Berry phase, Lorentz covariance, and anomalous velocity for Dirac and Weyl particles

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(Received 23 June 2014; published 6 January 2015)

We consider the relation between spin and the Berry-phase contribution to the anomalous velocity of massive and massless Dirac particles. We extend the Berry connection that depends only on the spatial components of the particle momentum to one that depends on the space and time components in a covariant manner. We show that this covariant Berry connection captures the Thomas-precession part of the Bargmann-Michel-Telegdi spin evolution, and contrast it with the traditional (unitary, but not naturally covariant) Berry connection that describes spin-orbit coupling. We then consider how the covariant connection enters the classical relativistic dynamics of spinning particles due to Mathisson, Papapetrou and Dixon. We discuss the problems that arise with Lorentz covariance in the massless case, and trace them mathematically to a failure of the Wigner-translation part of the massless-particle little group to be an exact gauge symmetry in the presence of interactions, and physically to the fact that the measured position of a massless spinning particle is necessarily observer dependent.

DOI: 10.1103/PhysRevD.91.025004 03.65.Vf, 04.20.-q

I. INTRODUCTION

There has been much recent interest in the fluid dynamics of systems possessing anomalous conservation laws [1–6]. An unexpected consequence of this work has been the discovery that anomalies, which are usually thought of as being purely quantum mechanical effects, can be extracted from the classical kinetic theory of a degenerate gas of Weyl fermions [7]. The incompressibility of phase space allows the anomalous inflow of particles from the negative-energy Dirac sea into the positive-energy Fermi sea [8–10] to be reliably counted by keeping track of the density flux near the Fermi surface where a classical Boltzmann equation becomes sufficiently accurate for this purpose. The only required quantum input is knowledge of how to normalize the phase space-measure and the inclusion of a Berry-phase effect. The Berry phase causes the velocity of the particle to no longer be parallel to its momentum. Instead an additional "anomalous velocity" appears as a momentum-space analogue of the Lorentz force in which the electromagnetic field tensor is replaced by the Berry curvature, and the particle velocity by $\dot{\mathbf{k}}$. The Berry phase also alters the classical canonical structure so that $\mathbf{x}$ and $\mathbf{k}$ are no longer conjugate variables, and d^3kd^3x is no longer the element of phase space volume [11,12].

It is possible to extend these derivations to the non-Abelian anomaly [13] and to higher dimensions [14], but the kinetic theory used in all these papers is based on Hamiltonian dynamics where time and space are treated very differently. It is therefore a challenge to make the formalism manifestly covariant so that a coupling to gravity might be included. Indeed it is not easy to see how even flat-space Lorentz invariance is realized in the Hamiltonian kinetic theory. This issue was raised in [15] and the curious manner in which the dynamical variables must transform was made clear in [16].

The most obvious problem with extending the three-dimensional Hamiltonian formulation to a covariant $3+1$ version is that the Berry curvature is a differential form in only the three spatial components of the momentum. In a formalism that treats space and time on an equivalent footing we would expect the connection to involve differentials of all four components of the energy-momentum vector. In this paper we show how to make such an extension, and in doing so we make a connection between the Hamiltonian formalism with its Berry phase modification and the relativistic classical mechanics of spinning particles.

In Sec. II we use a WKB solution to the massive-Dirac equation to motivate an unconventional, but covariant, Berry connection that captures the geometric Thomas precession of the spin. We contrast the properties of this Berry connection with the traditional, noncovariant Berry connection whose importance in the dynamics of charged Dirac particles was revealed in [17–19]. In Sec. III we introduce a classical mechanical action for a spinning particle interacting with a gravitational field. This manifestly covariant action gives rise to the well-known Mathisson-Papapetrou-Dixon equations [20–22], and we show how these equations can be recast to make explicit the role of the covariant Berry connection. In Sec. IV we discuss the problems that arise when the particle mass becomes zero, and show how these arise

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from a hidden gauge invariance of the free action. After selecting a natural gauge fixing condition, the covariant action reduces to the Berry-connection actions used in [7,13,14]. Mathematically, it is the necessity of gauge fixing that is responsible for the curious Lorentz transformation laws that appear in [16], physically it is because the "position" of a massless spinning particle is an observer-dependent concept. The gauge invariance of the massless action is only approximate in the presence of introduction of interactions and this leads to the gauge-fixed action not being exactly equivalent to the manifestly covariant action. We argue that this is perhaps not surprising as in a massless system the adiabatic approximation that is tacit in any system involving a Berry connection can be violated by a sufficiently large Lorentz transformation.

A discussion section addresses the physical origin of the anomalous velocity. Finally, several derivations that would be intrusive in the main text appear in Appendices A–D.

II. A COVARIANT BERRY CONNECTION

That a Berry phase gives rise to an anomalous velocity correction was first observed in the band theory of solids. We begin with a brief account of how the effect appears there, and why a similar correction is expected in the motion of Dirac particles.

A. Lorentz covariance versus the Berry phase

A semiclassical wave-packet analysis [23,24] shows that the motion of a charge-e Bloch electron in an energy band in a crystalline solid is governed by the equations

$$\dot{\mathbf{k}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + e(\dot{\mathbf{x}} \times \mathbf{B}), \tag{1}$$

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \Omega. \tag{2}$$

The effective Hamiltonian $\mathcal{H} = \varepsilon(\mathbf{k}) + e\phi(\mathbf{x})$ includes the band-energy $\varepsilon(\mathbf{k})$ as a function of the crystal momentum $\mathbf{k}$, together with the interaction with the scalar potential ϕ. The vector Ω with components $\Omega_i = \frac{1}{2}\epsilon_{ijk}\Omega_{jk}$ is a Berry curvature that accounts for the effects of all other energy bands. The magnetic field $\mathbf{B}$ is a function of $\mathbf{x}$ only, and Ω is a function of $\mathbf{k}$ only. The $-\dot{\mathbf{k}} \times \Omega$ term in (2) is the *anomalous velocity* correction to the naïve group velocity $\partial\varepsilon/\partial\mathbf{k}$. This correction arises because different momentum components of a localized wave-packet accumulate different geometric phases when $\mathbf{k}$ is changing and the Berry curvature is nonzero [25]. These $\mathbf{k}$-dependent geometric phases are just as significant in determining the wave-packet position as the $\mathbf{k}$-dependent dynamical phases arising from the dispersion equation $\omega = \varepsilon(\mathbf{k})$. A nice illustration of the effect of the anomalous velocity on a particle trajectory is to be found in [26].

