certain features of this background as, e.g., its dimension. For a spinless test particle performing a Brownian random walk on a Riemannian manifold with metric $g_{\mu\nu}$, the diffusion process is described by the heat kernel $K_g(x, x'; T)$ which gives the probability density for a particle diffusing from the point x to x' in the diffusion time T . The heat kernel satisfies the heat equation

$$\begin{aligned} (\partial_T + \Delta_g) K_g(x, x'; T) &= 0, \\ K_g(x, x'; 0) &= \delta(x - x'), \end{aligned} \quad (26)$$

where $\Delta_g \equiv -D^2$ is the Laplace-Beltrami operator. In flat space, the solution of this equation is

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot (x-x')} e^{-p^2 T}. \quad (27)$$

In general $K_g(x, x'; T)$ is the matrix element of the operator $\exp(-T\Delta_g)$. For the diffusion process, its trace per volume gives the averaged return probability

$$\begin{aligned} \mathcal{P}_g(T) &= V^{-1} \int d^d x \sqrt{g(x)} K_g(x, x; T) \\ &= V^{-1} \text{Tr} \exp(-T\Delta_g), \end{aligned} \quad (28)$$

measuring the probability that the particle returns to its origin after a diffusion time T . Here $V \equiv \int d^d x \sqrt{g(x)}$ denotes the total volume. For the flat-space solution (27)

$$\mathcal{P}(T) = (4\pi T)^{-d/2}. \quad (29)$$

The (standard) spectral dimension d_S is defined as the T -independent logarithmic derivative

$$d_S \equiv -2 \lim_{T \rightarrow 0} \frac{\partial \ln \mathcal{P}(T)}{\partial \ln T}. \quad (30)$$

On smooth manifolds d_S agrees with the topological dimension of the manifold d . In order to also capture the case of diffusion processes exhibiting multiple scaling regimes, it is natural to generalize the definition (30) to the T -dependent spectral dimension

$$D_S(T) \equiv -2 \frac{\partial \ln \mathcal{P}(T)}{\partial \ln T}. \quad (31)$$

In the classical spectral action (1) the propagation of the test particles on a flat Euclidean background is modified by the higher-derivative terms entering into the (inverse) propagators of the fields. In Eq. (26) this effect can readily be incorporated by replacing the Laplace-Beltrami operator by the inverse propagators

$$\begin{aligned} (\partial_T + F(-\partial^2)) K_g(x, x'; T) &= 0, \\ K_g(x, x'; 0) &= \delta(x - x'). \end{aligned} \quad (32)$$

The solution of this equation can again be given in terms of its Fourier transform

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d} e^{ip \cdot (x-x')} e^{-F(p^2)T}. \quad (33)$$

Notably, given a generic function $F(p^2)$ there is no guarantee that the resulting heat kernel is positive semi-definite thereby admitting an interpretation as probability density. This ‘‘negative probability problem’’ has been discussed in detail [45,64], concluding that the spectral dimension remains meaningful.

The $\mathcal{P}(T)$ resulting from (33) is given by

$$\mathcal{P}(T) = \int \frac{d^d p}{(2\pi)^d} e^{-F(p^2)T}, \quad (34)$$

and may still admit the interpretation of a (positive-semidefinite) return probability even in the case where a probability interpretation of $K(x, x'; T)$ fails. The generalized spectral dimension may then be obtained by substituting the inverse propagators from Eq. (11) and evaluating (31) for the corresponding return probabilities.

Following the ideas of [41,42] the spectral dimension arising from (34) permits an interpretation as the Hausdorff dimension of the theory’s momentum space. Provided that the change of coordinates $k^2 = F(p^2)$ is bijective, the inverse propagator in the exponential can be traded for a nontrivial measure on momentum space

$$P(T) = \frac{\text{Vol}_{S^d}}{(2\pi)^d} \int k dk \frac{(F^{-1}(k^2))^{(d-2)/2}}{F'(p^2)} e^{-Tk^2}. \quad (35)$$

Here we invoked the inverse function theorem where $F'(p^2)$ is understood as the derivative of $F(z)$ with respect to its argument, evaluated at $p^2 = F^{-1}(k^2)$. Thus Eq. (34) is equivalent to a particle with canonical inverse propagator, $F(p^2) \propto p^2$ in a momentum space with nontrivial measure. This picture also provides a meaningful interpretation of $D_S(T)$ even in the case where the model is purely classical so that the nontrivial spectral dimension cannot originate from properties of an effective quantum spacetime.

