Continuous-spin particle field theory with helicity correspondence

Philip Schuster[*] and Natalia Toro[†]

Perimeter Institute for Theoretical Physics, Ontario, Canada N2L 2Y5

We propose the first covariant local action describing the propagation of a *single* free continuous-spin degree of freedom. The theory is simply formulated as a gauge theory in a "vector superspace," but can also be formulated in terms of a tower of symmetric tensor gauge fields. When the spin invariant ρ vanishes, the helicity correspondence is manifest—familiar gauge theory actions are recovered and couplings to conserved currents can easily be introduced. For nonzero ρ, a tower of tensor currents must be present, of which only the lowest rank is exactly conserved. A paucity of local gauge-invariant operators for nonzero ρ suggests that the equations of motion in any interacting theory should be *covariant*, not invariant, under a generalization of the free theory's gauge symmetry.

I. INTRODUCTION

This paper presents a field theory for a single bosonic continuous-spin particle (CSP). The theory is defined by a simple free-particle action, which is local and bilinear in a gauge field and can be coupled to an appropriately conserved background current. The action is most simply specified using a "vector superspace" in which the orientation of an auxiliary 4-vector geometrizes spin. This formalism subsumes and simplifies the well-known Schwinger-Fronsdal description of high-spin bosons; CSPs emerge as a simple and natural generalization. Our focus here is primarily on the bosonic CSP in $3+1$ dimensions, but our results apply more generally. Of particular interest is the $2+1$-dimensional analogue of CSPs, which has a single polarization state with novel properties closely related to the fractional spin of massive anyons. Our result completes the program of obtaining local spacetime actions for free particles of every integer spin, and opens a new door to investigating all long-distance physics compatible with relativity and quantum mechanics.

A. Asymptotic states and quantum mechanics

Massive particles in $3+1$ dimensions transform in unitary representations of SO(3) and are classified, as is well known, by their spin eigenvalue $\vec{J}^2 = S(S+1)$ or, more covariantly, $W^2 = -m^2 S(S+1)$ where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$ and $m^2 = +P^2$, in the mostly negative metric convention. An action formalism for massive particles of arbitrary integer spin was formulated by Singh and Hagen in Ref. [1].

Massless particles, in contrast, transform in unitary representations of ISO(2), the isometry group of a plane [2]. In the generic faithful representation, $W^2 = -\rho^2$ takes

an arbitrary negative real eigenvalue (with dimensions of momentum squared) and the representation comprises infinitely many polarizations. These are the "continuous-spin" representations. The states can always be decomposed into a basis of eigenstates of a "helicity" operator $\vec{J} \cdot \hat{k}$ for 3-momentum $\vec{k}$. The two ISO(2) "translation" generators (i.e. combinations of transverse boosts and rotations) mix helicities, and can be grouped into raising and lowering operators with $T_\pm|h\rangle = \rho|h \pm 1\rangle$ (see e.g. Ref. [3] for a review). For a given spin scale ρ, there are two continuous-spin representations in $3+1$ dimensions: a "bosonic" one comprising states of all integer helicities, and a "fermionic" one comprising states of all half-integer helicities. In the degenerate case $\rho = 0$, each helicity state separates into an independent representation satisfying $(W^\mu - hP^\mu)|h\rangle = 0$. Action formalisms describing free particles of any helicity are known, with the most general high-spin case due to Fang and Fronsdal [4,5].

Quantum-mechanical consistency places severe constraints on interacting helicity degrees of freedom, suggesting that long-range forces in flat space can only be mediated by particles of helicity 2 or less [6–8]. But this finding crucially relies on the assumption that helicity is exactly invariant under boosts, i.e. $\rho = 0$. When that assumption is relaxed, quantum consistency permits covariant soft emission amplitudes for a CSP that yield finite differential cross sections and (for small enough ρ) viable approximate thermodynamics despite the infinite number of polarizations [3,9]. Moreover, at energies large compared to ρ or in the nonrelativistic limit, these amplitudes approach those of a helicity-0, -1, or -2 particle [10]. Thus, relativity and quantum mechanics alone imply that nontrivial CSP physics should mirror that of familiar gauge theories and Einstein's general relativity (GR) in the $\rho \to 0$ limit, raising the possibility that CSPs might offer consistent models of electromagnetism and gravity that differ from existing descriptions only deep in the infrared.

---

[*]pschuster@perimeterinstitute.ca
[†]ntoro@perimeterinstitute.ca

Up until now, it has not been possible to investigate these findings from the point of view of a spacetime field theory for a *single* CSP,[1] a situation that we are correcting with the formalism presented here.

B. Summary of the free field theory

In addition to familiar coordinate space x^μ, the free gauge potential $\Psi(\eta, x) \equiv \psi_0(x) + \eta_\mu \psi_1^\mu(x) + \cdots$ resides in a "vector superspace" with 4-vector coordinate η^μ. Only the orientation of η is physical, and in this space our action is

$$S = \frac{1}{2} \int d^4x [d^4\eta] (\delta'(\eta^2 + 1)(\partial_\alpha \Psi)^2 + \frac{1}{2} \delta(\eta^2 + 1)(\Delta \Psi)^2),$$

(1.1)

where $\Delta \Psi \equiv (\partial_\eta . \partial_x + \rho) \Psi$. Ψ is an analytic function in vector superspace, and for such functions the suitably defined η integrations yield tensor contractions among the components of Ψ. As we will show in this paper, this action describes a single CSP with spin invariant ρ. The whole tower of tensors $\psi_n^{\mu_1 \cdots \mu_n}(x)$ in Ψ is needed to describe the tower of massless integer-helicity polarization states present in the spectrum of a CSP. For $\rho \neq 0$, boosts mix the different spin states in exactly the manner expected for a particle with $W^2 = -\rho^2$. Moreover, for nonzero ρ the helicity-h eigenmode cannot be described by a single tensor component in any gauge, but must rather involve infinitely many nonzero tensor components; in this sense, the whole vector superspace is optional for $\rho = 0$ but essential for nonzero ρ.

The action is invariant under the gauge transformation $\delta \Psi_{\epsilon, \chi} = (\eta \cdot \partial_x - \frac{1}{2}(\eta^2 + 1)\Delta)\epsilon(\eta, x) + (\eta^2 + 1)^2 \chi(\eta, x)$ where ϵ and χ are arbitrary smooth functions. In an appropriate component form, the ϵ gauge transformation reproduces familiar gauge transformations, but with the addition of rank-mixing terms proportional to ρ, while χ can be used to fix a double-traceless gauge. For $\rho \to 0$, the action after this partial gauge fixing smoothly and precisely recovers the familiar Fronsdal form, thereby making the helicity correspondence conjectured in Ref. [10] manifest. Both the gauge potential Ψ and the gauge parameters ϵ and χ are unconstrained, which differs from the Fronsdal formalism where double-trace degrees of freedom must be removed by hand. Our formalism is naturally endowed

---

[1]Several early papers aimed to describe CSPs with fully covariant fields (rather than a gauge potential) and encountered difficulties [11–15]. A covariant equation of motion for a CSP gauge potential was found in Ref. [16], but was not of Lagrangian form. Although the authors of this work presented a field theory action recently in Ref. [17], it propagated a continuum of CSP species with *every* spin scale ρ rather than a single degree of freedom, and its $\rho \to 0$ limit could not readily be coupled to conserved currents.

with an enhanced gauge symmetry that simply decouples all such double traces.

For $\rho = 0$, familiar interactions are introduced with a coupling $S = \int d^4x[d^4\eta]\delta'(\eta^2 + 1)\Psi J$, where $J(\eta, x) = J_0(x) + \eta_\mu J_1^\mu(x) + (\eta_\mu \eta_\nu + \frac{1}{2}g_{\mu\nu})J_2^{\mu\nu} + \cdots$. Gauge invariance requires $\delta(\eta^2 + 1)\Delta J = 0$, which for $\rho = 0$ encodes the conservation of J_1 and J_2, and forces higher nonderivative currents to vanish. For nonzero ρ, the continuity condition can be satisfied for a tower of component currents obeying

$$\partial \cdot J_1 + \frac{\rho}{\sqrt{2}}\left(J_0 + \frac{1}{2}J_2'\right) = 0,$$

$$\partial \cdot J_2^\mu + \frac{\rho}{2\sqrt{2}}\left(J_1^\mu + \frac{1}{2}J_3'^\mu\right) = 0,$$

$$\left\langle \partial \cdot J_3^{\mu\nu} + \frac{\rho}{3\sqrt{2}}\left(J_2^{\mu\nu} + \frac{1}{2}J_4'^{\mu\nu}\right)\right\rangle = 0,$$

$$\cdots$$

$$\left\langle \partial \cdot J_n + \frac{\rho}{n\sqrt{2}}\left(J_{(n-1)} + \frac{1}{2}J_{(n+1)}'\right)\right\rangle = 0,$$

where $\langle \ldots \rangle$ denotes the traceless part of the enclosed tensor. Thus, Ψ can be sourced by a current with an exactly conserved lowest-rank current; for example a conserved J_1^μ component would yield a QED-like theory. For $\rho \neq 0$, gauge invariance implies that all higher-rank currents must be nonzero, are not conserved, and naively have ρ suppression increasing with rank. This is entirely compatible with helicity correspondence and general constraints on high-rank currents in flat space [6–8].

C. Potential applications

The action found in the present work does not address the question of CSP-matter interactions, or even CSP self-interactions. But it simultaneously sharpens the challenge of constructing such couplings and the motivations to do so. Like familiar helicity-1 and -2 gauge fields, our CSP field can couple only to appropriately conserved currents. The conservation condition is local and at zeroth order in ρ it is satisfied by scalar, conserved vector, and conserved tensor currents—echoing in the field theory the S-matrix *helicity correspondence* found in Ref. [10]. But it is not clear what symmetry the matter sector must possess, in order to satisfy the full ρ-dependent continuity condition in a local way (or whether this is even possible). Likewise, for nonzero ρ, the only gauge-invariant (GI) operators with at most two derivatives and that are linear in the gauge potential vanish on the equation of motion. But there are GI operators local in time, but spatially nonlocal, that recover familiar GI operators in the $\rho \to 0$ limit and appear causal even for nonzero ρ.

At the same time, we can clearly identify two things that are *not* gauge-invariant: a mass term for the CSP field, and a

tadpole-like coupling for the trace of the spin-2 mode. In other words, it may be consistent for a CSP to be scalar-like, in the sense that its high-energy matter amplitudes approach those of a generic scalar particle. Such a particle would look to an effective field theorist like a fine-tuned scalar field, while in fact its masslessness is protected by symmetry! Likewise, even if one can introduce graviton-like CSP-matter couplings, it is not consistent for that CSP to be sourced by vacuum energy, at least not without spontaneously breaking the underlying gauge symmetry. An effective field theorist describing the CSP as an Einstein graviton would certainly conclude that it is fine-tuned!

We cannot read too much into these putative violations of effective field theory intuition until we have an interacting theory in hand and can study its quantum consistency. Indeed, the pessimistic reading is that the above arguments just provide evidence against such interactions existing. Yet the properties encountered in the free theory and coupling to classical backgrounds hint at just the right kinds of structure to address well-known fine-tuning puzzles: a large deformation of the *infrared* degrees of freedom, new symmetries required in the matter sector, and possibly a mild degree of infrared nonlocality in the interactions. We believe this is a strong motivation to resolve the question of CSP couplings to matter one way or the other.

A very different motivation to study continuous-spin theories is their potential relevance to $2 + 1$-dimensional physics [18]. Continuous-spin particles have analogues in $2 + 1$ dimensions or higher (see Ref. [19] for a classification). Our action extends straightforwardly to higher and lower dimensions, though we have not fully explored the higher-dimensional state content. The $2 + 1$-dimensional CSP, unlike its higher-dimensional counterparts, has just two states, related by parity, that transform as $W|\psi\rangle = \pm\rho|\psi\rangle$ with $W = \frac{1}{2}\epsilon^{\mu\nu\rho}J_{\mu\nu}P_\rho$. To our knowledge, such states have not received any attention in the literature.

We know of no in-principle obstruction to the emergence of CSP quasiparticles in $2 + 1$-dimensional condensed matter systems. This would not be without precedent, as massive anyons (with $W|\psi\rangle = \rho|\psi\rangle$, with $\rho = m \cdot s$ arbitrary) play an important role in many such systems (see e.g. Refs. [20,21]). From at least one perspective, emergent CSPs would be less surprising than anyons; while anyon wave functions transform nontrivially under a *compact* rotation generator, and therefore must be multivalued, CSPs merely acquire nontrivial transformations under a non-compact boost + rotation generator. This is reflected in the simplicity of our action compared to the Jackiw-Nair action needed to represent anyons as elementary fields [22].

D. Outline

To set the stage, we begin in Sec. II by summarizing the Schwinger-Fronsdal actions for massless particles of arbitrary integer helicity. Section III introduces "vector-superspace," and motivates the introduction of gauge symmetries to allow unitary representations of helicity degrees of freedom in terms of covariant fields. The resulting action is then shown to be equivalent to a sum of Schwinger-Fronsdal actions, but greatly simplified. In Sec. IV, we present the action describing a single CSP, showing that the action propagates massless degrees of freedom with the correct polarization content. We also present the action and equation of motion in tensor form, thereby demonstrating that the free theory smoothly and transparently recovers the Schwinger-Fronsdal form as $\rho \to 0$. In Sec. V we consider couplings of the vector-superspace action to backgrounds currents. We illustrate how familiar matter couplings for helicities 0, 1, and 2 can be added to the vector-superspace action with $\rho = 0$, and examine the generalized continuity condition for nonzero ρ. In the latter case, gauge invariance is satisfied by a tower of related currents, of which only the lowest in rank is exactly conserved; we show how appropriately conserved and covariant ansatz currents reproduce the soft CSP emission amplitudes of Ref. [3]. In Sec. VI, we comment on several gauge-invariant and -noninvariant operators: the Poincaré generators are gauge invariant, but their spacetime *densities* are not, much like those of gravitational radiation in linearized GR. We then comment on the absence of simple, local gauge-invariant operators (excluding the equation of motion), and the causality properties of some nonlocal gauge invariants. We conclude in Sec. VII with a summary, discussion of open problems, and speculation on the physical significance of the paucity of gauge invariants encountered in Sec. VI.

II. REVIEW OF MASSLESS BOSONS IN THE SCHWINGER-FRONSDAL FORMALISM

The Schwinger-Fronsdal formalism describes a propagating helicity-h particle through the dynamics of a rank-h gauge field, which we will denote $\phi^{(h)}$. The formalism has been extensively reviewed (see e.g. Refs. [23,24]), but we summarize the essential results for completeness and to introduce our notation used to suppress explicit Lorentz indices, which differs slightly from the usual conventions in the high-spin literature.

Given two tensors $X^{(n)}$ and $Y^{(m)}$ of rank n and m, respectively, $X \circ Y$ is the symmetric rank-$(n + m)$ tensor obtained by summing over inequivalent orderings of the indices, with no explicit symmetry factor. For example, $(X \circ Y)^{\mu\nu} \equiv X^\mu Y^\nu + X^\nu Y^\mu$ for two rank-1 tensors, $(X \circ Y)^{\mu\rho\sigma} \equiv X^\mu Y^{\nu\rho} + X^\nu Y^{\rho\mu} + X^\rho Y^{\mu\nu}$ for a rank-1 with rank-2 tensor and so on. A contraction of two tensors is denoted by $X \cdot Y$, so that $X \cdot Y \equiv X^{\mu_1 \dots \mu_n} Y_{\mu_1 \dots \mu_n}$ for two rank-n tensors, $(X \cdot Y)_\mu \equiv X^\nu Y_{\nu\mu}$ for a rank-1 with rank-2 tensor and so on. In actions and other contexts where the end result is clearly a scalar, we will often write contractions of a tensor with itself as X^2, as was done above.