Now a Dirac Hamiltonian can be thought of as a Bloch system with two energy bands $\varepsilon(\mathbf{k}) \equiv E(\mathbf{k}) = \pm\sqrt{\mathbf{k}^2 + m^2}$, and each band possesses a nonzero Berry curvature [17–19]. Consequently (1) and (2) should also describe the semiclassical motion of a relativistic spin-$\frac{1}{2}$ particle. This raises an interesting issue. We expect that the equation of motion of a Dirac particle can be written in a manifestly Lorentz invariant form, but it is not immediately obvious how to massage the Dirac version of (1) and (2) into covariant expressions. When $-\partial\mathcal{H}/\partial\mathbf{x}$ is the force due to an electric field [27], the first line (1) can be written as $\dot{k}_\mu = eF_{\mu\nu}\dot{x}^\nu$, but for (2) how does one define a 3+1-dimensional analogue of the Maxwell tensor $F^{\mu\nu}$ for the intrinsically three-dimensional Berry curvature Ω_{ij}?

B. Covariant WKB approximation for the Dirac equation

In order to obtain a manifestly Lorentz invariant semiclassical equation of motion for a Dirac particle, we need to extend the noncovariant Berry connection to one in which space and time components are treated equally. Now the simplest semiclassical approximation to any wave equation is that of WKB. We therefore construct a WKB approximation to the Dirac equation coupled to an externally imposed Maxwell field. We maintain covariance at each step, anticipating that a covariant version of Berry curvature will play some role. WKB approximations to the Dirac equation have a long history, going back to W. Pauli in 1932 [28]. More recent references are [29–31]. None of these works make use of the particular covariant approach that we introduce here.

We take the particle to have charge e (a positive number when the charge is positive) and to have positive mass m. Let $x^\mu = (t, \mathbf{x})$, and seek a positive-energy WKB solution

$$\psi(x) = a(x)e^{-i\varphi(x)/\hbar},\qquad a = a_0 + \hbar a_1 + \hbar^2 a_2 + \cdots \tag{3}$$

to the Dirac equation

$$(i\hbar\gamma^\mu(\partial_\mu + ieA_\mu/\hbar) - m)\psi = 0. \tag{4}$$

Here $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$, and $A_\mu = (\phi, -\mathbf{A})$.

Setting $p_\mu \stackrel{\mathrm{def}}{=} \partial_\mu\varphi = (E, -\mathbf{p})$, we have at order $\hbar^0$

$$(\gamma^\mu(p_\mu - eA_\mu) - m)a_0 = 0. \tag{5}$$

We satisfy (5) by setting $a_0(x) = u_\alpha(k(x))C^\alpha(x)$ with $k_\mu = p_\mu - eA_\mu$ being the gauge-invariant kinetic momentum, and $u_\alpha(k)$ being a complete set of eigenspinor solutions to

$$(\gamma^\mu k_\mu - m)u_\alpha = 0. \tag{6}$$

In this equation the kinetic four-momentum $k_\mu = (E, -\mathbf{k})$ lies on the positive-energy mass shell: $E^2 = \mathbf{k}^2 + m^2$, $E > 0$. We take the eigenspinors to have the covariant normalization $\bar{u}_\alpha u_\beta = \delta_{\alpha\beta}$. (See Appendix A for details)

At order $\hbar^1$ we have

$$(\gamma^\mu k_\mu - m)a_1 + (i\gamma^\mu \partial_\mu)a_0 = 0. \tag{7}$$

Now if

$$(\gamma^\mu k_\mu - m)u_\alpha = 0, \tag{8}$$

then

$$\bar{u}_\alpha(\gamma^\mu k_\mu - m) = 0. \tag{9}$$

We can therefore eliminate the influence of the unknown coefficient a_1 and deduce that

$$\bar{u}_\beta \gamma^\mu \partial_\mu a_0 = \bar{u}_\beta \gamma^\mu \partial_\mu (u_\alpha C^\alpha) = 0. \tag{10}$$

Equation (10) tells us how both the amplitude and spin components evolve along the classical trajectory. We rewrite (10) as

$$\bar{u}_\beta \gamma^\mu u_\alpha (\partial_\mu C^\alpha) + (\bar{u}_\beta \gamma^\mu \partial_\mu u_\alpha) C^\alpha = 0, \tag{11}$$

and then use (B5) from Appendix B to write

$$\bar{u}_\beta \gamma^\mu u_\alpha = \delta_{\alpha\beta} \frac{k^\mu}{m} \equiv \delta_{\alpha\beta} V^\mu, \tag{12}$$

and so express the transport equation (10) as

$$\left[\delta_{\alpha\beta} V^\mu \frac{\partial}{\partial x^\mu} + M_{\alpha\beta}\right] C^\beta = 0. \tag{13}$$