Before embarking on the computation of the spectral dimensions resulting from the spectral action it is illustrative to consider the case where the inverse momentum-space propagator $F(p^2)$ contains a mass term $m^2 = F(p^2)|_{p^2=0}$. In this case it is useful to split off the massless part from $F(p^2)$ and write

$$F(p^2) = F^{(0)}(p^2) + m^2. \quad (36)$$

Based on $F^{(0)}(k^2)$ we can then introduce the return probability

$$P^{(0)}(T) \propto \int \frac{d^d p}{(2\pi)^d} e^{-TF^{(0)}(p^2)} \quad (37)$$

together with the spectral dimension seen by the massless field

$$D_S^{(0)}(T) \equiv -2T \frac{\partial}{\partial T} \ln P^{(0)}(T). \quad (38)$$

Substituting (36) into the return probability (34) and extracting the mass term from the integral it is straightforward to establish

$$D_S(T) = 2m^2 T + D_S^{(0)}(T). \quad (39)$$

Thus a mass term just leads to a linear contribution in $D_S(T)$ and does not encode nontrivial information on the

propagation of the particle. Thus we limit ourselves to the study of $D_S^{(0)}(T)$ in the sequel.

IV. THE SPECTRAL DIMENSION FROM THE SPECTRAL ACTION

Based on the discussion of the last two sections, it is now straightforward to compute the spectral dimensions from the spin-dependent propagators provided by the spectral action. We will start by investigating the truncated propagators based on Eqs. (11) and (24) in Sec. IV A before including the full-momentum dependence in Sec. IV B.

A. Effective field theory framework

In the effective field theory interpretation of (24) the functions $\mathcal{F}_{s,\mathcal{Z}}$ are truncated at a fixed power of the cutoff Λ . The resulting massless parts of the bosonic propagators then become polynomials in the particles momentum,

$$\mathcal{F}_s^{(0)}(p^2) = \sum_{n=1}^{N_{\max}} a_n^s (p^2)^n, \quad (40)$$

with obvious relations among the polynomial coefficients a_n and the numbers Q_n (25). Limiting the expansion to the marginal and relevant operators, coming with powers of the cutoff Λ^{2n} , $n \leq 2$, fixes $N_{\max} = 1$ and all propagators retain their standard p^2 -form. Also taking into account power-counting irrelevant terms containing inverse powers of the cutoff adds further powers to the polynomial (40). Thus the propagators include higher powers of momentum in this case.

A positive-semidefinite spectral dimension $D_S^{(0)}$ requires a positive function $\mathcal{F}_s^{(0)}$. This requirement puts constraints on the signs of the momenta Q_n appearing in Eq. (25). In particular $a_1^s > 0$ is required for obtaining classical propagators at low energies while $a_{N_{\max}}^s > 0$ is needed for stability at high energies.

The asymptotic behavior of $D_S^{(0)}$ for short (long) diffusion time T is governed by the highest (lowest) power of p^2 contained in (40). Evaluating (37) and (38) for the special cases $\mathcal{F}_s^{(0)}(p^2) \propto p^2$ and $F(p^2) \propto (p^2)^{N_{\max}}$, a simple scaling argument establishes

$$\begin{aligned} \lim_{T \rightarrow \infty} D_S^{(0)}(T) &= 4, \quad a_1 > 0, \\ \lim_{T \rightarrow 0} D_S^{(0)}(T) &= \frac{4}{N_{\max}}. \end{aligned} \quad (41)$$

Thus $a_1 > 0$ ensures that the spectral dimension seen by particles for long diffusion times coincides with the topological dimension of spacetime. Including higher powers of momenta decreases $D_S^{(0)}(T)$ for short diffusion times. The generalized spectral dimension then interpolates smoothly between these limits. This feature is illustrated in Fig. 3. The case $N_{\max} = 1$, $a_1 > 0$ leads to a spectral dimension which is independent of the diffusion time