However, powers of ∂_μ or the (mostly negative) flat metric $g_{\mu\nu}$ are always taken to mean outer products. Thus, for a rank-n tensor X, the rank-$(n-2)$ object $\partial^2 \cdot X = \partial_\mu \partial_\nu X^{\mu\nu\cdots}$ should never be confused with the rank-n object $\Box X = (\partial.\partial)X$. In addition, $X' = \mathrm{Tr}X = g \cdot X$, $X'' = \mathrm{Tr}^2 X = g^2 \cdot X$, etc.[2]

Using these conventions, we can write the Schwinger-Fronsdal action describing a helicity-h particle in d dimensions as

$$S_h = (-1)^h \int d^d x \frac{1}{2} (\partial_\alpha \phi)^2 - \frac{s}{2} (\partial \cdot \phi)^2$$
$$- \frac{s(s-1)}{2} \phi' \cdot (\partial \cdot \partial \cdot \phi) - \frac{s(s-1)}{4} (\partial_\alpha \phi')^2$$
$$- \frac{s(s-1)(s-2)}{8} (\partial \cdot \phi')^2 - \phi \cdot J^{(h)} \qquad (2.1)$$

$$= (-1)^h \int d^d x \frac{1}{2} (\partial_\alpha \phi)^2 - \frac{s(s-1)}{8} (\partial_\alpha \phi')^2$$
$$- \frac{s}{2} \left(\partial \cdot \phi - \frac{1}{2} \partial \circ \phi' \right)^2 - \phi \cdot J^{(h)}. \qquad (2.2)$$

The rank-h tensor field ϕ is restricted to be double traceless, $\phi'' = 0$, and the sign factor $(-1)^h$ ensures a canonical kinetic term for the physical components with our mostly negative metric. The free action ($J = 0$) is then invariant under gauge transformations

$$\delta \phi = \partial \circ \varepsilon, \qquad (2.3)$$

where ε is a *traceless* rank-$(h-1)$ tensor ($\varepsilon' = 0$), though in general the Lagrangian changes by a total derivative under gauge variations.

The trace condition on the gauge parameter ε is trivial for helicities $h \leq 2$ and the double-trace condition on the field ϕ is trivial for $h \leq 3$. In fact, for ranks 0, 1, and 2 the action above is equivalent (up to total derivatives) to the Klein-Gordon, Maxwell, or linearized Einstein actions, respectively. The action remains gauge invariant when coupled to a current $J(x)$ satisfying $J'' = 0$ and $\partial \cdot J - \frac{1}{2h+d-6} g \circ \partial \cdot J' = 0$; in other words, the traceless part of $\partial \cdot J$ must vanish. The equation of motion for ϕ is most simply written as

$$\mathcal{F} = \Box \phi - \partial \circ \partial \cdot \phi + \partial^2 \circ \phi' = 0, \qquad (2.4)$$

though in fact the variation of Eq. (2.2) with respect to ϕ yields the equivalent but trace-reversed equation $\mathcal{F} - \frac{1}{2} g \circ \mathcal{F}' = 0$.

---

[2]We caution that the notation $X^{(n)}$ is used here to denote the rank of X, *not* to denote an nth trace as is sometimes done in the literature. In addition, our use of a mostly negative metric leads to some overall sign discrepancies with most of the literature.

III. TO VECTOR SUPERSPACE AND BACK

Symmetric rank-n gauge potentials can locally encode the propagation of helicity-n radiation. To describe a bosonic CSP, for which all integer helicities mix under Lorentz transformations, we must treat all integer spins on an equal footing. A natural way to do this, which is quite standard in the high-spin literature, is to introduce a new 4-vector coordinate η^μ, and consider fields $\Psi(\eta, x)$ that are smooth in η, i.e. they have Taylor expansions

$$\Psi(\eta, x) = \sum_n \eta^{\mu_1} \ldots \eta^{\mu_n} \psi^{(n)}(x)_{\mu_1 \ldots \mu_n} = \sum_n \eta^n \cdot \psi^{(n)}(x), \qquad (3.1)$$

whose coefficients $\psi^{(n)}$ are precisely the rank-n symmetric tensors that encode spin-n degrees of freedom. The compact notation introduced in the last equality follows the conventions of the previous section. Lorentz transformations Λ are taken to act on η and x while translations a will only act on x, i.e. under Poincare transformations,

$$\Psi(\eta, x) \to \Psi(\Lambda^{-1}\eta, \Lambda^{-1}x + a), \qquad (3.2)$$

which implies the usual tensor transformation law of the coefficient functions $\Psi^{(n)}$. Thus, the *orientation* of η encodes spin, while the *scale* of η can be absorbed by a field redefinition. There is no dynamics in η-space; it is just a useful bookkeeping device for compactly manipulating many tensors simultaneously. Typically, this is done by constructing functions of $\Psi(\eta = 0, x)$ and its η derivatives, which is equivalent to a tensor contraction of the components $\psi^{(n)}$. While this bookkeeping has facilitated finding equations of motion for CSPs [16], no Lagrangian equations of this form have ever been found.

The new approach that we introduce here, which we refer to as "vector superspace," is to embrace the *geometry* of η space. Though the actions we consider are superficially quite similar to those found in Ref. [17], that work considered rather singular functions of η, while here we will consider smooth fields (3.1). The simplicity of the actions thus obtained is remarkable. The sum of free Schwinger-Fronsdal actions for integer helicities in four dimensions, usually written as a sum of five terms for each tensor field $\psi^{(n)}$, can be recast as an integral in η space localized to the neighborhood of the unit spacelike hyperboloid:

$$\int [d^4\eta] \delta'(\eta^2 + 1) \frac{1}{2} (\partial_\mu \Psi(\eta, x))^2 + \frac{1}{4} \delta(\eta^2 + 1)(\partial_\eta . \partial_x \Psi)^2. \qquad (3.3)$$

The notation $[d^4\eta]$ denotes that the integral should be regulated by analytic continuation and appropriately normalized as explained below, while $\delta'(a) = \frac{d}{da}\delta(a)$.

The generalization to a CSP with nonzero ρ is obtained simply by the substitution $\partial_\eta . \partial_x \equiv \partial_\eta . \partial_x + \rho$, resulting in a ρ-deformed gauge invariance.

The remainder of this section introduces the key features of the integer-helicity action (3.3) and demonstrates its equivalence to a sum of Schwinger-Fronsdal actions; the bosonic CSP generalization will be considered in detail in the next section. Sections III A and III B motivate the basic structure of the action—its localization to an η-space hyperboloid and the need for a gauge symmetry to encode the ISO(2) little group through the hyperboloid's SO(3,1) symmetry. Section III C explicitly constructs the Schwinger-Fronsdal action and equations of motion from a component decomposition of Eq. (3.3). Some computational details are deferred to Appendix A. Though we focus on $3 + 1$-dimensional theories throughout most of this paper, the results in the Appendix suggest simple generalizations to higher dimensions, as well as to $2 + 1$ dimensions as discussed in Ref. [18].

A. Spin as superspace orientation and the delta function

We have already noted the qualitative idea that the orientation of η should geometrize the notion of spin, while its magnitude, which could be absorbed into a field redefinition of the $\psi^{(n)}(x)$, is unimportant. This idea points to a restricted class of Lorentz invariants built out of Ψ. For example, we cannot simply integrate products of η space functions to form invariants, such as $\int d^4\eta \Psi_1(\eta)\Psi_2(\eta)$, as this is sensitive to the profile of Ψ as we rescale η, rather than just the orientation. Moreover, such integrals are not well defined for smooth wave functions Ψ.

The simplest invariants that are both well defined and only sensitive to the η-space orientation dependence of fields are integrals along hypersurfaces. To guarantee the Lorentz invariance of the resulting action, we focus on the hyperboloids $\eta^2 + \alpha^2 = 0$, with Lorentz-invariant integrals of the form

$$\int d^4\eta \, \delta(\eta^2 + \alpha^2) F[\Psi(\eta)], \qquad (3.4)$$

for some functional F. The magnitude of any nonzero α can be absorbed into a field redefinition of Ψ, so without loss of generality we take $\alpha = +1$, which corresponds to a unit *spacelike* norm of η in our mostly negative metric convention. Integrals of this form still appear to be ill-defined; for example, if $F[\Psi] = 1$ then Eq. (3.4) is simply the volume of the hyperboloid, which is clearly divergent. But integrals of the form (3.4) are uniquely defined by analytically continuing η to a Euclidean-signature variable $\bar\eta$ defined by $(\bar\eta^0, \bar\eta^i) = (i\eta^0, \eta^i)$. Upon continuation, Eq. (3.4) becomes an integral of a smooth function over the compact 3-sphere $\bar\eta^2 = 1$, which is convergent. Alternatively, one can define Eq. (3.4) directly in Minkowski space using only its symmetry properties, up

to an overall normalization. Other equally well-defined invariants can be formed, for example by integrating over derivatives of δ functions. These generalizations are discussed more fully in Appendix A.

Of course, integrals such as Eq. (3.4) can equally well be thought of as functionals of the tensor components $\psi^{(n)}(x)$. To make contact with this decomposition, it is useful to introduce the generating function $G(w) \equiv \int d^4\eta \delta(\eta^2 + 1)e^{-i\eta \cdot w} = 2J_1(x)/x|_{x=\sqrt{-w^2}}$, in terms of which (see also Appendix A)

$$\int [d^D\eta]\delta(\eta^2 + 1)F(\eta) = [F(i\partial_w)G(w)]_{w=0}$$

$$= [G(i\partial_\eta)F(\eta)]_{\eta=0} \equiv F * G. \qquad (3.5)$$

But the geometric form has at least two advantages: the action is quite simple, and many important results can be demonstrated using simple algebraic manipulations or integrations (see e.g. the derivations of the Pauli-Lubanski invariant for a CSP in Sec. IV A and of the orthonormality of basis states in Appendix A 3).

B. Gauge invariance in superspace from the bottom up

The physical motivation for keeping track of only the orientation of η is that it should encode spin; this is why the actions in vector superspace are supported in a neighborhood of a hyperboloid. In fact the hyperboloid's symmetry group SO(3,1) is the smallest group that contains an ISO(2) subgroup *and* is related by analytic continuation to a compact group (allowing the definition of surface integrals by analytic continuation). For a given null momentum k, the little group is spanned by the three Lorentz generators that leave $\eta \propto k$ invariant.

To specify the generators more explicitly, we introduce a set of null frame vectors $k, q, \epsilon_\pm$ with $q \cdot k = 1$, $\epsilon_+ \cdot \epsilon_- = -2$, and all other products among frame vectors vanishing.[3] The three little-group generators are

$$R = \frac{1}{2}(\eta \cdot \epsilon_- \epsilon_+ \cdot \partial_\eta - \eta \cdot \epsilon_+ \epsilon_- \cdot \partial_\eta), \qquad (3.6)$$

$$T_\pm = i(\eta \cdot k\epsilon_\pm \cdot \partial_\eta - \eta \cdot \epsilon_\pm k \cdot \partial_\eta). \qquad (3.7)$$

Here, R reduces to the familiar helicity operator if we choose $\epsilon_\pm^0 = 0$, while $T_\pm$ are the commuting translations [more precisely, $T_+ + T_-$ and $i(T_+ - T_-)$ are the Hermitian translation generators of ISO(2)]. The algebra $[R, T_\pm] = \pm T_\pm$ can be trivially verified from the above.[4]

---

[3]This normalization convention differs from that of Refs. [3,10,17], where we took $\epsilon_+ . \epsilon_- = -1$. The unconventional normalization used here implies $W^2 = -T_+T_-$.

[4]Upon analytically continuing $\eta^0 \to i\eta^0$, R remains Hermitian provided $\epsilon_\pm^0 = 0$, while $T_\pm$ continue to *complex* linear combinations of Hermitian SO(4) generators.

Although these differential operators realize ISO(2), smooth wave functions $\Psi(\eta, k)$ do not realize its unitary helicity-h representations, except for the trivial $h = 0$ representation. Helicity representations are eigenstates of R annihilated by both $T_\pm$. But for an eigenstate of R, the best we can do is construct a wave function annihilated by *one* of $T_\pm$. For example, $\Psi_h \sim (\eta \cdot \epsilon_+)^h$ is annihilated by T_+ but has $T_-\Psi_h \sim h\eta \cdot k(\eta \cdot \epsilon_+)^{h-1}$ [of course the antisymmetric field strength $F^{\mu\nu}$ and its higher-spin generalizations are annihilated by $T_\pm$, but because $\Psi(\eta, x)$ decomposes into *symmetric* tensors, they can be obtained only as gradients of Ψ, consistent with the above]. Moreover, the functional form of Ψ_h is not fully specified by its helicity eigenvalue; any function $\Psi \sim (\eta \cdot \epsilon_+)^h f(\eta.k, \eta.q)$ is also an R eigenstate.

Both of the above difficulties are resolved simultaneously by imposing a "transverse" requirement on the wave function and positing a gauge redundancy. For example, the transverse condition $k \cdot \partial_\eta \Psi = 0$ rules out $\eta.q$ dependence of Ψ, while a gauge redundancy $\Psi \simeq \Psi + \eta \cdot k\varepsilon(\eta, x)$ ensures that $T_-\Psi_h$ is pure gauge, as is the $\eta.k$ dependence of Ψ. But these two requirements are actually incompatible: generic gauge variations $\delta\Psi = \eta \cdot k\varepsilon(\eta, x)$ violate the transverse condition!

Fortunately, our physically motivated assumption that only the behavior of Ψ on the surface $\eta^2 + 1 = 0$ matters resolves this bind; for consistency, we should have demanded the weaker transverse condition

$$\delta(\eta^2 + 1)k \cdot \partial_\eta \Psi = 0 \qquad (3.8)$$

and the surface-localized gauge equivalence $\delta(\eta^2 + 1)\Psi \simeq \delta(\eta^2 + 1)(\Psi + \eta \cdot k\varepsilon(\eta))$. Back in position space, the covariant equations $\delta'(\eta^2 + 1)\Box_x\Psi(\eta, x) = 0$ and $\delta(\eta^2 + 1)\partial_x \cdot \partial_\eta\Psi(\eta, x) = 0$, with a gauge redundancy $\delta(\eta^2 + 1)\Psi \simeq \delta(\eta^2 + 1)(\Psi + \eta \cdot \partial_x\varepsilon(\eta, x))$ realize all the helicity representations; solutions to these equations decompose (on the δ-function support) into helicity eigenstates proportional to $(\eta.\epsilon_\pm)^{|h|}$ plus gauge. The derivative of a delta function, $\delta'(x) = \frac{d}{dx}\delta(x)$, was needed in the first equation because the transverse condition involves a derivative of Ψ, and therefore depends on Ψ in a first neighborhood of $\eta^2 + 1 = 0$.

What can we say about the gauge transformation of Ψ itself, away from the $\eta^2 + 1 = 0$ surface? The most general gauge transformation on Ψ that reduces to the above on the $\eta^2 + 1 = 1$ surface, $\delta_{\varepsilon,\tilde\varepsilon}\Psi = \eta \cdot k\varepsilon(\eta) + (\eta^2 + 1)\tilde\varepsilon(\eta)$, does not preserve the transverse condition $\delta(\eta^2 + 1)k \cdot \partial_\eta\Psi = 0$. It is easy to verify that the transverse condition *is* invariant under the restricted gauge transformation

$$\delta\Psi_{\varepsilon,\chi} = \left(k \cdot \eta - \frac{1}{2}(\eta^2 + 1)k \cdot \partial_\eta\right)\epsilon(\eta, x)$$
$$+ (\eta^2 + 1)^2\chi(\eta, x), \qquad (3.9)$$

and that this redundancy suffices to guarantee that $\delta(\eta^2 + 1)\Psi(\eta, x)$ realizes the integer-helicity representations.