Here

$$M_{\alpha\beta} = \bar{u}_\alpha \gamma^\mu \frac{\partial}{\partial x^\mu} u_\beta, \tag{14}$$

and $V^\mu = \gamma(1, \mathbf{v}) = k^\mu/m$ is the 4-velocity corresponding to the ray-tracing group velocity

$$\mathbf{v} = \frac{\partial E}{\partial \mathbf{k}}. \tag{15}$$

Thus the combination

$$V^\mu \frac{\partial}{\partial x^\mu} \equiv \frac{d}{d\tau} \tag{16}$$

is a convective derivative with respect to proper time along the particle's trajectory. The $(\mathbf{x}, \mathbf{k})$ trajectory itself is given by Hamilton's ray-tracing equations and coincides with that of a spinless charged particle in the background field. There is no sign of the anomalous velocity. As pointed out in [29], this absence is to be expected because both the intrinsic spin and magnetic moment of a Dirac particle are proportional to $\hbar$, and vanish in the classical limit. Thus leading-order WKB is not able to account for the effect of the spin on the particle's motion. Nonetheless the *ratio* of the magnetic moment to the spin angular momentum is independent of $\hbar$. As a consequence leading-order WKB *is* adequate for obtaining the Bargmann-Michel-Telegdi (BMT) equation [32] that describes the effect of the magnetic field on the spin evolution. A Berry connection is a key ingredient in this equation.

To isolate the Berry connection, we decompose

$$M_{\alpha\beta} = \frac{1}{2}(M_{\alpha\beta} + M^*_{\beta\alpha}) + \frac{1}{2}(M_{\alpha\beta} - M^*_{\beta\alpha}), \tag{17}$$

and, from equation (B5), recognize that

$$\frac{1}{2}(M_{\alpha\beta} + M^*_{\beta\alpha}) = \frac{1}{2}\delta_{\alpha\beta} \frac{\partial V^\mu}{\partial x^\mu}. \tag{18}$$

We now insert the completeness relation $\mathbb{I} = u_\lambda \bar{u}_\lambda - v_\lambda \bar{v}_\lambda$ as intermediate states in the definition of $M_{\alpha\beta}$. From the positive-energy $u_\lambda \bar{u}_\lambda$ terms we get

$$\begin{aligned}(\bar{u}_\alpha \gamma^\mu u_\lambda)\left(\bar{u}_\lambda \frac{\bar{\partial}}{\partial x^\mu} u_\beta\right) &= \left(V^\mu \frac{\partial k^\nu}{\partial x^\mu}\right)\left(\bar{u}_\alpha \frac{\partial}{\partial k^\nu} u_\beta\right) \\ &= -i\mathfrak{a}_{\alpha\beta,\nu} \frac{dk^\nu}{d\tau}.\end{aligned} \tag{19}$$

The quantity

$$\mathfrak{a}_{\alpha\beta,\nu} \stackrel{\text{def}}{=} i\bar{u}_\alpha \frac{\partial u_\beta}{\partial k^\nu} \tag{20}$$

is an unconventional Berry-phase-like connection. It is unitary only with respect the non-positive-definite inner product $\langle\psi|\chi\rangle \equiv \psi^\dagger \gamma^0 \chi$, but makes use of all four components of dk^ν and is constructed out of Lorentz-covariant objects. We will therefore refer to it as the *covariant Berry connection*.

The contribution of the negative energy intermediate states $-v_\lambda \bar{v}_\lambda$ is an example of Littlejohn's "no-name" phase [33]. After some labor, we find that their contribution is

$$-\frac{1}{2}\left(\bar{u}_\alpha \gamma^\mu v_\lambda \bar{v}_\lambda \frac{\partial}{\partial x^\mu} u_\beta - (\alpha \leftrightarrow \beta)^*\right) = \frac{ie}{2m} S^{\mu\nu}_{\alpha\beta} F_{\mu\nu} C^\beta, \tag{21}$$

where $\dot{k}^\nu = dk^\nu/d\tau$,

$$(S_{\mu\nu})_{\alpha\beta} = \bar{u}_\alpha \left(\frac{i}{4}[\gamma_\mu, \gamma_\nu]\right) u_\beta, \tag{22}$$

and we have used $k_\mu = \partial_\mu \varphi - eA_\mu$ to write

$$\partial_\mu k_\nu - \partial_\nu k_\mu = -eF_{\mu\nu}. \tag{23}$$

The combined contribution of both sets of intermediate states therefore leads to

$$\left[\delta_{\alpha\beta}\left(V^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{2}\frac{\partial V^\mu}{\partial x^\mu}\right) - i\dot{k}^\nu (\mathfrak{a}_\nu)_{\alpha\beta} + \frac{ie}{2m} S^{\mu\nu}_{\alpha\beta} F_{\mu\nu}\right] C^\beta = 0. \tag{24}$$

The divergence of the 4-velocity in (24) accounts for the change in amplitude due to geometric focussing. The remaining terms describe how the spin evolves through its interaction with the external field, and as a result of its parallel transport under the Berry connection.

The combination $S^{\mu\nu}F_{\mu\nu}$ is Lorentz invariant, so we can evaluate it in the particle's rest frame where

$$\left(\frac{e}{2m}\right)(S^{\mu\nu})_{\alpha\beta} F_{\mu\nu} \to -\left(\frac{e}{m}\right)\mathbf{B}\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta}. \tag{25}$$

Since the unitary operator for a rotation at angular velocity $\boldsymbol{\omega}$ is $U(t) = \exp\{-i\boldsymbol{\omega}\cdot(\boldsymbol{\sigma}/2)t\}$ we see that (25) accounts for the Larmor precession $\boldsymbol{\omega}_{\text{Larmor}} = -|\boldsymbol{\mu}|\mathbf{B}$ of the spin due to its $\boldsymbol{\mu} = (e/m)\mathbf{S}$ Dirac-value magnetic moment. The two-by-two matrix $(e/2m)\mathbf{B}\cdot\boldsymbol{\sigma}$ acts on the *polarization spinor* χ_α that is defined in (A2). Polarization is the spin measured in the rest frame of the particle [30].

