

Scale setting for self-consistent backgrounds

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The quest for finding self-consistent background solutions in quantum field theory is closely related to the way one decides to set the renormalization scale k . This freedom in the choice of the scale setting can lead to ambiguities and conceptual inconsistencies such as the nonconservation of the stress-energy tensor. In this paper a setting for the “scale field” is proposed at the level of effective action, which avoids such inconsistencies by construction. The mechanism and its potential is exemplified for scalar ϕ^4 theory and for Einstein-Hilbert-Maxwell theory.

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I. INTRODUCTION

The effective action approach [1] can be seen as an elegant way of defining a generating functional for one-particle-irreducible Green’s functions. Following Wilson’s idea [2] one can study the effect of integrated quantum degrees of freedom at different scales k . The scale dependent effective action Γ_k is to be understood as interpolation between the ultraviolet (UV) bare action Γ_∞ and the fully integrated action in the IR Γ_0 as it is sketched in Fig. 1. Γ_k contains scale dependent couplings g_k^a which are obtained from a suitable flow equation $k\partial_k\Gamma_k = \dots$. The space of solutions g_k^a is called the “coupling flow” [3]. A specific trajectory is selected out of this flow by imposing conditions for the couplings at an initial scale k_0 . The evaluation of the effective action Γ_k is typically hampered by various technical difficulties such as singularities, anomalies, and nonlocalities. However, in many cases those difficulties can be overcome by the “regularization-renormalization” technique, where infinities are absorbed in the initial conditions at a scale k_0 . The technical details of this procedure will not be presented here, since they are not relevant for the following discussion. It will be assumed that the effective action Γ_k has already been calculated.

Minimizing a given effective action with respect to variations of its (average) field content ϕ_a gives the equations of motion of the effective action

$$\frac{\delta\Gamma_k}{\delta\phi_a} = 0. \quad (1)$$

Those equations are typically nonlinear and sometimes nonlocal differential equations and are frequently referred to as “gap equations” [4]. Solutions of those equations have minimal energy (Gibbs free energy in statistical mechanics). Therefore, finding solutions for the gap equations is highly relevant for defining a self-consistent background in quantum field theory.

However, even if it is technically possible to solve the gap equations, the physical interpretation of such a solution

is still biased by the way the scale k is related to the quantities x_i, Q_i, \dots (for example positions and charges) that are used to describe the physical system. Choosing a relation $k = k(x_i, Q_i, \dots)$ is called “scale setting.” The main focus of this article is on the role of scale setting in the quest of finding self-consistent solutions of (1).

The paper is organized as follows: In Sec. II the approach of improving classical solutions is studied and a criterion for scale setting is proposed. Section III goes beyond improving classical solutions by studying the gap equations. A scale setting is proposed in terms of an additional equation of motion. The self-consistency and predictivity of this approach is then studied for specific examples such as scalar ϕ^4 theory in Sec. IV and the Einstein-Hilbert-Maxwell action in Sec. V. Conclusions can be found in Sec. VI.

II. A PROBLEM WITH SCALE SETTING IN IMPROVING SOLUTIONS SCHEMES

The method of improving solutions has been successfully applied in many different contexts [5–29]. This

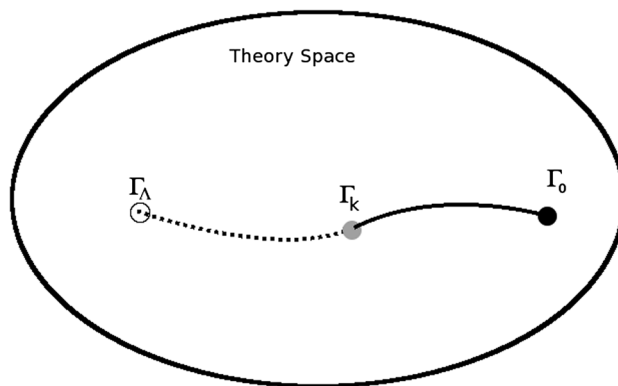


FIG. 1. Effective action Γ_k in theory space where ideally $\Lambda \rightarrow \infty$ and the dotted line indicates an integration of ϕ_a over momentum degrees of freedom with $p \geq k$.

intuitive method is potentially useful when perturbation theory has limited reliability such as for strong coupling or for nonrenormalizable theories.

Instead of turning to one of the particular examples, this approach will be discussed in quite generic fashion. Let us assume a quite general ϕ^4 quantum field theory with bare fields ϕ_i and generic couplings α, g with the equations of motion

$$\mathcal{D}(\alpha\phi_a) + \sum g^{abcd}(\phi_b\phi_c\phi_d) = 0, \quad (2)$$

where \mathcal{D} is some differential operator (for example $\partial_\mu\partial^\mu$). Let us further assume that the classical solution of (2) is given by

$$\phi_a = \phi_a^0(r, \alpha, g_n, A), \quad (3)$$

where r are the coordinates and A are integration constants. Renormalization methods allow one to calculate quantum corrections to the couplings at arbitrary scale k , such that the couplings of the bare theory become scale dependent quantities (α_k, g_k) . How do those supposedly small quantum corrections modify the form of the classical solution? One approach to address this question is given the improving solutions procedure [5,6], where one assumes that the classical solution is actually the quantum solution at a certain scale say $k = k_0$. For the case of simplicity let us choose $k_0 = 0$, just as it is done for the electromagnetic coupling in the standard model. At this scale the classical equation of motion,