Strictly speaking, the subset of ϵ transformations with $\epsilon(\eta, x) = (\eta^2 + 1)\xi(\eta, x)$ for some ξ are entirely equivalent to the gauge transformations generated by χ. But it will prove convenient to include both explicitly.

From these observations, it is easy to discover that the covariant equation of motion

$$\delta'(\eta^2 + 1)\Box_x\Psi - \frac{1}{2}\partial_x \cdot \partial_\eta(\delta(\eta^2 + 1)\partial_x \cdot \partial_\eta\Psi) = 0 \qquad (3.10)$$

is gauge invariant under Eq. (3.9) with Ψ and ε unconstrained, and precisely reduces to the on-shell form once we specialize to the transverse $\delta(\eta^2 + 1)\partial_x \cdot \partial_\eta\Psi = 0$ gauge.

To see geometrically how this equation of motion encodes a tower of helicities, it is useful to fully fix gauge by taking $\Psi(\eta, x)$ to satisfy $\partial_\eta^2\Psi = \nabla_\eta.\nabla_x\Psi = \partial_\eta^0\Psi = 0$ (all three conditions can only be satisfied simultaneously on the support of the equation of motion; we defer the proof that such a gauge can be reached to Sec. IV B). In this gauge, the physical plane-wave solutions with momentum $\vec{k}$ are specified by a harmonic function on the spatial 2-plane in η space transverse to $\vec{k}$ (e.g. the plane $\eta^0 = \vec{\eta}.\vec{k} = 0$); the helicity rotation operator (3.6) simply rotates this plane. Identifying the 2-plane with the complex z plane, the helicity operator is $z\partial_z - \bar{z}\bar\partial_z$ and the harmonic condition $\partial_z\bar\partial_z\Psi(z) = 0$ restricts Ψ to be a sum of holomorphic and antiholomorphic parts, which can be decomposed as a sum of monomials z^h and $\bar{z}^h$ with helicity $\pm h$, respectively. The physical data, i.e. coefficients of each monomial in a series expansion for $\Psi(z, \bar{z})$, can indeed be obtained by integrating $\Psi(z, \bar{z})$ against z^h (or $\bar{z}^{|h|}$) over the circle defined by the plane's intersection with the hyperboloid $\eta^2 + 1 = 0$. In fact, this identification of states with harmonic functions of z will persist in the generalization of Eq. (3.10) to CSPs with $\rho \neq 0$, with Ψ acquiring a nontrivial (but fully determined up to gauge) dependence on a direction transverse to the z plane.

C. The Fronsdal formalism simplified in vector superspace

By demanding that the spin of helicity-h particles be realized geometrically and covariantly in vector superspace, we derived the quadratic action (3.3) (with $\Delta \equiv \partial_\eta.\partial_x$), whose field variation is the unified equation of motion (3.10). The action and the equation of motion (but not the Lagrangian) are invariant under the gauge transformations (3.9).

It is not yet obvious that the simple action (3.3) is equivalent to the Schwinger-Fronsdal tensor actions. To see this connection, we will need to carry out the η integration explicitly. In terms of the generating functional $G'(w) = J_0(\sqrt{-w^2})$ (see Appendix A), we can write the action as

$$S = \int d^4x \left[\frac{1}{2} G'(i\partial_\eta)(\partial_\alpha \Psi)^2 + \frac{1}{4} G(i\partial_\eta)(\Delta \Psi)^2 \right]_{\eta=0}.$$

$$(3.11)$$

Evidently, although the action (3.11) is quadratic in fields, it is not rank diagonal when we expand Ψ into components as in Eq. (3.1); for example, the degree-$2n$ term in the Taylor expansion of G' provides a rank-diagonal kinetic term for the field $\psi^{(n)}$, but also mixes the "scalar" $\psi^{(0)}$ with the maximal trace of $\psi^{(2n)}$. The rank mixing is related to the fact that tensor components in Eq. (3.1) were defined via Taylor expansion about $\eta = 0$, while the action is supported at nonzero η. To remove the rank mixing, we must decompose $\Psi(\eta, x)$ into tensors in a different way.

In this section we present a simple decomposition that recovers a sum of Fronsdal actions, but is only rank diagonal after partially fixing gauge so that the tensor components are double traceless. An example of a different decomposition, which removes rank mixing to all orders in traces while maintaining the full gauge symmetry (3.9), is discussed in Appendix B.

The nature of the Fronsdal fields' gauge transformations $\delta\phi^{(n)} = \partial \circ \epsilon^{(n-1)}$ for traceless ϵ will be our guide to decomposing Ψ into Fronsdal-like components. We can parametrize arbitrary gauge transformations in terms of tensor components as

$$\epsilon(\eta, x) = \sum_n (n+1)c_{n+1}\eta^n \tilde{\epsilon}^{(n)}(x)$$
$$= c_1 \tilde{\epsilon}^{(0)} + 2c_2 \eta^\mu \tilde{\epsilon}^{(1)}_\mu + 3c_3 \eta^\mu \eta^\nu \tilde{\epsilon}^{(2)}_{\mu\nu} + \cdots, \quad (3.12)$$

$$\chi(\eta, x) = \sum_n \eta^n \chi^{(n)}(x) = \chi^{(0)} + \eta^\mu \chi^{(1)}_\mu + \eta^\mu \eta^\nu \chi^{(2)}_{\mu\nu} + \cdots,$$

$$(3.13)$$

where the numerical factors $c_n = 2^{n/2}$ are chosen for later convenience. We can further decompose each $\tilde{\epsilon}^{(n)} = \epsilon^{(n)} + g \circ \xi^{(n-2)}$ where $\epsilon^{(n)}$ is traceless. In fact, as was mentioned already in the vector-superspace language, this way of parametrizing gauge transformations is redundant. For example, any transformation of the form $\tilde{\epsilon}^{(2)} \propto g \circ \xi^{(0)}$ is equivalent to one generated by $\tilde{\epsilon}^{(0)} \propto \xi^{(0)}$. Likewise, a transformation with $\tilde{\epsilon}^{(3)} \propto g \circ \xi^{(1)}$ is equivalent to one generated by $\tilde{\epsilon}^{(1)} \propto \xi^{(1)}$ *and* $\chi^{(0)} \propto \partial \cdot \xi^{(1)}$. In general, gauge transformations generated by $\tilde{\epsilon}^{(n)} \propto g \circ \xi^{(n-2)}(x)$ are equivalent to $\tilde{\epsilon}^{(n-2)} \propto \xi^{(n-2)}$ with $\chi^{(n-3)} \propto \partial \cdot \xi^{(n-2)}$. Thus, starting at rank 2, the trace of $\tilde{\epsilon}^{(2)}$ is redundant with a lower-rank gauge transformation. At rank 3, the trace of $\tilde{\epsilon}^{(3)}$ is redundant with a lower-rank $\tilde{\epsilon}^{(1)}$ and $\chi^{(1)}$. By induction, the trace of $\tilde{\epsilon}^{(n)}$ will in general be redundant with a lower-rank traceless $\tilde{\epsilon}$ and χ. Thus, it suffices to consider the

gauge transformations generated by traceless $\epsilon^{(n)}$ and arbitrary $\chi^{(n)}$.

The gauge variation induced by the traceless gauge parameters $\epsilon^{(n)}$ can be written as

$$\delta\Psi(\eta, x) = \sum_{n \geq 1} c_n \left(\eta^{(n)} - \frac{1}{2}(\eta^2 + 1)g \circ \eta^{(n-2)} \right) \partial \circ \epsilon^{(n-1)}$$

$$= c_1 \eta^\mu \partial_\mu \epsilon^{(0)} + c_2 \left(\eta^\mu \eta^\nu - \frac{1}{2} g^{\mu\nu}(\eta^2 + 1) \right) \partial_{(\mu} \epsilon^{(1)}_{\nu)}$$

$$+ c_3 \left(\eta^\mu \eta^\nu \eta^\rho - \frac{1}{2}(g^{\mu\nu}\eta^\rho + g^{\mu\rho}\eta^\nu \right.$$

$$\left. + g^{\rho\nu}\eta^\mu)(\eta^2 + 1) \right) \partial_{(\mu} \epsilon^{(2)}_{\nu\rho)} + \cdots, \quad (3.14)$$

where $\partial_{(\mu}\epsilon_{\nu\ldots)}$ denotes the sum (without a combinatoric factor) over all permutations of the indices $\mu, \nu, \ldots$. The suggestive resemblance to the gauge transformation (2.3) of the Fronsdal fields motivates the decomposition of $\Psi(\eta, x)$ into tensor components as

$$\Psi(\eta, x) = \sum_n c_n \left(\eta^n - \frac{1}{2} g \circ \eta^{n-2}(\eta^2 + 1) \right) \phi^{(n)}, \quad (3.15)$$

which transform as $\delta\phi^{(n)} = \partial \circ \epsilon^{(n-1)}$. Up to now, the fields $\phi^{(n)}$ are unconstrained; in particular, they do not satisfy a double-trace condition. But each of the remaining $\chi^{(n)}$ gauge redundancies gives us precisely enough freedom to fix a gauge where $\phi^{(n+4)}$ is double traceless, i.e. $\phi^{(n+4)''} = 0$.

To summarize our results: we can parametrize all inequivalent gauge transformations by a sequence of traceless $\epsilon^{(n)}$ and unconstrained $\chi^{(n)}$. If we decompose the vector-superspace field $\Psi(\eta, x)$ *à la* Eq. (3.15) in terms of tensor components $\phi^{(n)}(x)$, we can always reach a gauge where the $\phi^{(n)}$ are double traceless. This requirement fully fixes the $\chi^{(n)}$ gauge symmetries leaving only the traceless $\epsilon^{(n)}$'s, which transform $\psi^{(n)}$ in precisely the same manner as Eq. (2.3), i.e. $\delta\psi^{(n)} = \partial \circ \epsilon^{(n)}$.

With this decomposition in hand, it is straightforward to verify that the action (3.3) is equivalent to a direct sum of Schwinger-Fronsdal actions for all n. The coefficients c_n have been chosen so that the component fields $\phi^{(n)}$ are canonically normalized in four dimensions. For example, to expand the term $\int [d^4\eta] \frac{1}{4} \delta(\eta^2 + 1)(\Delta\Psi(\eta, x))^2$ in components, we first note that

$$\Delta\Psi(\eta, x) = \partial_\eta \cdot \partial_x \Psi(\eta, x)$$
$$= \sum_{n \geq 1} nc_n \eta^{n-1} D^{(n-1)}(x) + (\eta^2 + 1)\ldots, \quad (3.16)$$

$$D^{(n-1)} \equiv \left(\partial \cdot \phi^{(n)} - \frac{1}{2} \partial \circ \phi^{(n)'} \right), \quad (3.17)$$

where the terms proportional to $(\eta^2 + 1)$ do not contribute to the action because of the δ function. Moreover, $D^{(n-1)}$ is traceless, and is therefore annihilated by ∂_η^2. Thus, upon expanding the generating function $G(i\partial_\eta)$ as a power series in ∂_η^2, the only surviving terms in Eq. (3.11) are those where each ∂_η^2 acts as $\partial_\eta^\mu \Delta\Psi \partial_{\eta_\mu} \Delta\Psi$, i.e. where two D-tensors of the same rank are fully contracted into each other:

$$\int [d^4\eta]\delta(\eta^2 + 1)(\partial_\eta \cdot \partial_x \Psi(\eta, x))^2$$

$$= G(i\partial_\eta)\left(\sum_{n\geq 1} 2^{n/2} n\eta^{n-1} D_{(n-1)}\right) \quad (3.18)$$

$$\left(\sum_{m\geq 1} 2^{m/2} m\eta^{m-1} D_{(m-1)}\right)\Big|_{\eta=0} \quad (3.19)$$

$$= \sum_n \frac{n}{2} D^{(n-1)}(x) \cdot D^{(n-1)}(x), \quad (3.20)$$

which precisely matches the third term in the Schwinger-Fronsdal action (2.2). The final result follows from Taylor expanding the Bessel function in G and a bit of careful counting; the factor of $n/2$ arises from the product of the Taylor coefficient in the Bessel function, the $2^p p!^2$ combinatoric factor from all the ways that $(\partial_\eta)^{2p}$ can act on the term in Eq. (3.19) with $n = m = p$, and the explicit factors $(2^{n/2}n) \times (2^{m/2}m)$. Completely analogous calculations show that the first two terms of Eq. (2.2) can be obtained by expanding the δ' term in the vector-superspace action (3.3).

This construction demonstrates that we can view the vector-superspace action (2.2) as a sum over helicities of helicity-n Schwinger-Fronsdal actions, with the tensor fields grouped together in a convenient way. Conversely, we can view the Schwinger-Fronsdal action for a single helicity as arising from a partial gauge fixing of the vector-superspace action, with the field Ψ restricted to a particular spin sector. It would be interesting to explore whether this repackaging of the integer-helicity Schwinger-Fronsdal actions in a simple geometric form may generalize to high-spin theories in de Sitter and anti–de Sitter backgrounds, and to other formulations of high-spin theories [24–27], but we do not pursue these questions here. Instead, we take the η-space action (3.3)—which propagates one state of every integer helicity and treats all spins on an equal footing—as a starting point for building a gauge theory for a free CSP.

IV. THEORY OF A SINGLE CSP

We have seen that the gauge potential Ψ with dynamics governed by Eq. (3.3) describes a tower of free helicity-h degrees of freedom, one for each integer helicity value. This is a natural starting point for describing a CSP with $\rho \neq 0$, as we expect such an object to involve a propagating tower

of integer-spaced helicity-like degrees of freedom that mix under boosts. In fact, a trivial generalization of this action does the job.

As was anticipated in the last section, the action for a CSP with nonzero spin invariant ρ is simply

$$S = \frac{1}{2}\int d^4x [d^4\eta]\left(\delta'(\eta^2 + 1)(\partial_\alpha \Psi)^2 + \frac{1}{2}\delta(\eta^2 + 1)(\Delta\Psi)^2\right),$$
(4.1)

where $\Delta\Psi \equiv (\partial_\eta.\partial_x + \rho)\Psi$ [which differs from Eq. (3.3) by only the "$+\rho$" term in Δ]. This action is invariant under gauge transformations

$$\delta_{\epsilon,\chi}\Psi \equiv \left(\eta.\partial_x - \frac{1}{2}(\eta^2 + 1)\Delta\right)\epsilon(\eta, x) + (\eta^2 + 1)^2\chi(\eta, x).$$
(4.2)

This section unpackages the physical content of this free action. We first demonstrate that this action indeed propagates massless degrees of freedom of the "continuous-spin" type; i.e. they transform faithfully under the little group, with a nonzero eigenvalue for the Pauli-Lubanski invariant. The argument closely parallels the discussions of the previous section, but it is necessary to keep track of what changes under the ρ deformation of the action and gauge symmetry, and what features are unaffected. To make the little-group transformation properties more explicit, and to show that the action propagates precisely one CSP (i.e. one state for each helicity eigenvalue), we construct an explicit basis of gauge-inequivalent wave functions that solve the equations of motion following from Eq. (4.1) and illustrate their transformation under the little group in Sec. IV B. In Sec. IV C, we recast this action and the resulting equation of motion in a tensor form similar to the Fronsdal action.

A. Mass and spin

To demonstrate that the action (4.1) propagates continuous-spin particles, we show that the propagating degrees of freedom have eigenvalues $P^2\Psi(\eta, x) = 0$ and $W^2\Psi(\eta, x) = -\rho^2\Psi(\eta, x)$ under the two invariants of the $3+1$-dimensional Poincaré group, where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma = \epsilon^{\mu\nu\rho\sigma}\eta_\nu\partial_{\eta_\rho}\partial_{x_\sigma}$ is the Pauli-Lubanski pseudovector. This is to be contrasted with massless helicity particles, which have $P^2 = W^2 = 0$, and with massive particles with spin, for which P^2 and W^2 are both nonzero. Of course, in a gauge theory we should only demand these equalities up to pure-gauge corrections:

$$P^2\Psi(\eta, x) = 0 + \text{gauge} \quad \text{and}$$
$$W^2\Psi(\eta, x) = -\rho^2\Psi(\eta, x) + \text{gauge}. \quad (4.3)$$

To show that the degrees of freedom are massless, we can simply reverse the argument of Sec. III B. But because the

$O(\rho^2)$ term in Eq. (4.1) looks superficially like a mass term, it is worth going through the logic explicitly.