To understand the origin of the Berry connection term we use the explicit formulas for $u_\alpha(\mathbf{k})$ given in (A1) to evaluate

$$\begin{aligned}\mathfrak{a}_{\alpha\beta,\nu}\dot{k}^\nu &= \frac{1}{m^2(1+\gamma)}(\mathbf{k}\times\dot{\mathbf{k}})\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta} \\ &= \frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta} \\ &= -\boldsymbol{\omega}_{\text{Thomas}}\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta}.\end{aligned} \tag{26}$$

Here $\boldsymbol{\beta} \equiv \mathbf{k}/E$, and

$$\boldsymbol{\omega}_{\text{Thomas}} = -\left(\frac{\gamma^2}{1+\gamma}\right)(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) \tag{27}$$

is a standard expression for the Thomas-precession angular velocity. Our covariant Berry transport is therefore nothing other than Thomas precession—i.e. parallel transport on the tangent bundle of the positive-mass hyperboloid embedded in Minkowski-signature momentum space [34]. The minus sign occurs because the mass-shell hyperboloid is a negative-curvature Lobachevskii space.

The matrix-valued connection one-form is defined by

$$\mathfrak{a} \stackrel{\text{def}}{=} \frac{1}{m^2(1+\gamma)}\left(\frac{\boldsymbol{\sigma}}{2}\right)\cdot(\mathbf{k}\times d\mathbf{k}), \tag{28}$$

and the associated matrix-valued curvature $\mathfrak{F} = d\mathfrak{a} - i\mathfrak{a}^2$ is

$$\mathfrak{F} = \frac{1}{2m^2\gamma}\left\{\frac{1}{2}\left(\boldsymbol{\sigma} + \frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)}\right)\right\}\cdot(d\mathbf{k}\times d\mathbf{k}). \tag{29}$$

The connection-form and the curvature do not *look* covariant as they involve only the spatial components of dk^μ. This is a consequence of the way we wrote $u_\alpha(\mathbf{k})$ in (A1). In Appendix B we avoid explicit formulas for u_α and use only general properties of the Dirac equation to obtain an expression for the curvature in arbitrary dimensions. We find that

$$\mathfrak{F}_{\alpha\beta} \equiv (d\mathfrak{a} - i\mathfrak{a}^2)_{\alpha\beta} = \frac{1}{2m^2}(S_{\mu\nu})_{\alpha\beta} dk^\mu \wedge dk^\nu, \tag{30}$$

where $(S_{\mu\nu})_{\alpha\beta}$ was defined in equation (22). This form of the curvature is manifestly covariant and contains both space and time components of dk^μ. The dk^μ are not independent however, but are constrained by the mass-shell condition $k^2 = m^2$. If we desire, therefore, we may eliminate dk^0 as $dk^0 = d\sqrt{\mathbf{k}^2 + m^2} = k^i dk^i/E = -k_i dk^i/E$ and find

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2}\left(S_{ij} - \frac{k_i}{E}S_{0j} - S_{i0}\frac{k_j}{E}\right)_{\alpha\beta} dk^i \wedge dk^j, \tag{31}$$

where i, j run over space indices only. Evaluation of the required $S_{\mu\nu}$ matrix elements confirms that this reduced expression coincides with (29). The combination of spin components in parentheses on the right-hand side is a general-dimension analogue of the space part of the $(3+1)$-dimensional Pauli-Lubansky vector. We will therefore refer to it as the Pauli-Lubanksy tensor (It is tensor only under space rotations. It is not a Lorentz tensor). It will appear frequently in the rest of the paper and its geometric and physical significance is further discussed in Appendixes A and D.

To verify that parallel transport *via* the covariant Berry connection is nothing other than Thomas precession, we show in Appendix C that under such transport (i.e. no external torque or Larmor precession) the WKB approximation to the Dirac-field angular momentum tensor $S_{\mu\nu} = \bar{\psi}(i[\gamma_\mu, \gamma_\nu]/4)\psi$ obeys

$$\frac{\partial S^{\mu\nu}}{\partial\tau} + V^\nu \frac{\partial V^\lambda}{\partial\tau} S^\mu{}_\lambda + V^\mu \frac{\partial V^\lambda}{\partial\tau} S_\lambda{}^\nu = 0. \tag{32}$$

Since (B9) tells us that $V^\mu S_{\mu\nu} = 0$, and so (32) states that $S_{\mu\nu}$ is Fermi-Walker transported along the particle trajectory. Thomas precession is simply the evolution under Fermi-Walker transport of vectors (such as the spin four-vector $\mathbf{S}$) that are perpendicular to the 4-velocity vector.

C. Comparison with the noncovariant WKB approximation

The traditional form of the WKB transport equation is obtained by expanding $\psi = u_\alpha(\mathbf{k})K^\alpha(x)$ where the u_α are given the *noncovariant* normalization $u^\dagger_\alpha u_\beta = \delta_{\alpha\beta}$, and paired with negative-energy solutions that, in terms of the covariant v_α, are given $\gamma^{-1}v_\alpha(-\mathbf{k})$. These noncovariant spinors have completeness relation $\mathbb{I} = u_\alpha u^\dagger_\alpha + v_\alpha v^\dagger_\alpha$. On using them as intermediate states we obtain the alternative form of transport equation found in [30,31]:

$$\left\{\delta_{\alpha\beta}\left(\frac{d}{dt} + \frac{1}{2}\mathrm{div}\mathbf{v}\right) + N_{\alpha\beta}\right\}K^\beta = 0. \tag{33}$$