$$\mathcal{D}(\alpha_0\phi_a^0) + \sum g_0^{abcd}(\phi_b^0\phi_c^0\phi_d^0) = 0, \quad (4)$$

is solved by ϕ_a^0 and the k dependence is a small correction to this solution. In this scheme one assumes that at first order the functional form of the solution stays unchanged, and only the couplings have to be replaced by the scale dependent couplings in ϕ :

$$(\alpha_0, g_0) \rightarrow (\alpha_k, g_k). \quad (5)$$

Now one makes the ansatz that the classical solution obtains its first order quantum correction only due to the scale dependence of the couplings

$$\begin{aligned} \phi_a &= \phi_a^0(r, \alpha_0, g_0, A) + \frac{d}{dk}\phi_a(r, \alpha_k, g_k, A)|_{k=0} \cdot k + \mathcal{O}(k^2), \\ \alpha_k &= \alpha_0 + \frac{d}{dk}\alpha_k|_{k=0} \cdot k + \mathcal{O}(k^2), \\ g_k &= g_0 + \frac{d}{dk}g_k|_{k=0} \cdot k + \mathcal{O}(k^2). \end{aligned} \quad (6)$$

The second step is to perform a scale setting which relates the arbitrary scale k to the physical coordinates,

$$k \rightarrow k(r). \quad (7)$$

The explicit form for this scale setting is however *a priori* not uniquely determined (in static spherically symmetric problems it has for example been proposed to use $k \sim 1/r$). One first consistency check for this procedure would be to improve the equations of motion (2) in the same way and to check whether the improved solution (6) is actually a solution (up to order k^1) of those equations. Inserting (6) into (2) one obtains

$$\begin{aligned} 0 &= \mathcal{D}(\alpha_0\phi_a^0) + \sum g_0^{abcd}(\phi_b^0\phi_c^0\phi_d^0) + \mathcal{D}\left[\left(k\frac{d}{dk}\right)(\alpha_k\phi_a)|_{k=0}\right] \\ &+ \left(k\frac{d}{dk}\right)\sum g_k^{abcd}(\phi_b^k\phi_c^k\phi_d^k)|_{k=0} + \mathcal{O}(k^2). \end{aligned} \quad (8)$$

Using (4) the first line is identically zero and one obtains that the second line has to be zero too, if one wants to insist on the improved equations. In most of the articles cited above it was not imposed that the improved equations of motion stay valid and the scale setting was performed basically based on dimensional analysis. At this point an important question arises: ‘‘Is the coupling α_k inside or outside the square brackets of the differential operator \mathcal{D} ?’’ This is *a priori* not clear since starting from the equation of motion (4) both alternatives would be equally possible. However, this question can be ‘‘answered’’ (or say evaded) if one imposes a particular scale setting (7) such that the differential operator commutes with the scale function

$$[\mathcal{D}, k(r)] = 0. \quad (9)$$

In this case the second line of (8) is equivalent to

$$0 = \left(k\frac{d}{dk}\right)\left[\mathcal{D}[\phi_a^k] + \sum \alpha_k^{-1}g_k^{abcd}(\phi_b^k\phi_c^k\phi_d^k)\right]|_{k=0} + \mathcal{O}(k^2). \quad (10)$$

Thus, the remaining task would be first solving (9) for k and then solving (10) for ϕ_i^k .

The fact that (9) can actually serve as a useful way of defining a scale setting can be seen from a simple example: For the three-dimensional Laplace operator with spherical symmetry, the condition (9) reads

$$\left[\frac{1}{r^2}\partial_r(r^2\partial_r), k(r)\right] = 0. \quad (11)$$

It is solved for $r \neq 0$ by

$$k(r) = \frac{\xi}{r}, \quad (12)$$

which indeed agrees perfectly with the *ad hoc* intuition coming from a dimensional analysis.

However, there is (at least) one other consistency condition one would like to impose, the conservation of the stress-energy tensor, even at the quantum-improved level. Let $T_{\mu\nu}^0$ be the classically conserved stress-energy tensor

$$\nabla^\mu T_{\mu\nu}^0 = 0, \quad (13)$$

then the straightforward improved stress-energy tensor would be taken to be

$$T_{\mu\nu} = T_{\mu\nu}^0 + \left(\frac{d}{dk} T_{\mu\nu}^k \Big|_{k=0} \right) \cdot k + \mathcal{O}(k^2). \quad (14)$$

Imposing conservation of (14) and using (13) one finds to leading order in k ,

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu \left(\left(\frac{d}{dk} T_{\mu\nu}^k \Big|_{k=0} \right) \cdot k \right) \equiv 0. \quad (15)$$

If one would try to solve this problem in the same spirit as (9) by imposing $[\nabla^\mu, k(r)] \equiv 0$ one easily finds that this is overly restrictive allowing only for trivial solutions. To circumvent this problem one can modify the definition (5) of the stress-energy tensor (for example by using a different equation of state [25]) but such an *ad hoc* redefinition is not completely satisfactory.

To summarize, one sees that the method of improved solutions (8) can be made consistent, even at the level of improved equations of motions if one imposes an adequate scale setting (9) (for example $k \sim 1/r$ for the spherical symmetric Laplacian). However, it has limitations in the sense that this procedure raises questions in the context of symmetries and conservation laws and that it is restricted to first order corrections only.

This can be taken as motivation for seeking a more elegant way for obtaining a description in the context of scale dependent couplings.