Varying the action (4.1) with respect to Ψ yields the free equation of motion

$$\delta'(\eta^2 + 1)\Box\Psi - \frac{1}{2}\Delta(\delta(\eta^2 + 1)\Delta\Psi(\eta, x)) = 0, \quad (4.4)$$

which is gauge invariant.[5] As is often the case in gauge theories, the equation of motion suggests a choice of "harmonic" gauge in which the Ψ equation of motion simplifies to a massless wave equation. Here, the gauge choice $\delta(\eta^2 + 1)\Delta\Psi = 0$ brings the equation of motion into the form $\delta'(\eta^2 + 1)\Box\Psi = 0$. To see that this gauge can be reached, note that under ϵ-type gauge transformations, $\delta_\epsilon\Delta\Psi(\eta, x) = -\frac{1}{2}(\eta^2 + 1)\Delta^2\epsilon(\eta, x) + \Box\epsilon(\eta, x)$. The first term vanishes on the support of $\delta(\eta^2 + 1)$, while the second can be used to gauge away $\delta(\eta^2 + 1)\Delta\Psi$ using $\epsilon = -\frac{1}{\Box}\Delta\Psi$. Writing $\Box\Psi = (\eta^2 + 1)^2\beta(\eta, x)$ for any solution to this equation of motion, we can further gauge away β by using a $\chi = -\frac{1}{\Box}\beta$ gauge transformation. Thus, any solution to the equation of motion (4.4) is gauge equivalent to a solution of

$$\Box\Psi(\eta, x) = 0 \quad \text{and} \quad \delta(\eta^2 + 1)\Delta\Psi = 0. \quad (4.5)$$

The first equation shows that the degrees of freedom propagated in Ψ are massless. The harmonic gauge choice leaves a residual gauge freedom generated by arbitrary $\chi(\eta, x)$ and $\epsilon(\eta, x)$ with $\Box\epsilon(\eta, x) = \Box\chi(\eta, x) = 0$.

Having shown that our action describes massless degrees of freedom, we can now study the action of the spin invariant W^2 on the field Ψ, working in harmonic gauge for simplicity. On the support of $\Box_x\Psi = 0$, the operator W^2 can be simplified to the form

$$W^2\Psi = [\eta^2(\partial_\eta.\partial_x)^2 + (\eta.\partial_x)^2\Box_\eta - 2\eta.\partial_x\partial_\eta.\partial_x\eta.\partial_\eta]\Psi \quad (4.6)$$

$$= [\eta^2(\Delta - \rho)^2 + (\eta.\partial_x)^2\Box_\eta - 2\eta.\partial_x(\Delta - \rho)\eta.\partial_\eta]\Psi \quad (4.7)$$

[5]As we noted in our earlier work [17], this equation is related to the equation of motion proposed in Ref. [16]. Specifically, if we define $\phi(w, x) \equiv \int d^4\eta\delta'(\eta^2 + 1)e^{iw.\eta}\Psi(\eta, x)$ (which can be inverted for any function ϕ satisfying an appropriate double-trace condition), then ϕ satisfies the non-Lagrangian equation of motion of Ref. [16]. This connection motivated generalizing our action in Ref. [17] from $\eta^2 = 0$ localization to $\eta^2 \pm 1 = 0$, and we thank X. Bekaert and J. Mourad for encouraging us to do so. Since this paper appeared, the connection has been further discussed in Ref. [28].

$$= [\eta^2(\Delta - \rho)^2 + (\eta.\partial_x)^2\Box_\eta - 2\eta.\partial_x(\eta.\partial_\eta + 1) \\ \times (\Delta - \rho)]\Psi, \quad (4.8)$$

where in the last line we have used the commutator $[\Delta, \eta.\partial_\eta] = \Delta - \rho$. Other useful commutation relations are

$$[\Delta, \eta^2] = 2\eta.\partial_x, \qquad [\eta.\partial_\eta, \eta^2] = 2\eta^2,$$
$$[\partial_\eta^2, \eta^2] = 4(\eta.\partial_\eta + 2), \quad (4.9)$$

of which the last is specific to four dimensions. For Ψ in harmonic gauge, we can write

$$\Delta\Psi = (\eta^2 + 1)\alpha(\eta, x), \quad (4.10)$$

where the residual α is unconstrained.

Using this and the commutator relations (4.9) repeatedly, we obtain

$$W^2\Psi = -\rho^2\Psi + \delta_\epsilon[(\eta.\partial_x\partial_\eta^2 - 2\partial_\eta.\partial_x\eta.\partial_\eta)\Psi - 2\alpha] \\ + \delta_\chi\left[\frac{1}{2}(\eta.\partial_x\partial_\eta^2 - 2\partial_\eta.\partial_x(\eta.\partial_\eta + 2))\alpha\right], \quad (4.11)$$

where α is the solution of Eq. (4.10), and $\delta_\chi[\chi_T]$ denotes the gauge transformations obtained from Eq. (4.2) with $\chi(\eta, x) = \chi_T$ (and similarly for δ_ϵ). We note that the pure-gauge residual is independent of ρ!

We thus see that $W^2\Psi = -\rho^2\Psi$ up to gauge transformations, as claimed. This shows that the action (4.1) propagates CSP degrees of freedom with a definite spin scale ρ—a considerable advance over [17], which propagated a continuum of CSPs of *every* real ρ. We will see below, by constructing an explicit basis of states, that Ψ represents precisely one CSP.

B. Plane waves and polarizations

To identify the polarization content in Ψ more explicitly, we now exhibit a convenient basis of solutions (up to gauge) to the harmonic-gauge equations of motion (4.5). We will decompose the solutions into helicity eigenmodes, as can always be done in asymptotically flat space. When the Pauli-Lubanski spin invariant W^2 vanishes, this decomposition is Lorentz invariant. When W^2 is nonzero, as we expect it is for $\rho \neq 0$, the helicity modes will mix under boosts. We will see this explicitly in the transformation of our basis wave functions.

The helicity decomposition is readily carried out in a plane wave basis, $\Psi_k(\eta, x) = \psi(\eta, k)e^{-ik\cdot x}$, where of course $k^2 = 0$ and the function $\psi(\eta, k)$ is analytic in both arguments and satisfies

$$\delta(\eta^2 + 1)(ik \cdot \partial_\eta + \rho)\psi(\eta, k) = 0. \quad (4.12)$$

As before, we can introduce a set of null frame vectors $q, \epsilon_\pm$ with $q \cdot k = 1$, $\epsilon_+ \cdot \epsilon_- = -2$, and all other products vanishing. The helicity operator is just

$$R\Psi = \frac{1}{2}(\eta \cdot \epsilon_- \epsilon_+ \cdot \partial_\eta - \eta \cdot \epsilon_+ \epsilon_- \cdot \partial_\eta)\Psi. \qquad (4.13)$$

For functions analytic in η, this operator necessarily has integer eigenvalues.

One set of eigenstates satisfying the harmonic gauge condition—which actually furnish a basis up to gauge transformations—is

$$\psi_{+h}(\eta, k) = (\eta \cdot \epsilon_+)^h e^{i\rho\eta \cdot q} \psi_{-h}(\eta, k) = (\eta \cdot \epsilon_-)^{|h|} e^{i\rho\eta \cdot q}. \qquad (4.14)$$

To see that these are a basis, we first consider an arbitrary function $\psi(\eta.k, \eta.q, \eta.\epsilon_+, \eta.\epsilon_-)$ at null momentum k (we leave the k dependence of ψ implicit). The harmonic gauge condition implies $(ik.\partial_\eta + \rho)\psi(\eta) = (\eta^2 + 1)\alpha(\eta)$; when $k^2 = 0$, transforming by a gauge parameter ϵ satisfying $(ik.\partial_\eta + \rho)^2 \epsilon = 2\alpha$ brings ψ into a gauge where $(ik.\partial_\eta + \rho)\psi = 0$ everywhere (not just on the $\eta^2 + 1 = 0$ surface), so that we can write $\psi = e^{i\rho\eta \cdot q}[f(\eta.\epsilon_+, \eta.\epsilon_-) + \eta. kg(\eta.k, \eta.\epsilon_+, \eta.\epsilon_-)]$, where we have explicitly separated out the $\eta.k$-independent part of ψ. Transforming by $\epsilon = -e^{i\rho\eta \cdot q}g$ removes all $\eta.k$ dependence, so that $q.\partial_\eta \psi = 0$. At this point, the remaining gauge freedom that preserves the conditions $(ik.\partial_\eta + \rho)\psi = q.\partial_\eta \psi = 0$ are those generated by $\epsilon = e^{i\rho\eta \cdot q} q.\eta h(\eta.\epsilon_+, \eta.\epsilon_-)$, which generate $\delta\psi = \frac{1}{2}e^{i\rho\eta \cdot q}(\eta.\epsilon_+\eta.\epsilon_- - 1)h(\eta.\epsilon_+, \eta.\epsilon_-)$. These can be used to remove any dependence of ψ on the product $\eta.\epsilon_+\eta.\epsilon_-$, so that all that remains is a sum of the basis functions (4.14), $\psi(\eta) = e^{i\rho\eta \cdot q}(f_+(\eta.\epsilon_+) + f_-(\eta.\epsilon_-))$. Thus, any function satisfying the harmonic gauge conditions and equation of motion can be decomposed into a linear combination of basis functions of the form (4.14) plus pure gauge, and we have already argued that a general solution to the covariant equation of motion (4.4) can be brought into harmonic gauge. In fact, the argument above applies whether or not q is null, so we could for example reach a "Coulomb-like" gauge where $q^\mu = (1/k^0, 0, 0, 0)$ and $\nabla_\eta.\nabla_x \psi = 0$. This construction also justifies the description of ψ as a sum of holomorphic and antiholomorphic functions on a complex plane introduced in Sec. III B.

The basis functions (4.14) are also orthonormal under the natural inner product

$$\langle \psi_h(k)|\psi_{h'}(k)\rangle \equiv \int d^4\eta \delta'(\eta^2 + 1)\psi_h(\eta, k)^* \psi_{h'}(\eta, k)$$

$$= \delta_{h,h'}. \qquad (4.15)$$

Evaluating this integral using the Euclidean η-space techniques is very simple, and is done explicitly in Appendix A 3.

Given an arbitrary wave function $\Psi(\eta, k)$ in momentum space satisfying the equation of motion, we can project out helicity components $a_h^*(k)$ (for positive energy null $k^0 > 0$) as

$$\frac{a_h^*(k)}{2|k|} = \int d^4\eta \delta'(\eta^2 + 1)\psi_h(\eta, k)\Psi(\eta, -k) \qquad (4.16)$$

where the normalization is chosen to coincide with relativistic conventions. This projection is gauge invariant. The variation of Eq. (4.16) under a gauge transformation is

$$\int d^4\eta \delta'(\eta^2 + 1)\psi_h$$

$$\times (\eta, k)\left(-ik \cdot \eta - \frac{1}{2}(\eta^2 + 1)(-ik \cdot \partial_\eta + \rho)\right)\epsilon$$

$$= \int d^4\eta \psi_h(\eta, k)(-ik \cdot \partial_\eta + \rho)(\delta(\eta^2 + 1)\epsilon) \qquad (4.17)$$

which vanishes after integrating by parts in η because $\delta(\eta^2 + 1)(ik \cdot \partial_\eta + \rho)\psi_h(\eta, k) = 0$ (i.e. ψ_h satisfies the harmonic-gauge condition).

The helicity decomposition for a real field $\Psi(\eta, x)$ is therefore

$$\Psi(\eta, x) = \int \frac{d^3k}{2|k|}$$

$$\times \sum_h (a_h(k)\psi_h(\eta, k)e^{-ik \cdot x} + a_h^*(k)\psi_h^*(\eta, k)e^{ik \cdot x}) \qquad (4.18)$$

where $a_h^*(k)$ denotes complex conjugation and we are allowed of course to add a pure-gauge term to the right-hand side.

Now we can see how the gauge symmetry allows CSPs to be encoded in the wave function Ψ. Let us consider how $a_h^*(k)$ defined by Eq. (4.16) transforms under the combinations of rotations and boosts that leave k invariant, namely transformations generated by $T_\pm = \epsilon_\pm \cdot W$. These generators should act like raising and lowering operators on $a_h^*(k)$. Starting from the covariant transformation rule $U(\Lambda)^\dagger \Psi(\eta, k)U(\Lambda) \equiv \Psi(\Lambda^{-1}\eta, \Lambda^{-1}k)$, we have

$$U(\Lambda)^\dagger \frac{a_h^*(k)}{2|k|} U(\Lambda)$$

$$\equiv \int d^4\eta \delta'(\eta^2 + 1)\psi_h(\eta, k)\Psi(\Lambda^{-1}\eta, -\Lambda^{-1}k)$$

$$= \int d^4\eta \delta'(\eta^2 + 1)\psi_h(\Lambda\eta, k)\Psi(\eta, -\Lambda^{-1}k).$$

Specializing to transformations $U(\Lambda) = e^{i(\beta_+ T_+ + \beta_- T_-)}$ generated by $T_\pm$, and working infinitesimally, we have

$$[T_\pm, a_h^*(k)] = \int d^4\eta \delta'(\eta^2 + 1)[T_\pm, \psi_h(\eta, k)]\Psi(\eta, -k), \qquad (4.19)$$

where we have used the fact that $T_\pm$ annihilates k. We are free to work in the harmonic gauge $\delta(\eta^2 + 1)\Delta\Psi = 0$, where any pure-gauge terms in $[T_\pm, \psi_h(\eta, k)]$ give vanishing contributions to the integral (4.19), by an integration-by-parts argument analogous to that used in deriving Eq. (4.18). For $h > 0$, we have $\psi_h(\eta, k) = (\eta \cdot \epsilon_+)^h e^{i\rho\eta \cdot q}$ and

$$[T_+, \psi_h(\eta, k)] = i(\eta \cdot k\epsilon_+ \cdot \partial_\eta - \eta \cdot \epsilon_+ k \cdot \partial_\eta)\psi_h(\eta, k)$$
$$= \rho\eta \cdot \epsilon_+ \psi_h(\eta, k) = \rho\psi_{h+1}(\eta, k),$$

so that T_+ acts as expected on the mode $a_h^*(k)$, i.e. $[T_+, a_h^*(k)] = \rho a_{h+1}^*(k)$. On the other hand

$$[T_-, \psi_h(\eta, k)] = i(\eta \cdot k\epsilon_- \cdot \partial_\eta - \eta \cdot \epsilon_- k \cdot \partial_\eta)\psi_h(\eta, k)$$
$$= -2ih\eta.k\psi_{h-1}(\eta, x) + \rho\eta \cdot \epsilon_- \psi_h \qquad (4.20)$$
$$= \rho\psi_{h-1} + \delta_\epsilon[-2(h + i\eta \cdot q)\psi_{h-1}]. \qquad (4.21)$$

To see that the last two lines are indeed equivalent up to pure-gauge terms [which contribute nothing to the integral (4.19)], it is useful to note that $g^{\mu\nu} = -\frac{1}{2}\epsilon_+^{(\mu}\epsilon_-^{\nu)} + q^{(\mu}k^{\nu)}$ so that $\eta^2 = -\eta.\epsilon_+\eta.\epsilon_- + 2\eta.q\eta.k$ and therefore $\epsilon = -2i(q \cdot \eta)\psi_{h-1}$ generates the gauge transformation

$$\delta\psi = \left(ik \cdot \eta - \frac{1}{2}(\eta^2 + 1)(ik \cdot \partial_\eta + \rho)\right)\epsilon$$
$$= 2k \cdot \eta q \cdot \eta\psi_{h-1} - (\eta^2 + 1)\psi_{h-1}$$
$$= (2k \cdot \eta q \cdot \eta - \eta^2)\psi_{h-1} - \psi_{h-1}$$
$$= \epsilon_+ \cdot \eta\epsilon_- \cdot \eta\psi_{h-1} - \psi_{h-1},$$

while $\epsilon = -2h\psi_{h-1}$ simply generates $\delta\psi = -2ih\eta.k\psi_{h-1}$. We therefore obtain

$$[T_-, \psi_h(\eta, k)] \simeq \rho\psi_{h-1}, \qquad (4.22)$$

so that going back to Eq. (4.19) we find

$$[T_\pm, a_h^*(k)] = \rho a_{h\pm1}^*(k) \qquad (4.23)$$

as expected for the polarizations of a CSP with a spin scale $W^2 = -\rho^2$. Similar manipulations show that $T_\pm$ act correctly for $h \leq 0$ as well.