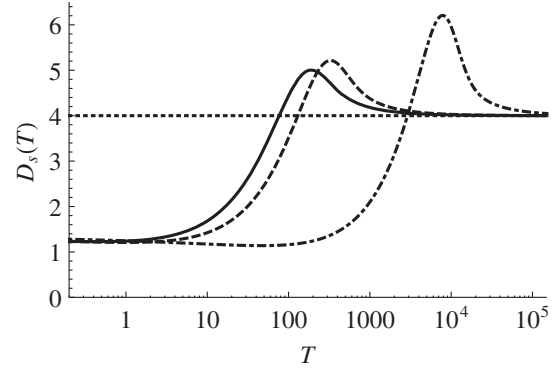


FIG. 3. The spin-dependent spectral dimension $D_S^{(0)}(T)$ obtained from (40) with $N_{\max} = 1$ (dotted line) and $N_{\max} = 3$, $Q_n = 1$, $n = 1, 0, -1, -2$ for spin 0: solid line, spin 1: dashed line, and spin 2: dashed-dotted line. The crossover scale is set by $t = \Lambda^{-2}$ and has been normalized to $t = 1$.

(dashed line). The spectral dimension obtained for $N_{\max} = 3$ and momenta $Q_n = 1$, $n = 1, 0, -1, -2$. In this case, $D_S^{(0)}(T)$ interpolates between 4 at large T and $4/3$ for small T respectively. The crossover occurs for $T/t \approx (4\pi)^2$ (spin 0 and 1), respectively, $T/t^2 \approx 10 \times (4\pi)^2$ (spin 2). Notably, it is only the shape of this crossover, which depends on the spin of the particle, while the asymptotic limits are universal for all spins.

B. Propagators with full-momentum dependence

We now take the step beyond effective field theory and investigate the spectral dimension arising from the inverse propagators (14) including the full-momentum dependence. Comparing Figs. 1 and 2 the structural difference is immediate: in the effective field theory framework $F_s(z)$ diverges as $z \rightarrow \infty$ while the inclusion of the full-momentum dependence renders $\lim_{z \rightarrow \infty} F_s(z)$ finite with the limit given by the leading term in (21). As a consequence of the modified asymptotics, the integral (37) diverges at the upper boundary, since the contribution of large momenta is no longer exponentially suppressed once the full-momentum dependent propagators are considered.

In order to still be able to analyze the spectral dimension arising in this framework, we regulate (37) by introducing a UV cutoff Λ_{UV} ,

$$P^{(0)}(T; \Lambda_{UV}) = \int^{\Lambda_{UV}} \frac{d^4 p}{(2\pi)^4} e^{-TF^{(0)}(p^2)}. \quad (42)$$

In the spirit of the discussion leading to Eq. (39), we consider the “massless” structure functions $F_s^{(0)}(z)$ where the constant terms appearing in the late-time expansion (21) have been removed. The return probability (42) then allows us to construct the spectral dimension as a function of Λ_{UV}

$$D_S^{(0)}(T; \Lambda_{UV}) = -2T \partial_T \ln(P^{(0)}(T; \Lambda_{UV})). \quad (43)$$

The scale Λ appearing in the spectral action (1) is thereby held fixed and sets the transition scale between the UV and IR regime. A detailed analytical and numerical analysis based on the expansion (21) then establishes

$$\lim_{\Lambda_{UV} \rightarrow \infty} D_S^{(0)}(T; \Lambda_{UV}) = \lim_{\Lambda_{UV} \rightarrow \infty} 4TF_S^{(0)}(\Lambda_{UV}). \quad (44)$$

Based on the late-time expansion (21), we thus conclude that including the full-momentum dependence in the structure function leads to a spectral dimension which *vanishes for all diffusion times T* :

$$D_S^{(0)}(T) = \lim_{\Lambda_{UV} \rightarrow \infty} D_S^{(0)}(T; \Lambda_{UV}) = 0. \quad (45)$$

This result entails in particular that there is no scaling regime for large diffusion time where the spectral dimension matches the topological dimension. Thus including the full-momentum dependence, one does not recover a “classical regime” where the spectral dimension would indicate the onset of classical low-energy physics.