III. THE PROPOSAL: SCALE-FIELD SETTING AT THE LEVEL OF EFFECTIVE ACTION

In the previous section it was shown how scale setting can be realized in the improving solutions approach. In this section a more general scale setting at the level of effective action will be proposed.

Let us assume that within the quantum field theoretical model it was possible to evaluate the corresponding coupling flow and to select a particular trajectory due to the choice of initial conditions $[g_i(k_0) = g_{i,0}]$. Thus, one can start with the effective quantum action [1]

$$\Gamma_k(\phi_a(x), \partial\phi_a(x), g_k^a) = \int d^4x \sqrt{-g} \mathcal{L}(\phi_a(x), \partial\phi_a(x), g_k^a), \quad (16)$$

where ϕ_a are actually the expectation values of the quantum fields and g_k^a are the scale dependent couplings, including the coupling multiplying the kinetic term that is frequently expressed in terms of field renormalization (see the first two subsections of IV).

Note that doing this, one frequently has to truncate higher order or nonlocal couplings [30–32] from the model, which might appear due to the quantum integration procedure. In the following discussion it will however be assumed that all relevant couplings are taken into account. Now, one can derive the equations of motion for the average quantum fields ϕ_a from

$$\frac{\delta\Gamma_k}{\delta\phi_a} = 0. \quad (17)$$

As mentioned in the Introduction, the solutions $\bar{\phi}_a(x, k)$ of those gap equations will also be functions of the arbitrary scale k . From a physical point of view this is however not yet satisfactory since no possible observable can be a function of an *a priori* arbitrary scale. In order to obtain a physical quantity one has to define some kind of scale setting procedure, which establishes a relation between the physical quantities (charges Q_i and positions x_j) of a given problem and the scale k . When doing this one can borrow an idea from the calculation of observables $\langle T\phi(x_i) \dots \rangle_k$ in standard quantum field theory. Also there, the observables turn out to be scale k dependent quantities.¹ Subsequently, the scale setting for those observables in terms of initial conditions and kinematical variables $k = k(x_i; Q_i \dots)$ is chosen such that any k dependence of the time ordered correlation function is minimized:

$$\frac{d}{dk} \langle T\phi_1(x_1)\phi_2(x_2) \dots \rangle_k \Big|_{k=k_{\text{opt}}} \equiv 0. \quad (18)$$

This is the key philosophy that is used when deriving the ‘‘Callan-Symanzik’’ equations [33,34], the minimal sensitivity setting, or the ‘‘principle of maximal conformality’’ [35,36].

It is proposed to implement an analogous philosophy at the level of the effective action Γ_k . This means that one should choose an optimal scale setting prescription for which a variation of k has a minimal impact on the self-consistent background $\bar{\phi}_i$. This principle can be implemented by promoting the *a priori* arbitrary scale to a physical scale-field in the effective quantum action

$$\Gamma_k(\phi_a(x), \partial\phi_a(x), g_k^a) \rightarrow \Gamma(\phi_a(x), \partial\phi_a(x), k(x), g_k^a). \quad (19)$$

This leads to the coupled equations of motion

¹It is argued that this k dependence is an artifact of the truncation in the loop expansion.

$$\frac{\delta\Gamma}{\delta\phi_a} = 0, \quad \frac{d}{dk}\mathcal{L}(\phi_a(x), \partial\phi_a(x), k(x), g_k^a)\Big|_{k=k_{\text{opt}}} = 0. \quad (20)$$

Clearly it is not guaranteed that a solution for (20) can be found, but such a prescription is not limited to be a variation of a classical solution or to a saddle point approximation. The procedure (20) has already been applied for some particular gravitational actions [13,28,37–39] but in this paper it is discussed in a broader context. A nice feature of such a procedure is that any solution of the equations (20) is automatically independent of k , which is actually the fundamental precondition for a physical observable in the language of the renormalization group approach.

Promoting the scale k to a scale field $k(x)$ raises the question whether this new field only appears in the couplings g_k^a , or whether it has to be equipped with other additional couplings, for instance a proper kinetic term. A standard procedure when introducing new fields into a Lagrangian is to incorporate actually all couplings that are in agreement with the symmetry of the Lagrangian. This abundant freedom is then restricted by imposing some other additional conditions such as renormalizability, simplicity, and/or agreement with experimental constraints. However, in the presented approach the philosophy is different. The scale field k is understood to have its origin in the process of renormalization and, throughout this process, no such extra couplings are taken into account. In particular, the beta functions of the couplings g_k^a are calculated without any additional couplings. Therefore, the presented version of scale-field setting is chosen in a sense “minimal,” since it contemplates the appearance of $k(x)$ only as dictated by the running couplings g_k^a .

Even though the prescription (19) looks quite convincing from this perspective, it might be insufficient for example in the sense that the space of solutions of (20) is actually empty, apart from a trivial configuration or it is insufficient in the sense that the conservation of the stress-energy tensor cannot be guaranteed either. Therefore, the idea will be studied for some examples, where self-consistency of the approach can be shown explicitly.

IV. SCALE-FIELD SETTING FOR SCALAR ϕ^4 THEORY

As a most simple example without any further complications due to gauge symmetry let us study the scale-field setting procedure for scalar ϕ^4 theory. There are various ways of writing the effective action for ϕ^4 theory. One of them is in terms of a scale dependent wave function renormalization Z_k , running mass m_k , and running quartic coupling g_k . The other way of writing this action is terms of separate couplings for every term appearing in the Lagrangian, which are a coupling for the kinetic term α_k , a coupling for the ϕ^2 term \tilde{m}_k^2 , and a coupling for the

quartic term \tilde{g}_k . As long as the scale k is assumed to be fixed, the formalism for both is exactly equivalent. However, in the context of scale-field setting $k \rightarrow k(x)$, derivatives do not necessarily commute with $k(x)$ and both formulations could be treated differently. This subtlety will be exemplified in the following subsection, before applying the scale setting to ϕ^4 theory at the one-loop level.