Here t is the lab-frame time, $\mathbf{v} = \boldsymbol{\beta}$ is the 3-velocity, and

$$\begin{aligned} N_{\alpha\beta} &= -ia_{\alpha\beta,i}\dot{k}^i - i\left(\frac{e}{m}\right)\mathbf{B}\cdot\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{\gamma+1}\right)_{\alpha\beta}\frac{1}{2\gamma^2} \\ &= -ia_{\alpha\beta,i}\dot{k}^i - i\left(\frac{e}{m}\right)\frac{1}{\gamma^2}\mathbf{B}\cdot(\mathbf{S}_{\mathrm{lab}})_{\alpha\beta}. \end{aligned} \tag{34}$$

The term with the magnetic field $\mathbf{B}$ is again a "no-name" phase that arises from the negative-energy intermediate states [33]. The Berry connection $a_{\alpha\beta,i}$ is here of conventional form

$$\begin{aligned} a_{\alpha\beta,i}dk^i &\stackrel{\mathrm{def}}{=} iu^\dagger_\alpha\frac{\partial u_\beta}{\partial k^i}dk^i \\ &= -\frac{\gamma}{1+\gamma}(\boldsymbol{\beta}\times d\boldsymbol{\beta})\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta}. \end{aligned} \tag{35}$$

Compared to the covariant connection, (35) lacks one power of γ. More importantly, it has the *opposite sign*. The associated matrix-valued curvature is [19]

$$\begin{aligned} \mathcal{F} &= da - ia^2 \\ &= -\frac{1}{4m^2\gamma^3}\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{\gamma+1}\right)\cdot(d\mathbf{k}\times d\mathbf{k}). \end{aligned} \tag{36}$$

Again compared to the covariant expression $\mathfrak{F}$, the noncovariant Berry curvature $\mathcal{F}$ lacks two powers of γ, and again has the opposite sign.

Both the covariant and the noncovariant transport equation lead to the same BMT equation, but there is a different distribution between terms of the dynamical Larmor precession and the geometric parallel transport. In the covariant formulation we have precession of the rest-frame polarization $\mathbf{s}$ due to the magnetic field as seen by the particle in its *rest frame*, and augmented by the geometric Thomas precession factor. This is how the BMT equation is broken up in Jackson [35], in his Eq. (11.166). In the noncovariant formulation we have precession of the same rest-frame polarization $\mathbf{s}$, but now due to the magnetic field as seen by the spin in the *lab frame* and augmented by the conventional Berry transport term. This is how the BMT equation is decomposed in [19], where the connection (35) and curvature (36) are obtained from a wave-packet approach.

The difference in sign between the two connections is accounted for by the different physical effects that they capture. The covariant connection provides the purely geometric Thomas precession effect. The noncovariant Berry connection implements the spin-orbit coupling due to the particle's motion viewed from the lab frame [36]. As was famously explained by Llewellyn Thomas [37], this spin-orbit coupling comes from two competing effects: firstly the Lorentz transform of the external field that leads to the motion through an $\mathbf{E}$ field being perceived as a $\mathbf{B}$ field, and secondly the Thomas precession that half-undoes the Lorentz transformation contribution. The net precession rate therefore has opposite sign to its Thomas-precession component.

III. CLASSICAL MOTION OF PARTICLES WITH SPIN

Rather than attempt to extend the WKB approximation to higher order, we will use symmetry consideration to construct a Hamiltonian action-principle version of the dynamics that is manifestly covariant, gives the same spin transport as the WKB approximation, but also gives us an anomalous-velocity correction. As our ultimate goal is to understand the effect of gravity on the particle, we will from the outset take our space-time to be curved.

A. Mathisson-Papapetrou-Dixon equations

There is an extensive literature on the relativistic classical dynamics of particles with spin, but a desire to make contact with the Berry phase methods of [7,13,14] suggests that we follow the particular approach of [38–40] and take as our dynamical degrees of freedom the position $x \in M$ (where M is the d-dimensional space-time manifold) and a vielbein frame $\tilde{\mathbf{e}}_a$ with $\tilde{e}^\mu_a\tilde{e}^\nu_b g_{\mu\nu} = \eta_{ab}$ where $\eta_{ab} = \mathrm{diag}(+,-,-,\ldots,-)$. Our phase space is then the total space P of a Lorentz-frame bundle $\pi{:}\,P \to M$ equipped with local coordinates $(x^\mu, \tilde{e}^\mu_a)$ and structure group $\mathrm{SO}(1, d-1)$.

It is convenient to introduce a *reference vielbein* $\mathbf{e}_a$, again with $e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}$. This reference frame allows us to write

$$\tilde{\mathbf{e}}_a = \mathbf{e}_b\Lambda^b{}_a, \qquad \Lambda \in \mathrm{SO}(1, d-1), \tag{37}$$

and so equivalently regard the dynamical degrees of freedom to be $x \in M$ and the Lorentz transformation Λ.

We assume that the space-time M is equipped with a Riemann connection—and hence with covariant derivatives ∇_μ. The reference vielbein then defines the components of the spin connection $\omega^a{}_{b\mu}$ by

$$\nabla_\mu \mathbf{e}_a = \mathbf{e}_b \omega^b{}_{a\mu}. \tag{38}$$

We use these components to assemble the spin-connection one-form

$$\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu, \tag{39}$$

which lives on the base-space M. The associated Riemann curvature is the base-space two-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{40}$$

As with any frame bundle, the connection on the base space automatically provides a decomposition of the tangent space at each point p in the total space P of the bundle into horizontal and vertical subspaces: $T(P) = H \oplus V$.