We will take up a covariant quantization of this theory in Ref. [29], but much of the free quantum theory is already easy to anticipate from the mode expansion. The coefficients $a_h(k)$ will become annihilation operators with commutation relations

$$[a_h(k), a_{h'}(k')^*] = 2|k|\delta^3(k - k')\delta_{hh'}, \qquad (4.24)$$

with the interpretation of creating or annihilating states with momentum k and polarization h. This fact, together

with Eq. (4.23), shows that the quantum theory indeed describes a single CSP.

C. The action and equation of motion in tensor form

To gain more intuition, it is useful to decompose the action into tensor fields as we did for $\rho = 0$ in Sec. III C. As before, we decompose $\Psi(\eta, x)$ into tensor fields as

$$\Psi(\eta, x) = \sum_n c_n\left(\eta^n - \frac{1}{2}g \circ \eta^{n-2}(\eta^2 + 1)\right)\phi^{(n)}, \qquad (4.25)$$

with the coefficients $c_n = 2^{n/2}$ chosen so that the fields $\phi^{(n)}$ are canonically normalized. We can again fix a gauge where the $\phi^{(n)}$ are double traceless using χ gauge invariance; the gauge transformations that preserve this double-traceless gauge are parametrized by $\epsilon(\eta, x) = \sum_n(n + 1)c_{n+1}\eta^n\epsilon^{(n)}(x)$ where $\epsilon^{(n)}(x)$ is an arbitrary *traceless* rank-n tensor. The component fields vary under these gauge transformations as

$$\delta\phi^{(n)} = \partial \circ \epsilon^{(n-1)} + \frac{\rho}{\sqrt{2}}\left[\epsilon^{(n)} + \frac{1}{n(n-1)}g \circ \epsilon^{(n-2)}\right]. \qquad (4.26)$$

The action for the Fronsdal-like fields can be found by the same methods used in Sec. III C. Here

$$\Delta\Psi \simeq \sum_{n \geq 1}\eta^{n-1} \cdot \left(nc_n\left(\partial_x \cdot \phi^{(n)} - \frac{1}{2}\partial_x \cdot \phi^{(n)\prime}\right) + \rho c_{n-1}\phi^{(n-1)}\right), \qquad (4.27)$$

where $\phi^{(n)\prime} = \text{Tr}\phi^{(n)}$ and $\simeq$ denotes that terms proportional to $\eta^2 + 1$, which vanish on the δ-function support, have been dropped. With the inclusion of the $O(\rho)$ term this is no longer traceless (but it is double traceless), so when we compute $G(i\partial_\eta)(\Delta\Psi)^2$ as in Sec. III C using the series expansion for the generating function $G(i\partial_\eta) = J_0(\sqrt{\partial_\eta^2})$ we must keep terms where zero or one ∂_η^2 act on the series expansion of each $\Delta\Psi$. The resulting action can be written as

$$S = \sum_n(-1)^n\int d^4x\frac{1}{2}(\partial_\mu\phi^{(n)})^2 - \frac{n(n-1)}{8}(\partial_\mu\phi^{(n)\prime})^2$$
$$- \frac{n}{2}(D_\rho^{(n-1)})^2 \qquad (4.28)$$

where

$$D_\rho^{(n-1)}(x) \equiv \partial_x \cdot \phi^{(n)} - \frac{1}{2}\partial_x \circ \phi^{(n)\prime}$$
$$+ \frac{\rho}{\sqrt{2n}}\left(\phi^{(n-1)} - \frac{1}{2(n-1)}g \circ \phi^{(n-1)\prime}\right)$$
$$- \frac{\rho}{2\sqrt{2}}\phi^{(n+1)\prime}. \tag{4.29}$$

The action, which was rank diagonal at $O(\rho^0)$, exhibits rank mixing at $O(\rho)$. Of course, the action becomes rank diagonal in harmonic gauge (which corresponds to $D_\rho^{(n-1)} = 0$), but then the gauge condition itself mixes ranks.

Similarly, the CSP equation of motion mixes tensors of different ranks:

$$- \Box\phi^{(n)} + \partial \circ D_\rho^{(n-1)} + \frac{\rho}{\sqrt{2}}D_\rho^{(n)}$$
$$+ \frac{\rho}{\sqrt{2}n(n-1)}g \circ D_\rho^{(n-2)} = 0. \tag{4.30}$$

While the tensor decomposition makes manifest that we smoothly recover the Schwinger-Fronsdal equations and action in the $\rho \to 0$ limit, it is more cumbersome for nonzero ρ than the simple vector-superspace description. Firstly, the equation of motion (4.30) is a bit complicated (this is, of course, not invariant under field redefinitions; for example, tensors defined as Taylor coefficients of $\tilde{\Psi}(\omega, x) \equiv \int d^4\eta\delta'(\eta^2 + 1)e^{i\eta\cdot\omega}\Psi(\eta, x)$, which by definition respects a double-trace condition $(\Box_\omega - 1)^2\tilde{\Psi} = 0$, satisfy a different equation of motion of the form found in Ref. [16]). Another drawback—and a much more invariant one—is that our helicity eigenmodes $\psi(\eta, k) = (\eta.\epsilon_\pm)^{|h|}e^{i\rho\eta.q}$ (for q with $q \cdot k = 1$ and $q \cdot \epsilon_\pm = 0$) from Sec. IV B do not correspond to a single tensor mode when $\rho \neq 0$. For example, for $h = 0$ we have $\phi^{(0)} = 0$, $\phi_\mu^{(1)} = i\rho q_\mu/\sqrt{2}$, $\phi_{\mu\nu}^{(2)} = -\rho^2 q_\mu q_\nu/2$, etc. It is tempting to look for a Lorentz-covariant field redefinition where the helicity-$\pm h$ modes are encoded only in a rank-h tensor, but in fact such a representation is at odds with the action of the little group on single-particle CSP states. In particular, any rank-h tensor is annihilated by $(T_\pm)^{h+1}$, while this would simply raise (lower) the helicity-h mode of a CSP to a mode with helicity $(2h + 1)$ or -1. Thus, a covariant representation of *any* CSP state in terms of tensors must involve infinitely many nonzero tensor components, so that $T_\pm$ can act arbitrarily many times without annihilating the state. These and other considerations underscore the value of the vector-superspace formulation emphasized in this paper.

V. A WINDOW ON CSP INTERACTIONS

Though we have thus far focused on the free action, it is straightforward to couple the gauge field Ψ to a background current by adding a source term

$$S_{\text{int}} = \int d^4x[d^4\eta]\delta'(\eta^2 + 1)\Psi(\eta, x)J(\eta, x). \tag{5.1}$$

This is, in fact, the most general linear current that we can couple to the degrees of freedom of Ψ in the first neighborhood of the $\eta^2 + 1 = 0$ surface. The coupling is allowed by gauge invariance so long as J satisfies a "continuity condition"

$$\delta(\eta^2 + 1)(\partial_x.\partial_\eta + \rho)J = 0, \tag{5.2}$$

or in other words

$$(\partial_x.\partial_\eta + \rho)J = (\eta^2 + 1)\alpha(\eta, x), \tag{5.3}$$

for arbitrary analytic $\alpha(\eta, x)$.

This section explores some simple features of this current coupling, for $\rho = 0$ and nonzero ρ. In Sec. V A we sketch how standard scattering amplitudes involving helicities 0, 1, and 2 can be obtained in the vector-superspace formalism. Section V B describes how the usual conservation conditions are modified with nonzero ρ, and the pattern of nonzero currents at all ranks suggested by the helicity correspondence of CSP-emission amplitudes. Finally, Sec. V C demonstrates the consistency of this formalism with the candidate single-CSP amplitudes found in Refs. [3,10], albeit with an *ad hoc* and nonlocal ansatz for current matrix elements.

A. Helicity interactions in vector superspace

It is straightforward to calculate simple helicity amplitudes using η space, and in fact this formalism is appealingly universal, regardless of whether the emitted particle has helicity 0, 1, or 2. Before doing these calculations, however, we pause to make contact between the vector-superspace continuity condition (5.3) and the more familiar conservation conditions, and in particular to comment on the significance of the arbitrary function α.

It will be convenient to expand $J(\eta, x)$ into tensor components so that the current coupling in tensor components is rank diagonal, i.e. so that Eq. (5.1) reduces to $\sum_n(-1)^n\phi^{(n)}(x) \cdot J^{(n)}(x)$. This is achieved by decomposing $J(\eta, x)$ as

$$J(\eta, x) = \sum_n 2^{n/2}\left(\eta^n + \frac{1}{2(n-1)}\eta^{(n-2)} \circ g\right)J^{(n)}$$
$$= \sum_n 2^{n/2}\eta^n\left(J^{(n)} + \frac{n+2}{2}J^{(n+2)\prime}\right). \tag{5.4}$$

In this case, allowing nonzero α is analogous (and, for $\rho = 0$, equivalent) to the well-known consistency of coupling high-spin fields to currents that are only conserved up to traces (i.e. "weakly conserved"), e.g. $\partial^\mu J_{\mu\nu\rho}^{(3)}(x) = g_{\nu\rho}\alpha(x)$ with arbitrary α. Thus, the possibility

of nonzero α can be ignored for helicities ≤ 2. For $\rho = 0$, the continuity condition does not constrain $J^{(0)}$ and imposes the usual conservation requirements $\partial . J^{(1)} = 0$, and $\partial . J^{(2)}$ for the rank-1 and -2 currents.

Returning to the vector-superspace formalism, it is simple to calculate emission amplitudes for a polarization-h component of Ψ with momentum k in the presence of the background $J(\eta, x)$ which we assume to have only $J^{(0)}$, $J^{(1)}$, and/or $J^{(2)}$ nonzero, i.e. $J(\eta, x) = J^{(0)} + \sqrt{2}\eta \cdot J^{(1)} + (2\eta^2 \cdot J^{(2)} + J^{(2)\prime})$. Representing the single-particle states as $a_h^\dagger(k)|0\rangle$, the amplitude is just

$$
\begin{aligned}
\mathcal{A}_J(\{k, h\}) &= \int d^4x d^4\eta \, \delta'(\eta^2 + 1)\langle k, h|\Psi(\eta, x)|0\rangle J(\eta, x) \\
&= \int d^4x d^4\eta \, \delta'(\eta^2 + 1)\psi_h^*(\eta, k)e^{ik\cdot x}J(\eta, x) \\
&= \int d^4\eta \, \delta'(\eta^2 + 1)\psi_h^*(\eta, k)\tilde{J}(\eta, -k) \quad (5.5)
\end{aligned}
$$

where $\tilde{J}(\eta, -k)$ is the Fourier transform of $J(\eta, x)$. For the current $J(\eta, x)$ given above, we find that all helicity modes with $|h| > 2$ have vanishing emission amplitude, while for $|h| \leq 2$ we obtain

$$
\mathcal{A}_J(\{k, h = 0\}) = \tilde{J}_0(-k), \quad (5.6)
$$

$$
\mathcal{A}_J(\{k, h = 1\}) = \epsilon_{+,\mu}(k)\tilde{J}_1^\mu(-k), \quad (5.7)
$$

$$
\mathcal{A}_J(\{k, h = 2\}) = \epsilon_{+,\mu}(k)\epsilon_{+,\nu}(k)\tilde{J}_2^{\mu\nu}(-k) \quad (5.8)
$$

which precisely reproduces familiar results for helicity degrees of freedom.

We can similarly calculate current-current correlators, and hence matter amplitudes provided the matter sector can furnish an appropriately conserved current (which we know how to do for $\rho = 0$ of course). As will be discussed in Ref. [29], the harmonic-gauge propagator for Ψ is

$$
\begin{aligned}
\delta'(\eta^2 + \alpha)\delta'(\bar{\eta}^2 + \alpha)&\langle 0|T\Psi(\eta, x)\Psi(\bar{\eta}, y)|0\rangle \\
&= \delta'(\eta^2 + \alpha)\delta^{(4)}(\eta - \bar{\eta})D_F(x - y), \quad (5.9)
\end{aligned}
$$

where $D_F(x - y)$ is the Feynman propagator. Suppose we have a particle of type A with current $J_A(\eta, x)$ and of type B with current $J_B(\eta, x)$. Then the first-order scattering amplitude $A(A, B \rightarrow A'B')$ mediated by Ψ exchange is

$$
\begin{aligned}
\mathcal{A}(A, B \rightarrow A'B') &= \int d^4x d^4\eta \, \delta'(\eta^2 + 1) \int d^4y d^4\eta' \delta'(\eta'^2 + 1)\langle A'|J_A(\eta, x)|A\rangle\langle 0|T\Psi(\eta, x)\Psi(\eta', y)|0\rangle\langle B'|J_B(\eta', y)|B\rangle \\
&= \int d^4x d^4y d^4\eta \, \delta'(\eta^2 + 1)\langle A'|J_A(\eta, x)|A\rangle D_F(x - y)\langle B'|J_B(\eta, y)|B\rangle.
\end{aligned}
$$

For a scalar current $J(\eta, x) = J_0(x)$, this obviously reproduces standard results. For a vector current $J(\eta, x) = \eta \cdot J_1(x)$, the η integration yields a $g^{\mu\nu}$ contraction between $J_1^\mu(x)$ and $J_1^\nu(y)$, again as expected. For a tensor current, the η integration yields the expected symmetric tensor contractions. In fact, it is straightforward to recast simple theories like QED in this language, and calculations of correlators and scattering amplitudes are straightforward, as the above examples illustrate. It would be interesting to express the self-interactions of Yang-Mills theories and general relativity in vector-superspace language, but we have not done so.