A. Consistency in scalar ϕ^4 theory

If one works with separate couplings for every term appearing in the Lagrangian, including the kinetic term, the effective action is

$$\Gamma = \int d^4x \left(\frac{\alpha_k}{2} (\partial\phi)^2 - \frac{\tilde{m}_k^2}{2} \phi^2 - \frac{\tilde{g}_k}{4!} \phi^4 \right) \quad (21)$$

with two fields ϕ and k . The couplings α_k , \tilde{m}_k^2 , and \tilde{g}_k are functions of the field k . This implies an equation of motion for $\delta\phi$:

$$\partial_\mu(\alpha_k \partial^\mu \phi) + \tilde{m}_k^2 \phi + \frac{\tilde{g}_k}{6} \phi^3 = 0, \quad (22)$$

and another equation of motion for k :

$$\alpha'_k (\partial\phi)^2 - (\tilde{m}_k^2)' \phi^2 - \frac{1}{12} \tilde{g}'_k \phi^4 = 0, \quad (23)$$

where $\alpha' = \partial_k \alpha = (\partial_x \alpha) dx/dk$.

The conserved energy momentum tensor is obtained as a variation with respect to the metric tensor,

$$T_{\mu\nu} = \alpha_k (\partial_\mu \phi) (\partial_\nu \phi) - g_{\mu\nu} \left(\frac{\alpha_k}{2} (\partial\phi)^2 - \frac{\tilde{m}_k^2}{2} \phi^2 - \frac{\tilde{g}_k}{4!} \phi^4 \right). \quad (24)$$

The corresponding conservation law reads

$$\begin{aligned} 0 &= \partial^\mu T_{\mu\nu} \\ &= \partial^\mu (\alpha_k \partial_\mu \phi) \partial_\nu \phi + \alpha_k (\partial_\mu \phi) \partial^\mu \partial_\nu \phi \\ &\quad - \frac{1}{2} \partial_\nu \left(\alpha_k (\partial\phi)^2 - \tilde{m}_k^2 \phi^2 - \frac{\tilde{g}_k}{12} \phi^4 \right) \\ &= -\frac{1}{2} (\alpha'_k (\partial\phi)^2 - (\tilde{m}_k^2)' \phi^2 - \frac{1}{12} \tilde{g}'_k \phi^4) \cdot \partial_\nu k, \end{aligned} \quad (25)$$

where in the second line the ϕ equation of motion (22) was used. One easily observes that (25) is identical to the equation of motion for k (23). This shows that the approach at the level of effective action (20) is on the one hand implementing the idea of minimal scale dependence and on the other hand maintaining the validity of improved equations of motion and the fundamental conservation law (see Sec. II).

Instead of writing a coupling for the kinetic term one frequently works with wave function renormalization where the bare field is $\phi_B = \sqrt{Z_k}\phi$. In this case one could simply identify $\alpha_k = Z_k$, $\tilde{m}_k^2 = Z_k m_k^2$, and $\tilde{g}_k = g_k Z_k^2$ and observe that the corresponding effective action is completely equivalent to (21). However, if one allows for field valued scales $k = k(x)$, this identification is not the only possibility, since the derivatives of the kinetic term acting on the scale field might contribute to the action. Still, even in this case it can be shown that the scale setting procedure is consistent with the conservation law, just as in (25).

B. Scale-field setting in the one-loop expansion of ϕ^4 theory

The loop expansion of ϕ^4 theory has been calculated up to high order in perturbation theory [40]. The following example will be restricted to the one-loop expansion of the beta functions (see [41]):

$$\begin{aligned}\gamma_Z &= \frac{d \ln Z_k}{d \ln k^2} = 0, \\ \beta_g &= \frac{dg_k}{d \ln k} = \frac{3}{16\pi^2} g_k^2, \\ \beta_{m^2} &= \frac{dm_k^2}{d \ln k} = \left(-2 + \frac{g_k}{16\pi^2}\right) m_k^2.\end{aligned}\quad (26)$$

Now one can integrate those flow equations with initial conditions for $k = k_0$:

$$Z_{k_0} \equiv 1 \quad \text{and} \quad g_{k_0} = g_0 \quad \text{and} \quad m_{k_0}^2 = m_0^2. \quad (27)$$

This determines the particular flow trajectory:

$$\begin{aligned}A_k \left[(4(4\pi)^8 + 11(4\pi)^6 g - \frac{(4\pi)^4}{12} g^2 + 40g^3 - g^4) (\partial k)^2 - 8\pi^2 \left(\frac{(4\pi)^6}{4} + \frac{(4\pi)^4}{6} g - 96\pi^2 g^2 + g^3 \right) k \cdot \partial^2 k \right] \\ + \frac{2k_0^3 m_0^2 (g + 64\pi^2)}{3g^2 (g_0/g)^{1/3} k^3} \sqrt{(m_0^2 (32\pi^2 - g))(g/g_0)^{1/3}} = 0,\end{aligned}\quad (32)$$

where

$$A_k = \frac{k_0}{64\pi^2 g^{5/6} k^3 (-32\pi^2 + g)^2} \sqrt{\frac{m_0^2 (32\pi^2 - g)}{g_0^{1/3}}}. \quad (33)$$