V. CONCLUSIONS AND OUTLOOK

In this work we constructed the generalized spectral dimension $D_S(T)$ describing the propagation of (massless) scalars, vectors and gravitons based on the *classical* spectral action (1). Our results distinguish three cases:

- (1) If the spectral action is interpreted as an effective field theory restricted to the power-counting relevant and marginal terms, the generalized spectral dimension is independent of the diffusion time T and matches the topological dimension of spacetime $D_S(T) = 4$.
- (2) If the effective field theory framework is extended to also include power-counting irrelevant terms, the generalized spectral action interpolates between $D_S(T) = 4$ for long diffusion time and $D_S(T) = 4/N_{\max}$ for short diffusion times. N_{\max} is determined by the highest power of momentum contained in the propagator, $(p^2)^{N_{\max}}$. The crossover between these two asymptotic regimes is set by the cutoff Λ and its shape explicitly depends on the spin of the propagating particle.
- (3) If the full-momentum dependence of the propagators is taken into account, the generalized spectral dimension becomes independent of the spin and vanishes identically $D_S(T) = 0$.

The last feature can be traced back to the fact that the full propagators approach a constant for momenta much larger than the characteristic cutoff scale Λ . In Ref. [31] the peculiar behavior was summarized by the catchy phrase that “high-energy bosons do not propagate.” Indeed the vanishing of the spectral dimension suggests that the momentum space of the theory resembles the one of a

zero-dimensional field theory. In this sense the situation resembles a picture where spacetime fragments into isolated points which do not communicate.

We stress, however, that all computations carried out in this work are at the classical level. In particular the spacetime is given by classical Euclidean space. All effects are due to the change of measure in momentum space reflected by noncanonical form of the classical propagators and thus do not capture properties of an underlying quantum spacetime. Nevertheless, we believe that the spectral action, comprising the three scenarios discussed above, provides a valuable test bed for new candidates for quantum gravity observables generalizing the spectral dimension.

The vanishing of the spectral dimension, $D_S(T) = 0$, is in agreement with the previous computations obtained for other noncommutative spacetimes [58]. This is a welcome and surprising result, since the noncommutative nature of our spectral triple construction differs substantially from that of [58]. Aside for the limiting case of vanishing spectral dimension, both results display the same qualitative features: The spectral dimension interpolates between the topological dimension and zero, and has a local maximum situated close to a transition scale. Additionally, the onset of the transition is, in both cases, controlled by the parameter which is introduced by the noncommutative description (Λ in our computation and the parameter κ that governs the noncommutativity of the coordinates in [58]). Therefore, the transition is truly representative of a regime in which the noncommutative features of the spacetime become important.

In agreement with the conjecture of “asymptotic silence” pushed forward in [36], it is a general feature of our computation that the spectral dimension decreases in the ultraviolet, and thus describes a spacetimes that breaks into disconnected regions at high energies. However, differently from [36] and [65], it is not immediately evident what is the physical mechanism underlying this scenario. Furthermore, our results are instead in stark contrast to the generalized spectral dimension typically obtained within quantum gravity approaches where $D_S(T)$ typically interpolates between $D_S(T) = 4$ (classical, macroscopic phase) and $D_S(T) < 4$ at microscopic scales [36].

At this stage, it is tempting to speculate that the vanishing spectral dimension is an artifact of extrapolating the classical spectral action into the trans-Planckian regime without taking quantization effects into account. In other words, vacuum fluctuations seem to be fundamental to balance the fragmentation of spacetime in the ultraviolet [37]. An UV completion of the spectral action could be achieved through the asymptotic safety mechanism [47,66], which seems a natural choice given the field content and symmetries of the model, may lead to $D_S(T)$ interpolating from four to two, while still keeping the nonpolynomial momentum dependence in the propagators. We hope to come back to this intriguing possibility in the future.

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