B. Deformed conservation conditions for currents coupled to CSPs

For $\rho = 0$, we could consistently consider sources with only one nonzero $J^{(n)}$ at a time. Of course, in flat space nontrivial interacting conserved currents only exist up to rank 2. For $\rho \neq 0$, the continuity condition (5.3) in general requires that if any single component of $J(\eta, x)$ is nonzero, all higher-rank components must be nonzero as well. This is easy to see by breaking down Eq. (5.3) into components:

$$
\partial \cdot J_1 + \frac{\rho}{\sqrt{2}}\left(J_0 + \frac{1}{2}J_2'\right) = 0,
$$

$$
\partial \cdot J_2^\mu + \frac{\rho}{2\sqrt{2}}\left(J_1^\mu + \frac{1}{2}J_3'^\mu\right) = 0,
$$

$$
\left\langle \partial \cdot J_3^{\mu\nu} + \frac{\rho}{3\sqrt{2}}\left(J_2^{\mu\nu} + \frac{1}{2}J_4'^{\mu\nu}\right)\right\rangle = 0,
$$

$$
\cdots
$$

$$
\left\langle \partial \cdot J_n + \frac{\rho}{n\sqrt{2}}\left(J_{(n-1)} + \frac{1}{2}J_{(n+1)}'\right)\right\rangle = 0,
$$

where $\langle \ldots \rangle$ denotes the traceless part of the enclosed tensor. If the scalar current $J_0(x)$ is nonzero, the first continuity condition requires a nonzero $J_1^\mu(x)$ at $O(\rho)$, with nonzero divergence. The next continuity condition in turn implies a nonzero $J_2^{\mu\nu}(x)$ at $O(\rho^2)$, and so on. If we instead started with a *conserved* vector current $J_1^\mu(x)$ at $O(\rho^0)$, the continuity condition would imply that there is a nonzero $J_2^{\mu\nu}(x)$ at $O(\rho)$ with nontrivial divergence, though J_0 can consistently vanish. Finally, a similar pattern of currents can be initiated by a *conserved* rank-2 tensor, with

vanishing scalar and vector current, and all higher ranks of successively higher order in ρ. This structure is a field theory version of the *helicity correspondence* discovered in Ref. [10]: all tree-level amplitudes that consistently factorize and obey perturbative unitarity constraints all approach helicity-0, -1, or -2 amplitudes in the high-energy or nonrelativistic limit (or $\rho \to 0$ limit).

C. Making contact with soft-emission amplitudes

We do not yet know of any *local* matter sector that furnishes an appropriately conserved current that can serve as a source for Eq. (5.1) when $\rho \neq 0$. This is the central open problem to solve in formulating a complete CSP theory. If we sacrifice manifest locality, then we can readily guess appropriate current matrix elements, and they will in general (as required by Lorentz symmetry) reproduce the soft factors found in Refs. [3,10]. As a specific example, we consider an ansatz for a correlation function $\langle p'|J(\eta, x)|p^*\rangle$. Here, the notation $|p^*\rangle$ means that instead of a matrix element with an external state, we are instead considering the correlator of a matter field at a (slightly) off-shell momentum p^* with the current. We then expect that the soft factor for emission of a soft CSP with momentum k and polarization h is of the form

$$S(\{k, h\}, p) = \int d^4x\, d^4\eta\, \delta'(\eta^2 + 1)$$
$$\times \langle k, h|\Psi(\eta, x)|0\rangle\langle p'|J(\eta, x)|p^*\rangle \quad (5.10)$$

where $k + p' = p^*$. One guess that satisfies the continuity condition (5.3) is

$$\langle p'|J(\eta, x)|p^*\rangle = e^{iq\cdot x}e^{i\rho\frac{\eta\cdot(p^*+p')}{q\cdot(p^*+p')}}|_{q=p^*-p'}. \quad (5.11)$$

This guess is one of many that satisfy Eq. (5.3), and actually satisfies a stronger continuity condition than is required, since $\Delta J = 0$ vanishes even off the δ-function support (i.e. it has $\alpha = 0$).[6] Integrating, we obtain (for $h \geq 0$)

$$S(\{k, h\}, p) = \int d^4\eta\, \delta'(\eta^2 + 1)(\eta \cdot \epsilon_+(k))^h e^{i\rho\frac{\eta\cdot p}{q\cdot p}}.$$

These integrals are easy to do using the generating function formula, or by direct integration in Euclidean η space as is illustrated by example in Appendix A 3. Using standard integral representations of Bessel functions, we have

$$S(\{k, h\}, p) = e^{ih\,\arg z}J_h(\rho z), \quad (5.12)$$

where $z \equiv \frac{\epsilon_+\cdot p}{k\cdot p}$. This is precisely the soft factor found in Refs. [3,10]! The underlying gauge symmetry of our field

---

[6]However, this guess cannot be directly applied to matter-matter scattering mediated by a CSP.

theory formalism is maintained by the ansatz above, so this answer was essentially guaranteed by Lorentz symmetry, but it is reassuring to see that such CSP soft factors can be obtained from our field theory.

VI. COMMENTS ON GAUGE-INVARIANT OPERATORS

While the preceding discussion focused on the consistency requirements for a CSP to couple to a background current, we turn now to explore physically significant quantities that can be constructed from the CSP field $\Psi(\eta, x)$ itself. Our focus in Sec. VI A is on the stress-energy tensor, which is not gauge invariant, and its spatial integrals, which *are* gauge invariant. This situation is quite reminiscent of general relativity, where the energy density of a gravitational wave at a point is not physically observable, but the average energy density in a sufficiently large physical region is [30]. In Sec. VI B we turn to single-field gauge-invariant operators. It is straightforward to show that no *local* gauge-invariant operators exist that involve two or fewer derivatives, except for those which are proportional to the free equation of motion and therefore vanish on radiation. This is again reminiscent of GR, albeit slightly *less* local (the Riemann curvature $R^{\mu\nu\rho\sigma}$ is not invariant under a a full coordinate transformation, but it *is* invariant under the linearized gauge transformations).

But then, in light of the helicity correspondence, what happens to operators like $F^{\mu\nu}(x)$? We will show that there are operators local in time, but noncovariant and spatially nonlocal, that approach these local field strengths in the $\rho \to 0$ limit. In fact, with a covariant quantization one can show that these objects commute at arbitrary spacelike separation, and are therefore causal. It is still not clear that these are physically relevant objects, but they do indicate that Cauchy problems involving only physical data continue to be well posed.

A. Energy, momentum, and angular momentum of a CSP

Canonical Poincaré generators can be obtained in the usual way from the Belinfante stress-energy tensor

$$\Theta^{\mu\nu}(x) \equiv -g^{\mu\nu}\mathcal{L}(x) + \int d^4\eta \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}(\partial^\nu\psi)$$
$$+ \partial_\kappa(A^{\kappa\mu\nu} - A^{\mu\kappa\nu} - A^{\nu\kappa\mu}) \quad (6.1)$$

where

$$A^{\kappa\mu\nu} \equiv \frac{i}{2}\int d^4\eta \frac{\delta\mathcal{L}}{\delta(\partial_\kappa\psi)}\mathcal{J}^{\mu\nu}\psi(\eta, x), \quad (6.2)$$

and $\mathcal{J}^{\mu\nu} \equiv i\eta^{[\mu}\partial_\eta^{\nu]}$ are the Lorentz generators on the η space. Here and throughout this section, we use $\eta^{[\mu}\partial_\eta^{\nu]} = \eta^\mu\partial_\eta^\nu - \eta^\nu\partial_\eta^\mu$. Because $\Theta^{\mu\nu}$ is conserved and symmetric, the 3-tensor

$$\mathcal{M}^{\kappa\mu\nu} \equiv x^{\mu}\Theta^{\kappa\nu} - x^{\nu}\Theta^{\kappa\mu} \qquad (6.3)$$

is also conserved. Its conserved charges

$$J^{\mu\nu} \equiv \int d^3 x \mathcal{M}^{0\mu\nu} = \int d^3 x (x^{[\mu}T^{0\nu]} - 2A^{0\mu\nu}) \qquad (6.4)$$

$$= -\int d^3 x x^{[\mu}g^{0\nu]}\mathcal{L}(x)$$

$$+ \int d^3 x d^4\eta \frac{\delta\mathcal{L}}{\delta\dot{\psi}}(x^{[\mu}\partial^{\nu]} + \eta^{[\mu}\partial_\eta^{\nu]})\psi(\eta,x) \qquad (6.5)$$

of course generate homogeneous Lorentz transformations on fields. The total energy, momentum, and angular momentum in harmonic gauge are

$$H = \int d^3 x \Theta^{00}$$

$$= \int d^3 x [d^4\eta]\delta'(\eta^2 + 1)\left(\frac{1}{2}\dot{\psi}^2 + \frac{1}{2}|\nabla\psi|^2\right), \qquad (6.6)$$

$$P^i = \int d^3 x \Theta^{0i} = \int d^3 x [d^4\eta]\delta'(\eta^2 + 1)\dot{\psi}\partial^i\psi, \qquad (6.7)$$

$$J^{ij} = \int d^3 x \Theta^{ij}$$

$$= -\int d^3 x [d^4\eta]\delta'(\eta^2 + 1)\dot{\psi}(x^{[i}\partial^{j]} + \eta^{[i}\partial_\eta^{j]})\psi. \qquad (6.8)$$

The Belinfante tensor $\Theta^{\mu\nu}$ and the associated $\mathcal{M}^{\mu\nu\rho}$ change under gauge transformations (even those that maintain harmonic gauge), but only by total derivatives so that these spatial integrals are gauge invariant. For example, under gauge transformations that maintain harmonic gauge

$$\delta\mathcal{M}^{0ij}(x) = \int [d^4\eta]\dot{\psi}(\eta,x)R^{ij}\Delta(\delta(\eta^2 + 1)\epsilon(\eta,x))$$

$$+ \Delta(\delta(\eta^2 + 1)\dot{\epsilon}(\eta,x))R^{ij}\psi(\eta,x),$$

where $R^{ij} = x^{[i}\partial_x^{j]} + \eta^{[i}\partial_\eta^{j]}$. Because $\delta(\eta^2 + 1)\Delta\psi = 0$ in harmonic gauge, this can be rewritten as a total derivative $-\partial_{x\mu}\int [d^4\eta]\delta(\eta^2 + 1)((\partial_\eta^{\ \mu}\dot{\psi}(\eta,x))R^{ij}\epsilon(\eta,x) + \dot{\epsilon}(\eta,x))\partial_\eta^\mu R^{ij}\psi(\eta,x))$. As usual, although the total time-derivative term cannot be removed by integration by parts in the definition of J^{ij}, it can be rewritten in terms of total spatial derivatives using the equation of motion $\Box\psi = 0$ and the restriction $\Box\epsilon = 0$ on the gauge parameter.

The noncovariance of the Belinfante stress-energy tensor is not surprising. Even for $\rho = 0$, the same phenomenon is observed in linearized general relativity, where gravitational waves do not have a well-defined stress-energy density. The absence of a conserved, gauge-invariant stress-energy density for CSPs does motivate the expectation that, like in GR, the flat coordinate space x^μ used in our free formulation will have no invariant meaning in the presence of CSP interactions.

In GR, gravitational waves do of course carry energy-momentum, and if we integrate $T^{\mu\nu}$ over a spacetime region of size L much larger than the wavelength λ of gravitational radiation, then the spatial average over this region of $T^{\mu\nu}$ is gauge invariant up to effects suppressed by λ/L. This follows simply from the fact that $T^{\mu\nu}$ is gauge invariant up to spatial total derivatives (after using the equation of motion to remove time total derivatives). Because the same is true for the radiation in our CSP theory, there is no obstacle to answering the physical question of how much energy a CSP wave carries through a large spatial region, at least at the free level.

We can also ask if the spatially averaged stress-energy tensor $T^{\mu\nu}_{\text{avg}}(x)$ has properties compatible with causality in the quantum theory. In the quantum theory, it follows from Eq. (4.24) that the gauge potentials $\Psi(\eta,x)$ (in a Feynman-like gauge) satisfy familiar commutation relations

$$\delta'(\eta^2 + 1)\delta'(\eta'^2 + 1)[\Phi(\eta,x), \Phi(\eta',x')]$$

$$= \delta'(\eta^2 + 1)\delta^{(4)}(\eta' - \eta)\Delta(x - y), \qquad (6.9)$$

where $\Delta(x - y) = \int \frac{d^3 p}{2|p|}[e^{-ip.(x-y)} - \text{c.c.}]$ is the usual scalar-field commutator. It follows that in this gauge $[T^{\mu\nu}(x), T^{\rho\sigma}(x')] = 0$ for $x - x'$ spacelike, and therefore that the spatial averages commute in any gauge, provided the averaging regions are small compared to $|x - x'|$.

B. Some single-field gauge-invariant operators

Besides the stress-energy carried by a wave, another very important class of physical objects in familiar gauge theories are the field strengths. Of course, in non-Abelian gauge theories and in GR these are not gauge invariant beyond the linear level. But $F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$ is invariant under the free gauge transformation of a vector field, and so is the linearized curvature for a helicity-2 field, $R^{\mu_1\mu_2\nu_1\nu_2} = \partial^{[\mu_1}\partial^{[\mu_2}h^{\nu_1]\nu_2]}$, where we antisymmetrize μ_i with ν_i, but not with ν_j. In fact, all of the higher-spin field strengths in the $\rho = 0$ theory can be grouped into a single object

$$R(\xi,x) = \int d^4\eta \delta'(\eta^2 + 1)e^{\xi_{\mu\nu}\eta^\mu\partial^\nu}\Psi(\eta,x), \qquad (6.10)$$

where $\xi_{\mu\nu}$ is an antisymmetric matrix variable. The nth Taylor component of R is precisely the rank-$2n$ mixed-symmetry field strength for the rank-n gauge field.

Are there analogous gauge-invariant, and hence possibly physical, local objects in the theory with $\rho \neq 0$?

It is straightforward to show that the *only* local and covariant gauge-invariant objects in the CSP theory involving two or fewer derivatives of $\Psi(\eta,x)$ are equivalent to the equations of motion. To show this explicitly, one may note that a completely general function $F(\xi,x) = \int [d^4\eta]\delta'(\eta^2+1)f(\eta,\xi,\partial_x)\Psi(\eta,x)$ has gauge variation $\delta F(\xi,x) = \int [d^4\eta]\delta(\eta^2+1)(-\partial_{\eta\mu}+\rho)f(\eta,\xi,\partial_x)\partial_x^\mu\epsilon(\eta,x))$;

requiring the latter to vanish, we find that F must be a multiple of the equation of motion (4.4). This result is not surprising, since a CSP's helicity states cannot be invariantly distinguished and the high-helicity modes in Eq. (6.10) require increasingly high powers of ∂_x to build gauge invariants.

On the other hand, the preceding result raises a basic question: if CSP theories are continuously connected to familiar gauge theories, then what has happened to the linear-level gauge invariants $\phi^{(0)}$, $F^{\mu\nu}$, $R^{\mu\nu\rho\sigma}$, etc.? It is possible to build gauge invariants that approach these in the $\rho \to 0$ limit, but the covariant ones are spatially and time-nonlocal functions of fields. We can invent such an operator, but to do so requires that we introduce an unphysical auxiliary object. For example, if we introduce another 4-vector ω^μ, we can construct invariants

$$\phi(\omega, x) = \int d^4\eta d^4 y \delta'(\eta^2 + 1) f(\omega, \eta, x - y) \Psi(\eta, y),$$

$$F^{\mu\nu}(\omega, x) = \int d^4\eta d^4 y \delta'(\eta^2 + 1) f(\omega, \eta, x - y)$$
$$\times \eta^{[\mu} \partial^{\nu]} \Psi(\eta, y), \dots \quad (6.11)$$

with

$$f(\omega, \eta, x - y) = \int d^4 k e^{ik \cdot (x-y)} e^{\rho \frac{\eta \cdot \omega}{k \cdot \omega}}. \quad (6.12)$$

Because they are so nonlocal, it is completely unsurprising that $[\phi(\omega, x), \phi(\omega', x')]$ does not vanish for spacelike-separated $x - x'$, except for special ω and ω' such as $\omega = \omega'$. Note that ω does not package the spin content of the CSP the way that vector superspace does; these functions are not analytic in ω, and it is not clear what physical meaning the ω space (or spinor analogues of it) has, if any. However, the fact that such covariant operators do not simply commute for spacelike-separated $x - x'$ for *arbitrary* ω and ω' was the cause for concern flagged by several earlier authors [11,12,14,15,31,32].