Within the validity of the beta functions (28), which corresponds to first order in g_0 , this equation simplifies after the cancellation of a global factor to

$$\begin{aligned}k_0^2 m_0^2 (64\pi^2 - 3g_0 - 56g_0 \ln(k/k_0)) \\ + 6g_0 (\partial k)^2 - 3g_0 k \partial^2 k = 0.\end{aligned}\quad (34)$$

$$Z_k = 1,$$

$$\begin{aligned}g_k &= \frac{g_0}{1 - \frac{3}{16\pi^2} g_0 \ln(k/k_0)}, \\ m_k^2 &= \frac{k_0^2 m_0^2}{k^2 \left(1 - \frac{3}{16\pi^2} g_0 \log(k/k_0)\right)^{1/3}} = \frac{k_0^2 m_0^2}{k^2} \left(\frac{g_k}{g_0}\right)^{1/3}.\end{aligned}\quad (28)$$

In order to maintain legibility, the subscript of the scale dependent couplings will be omitted ($m_k^2 \rightarrow m^2$ and $g_k \rightarrow g$) in the following calculation. Due to the constant wave function renormalization, the equation of motion for ϕ simplifies to

$$\partial^2 \phi + m^2 \phi + \frac{g}{6} \phi^3 = 0 \quad (29)$$

and the scale setting equation of motion (23) simplifies to

$$\phi^3 g' + 12\phi(m^2)' = 0. \quad (30)$$

The two equations of motion [(29) and (30)] have to be solved for the two functions k and ϕ . One way to approach this is to first solve (30) as a nondifferential equation for ϕ^2 giving

$$\phi^2 = 4 \frac{k_0^2 m_0^2 (32\pi^2 - g)}{k^2 g^2} \left(\frac{g}{g_0}\right)^{(1/3)}. \quad (31)$$

This can now be inserted into (29) inducing a second order differential equation for the scale setting, which after using the running couplings (28) reads

The use of this relation, in terms of scale setting, will now be exemplified for a specific system.

For static spherical symmetry the only allowed coordinate dependence is with respect to the radial distance $k = k(r)$. In this case (34) reads

$$\begin{aligned}k_0^2 m_0^2 (64\pi^2 - 3g_0 - 56g_0 \ln(k/k_0)) - 6g_0 (\partial_r k)^2 \\ + \frac{3g_0 k}{r^2} (2r \partial_r k + r^2 \partial_r^2 k) = 0.\end{aligned}\quad (35)$$

Due to the nontrivial structure of the differential operator it is hard to find a general solution of (35) but one observes that, if $k_0 \neq 0$, this equation has actually a constant solution,

$$k_i = k_0 \exp\left(-\frac{3}{56} + \frac{8\pi^2}{7g_0}\right). \quad (36)$$

From this expression one finds that demanding that $k_i = k_0 \neq 0$ would imply $g_0 \approx 210$, which is clearly not in the validity range of the small g_0 approximation. Thus, it is safe to say that $k_i \neq k_0$. One can now investigate the scale setting in the vicinity of this constant scale k_i :

$$k(r) = k_i + \delta k(r) + \mathcal{O}(\delta k^2). \quad (37)$$

Inserting this in (35) and expanding to first order in δk one obtains a simpler differential equation for the radial dependence of the scale field,

$$56m_0^2 r \delta k - 3 \frac{k_i^2}{k_0^2} (2\partial_r \delta k + r \partial_r^2 \delta k) = 0. \quad (38)$$

This differential equation can be directly solved by

$$\begin{aligned} \delta k(r) = & \exp\left(-2\sqrt{\frac{14}{3}} \frac{k_0}{k_i} m_0 r\right) \frac{c_1}{r} \\ & + \exp\left(+2\sqrt{\frac{14}{3}} \frac{k_0}{k_i} m_0 r\right) \frac{c_2}{r}, \end{aligned} \quad (39)$$

where c_1 and c_2 are the constants of integration. They have to be set by additional conditions, for example one might impose that δk does not diverge for large radii, which implies that $c_2 = 0$. The scale-field setting is then

$$k(r) = k_i + \exp\left(-2\sqrt{\frac{14}{3}} \frac{k_0}{k_i} m_0 r\right) \cdot \frac{c_1}{r}. \quad (40)$$

One notes that (40) actually reproduces the standard $1/r$ behavior for very small radii, but it has two additional features with respect to the naive guess. The first difference consists in the constant factor, which might be suppressed for the case of a small scale k_0 in the initial conditions (27). The second difference is an exponential suppression factor which is controlled by the mass m_0 and by the value of g_0 . In Fig. 2 this r dependence of the scale setting is shown in comparison to the usual setting $k \sim 1/r$. The figure confirms the intuitive behavior for small r and shows the exponential suppression for larger r .