We begin with particles with a nonzero mass m and orient the frame so that $k \equiv m\tilde{\mathbf{e}}_0$ is the 4-momentum. Thus $k^b = m\Lambda^b{}_0$ are the vielbein components of the momentum and $k^\mu = m e^\mu_a \Lambda^a{}_0$ are its coordinate components. We also introduce a co-frame of one-forms

$$\mathbf{e}^{*a} = e^{*a}_\mu dx^\mu \tag{41}$$

where $\mathbf{e}^{*a}(\mathbf{e}_b) \equiv e^{*a}_\mu e^\mu_b = \delta^a_b$ and $e^{*a}_\mu = g_{\mu\nu}\eta^{ab} e^\nu_b$. We then set $\tilde{\mathbf{e}}^{*a} = (\Lambda^{-1})^a{}_b \mathbf{e}^{*b}$. With our $\eta_{ab} = \mathrm{diag}(+,-,-,\ldots,-)$ signature we have $m\tilde{\mathbf{e}}^{*0} = k_\mu dx^\mu = k_a \mathbf{e}^{*a}$.

In [13,14], the action integral was written in terms of traces over some faithful representation of the spin or gauge groups. In the present case we could use any faithful representation of the Lorentz group, but it seems natural to make use of Dirac matrices γ_a and the Dirac representation $\Lambda \mapsto D(\Lambda)$ that acts on them as

$$D(\Lambda)\gamma_a D(\Lambda^{-1}) = \gamma_b \Lambda^b{}_a. \tag{42}$$

We will simplify the notation by setting $\lambda = D(\Lambda)$. In this section we use the matrices

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] \tag{43}$$

as the Lorentz generators. These matrices obey

$$[\sigma_{ij}, \sigma_{mn}] = \eta_{jm}\sigma_{in} - \eta_{im}\sigma_{jn} - \eta_{jn}\sigma_{im} + \eta_{in}\sigma_{jm}, \tag{44}$$

and

$$[\sigma_{ab}, \gamma_c] = \gamma_a \eta_{bc} - \gamma_b \eta_{ac}. \tag{45}$$

We also have

$$\mathrm{tr}\{\sigma_{ab}\sigma_{cd}\} = -\frac{1}{4}\mathrm{tr}(\mathbb{I})(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}). \tag{46}$$

The covariant derivative acting on a spin field is

$$\nabla_\mu \psi = \left(\frac{\partial}{\partial x^\mu} + \frac{1}{2}\sigma_{ab}\omega^{ab}{}_\mu\right)\psi, \tag{47}$$

and, as usual, we regard the spin connection in the Dirac representation

$$\omega \equiv \frac{1}{2}\sigma_{ab}\omega^{ab}{}_\mu dx^\mu \tag{48}$$

as a matrix-valued one-form.

We can use the Lorentz transformation matrix λ to write

$$k_a = \mathrm{tr}\{\kappa\lambda^{-1}\gamma_a\lambda\}, \tag{49}$$

where $\kappa = m\gamma^0/\mathrm{tr}(\mathbb{I})$. Similarly, we define a classical spin angular-momentum tensor

$$S_{ab} = \mathrm{tr}\{\Sigma\lambda^{-1}\sigma_{ab}\lambda\}, \tag{50}$$

where $\Sigma = \frac{1}{2}\Sigma^{ab}\sigma_{ab}$.

The quantities k_a and S_{ab} are the true dynamical variables of the system. They are coordinates on the orbit of κ and Σ under the co-adjoint action of the Lorentz group, and the reduced phase space is the cartesian product of M with this co-adjoint orbit [41]. After quantization of the co-adjoint orbit, the quantities κ and Σ will define the highest weights in the resulting representation of the Poincare group [39]. Different choices of the matrix Σ^{ab} lead to different values for the intrinsic spin of the particle. Similarly different choices for the matrix $\kappa = \kappa^a\gamma_a$ allow us to consider both massive and massless particles within one formalism.

If we compute

$$[\Sigma, \kappa] = \gamma_a \Sigma^{ab}\kappa_b, \tag{51}$$

we see that $[\Sigma, \kappa] = 0$ is equivalent to $\Sigma^{ab}\kappa_b = 0$, and by Lorentz covariance this is in turn equivalent to $S_{ab}k^b = 0$. But $[\Sigma, \kappa] = 0$ means that Σ lies in the Lie algebra of the little-group of κ. As $S_{ab}k^b = 0$ is a property possessed by the Dirac angular momentum $S_{ab} = i\bar{\psi}\sigma_{ab}\psi$ [see Eq. (B9)] we will accept this little-group property as a natural constraint on the spin tensor. In the relativity literature it is known as the Tulczyjew-Dixon condition [22,42]. It is to be contrasted with the rival Mathisson-Pirani [20,43] condition $S_{ab}\dot{x}^b = 0$, where

$$\dot{x}^b = e^b_\mu \frac{dx^\mu}{d\tau}. \tag{52}$$

Here τ can be any coordinate that parametrizes the space-time trajectory $x^\mu(\tau)$. It does not have to be the proper time.

When λ depends on τ we have

$$\begin{aligned}\frac{d}{d\tau}S_{ab} &= -\mathrm{tr}\{[\Sigma, \lambda^{-1}\dot{\lambda}]\lambda^{-1}\sigma_{ab}\lambda\} \\ &= -\mathrm{tr}\{[\lambda\Sigma\lambda^{-1}, \dot{\lambda}\lambda^{-1}]\sigma_{ab}\}.\end{aligned} \tag{53}$$

The covariant derivative of S_{ab} along the trajectory $x^\mu(\tau)$ is therefore given by

$$\frac{D}{D\tau}S_{ab} \stackrel{\text{def}}{=} \frac{d}{d\tau}S_{ab} - (S_{cb}\omega^c{}_{a\mu} + S_{ac}\omega^c{}_{b\mu})\dot{x}^\mu \\ = -\text{tr}\{[\Sigma, (\lambda^{-1}\dot{\lambda} + \lambda^{-1}(\omega_\mu \dot{x}^\mu)\lambda)]\lambda^{-1}\sigma_{ab}\lambda\}. \\ = -\text{tr}\{[\lambda\Sigma\lambda^{-1}, \dot{\lambda}\lambda^{-1} + \omega_\mu \dot{x}^\mu]\sigma_{ab}\}. \tag{54}$$