If we relax the requirement of manifest Lorentz covariance, we can construct gauge-invariant objects that are at least local in time. The existence of such objects is a prerequisite for setting up the Cauchy problem in an interacting theory, starting only from physical initial data. If we single out a time direction, we can define

$$f(\eta, \vec{x} - \vec{y}) = \int d^3 \vec{k} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} e^{\rho \frac{\vec{\eta} \cdot \vec{k}}{\vec{k}^2}} \quad (6.13)$$

and then construct noncovariant but time-local and gauge-invariant "curvature" tensors

$$\phi(\vec{x}, t) \equiv \int d^3 \vec{y} [d^4\eta] \delta'(\eta^2 + 1) f(\eta, \vec{x} - \vec{y}) \Psi(\eta, \vec{y}, t),$$
$$(6.14)$$

$$F^{\mu\nu}(\vec{x}, t) \equiv \int d^3 \vec{y} [d^4\eta] \delta'(\eta^2 + 1) f(\eta, \vec{x} - \vec{y}) \eta^{[\mu} \partial^{\nu]} \Psi(\eta, \vec{y}, t),$$
$$(6.15)$$

$$R(\xi, \vec{x}, t) = \int d^3 \vec{y} [d^4\eta] \delta'(\eta^2 + 1) e^{\xi_{\mu\nu} \eta^\mu \partial_x^\nu} f(\eta, \vec{x} - \vec{y}) \Psi(\eta, \vec{y}, t),$$
$$(6.16)$$

where the last line generalizes the series of curvatures in Eq. (6.10). We have explicitly separated the time and spatial dependencies to underscore that the objects above are spatially nonlocal functions of Ψ, but local in time. In the $\rho \to 0$ limit, $f(\eta, \vec{x} - \vec{y}) \to \delta^3(\vec{x} - \vec{y})$ so that we recover the familiar curvature forms in that limit. Moreover, even though these are spatially nonlocal we nevertheless find $[R(\xi, \vec{x}, t), R(\xi', \vec{x}', t')] = 0$ for spatially separated $(\vec{x}, t)$ and $(\vec{x}', t')$, and arbitrary ξ, ξ' by using the commutator (6.9).

In closing this section, we should emphasize that the generalized curvatures considered here are simply motivated by their relation to familiar field strengths; even though they obey nice causality properties, they are not especially natural objects in the vector-superspace formalism, and may not be physically significant at all in the context of CSP interactions. This is an important open question.

VII. DISCUSSION AND CONCLUSION

In this paper, we have presented a Lorentz-invariant, local action for a free continuous-spin degree of freedom. The action naturally generalizes to allow couplings of the CSP to a suitably conserved background current. The continuity condition allows a tower of tensor currents $J^{(n)}$, of which only the lowest-rank component is conserved; this structure is compatible with no-go theorems against high-rank conserved currents in $3 + 1$-dimensional Minkowski space, and with the covariant soft factors found in Refs. [3,10]. Like other local field theories for massless particles, it is a gauge theory; the canonical field from which the action and equations of motion are built is not directly observable. This feature is a key difference between our theory and various failed attempts in the literature [11–15].

Although the action is superficially similar to the one considered in our earlier work [17], the new action is superior in two dramatic ways. First, the action (4.1) propagates a single species of CSP (with an arbitrary value of the Pauli-Lubanski spin invariant ρ), whereas the theory in Ref. [17] propagated a continuum of CSPs with *every* ρ. Second, the $\rho \to 0$ limit of this theory exhibits the expected helicity correspondence [10] in a much sharper way than Ref. [17]; it becomes a direct sum of Schwinger-Fronsdal actions for integer-helicity particles, of which the helicity-0, -1, and -2 modes can consistently couple to the usual conserved currents.

These two advances are made possible by the introduction of a new kind of auxiliary space, which we call "vector superspace." Vector superspace simply unifies all integer-helicity actions in flat space, with a straightforward generalization to a CSP with nonzero spin scale ρ. Vector superspace starts from the old idea of grouping tensors of different ranks by contracting them into an auxiliary 4-vector η^μ, then viewing their sum as a Taylor series for a function of x and η. We take this basic idea further, showing that spin can be encoded by geometry in η space, and that integrals localized on invariant η-space hyperboloids can be unambiguously defined by analytic continuation of η. For ordinary ($\rho = 0$) integer-helicity particles, the vector-superspace action is an almost trivial rewriting of a sum of Schwinger-Fronsdal actions, though simplified and formulated in terms of *unconstrained* fields and gauge variations. A simple relevant deformation of the integer-helicity action in vector superspace, which deforms the gauge invariance rather than breaking it, turns the tower of helicities into a CSP with nonzero ρ. In this case, the vector-superspace form is much simpler than its tensor equivalent; the tensor action and equations of motion are not rank diagonal, but feature rank-mixing terms of $O(\rho)$. Likewise, even modes that are helicity eigenstates involve nonzero tensor components of arbitrarily high ranks.

The vector-superspace action has a straightforward generalization to higher and lower dimensions, but the resulting representation content has not been thoroughly explored. In the $2 + 1$-dimensional case the CSP has just two degrees of freedom related by parity, which to our knowledge has not been discussed previously in the literature; it may be possible for such degrees of freedom to arise in gapless condensed matter systems, and they display an intriguing connection to anyons at the kinematical level. The action likely has simple generalizations to describe fermionic and/or supersymmetric CSPs, which have not yet been explored. Another interesting open question is whether Fronsdal-type equations for massless particles in de Sitter or anti–de Sitter spacetimes can be recast in a vector superspace, and whether analogous ρ deformations of these theories have a sensible physical interpretation.

A. In search of a covariance principle

A few further comments—and a good deal more exploration—is called for on the questions of locality, causality, and the general form that we might expect an interacting CSP theory to take.

Past work has presented general arguments against covariant, single-particle gauge-invariant operators with microcausal commutation relations [11,12,14,15,31,32]. Yet here, we have a local gauge theory action where covariantly quantized fields (see Ref. [29]) have standard, causal commutation relations. These two findings are, of course, not incompatible, since our fields have nontrivial

gauge variation and are therefore not directly observable. Indeed, one can show that no local, single-field gauge-invariant operators with two or fewer derivatives exist, except those that are proportional to the CSP equation of motion and therefore vanish on physical states. Though we have not carried out the general analysis explicitly, this trend likely continues to any finite order in derivatives, and is consistent with the absence of gauge invariants with fewer than h derivatives for helicity-h degrees of freedom, even for $\rho = 0$.

Covariant GI operators [Eq. (6.11)] in our theory, which Lorentz transform like the operators considered in earlier work, can be constructed at the price of introducing nonlocality in both space and time. Moreover, these are not natural objects to write down in vector superspace, and likely have no physical meaning. So it is hardly surprising that they do not have standard commutation relations. We can, however, construct gauge-invariant operators that are local in time (but still nonlocal in space), at the expense of manifest Lorentz covariance. Remarkably, the operators in this class become local in the $\rho \to 0$ limit, and commute at arbitrary spacelike separations even for nonzero ρ [29]! Nonetheless, they too are somewhat artificial in vector superspace, and may well not be particularly relevant to an interacting theory.

Interpreted from the point of view of our free theory, these findings offer a clue. It seems likely that no physical role is played by operators linear in the gauge fields that are *invariant* under the gauge transformations. This would have strong precedent in non-Abelian gauge theories and general relativity, which also have no local single-field gauge-invariant operators (and in GR, no local gauge-invariant operators at all). Instead, physics in those theories is *covariant* under general coordinate and gauge transformations. For example, the equation of motion for matter in GR is not gauge invariant even at the linear level, and this noninvariance reflects the covariant transformation law of the spacetime coordinates used to describe the physics. This is to be contrasted with QED for example, where the simplest interactions exploit the existence of gauge-invariant electric and magnetic fields, and no broader covariance principle is needed.

Likewise, we should perhaps expect that the gauge invariance of the CSP action should be interpreted as a linearized version of some coordinate transformation—perhaps one that acts on the vector superspace (or some generalization thereof), as well as coordinate space. Understanding consistent CSP self-interactions or couplings to matter is the next important step towards finding such a covariance principle for CSPs. It may be fruitful to "bootstrap" up such interactions by expanding both in coupling and in ρ, starting from familiar self-interacting theories. We expect that the free theory formulated in this paper will be a useful stepping stone in that direction.

Note added.—After this paper appeared, we learned of a related formalism for high-spin fields on constant-curvature spacetimes [33]. Indeed, the action (52) of Ref. [33] is also localized on the first neighborhood of an auxiliary space, and its flat-space limit is readily seen to be the same as the $\rho = 0$ limit of our action (4.1) after an integration by parts in the auxiliary space.

APPENDIX A: DEFINITION AND EXAMPLES OF REGULATED VECTOR-SUPERSPACE INTEGRALS

In Sec. III, we noted that integrals of the form $\int d^D \eta \delta(\eta^2 + 1) F(\eta)$ diverge for generic smooth functions $F(\eta)$ [and even for simple cases, e.g. $F(\eta) = $ constant] because the hyperboloid on which the integral is evaluated has infinite volume. Nonetheless, these integrals can be computed either by analytic continuation in η space (which we take as a definition) or equivalently, up to normalization, directly from their symmetry properties in Minkowski space. This appendix carefully defines these measures and presents useful results and examples. Section A 1 defines the measure $[d^D \eta]$ using analytic continuation and derives generating functions for integrals over both $\delta(\eta^2 + 1)$ and $\delta'(\eta^2 + 1)$. In Sec. A 2, we explain how the same integrals could alternately be derived (up to an overall normalization) using symmetries and integration by parts directly in Minkowski space, and present a simple example. As a further illustrative example of η-space computations, Sec. A 3 demonstrates the orthonormality of the basis wave functions for CSPs introduced in Sec. IV B.

1. Regulated measure over vector superspace from analytic continuation

For any smooth function $F(\eta)$ we define η integration over the hyperboloid $\eta^2 + 1 = 0$ via analytic continuation. Specifically,

$$\int [d^D \eta] \delta(\eta^2 + 1) F(\eta) \equiv \frac{2}{S_{D-1}} \int d^D \bar{\eta} \delta(\bar{\eta}^2 - 1) F(\bar{\eta}), \tag{A1}$$

where $\bar{\eta}^\mu = (i\eta^0, \eta^1, \ldots \eta^{D-1})$ are Wick-rotated coordinates, $F(\bar{\eta})$ is defined by analytic continuation from real η, and S_{D-1} is the surface area of a unit $(D-1)$-sphere, so that

with the normalization above $\int [d^D \eta] \delta(\eta^2 + 1) = 1$. Similarly, integrals over a δ' are given by

$$\begin{aligned}
& \int [d^D \eta] \delta'(\eta^2 + 1) F(\eta) \\
& \equiv \frac{2}{S_{D-1}} \int d^D \bar{\eta} \delta'(-\bar{\eta}^2 + 1) F(\bar{\eta}) \\
& = -\frac{1}{S_{D-1}} \int d^D \bar{\eta} \delta'(\bar{\eta}^2 - 1) F(\bar{\eta}), \tag{A2}
\end{aligned}$$

where we have used $\delta'(-x) = -\delta(x)$. The above expressions are clearly finite for any smooth F, since they are integrals of smooth functions over a compact surface.

In fact, since any smooth $F(\eta) = [F(i\partial_w)e^{i\eta \cdot w}]_{w=0}$, we can rewrite the Wick-rotated integrals (A1) and (A2) as

$$\begin{aligned}
\int [d^D \eta] \delta(\eta^2 + 1) F(\eta) &= [F(i\partial_w)G(w)]_{w=0} \\
&= [G(i\partial_\eta)F(\eta)]_{\eta=0} \equiv F * G, \tag{A3}
\end{aligned}$$

$$\begin{aligned}
\int [d^D \eta] \delta'(\eta^2 + 1) F(\eta) &= [F(i\partial_w)G'(w)]_{w=0} \\
&= [G'(i\partial_\eta)F(\eta)]_{\eta=0} \equiv F * G' \tag{A4}
\end{aligned}$$

where $F * G$ is a common shorthand for the preceding expressions. The generating functions G and G' are defined via analytic continuation as

$$\begin{aligned}
G(w) &\equiv \frac{2}{S_{D-1}} \int d^D \bar{\eta} \delta(\bar{\eta}^2 - 1) e^{-i\bar{\eta} \cdot \bar{w}}|_{\bar{w}=iw^0, w^i} \\
&= \Gamma\left(\frac{D}{2}\right)(r/2)^{-\frac{D-2}{2}} J_{\frac{D-2}{2}}(r)|_{r=\sqrt{-w^2}}, \tag{A5}
\end{aligned}$$

$$\begin{aligned}
G'(w) &\equiv \frac{2}{S_{D-1}} \int d^D \bar{\eta} \delta'(\bar{\eta}^2 - 1) e^{-i\bar{\eta} \cdot \bar{w}}|_{\bar{w}=iw^0, w^i} \\
&= \Gamma\left(\frac{D}{2}\right)(r/2)^{-\frac{D-4}{2}} J_{\frac{D-4}{2}}(r)|_{r=\sqrt{-w^2}}. \tag{A6}
\end{aligned}$$

That these are built from Bessel functions is not surprising, since the condition $\int d^4 \bar{\eta} \delta(\eta^2 + 1)(\eta^2 + 1)e^{-i\eta \cdot w}$ is equivalent to $(\partial_w^2 - 1)G(w) = 0$. Using the invariance of G under Lorentz transformations of w, we can write $G(w) = G(r)$ with $r = \sqrt{-w^2}$; this implies $d^2 G/dr^2 + \frac{D-1}{r} dG/dr - G = 0$. This is precisely the D-dimensional analogue of the Bessel equation!

For $D = 4$ and $D = 3$, the general formulas above reduce to

$$G_{4d}(r) = 2J_1(r)/r, \qquad G'_{4d}(r) = J_0(r), \tag{A7}$$

$$G_{3d}(r) = j_0(r) = \sin(r)/r, \qquad G'_{3d}(r) = \cos(r)/2. \tag{A8}$$

Thus, for example, we can easily verify that in four spacetime dimensions

$$\int [d^4\eta]\delta'(\eta^2+1)\eta^\mu\eta^\nu A_{\mu\nu}$$

$$= \left[A_{\mu\nu}(i\partial_w^\mu)(i\partial_w^\nu)J_0\left(\sqrt{-w^2}\right)\right]_{w=0} \quad (A9)$$

$$= A_{\mu\nu}[-g^{\mu\nu}J_1(r)/r + w^\mu w^\nu J_2(r)/r^2]_{w=0}$$

$$= -A_\mu^\mu/2. \quad (A10)$$

With Eqs. (A5) and (A6) in hand, we can reexpress the action (3.3) as

$$S = \left[G(i\partial_\eta)\frac{1}{2}(\partial_\alpha\Psi)^2 - \frac{1}{4}(\eta^2+1)(\Delta\Psi)^2\right]_{\eta=0}. \quad (A11)$$

The qualitative resemblance of our η space to superspace (where integration in the fermonic coordinate θ can also be reexpressed as differentiation, there owing to the Grassmann nature of θ) is apparent. But here, because η is bosonic, we are led to include an *infinite* tower of same-statistics fields rather than a finite tower of fields with alternating statistics.

2. Vector superspace integrals from symmetry arguments

The regulated η-space measure defined by Eq. (A1) respects the δ-function and integration-by-parts identities

$$\int [d^D\eta]\delta(\eta^2+1)(\eta^2+1)F(\eta) = 0,$$

$$\int [d^D\eta]\partial_\eta^\mu(\delta(\eta^2+1)F(\eta)) = 0, \quad (A12)$$

which in turn guarantee

$$\int [d^D\eta]\delta'(\eta^2+1)F(\eta)$$

$$= \int [d^4\eta]\delta(\eta^2+1)\frac{1}{2}(D-2+\eta\cdot\partial_\eta)F(\eta). \quad (A13)$$

In fact, we can essentially *derive* the correct measure, up to an overall normalization, from these identities. To take a concrete example, if we define a constant V by

$$V \equiv \int [d^4\eta]\delta(\eta^2+1), \quad (A14)$$

then by symmetry

$$\int [d^4\eta]\delta'(\eta^2+1)\eta^\mu\eta^\nu = \int [d^4\eta]\delta(\eta^2+1)\left(1+\frac{1}{2}\eta\cdot\partial_\eta\right)\eta^\mu\eta^\nu$$

$$= 2\int [d^4\eta]\delta(\eta^2+1)\eta^\mu\eta^\nu$$

$$= \frac{1}{2}\int [d^4\eta]\delta(\eta^2+1)\eta^2 g^{\mu\nu}$$

$$= \frac{1}{2}g^{\mu\nu}\int [d^4\eta]\delta(\eta^2+1)(-1)$$

$$= -\frac{1}{2}g^{\mu\nu}V, \quad (A15)$$

which is readily seen to agree with the result of Eq. (A10) when we adopt the normalization $V = 1$.