V. SCALE-FIELD SETTING FOR EINSTEIN-HILBERT-MAXWELL ACTION

A. Consistency in Einstein-Hilbert-Maxwell case

In order to show the consistency of the proposed scale setting, with conservation laws in a less-trivial example, one can study the approach for gravity coupled to a $U(1)$ gauge field and to a cosmological constant. Gravity is exemplary for a nontrivial field theory that is notoriously

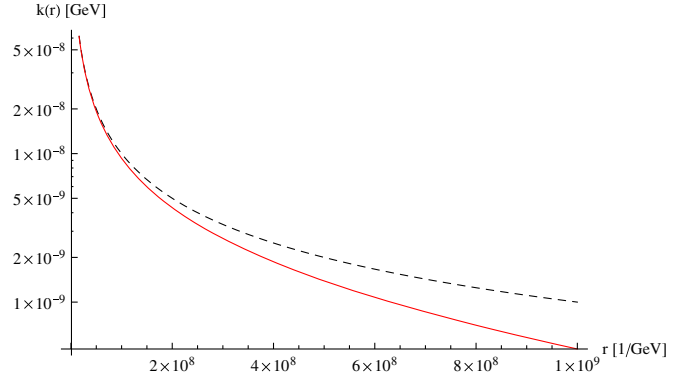


FIG. 2 (color online). Radial dependence of the scale setting (40) in comparison to the naive $1/r$ setting (dashed curve). The parameters used for the plot are $m_0 = 1$ GeV, $g_0 = 0.5$, $c_1 = 1$, $c_2 = 0$, and $k_0 = 0$.

perturbatively not renormalizable and the situation becomes even less favorable when it is coupled to matter. Still, there exist nonperturbative methods that allow one to calculate effective actions and scale dependent couplings for this theory [31,42–60].

Therefore, it is reasonable to investigate the proposed scale setting procedure in the context of a gravitational action coupled to matter. As an example for such a coupled gravitational system the Einstein-Hilbert-Maxwell action will be discussed:

$$\Gamma_k[g_{\mu\nu}, A_\alpha] = \int_M d^4x \sqrt{-g} \left(\frac{R - 2\Lambda_k}{16\pi G_k} - \frac{1}{4e_k^2} F_{\mu\nu} F^{\mu\nu} \right), \quad (41)$$

where R is the Ricci scalar and $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ is the antisymmetric electromagnetic field strength tensor. The scale dependent couplings are thus, Newton's coupling G_k , the cosmological coupling Λ_k , and the electromagnetic coupling e_k . Please note that the flow of those couplings has been derived nonperturbatively in [58]. As in (19) and (20) the scale k^2 will be considered as a field without kinetic term. The equations of motion for the metric field in (41) are

$$G_{\mu\nu} = -g_{\mu\nu} \Lambda_k - \Delta t_{\mu\nu} + 8 \frac{\pi G_k}{e_k^2} T_{\mu\nu}, \quad (42)$$

where the possible coordinate dependence of G_k induces an additional contribution to the stress-energy tensor [47]:

$$\Delta t_{\mu\nu} = G_k (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \frac{1}{G_k}. \quad (43)$$

Further, the stress-energy tensor for the electromagnetic part is given by

$$T_{\mu\nu} = F_\nu^\alpha F_{\mu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\mu\nu} F^{\mu\nu}. \quad (44)$$

The equations of motion (Maxwell equations) for this $U(1)$ gauge field are

$$D_\mu \left(\frac{1}{e_k^2} F^{\mu\nu} \right) = 0, \quad (45)$$

and finally the equations of motion for the scale field k are

$$\left[R \nabla_\mu \left(\frac{1}{G_k} \right) - 2 \nabla_\mu \left(\frac{\Lambda_k}{G_k} \right) - \nabla_\mu \left(\frac{4\pi}{e_k^2} F_{\alpha\beta} F^{\alpha\beta} \right) \right] \cdot (\partial^\mu k) = 0. \quad (46)$$

The above equations of motion are complemented by the relations corresponding to gauge invariance of the system. For the case of diffeomorphism invariance one has

$$\nabla^\mu G_{\mu\nu} = 0 \quad (47)$$

and for the internal $U(1)$ gauge symmetry the corresponding equations are

$$\nabla_{[\mu} F_{\alpha\beta]} = 0. \quad (48)$$

Please note that one has to work with (48) and not with $\nabla_{[\mu} e^{-1} F_{\alpha\beta]} = 0$ since the factor e^{-2} appears explicitly in the action (41) and not in the covariant derivatives.

The new ingredient due to the scale field is the equation (46), therefore it is important to check whether this equation is actually nontrivial and consistent with the Eqs. (42) and (45). The consistency can be shown by explicitly deriving (46) from (42) and (45) and further imposing that the gauge symmetries reflected by (47) and (48) are maintained. Starting from (42) one imposes (47).

$$\nabla^\mu G_{\mu\nu} = 0 = -g_{\mu\nu} \Lambda'_k \partial^\mu K + \nabla^\mu \Delta t_{\mu\nu} + 8\pi G'_k e_k^{-2} T_{\mu\nu} \partial^\mu K + 8\pi G_k \nabla^\mu (e_k^{-2} T_{\mu\nu}) \quad (49)$$

By using

$$\nabla^\mu (\nabla_\mu \nabla_\nu - \square g_{\mu\nu}) \frac{1}{G_k} = R_{\mu\nu} \nabla^\mu \frac{1}{G_k} \quad (50)$$

one can factorize a $(\partial^\mu k)$ in the whole expression except of a single term involving the electromagnetic stress-energy tensor

$$0 = [-g_{\mu\nu} G_k \Lambda'_k + G'_k (G_{\mu\nu} + \Lambda_k g_{\mu\nu} - 8\pi e_k^{-2} G_k T_{\mu\nu}) - R_{\mu\nu} G'_k + 8\pi G'_k G_k e_k^{-2} T_{\mu\nu}] (\partial^\mu k) + 8\pi G_k^2 \nabla^\mu (e_k^{-2} T_{\mu\nu}). \quad (51)$$