Similarly, from $k_a = m\text{tr}\{\kappa\lambda^{-1}\gamma_a\lambda\}$, we get

$$\frac{dk_a}{d\tau} = -\text{tr}\{[\lambda\kappa\lambda^{-1}, \dot{\lambda}\lambda^{-1}]\gamma_a\} \tag{55}$$

and hence

$$\frac{D}{D\tau}k_a = \frac{d}{d\tau}k_a - k_c\omega^c{}_{a\mu}\dot{x}^\mu \\ = -\text{tr}\{[\lambda\kappa\lambda^{-1}, \dot{\lambda}\lambda^{-1} + \omega_\mu\dot{x}^\mu]\gamma_a\}. \tag{56}$$

Now we introduce some one-forms that we will use to build the classical action functional for our particle. Let $e^* = \mathbf{e}^{*a}\gamma_a$ so we can write $\tilde{\mathbf{e}}^{*a} = [\Lambda^{-1}]^a{}_b\mathbf{e}^{*b}$ as $\tilde{e}^* = \lambda^{-1}e^*\lambda$. We use this to write

$$k_\mu dx^\mu = \text{tr}\{\kappa\lambda^{-1}e^*\lambda\} \stackrel{\text{def}}{=} \Omega_1. \tag{57}$$

which is to be considered as a one-form on the total space P, rather than on the base space M.

Next define

$$\tilde{\omega} = \frac{1}{2}\sigma_{ab}\tilde{\omega}^{ab} \stackrel{\text{def}}{=} \lambda^{-1}\left(d + \frac{1}{2}\sigma_{ab}\omega^{ab}\right)\lambda = \lambda^{-1}(d+\omega)\lambda. \tag{58}$$

This is again 1-form on the total space of the bundle π: $P \to M$. The $\tilde{\omega}^{ab}$ are zero on the horizontal subspace of $H \subset T(P)$ each point on the fibre, while the $\tilde{\mathbf{e}}^{*a}$ are zero on the vertical subspace of $V \subset T(P)$. We use these forms to define

$$\Omega_2 = \text{tr}\{\Sigma\lambda^{-1}(d+\omega)\lambda\}. \tag{59}$$

We take as the action functional

$$S[x,\lambda] = \int \Omega, \tag{60}$$

where

$$\Omega = \Omega_1 - \Omega_2, \tag{61}$$

and the integral is taken along the curve parametrized by τ. As shown in [14], the equations of motion are

$$i_X d\Omega = 0. \tag{62}$$

where X is a vector field tangential to the trajectory in P.

To compute $d\Omega_1$ we will assume that the spin connection is torsion free, so that

$$d\mathbf{e}^{*a} + \omega^a{}_b \wedge \mathbf{e}^{*b} = 0. \tag{63}$$

We can then use

$$[\sigma_{ab}, \gamma_c] = (\gamma_a\eta_{bc} - \gamma_b\eta_{ac}) \tag{64}$$

to see that

$$d\Omega_1 = d\text{tr}\{\kappa\lambda^{-1}e^*\lambda\} \\ = -\text{tr}\left\{\left[\lambda\kappa\lambda^{-1}, d\lambda\lambda^{-1} + \frac{1}{2}\sigma_{ab}\omega^{ab}\right]e^*\right\}. \tag{65}$$

For $d\Omega_2$ we need the matrix-valued Riemann curvature tensor

$$d\omega + \omega\wedge\omega = \frac{1}{2}\left(\frac{1}{2}\sigma_{ab}R^{ab}\right)_{\mu\nu}dx^\mu dx^\nu \equiv R, \tag{66}$$

and observe that if $\tilde{\omega} = \lambda^{-1}(d + \frac{1}{2}\sigma_{ab}\omega^{ab})\lambda$ we have

$$d\tilde{\omega} + \tilde{\omega}\wedge\tilde{\omega} = \lambda^{-1}\left(\frac{1}{2}\sigma_{ab}R^{ab}\right)\lambda \equiv \lambda^{-1}R\lambda. \tag{67}$$

Consequently

$$d\Omega_2 = d\text{tr}\left\{\Sigma\lambda^{-1}\left(d + \frac{1}{2}\sigma_{ab}\omega^{ab}\right)\lambda\right\} \\ = \text{tr}\{\lambda\Sigma\lambda^{-1}R)\} - \text{tr}\left\{\lambda\Sigma\lambda^{-1}\left(d\lambda\lambda^{-1} + \frac{1}{2}\sigma_{ab}\omega^{ab}\right)^2\right\}. \tag{68}$$

We will write $d\lambda\lambda^{-1} + \omega = \tilde{\omega}_\text{R} = \frac{1}{2}\sigma_{ab}\tilde{\omega}^{ab}_\text{R}$. (The subscript "R" is because $\tilde{\omega}^{ab}_\text{R}$ includes the right-invariant Maurer-Cartan form $d\lambda\lambda^{-1}$.) We note that e^{*a} and $\tilde{\omega}^{ab}_\text{R}$ are linearly independent and between them span $\text{T}^*(P)$.