The style of argument in Eq. (A15) is readily generalized to relate the rank-n tensor $\int [d^4\eta]\delta(\eta^2+1)\eta^n$ to $\int [d^4\eta]\delta(\eta^2+1)\eta^{n-2}$ and similarly for the integrals over δ'. The results of this argument precisely reproduce what is expected from a series expansion of Eqs. (A3)–(A6).

3. Orthonormality of basis wave functions

As an example of vector-superspace computations, we explicitly compute the orthonormality relation

$$\langle\psi_h(k)|\psi_{h'}(k)\rangle \equiv \int [d^4\eta]\delta'(\eta^2+1)\psi_h(\eta,k)^*\psi_{h'}(\eta,k)$$

$$= \delta_{h,h'} \quad (A16)$$

for the basis wave functions

$$\psi_{+h}(\eta,k) = (\eta\cdot\epsilon_+)^h e^{i\rho\eta\cdot q}$$

$$\psi_{-h}(\eta,k) = (\eta\cdot\epsilon_-)^{|h|} e^{i\rho\eta\cdot q}. \quad (A17)$$

Upon analytic continuation, we can choose a coordinate system with

$$\eta_E^\mu \equiv (\eta_\perp\cos\theta, \eta_\perp\sin\theta, \tilde{\eta}\cos\phi, \tilde{\eta}\sin\phi),$$

where $\eta_\perp^2 = \eta\cdot\epsilon_+\eta\cdot\epsilon_-$ labels the magnitude of η_E in the plane spanned by $\epsilon_\pm$. In this coordinate system, we can write $\eta\cdot\epsilon_+ = \eta_\perp e^{i\theta}$. Using $[d^4\eta] = \frac{1}{\pi^2}d^4\eta_E$, we can write the integral measure as $\frac{1}{\pi^2}\eta_\perp d\eta_\perp\tilde{\eta}d\tilde{\eta}d\theta d\phi$ and easily evaluate Eq. (A16) as

$$\int [d^4\eta]\delta'(\eta^2+1)\psi_h(\eta,k)^*\psi_{h'}(\eta,k) = -\frac{1}{\pi^2}\int \eta_\perp d\eta_\perp\tilde{\eta}d\tilde{\eta}d\theta d\phi\delta'(\eta_\perp^2+\tilde{\eta}^2-1)\eta_\perp^{|h|+|h'|}e^{(h'-h)\theta}$$

$$= -\delta_{h,h'}\frac{1}{\pi^2}\int \eta_\perp d\eta_\perp\tilde{\eta}d\tilde{\eta}d\theta d\phi\delta'(\eta_\perp^2+\tilde{\eta}^2-1)\eta_\perp^{2h}$$

$$= -\delta_{h,h'}\int_0^\infty du\int_0^\infty \delta'(u+v-1)u^h$$

$$= \delta_{h,h'}\int_0^\infty du\delta(u-1)u^h = \delta_{h,h'}. \quad (A18)$$

Similar calculations in vector superspace are carried out in the same manner.

APPENDIX B: DIAGONALIZING THE VECTOR-SUPERSPACE ACTION WITHOUT CONSTRAINTS

The results of the preceding appendix make clear that, if we expand Ψ about $\eta = 0$ as $\Psi(\eta, x) = \psi^{(0)}(x) + \eta^\mu \psi_\mu^{(1)}(x) + \eta^\mu \eta^\nu \psi_{\mu\nu}^{(2)}(x) + \cdots$, with the tensor fields $\psi^{(n)}$ unconstrained, the action (A11) will mix $\psi^{(n)}$ with $\mathrm{Tr}^t \psi^{(n+2t)}$ for all integer t, even for $\rho = 0$.

This rank mixing is related to the fact that we defined our polynomials via Taylor expansion about $\eta = 0$, but in fact the action is evaluated at nonzero η. It can be simply removed by a (nonunique) change of variables. We have shown already in Sec. III C that an appropriate decomposition of Ψ in terms of the tensor fields (3.15) yields a rank-diagonal action, provided we first partially fix gauge so that the tensor fields $\phi^{(n)}$ are double traceless. It is, however, possible to construct tensor fields from Ψ such that the action is both simple and rank diagonal *before any gauge fixing*.

Concretely, we can introduce unconstrained tensor fields

$$A^{(n)}(x) = \sum_{t=0}^{\infty} \frac{(-1)^t (2n + D - 4)!! (2t + n)!}{2^{n/2+t}(2n + 2t + D - 4)!! t! n!} \, \mathrm{Tr}^t \psi^{(n+2t)}(x)$$

(B1)

$$= \sum_{t=0}^{\infty} \frac{(-1)^t (2n + D - 4)!! (\partial_\eta^2)^t \partial_\eta^{\mu_1} \ldots \partial_\eta^{\mu_n}}{2^t (2n + 2t + D - 4)!! t! n!}$$

$$\times \Psi(\eta, x)|_{\eta=0}$$

(B2)

in D spacetime dimensions. For $D = 4$, these decompositions simplify to

$$A_{4d}^{(n)}(x) = \sum_{t=0}^{\infty} \frac{(-\partial_\eta^2/4)^t \partial_\eta^{\mu_1} \ldots \partial_\eta^{\mu_n}}{2^{n/2} t! (t + n)!} \Psi(\eta, x)|_{\eta=0},$$

(B3)

which is precisely the Taylor expansion of $[J_n(\sqrt{\partial_\eta^2}) \Psi (\eta, x)]|_{\eta=0}$ up to the normalization and additional Lorentz index structure. In terms of these new fields, the $\rho = 0$ action (3.11) becomes rank diagonal

$$S = \sum_n S_n$$

(B4)

where (for $D = 4$)

$$S_n = (-1)^n \int d^4 x \frac{1}{2} (\partial_\alpha \langle A^{(n)} \rangle)^2 - \frac{1}{8} (\partial_\alpha \langle A^{(n)'} \rangle)^2$$

$$- \frac{n}{2} \langle \partial \cdot A^{(n)} \rangle^2,$$

(B5)

where $A' = \mathrm{Tr} A$ and $\langle A \rangle$ denotes the "trace subtraction" of A, i.e. the *unique* fully symmetric, traceless rank-n tensor built from only A itself. For example, in $3 + 1$ dimensions we have $\langle A \rangle^{\mu\nu} \equiv A^{\mu\nu} - \frac{1}{4} g^{\mu\nu} A'$, $\langle A \rangle^{\mu\nu\rho} \equiv A^{\mu\nu\rho} - \frac{1}{6}(g^{\mu\nu} A'^\rho + g^{\rho\mu} A'^\nu + g^{\rho\nu} A'^\mu)$ and so on, with the general form

$$\langle X \rangle \equiv X + \sum_{t=1}^{n/2} \frac{(-1)^t (n - t)!}{2^t n!} \eta^{(t)} \circ \mathrm{Tr}^t X.$$

(B6)

Appropriate decompositions of $\varepsilon(\eta, x)$ and $\chi(\eta, x)$ into components $\varepsilon^{(n)}$ and $\chi^{(n)}$ relates the η-space gauge variations (3.9) to component gauge variations

$$\delta A^{(n)} = \partial \circ \varepsilon^{(n-1)} - g \circ (\partial \cdot \varepsilon^{(n-1)}) + g \circ g \circ \chi^{(n-4)}, \quad (B7)$$

which are easily seen to leave the action (B5) invariant. The gauge-invariant equation of motion is

$$-\Box \langle A \rangle + \frac{1}{2s(s - 1)} \eta \circ \Box \langle A' \rangle + \partial \circ \langle \partial \cdot A \rangle = 0. \quad (B8)$$

This is readily seen by using the tracelessness of $\langle \ldots \rangle$, which implies $\langle \delta A \rangle . \langle A \rangle = \delta A . \langle A \rangle$ (since $\langle \delta A \rangle - \delta A$ is proportional to the metric, and hence annihilates $\langle A \rangle$ upon contraction), and similarly for the other two terms in the variation of the action.

As was noted earlier, the gauge transformation (B7) is redundant because any transformation generated by χ can also be generated by $\epsilon = g \circ \xi$ with $\partial \cdot \xi = -\chi$. In other words, we can parametrize an arbitrary gauge variation *either* by ϵ alone (with $\chi = 0$) or by a *traceless* ϵ and an arbitrary χ.

We can again check that we have only two massless propagating degrees of freedom permitted by this system of equations subject to the gauge redundancies generated by $\langle \epsilon \rangle$. A double traceless rank-n field has $\frac{1}{6}[(n+3)(n+2)(n+1) - (n-1)(n-2)(n-3)]$ independent components. The constraint $\langle \partial \cdot A \rangle = 0$ and the remaining gauge parameters $\langle \epsilon \rangle$ both have $\frac{n}{6}[(n+2)(n+1) - (n-1)(n-2)]$ independent components. The number of field components minus the number of constraints and gauge parameters is therefore 2, as we would expect for a massless helicity-n degree of freedom.

This action can be rewritten in the Schwinger-Fronsdal form by first using the χ gauge invariance to fix a double-traceless gauge for $A^{(n)}$, then rewriting the action in terms of a trace-reversed field $\phi^{(n)} \equiv \langle A \rangle - \frac{1}{2(s-1)} g \circ \langle A' \rangle$ (this is analogous to the gauge-fixing used in Sec. III C), which is readily seen to transform in the usual way under the remaining gauge variations, parametrized by traceless ϵ. Incidentally, using the relation

$$\langle \partial . T \rangle = \partial . \langle T \rangle + \frac{1}{2n} \left[\partial \circ \langle T' \rangle - \frac{1}{n-1} g \circ \partial . \langle T' \rangle \right], \quad (B9)$$

the equation of motion (B8) (the variation of the action with respect to $A^{(n)}$, not $\phi^{(n)}$) can be written as

$$(\Box - \partial \circ \partial \cdot + \partial^2 \circ g \cdot)\phi^{(n)} = 0, \qquad \text{(B10)}$$

which is precisely the Fronsdal equation for $\phi^{(n)}$.

In the $A^{(n)}$ language, Fronsdal's double-trace condition is most naturally viewed as a partial gauge fixing. The restriction to traceless ϵ is replaced by the statement that trace terms in ϵ either have trivial action on the fields $A^{(n)}$ or violate the double-trace condition. It is, of course, well known that trace conditions can be replaced by projections

in the action, which is essentially what we have done here, but in a convenient basis for making contact with the vector-superspace action.

For nonzero ρ, the action (B5) changes only by the substitution

$$\langle \partial . A \rangle \to \left\langle \partial . A^{(n)} + \frac{\rho}{\sqrt{2n}}\left(A^{(n-1)} + \frac{1}{2}A^{(n+1)\prime}\right)\right\rangle,$$

with a corresponding modification to both the gauge transformation (B7) and the equation of motion (B8).

[1] L. P. S. Singh and C. R. Hagen, Lagrangian formulation for arbitrary spin. 1. The boson case, Phys. Rev. D **9**, 898 (1974).

[2] E. P. Wigner, On unitary representations of the inhomogeneous lorentz group, Ann. Math. **40**, 149 (1939).

[3] P. Schuster and N. Toro, On the theory of continuous-spin particles: wavefunctions and soft-factor scattering amplitudes, J. High Energy Phys. 09 (2013) 104.

[4] C. Fronsdal, Massless fields with integer spin, Phys. Rev. D **18**, 3624 (1978).

[5] J. Fang and C. Fronsdal, Massless fields with half integral spin, Phys. Rev. D **18**, 3630 (1978).

[6] S. Weinberg, Photons and gravitons in s matrix theory: derivation of charge conservation and equality of gravitational and inertial mass, Phys. Rev. **135**, B1049 (1964).

[7] S. Weinberg, Photons and gravitons in perturbation theory: derivation of Maxwell's and Einstein's equations, Phys. Rev. **138**, B988 (1965).

[8] M. Porrati, Old and new no go theorems on interacting massless particles in flat space, arXiv:1209.4876.

[9] P. Schuster and N. Toro, Simple experimental consequences of a CSP photon (to be published).

[10] P. Schuster and N. Toro, On the theory of continuous-spin particles: helicity correspondence in radiation and forces, J. High Energy Phys. 09 (2013) 105.

[11] J. Yngvason, Zero-mass infinite spin representations of the poincare group and quantum field theory, Commun. Math. Phys. **18**, 195 (1970).

[12] G. J. Iverson and G. Mack, Quantum fields and interactions of massless particles—the continuous spin case, Ann. Phys. (N.Y.) **64**, 211 (1971).

[13] A. Chakrabarti, Remarks on lightlike continuous spin and spacelike representations of the Poincare group, J. Math. Phys. (N.Y.) **12**, 1813 (1971).

[14] L. F. Abbott, Massless particles with continuous spin indices, Phys. Rev. D **13**, 2291 (1976).

[15] K. Hirata, Quantization of massless fields with continuous spin, Prog. Theor. Phys. **58**, 652 (1977).

[16] X. Bekaert and J. Mourad, The Continuous spin limit of higher spin field equations, J. High Energy Phys. 01 (2006) 115.

[17] P. Schuster and N. Toro, A gauge field theory of continuous-spin particles, J. High Energy Phys. 10 (2013) 061.

[18] P. Schuster and N. Toro, A new class of particle in $2 + 1$ dimensions, arXiv:1404.1076.

[19] L. Brink, A. M. Khan, P. Ramond, and X.-z. Xiong, Continuous spin representations of the Poincare and super-Poincare groups, J. Math. Phys. (N.Y.) **43**, 6279 (2002).

[20] A. Stern, Anyons and the quantum Hall effect—a pedagogical review, Ann. Phys. (Amsterdam) **323**, 204 (2008).

[21] F. Wilczek, *Fractional statistics and anyon superconductivity*, edited by F. Wilczek (World Scientific, Singapore, 1990).

[22] R. Jackiw and V. P. Nair, Relativistic wave equations for anyons, Phys. Rev. D **43**, 1933 (1991).

[23] D. Sorokin, Introduction to the classical theory of higher spins, AIP Conf. Proc. **767**, 172 (2005).

[24] N. Bouatta, G. Compere, and A. Sagnotti, An introduction to free higher-spin fields, arXiv:hep-th/0409068.

[25] M. A. Vasiliev, Equations of motion of interacting massless fields of all spins as a free differential algebra, Phys. Lett. B **209**, 491 (1988).

[26] M. A. Vasiliev, Consistent equations for interacting massless fields of all spins in the first order in curvatures, Ann. Phys. (N.Y.) **190**, 59 (1989).

[27] M. A. Vasiliev, Higher spin gauge theories in various dimensions, Fortschr. Phys. **52**, 702 (2004).

[28] V. O. Rivelles, Gauge theory formulations for continuous and higher spin fields, arXiv:1408.3576.

[29] P. Schuster et al., BRST quantization of continuous spin field theories (to be published).

[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[31] B. Schroer, Dark matter and Wigner's third positive-energy representation class, arXiv:1306.3876.

[32] T. Banks (private communication).

[33] A. Yu. Segal, A generating formulation for free higher spin massless fields, arXiv:hep-th/0103028.