However, by using (48) and the antisymmetry of $F_{\mu\nu}$ one can show that $(\partial^\mu k)$ can also be factorized from this term:

$$\nabla^\mu (e_k^{-2} T_{\mu\nu}) = \left[-\frac{g_{\mu\nu}}{4} (e_k^{-2})' F_{\alpha\beta} F^{\alpha\beta} \right] (\partial^\mu k). \quad (52)$$

Thus, one has

$$0 = \left[-g_{\mu\nu} G_k \Lambda'_k + G'_k (G_{\mu\nu} + \Lambda_k g_{\mu\nu} - 8\pi e_k^{-2} G_k T_{\mu\nu}) - R_{\mu\nu} G'_k + 8\pi G'_k G_k e_k^{-2} T_{\mu\nu} - 8\pi G_k^2 \frac{g_{\mu\nu}}{4} (e_k^{-2})' F_{\alpha\beta} F^{\alpha\beta} \right] (\partial^\mu k). \quad (53)$$

Since in this proof no scale setting $k \rightarrow k(x)$ has been applied yet, one can choose any direction for the vector $(\partial^\mu k)$ and still the above relation has to hold. The only nontrivial way for this to happen is that the quantity in squared brackets has to vanish. Tracing this quantity over its two indices one gets

$$0 = 2 \frac{\Lambda'_k}{G_k} + \frac{G'_k (R - 2\Lambda_k)}{G_k^2} + 4\pi ((e_k^{-2})' F^2), \quad (54)$$

which is indeed identical to (46).

Therefore, the equation of motion (46) is indeed consistent with the other equations of motion of the system [(42) and (45)] in combination with the symmetry relations [(47) and (48)]. This consistency does not guarantee that the system has physically reasonable and nontrivial solutions. But it confirms again that, even for gauge and gravitational systems, the approach at the level of effective action (20) is on the one hand an elegant way of minimizing scale setting ambiguities and on the other hand maintaining the validity of improved equations of motion and the fundamental conservation laws of the effective action (see Sec. II).

Please note that given the fact that the functional form of the scale dependent couplings G_k , Λ_k , and e_k is in most known cases highly nontrivial [58] one can hardly expect to obtain an analytical solution of the equations (42) and (45). However, in [61] it will be shown that with just one simplifying assumption it is actually possible to find meaningful black hole solutions for two different truncations of (41).

B. Integrating out the scale field: UV scale-field setting for Einstein-Hilbert-Maxwell action in asymptotic safety

Since in the proposed method, the scale k is actually a scale field and since fields can be integrated out of an effective action by solving their equations of motion and inserting back into the action, $k(x)$ can be treated in the same way. By this, the method (19) can actually be used in order to construct a scale-independent effective action $\tilde{\Gamma}$, by integrating out $k(x)$ of the scale dependent action Γ_k . In a general setting one proceeds by first solving the equation of motion for k in momentum space,

$$\frac{\partial \mathcal{L}}{\partial k} = \partial_\mu \frac{\partial \mathcal{L}}{\partial k_{,\mu}}, \quad (55)$$

giving $k = k(\phi)$. This can be inserted back into the effective action which then gives

$$\Gamma(\phi, k) \rightarrow \Gamma(\phi, k(\phi)) = \tilde{\Gamma}(\phi). \quad (56)$$

By this procedure one can “integrate out” the scale field k and one obtains a new effective action $\tilde{\Gamma}(\phi)$ which is a function of ϕ only. This new scale free effective action differs from the original effective action $\Gamma_k(\phi)$ in the sense that it automatically contains a self-consistent scale setting.

Let us exemplify this with the Einstein-Hilbert-Maxwell action (41). In the deep UV limit ($k \rightarrow \infty$) there is strong evidence [43,44,46] for the existence of a non-Gaussian fixed point for the two gravitational couplings,

$$\begin{aligned} G_k &\approx \frac{g^*}{k^2}, \\ \Lambda_k &\approx \lambda^* k^2, \end{aligned} \quad (57)$$

and there exists further evidence for (at least) one UV fixed point for the electromagnetic couplings [58]:

$$\lim_{k \rightarrow \infty} \frac{1}{e_{k,2}^2} \approx \frac{1}{e_2^{*2}}. \quad (58)$$

Since this fixed point in the electromagnetic coupling is not an attractor it is only approached by particular trajectories in the corresponding flow. Other trajectories either run into a Landau pole type of divergence at finite k , or they run to vanishing values of $e_{k,1}$ at infinite k [58]:

$$\lim_{k \rightarrow \infty} \frac{1}{e_{k,1}^2} \approx \frac{1}{e_1^{*2}} \cdot (k^2)^B. \quad (59)$$

The value of the exponentiating factor B for the second fixed point depends on the method of calculation. Since the numerical value of B ranges from 0.8 to 1.6 [58], in this example the simplest possibility of $B \equiv 1$ will be chosen. The behavior of this flow projection using the functions of [58] is shown in Fig. 3. One observes three fixed points in this flow: apart from the Gaussian fixed point at vanishing couplings one sees, one fixed point with finite g^* but vanishing e^2 (labeled Γ_1), which corresponds to the limit (59), and another fixed point with finite g^* and e^2 (labeled Γ_2), which corresponds to the limit (58). In order to integrate out the scale field from the effective action (41) one has to solve the corresponding equation of motion (46) for k^2 . In the UV limit (57) one finds for the fixed point in (58)

$$k_2^2|_{\text{UV}} = \frac{R}{4\lambda^*} \quad (60)$$

and for the asymptotic behavior (59) one finds in the same limit

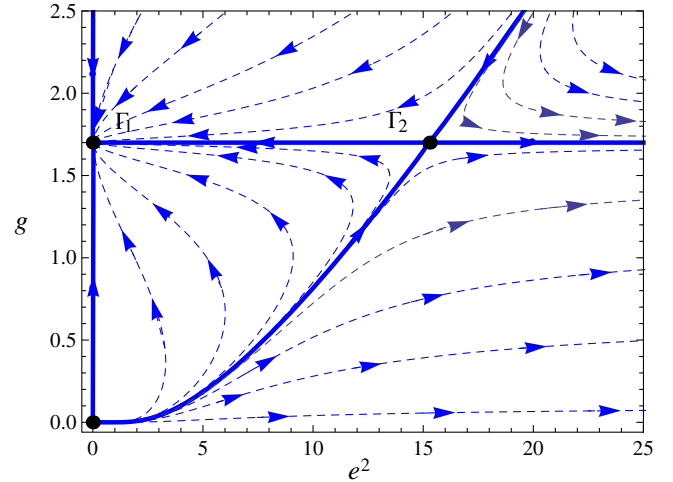


FIG. 3 (color online). Coupling flow projected to the dimensionless gravitational coupling g and the electromagnetic coupling e^2 . The arrows point in the direction of increasing k .

$$k_1^2|_{\text{UV}} = \frac{R - \frac{4\pi g^*}{e_1^{*2}} F^2}{4\lambda^*}. \quad (61)$$

Those field-scale settings relate the UV scale k^2 proportional to the curvature scalar R , in agreement to what is frequently intuited in the literature [29,62–65]. But the relations (60) and (61) go beyond this intuition since they also determine the constant proportionality factor and possible modifications due to the electromagnetic field strength.

Now follows the integrating out (56), where the scale field is eliminated from the original action. The effective actions valid in the deep UV are obtained, from (41) after using again the approximation (57). For the fixed point in (58) and the corresponding scale setting (60) this gives

$$\tilde{\Gamma}_{\text{UV},2} = \int d^4x \sqrt{-g} \left[\frac{R^2}{128\pi g^* \lambda^*} - \frac{F^2}{4e_2^{*2}} \right]. \quad (62)$$

For the asymptotic behavior (59) and the corresponding scale setting (61) the UV effective action results to be

$$\tilde{\Gamma}_{\text{UV},1} = \int d^4x \sqrt{-g} \left[\frac{(R - \frac{4\pi g^*}{e_1^{*2}} F^2)^2}{128\pi g^* \lambda^*} \right]. \quad (63)$$

One observes that for vanishing $F^2 = 0$, the R^2 dependence of the UV effective action in asymptotic safe gravity, which is indeed renormalizable [66], is recovered in agreement with other studies in the literature [29,63,64]. However, in addition to this expectable result, the scale-field procedure in combination with integrating out the additional k -field dependence, allowed one to predict a generalization of those results to the coupling to a finite electromagnetic field

strength $F^2 \neq 0$. Those UV results confirm that the fixed points (58) and (59) correspond to different physical systems with different effective equations of motion for the background. Therefore, the UV behavior of the self-consistent background, which is the solution of one of those equations of motion, can be largely different for different trajectories, depending on the fixed point which is approached by a specific trajectory in the RG flow (see Fig. 3).

VI. SUMMARY AND CONCLUSION

In this paper the problem of scale setting in the context of finding self-consistent solutions of the effective action was discussed. First, the procedure of improving solutions for nonperturbative problems was reviewed by the use of a quite generic example. It was shown that one might define a scale setting (9) that keeps consistency at the level of improved equations of motion, but it was also shown that usually the conservation of the stress-energy tensor cannot be guaranteed throughout the improving solutions procedure.

Then, in Sec. III a novel procedure for the scale setting is proposed, by promoting the scale k to a scale-field $k(x)$ at the level of effective action Γ . This proposal is the essential idea of the presented paper.

In order to demonstrate the functionality of the new procedure, the following sections are devoted to discuss the approach for emblematic field theories. As a first example scalar ϕ^4 theory is discussed and the consistent conservation of the stress-energy tensor, even after the scale setting, is shown explicitly. The self-consistency is shown for two common forms of the ϕ^4 effective action. In order to complement this general result, by a more practical

example, an approximated self-consistent scale setting for spherically symmetric backgrounds in ϕ^4 theory is calculated at the one-loop level.

In its generality the scale-field method is not limited to the simple scalar ϕ^4 model. Instead it is expected to work for a very broad class of theories. For example, it also is meant to work in the context of much richer gauge theories. This much broader applicability is exemplified by studying the scale setting prescription in gravity coupled to an electromagnetic stress-energy tensor, represented by Einstein-Hilbert-Maxwell theory. It is explicitly shown that also in this example the conservation of a generalized stress-energy tensor is guaranteed by the scale-field setting. As application, the UV scale-field setting of asymptotically safe gravity coupled to an electromagnetic field strength is calculated and the scale independent effective action (valid in the UV) of this theory is derived by integrating the scale field out. Finally, we would like to mention that the idea is not limited to background calculations, but it should also be applicable to scale setting problems in scattering theory in the conventional Feynman sense.

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