We can evaluate the contractions $i_X d\Omega \equiv d\Omega(X)$ by using

$$e^*(X) = \dot{x}^a\gamma_a = \dot{x}^\mu e^{*a}_\mu\gamma_a, \\ d\lambda\lambda^{-1}(X) = \dot{\lambda}\lambda^{-1}, \\ R(X) = -\frac{1}{2}\sigma_{ab}R^{ab}{}_{\mu\nu}dx^\mu\dot{x}^\nu = -\frac{1}{2}\sigma_{ab}R^{ab}{}_{\mu\nu}\dot{x}^\nu e^\mu_a e^{*a}, \\ \omega(X) = \frac{1}{2}\sigma_{ab}\omega^{ab}{}_\mu\dot{x}^\mu. \tag{69}$$

Here $\dot{x}^\mu$ denotes $dx^\mu/d\tau$. We find that

$$
\begin{aligned}
i_X d\Omega_1 &= -\mathrm{tr}\{[\lambda\kappa\lambda^{-1}, \dot\lambda\lambda^{-1} + \omega_\mu \dot x^\mu]\gamma_a\} e^{*a} \\
&\quad + \mathrm{tr}\{\lambda\kappa\lambda^{-1}[\sigma_{ab}, \gamma_c]\}\dot x^c \tilde\omega^{ab}/2 \\
&= -\mathrm{tr}\{[\lambda\kappa\lambda^{-1}, \dot\lambda\lambda^{-1} + \omega_\mu \dot x^\mu]\gamma_a\} e^{*a} \\
&\quad + \mathrm{tr}\{\lambda\kappa\lambda^{-1}(\gamma_a\eta_{bc} - \gamma_b\eta_{ac})\dot x^c\}\frac{1}{2}\tilde\omega_R^{ab} \\
&= \left(\frac{Dk_a}{D\tau}\right) e^{*a} + (k_a \dot x_b - k_b \dot x_a)\frac{1}{2}\tilde\omega_R^{ab} \qquad (70)
\end{aligned}
$$

and

$$
\begin{aligned}
i_X d\Omega_2 &= -\mathrm{tr}\{\lambda\Sigma\lambda^{-1} R_{\mu\nu}\dot x^\nu e_a^\mu\} e^{*a} \\
&\quad - \mathrm{tr}\{\lambda\Sigma\lambda^{-1}, \dot\lambda\lambda^{-1} + \omega_\mu \dot x^\mu]\sigma_{ab}\}\tilde\omega_R^{ab}/2 \\
&= \left(-\frac{1}{2}S_{mn}R^{mn}{}_{\mu\nu} e_a^\mu \dot x^\nu\right) e^{*a} + \left(\frac{DS_{ab}}{D\tau}\right)\frac{1}{2}\tilde\omega_R^{ab}. \qquad (71)
\end{aligned}
$$

The contraction $i_X d\Omega$ is therefore a position-dependent combination of e^{*a} and $\tilde\omega_R^{ab}$. For it to be zero, we need the coefficients of these forms be separately zero. Requiring the vanishing of the coefficient of e^{*a} yields

$$
\frac{D}{D\tau}k_c + \frac{1}{2}S_{ab}R^{ab}{}_{\mu\nu}\dot x^\nu e_c^\mu = 0. \qquad (72)
$$

Similarly, the vanishing of the coefficient of $\tilde\omega_R^{ab}$ gives

$$
\frac{D}{D\tau}S_{ab} + \dot x_a k_b - k_a \dot x_b = 0. \qquad (73)
$$

These are the Mathisson-Papapetrou-Dixon [20–22] equations. The momentum equation (72) exhibits a gravitational analogue of the Lorentz force, while (73) expresses the conservation of total (spin and orbital) angular momentum. It is well known that to obtain a closed system these two equations have to be supplemented by a condition on the spin such as our Tulczyjew-Dixon condition $k^a S_{ab} = 0$. It is explained in Appendix D that this condition means that $x^\mu(\tau)$ is the worldline of the particle's center of mass.

Before we proceed there is a necessary consistency check. Our entire action principle is built on the assumption that $k^2 = m^2$ is fixed—but the RHS of (72) does not immediately seem to ensure that $k^a k_a$ is a constant of the motion. To verify that it is, we can write $k_a = m u_a$ where $u_a u^a = 1$. We then contract the both sides of the momentum equation with $v^c = \dot x^c$ and use the antisymmetry of the curvature tensor to see that

$$
m \dot x^a \dot u_a + \dot m \dot x^a u_a = 0. \qquad (74)
$$

Now from

$$
u_a S^{ab} = 0 \qquad (75)
$$

we get

$$
u_a \dot u_b \dot S^{ab} = -\dot u_a \dot u_b S^{ab} = 0, \qquad (76)
$$

and hence from the angular momentum equation we find that

$$
0 = u_a \dot u_b (k^a \dot x^b - \dot x^a k^b) = m(\dot u_b \dot x^b - \dot u_b u^b u_a \dot x^a) = m \dot u^b \dot x_b. \qquad (77)
$$

Thus $0 = \dot m(\dot x^a u_a)$ and the mass is indeed a constant of the motion. This constancy continues when we include a Lorentz force. It would *not* survive were we to include an explicit magnetic moment. In that case the action would need to be extended to accommodate a modified mass-shell condition [44].

B. The anomalous velocity due to spin

It is the Mathisson-Papapetrou-Dixon angular-momentum equation (73), with its implication that $\dot x^a$ is no longer parallel to k^a, that gives us the anomalous velocity. From Eq. (73) and the Tulczyjew-Dixon little-group condition $k^a S_{ab} = 0$ we deduce that

$$
-\frac{Dk^a}{D\tau}S_{ab} = k^2 \dot x_b - k_b(\dot x \cdot k). \qquad (78)
$$

or

$$
\dot x_a = \frac{1}{m^2}\left(k_a(\dot x \cdot k) + S_{ac}\frac{Dk^c}{D\tau}\right). \qquad (79)
$$

There are several things that we can do with this result.

Firstly, substituting (79) into the angular momentum conservation law (73) we find

$$
\frac{DS_{ab}}{D\tau} + \frac{1}{m^2}\left(S_{ac}k_b\frac{Dk^c}{D\tau} + S_{cb}k_a\frac{Dk^c}{D\tau}\right) = 0. \qquad (80)
$$

This is Fermi-Walker transport of the spin angular-momentum tensor along the trajectory whose tangent vector is k^μ/m rather than $\dot x^\mu$. Dixon [22] calls this *M-transport*.

Secondly we can find the "anomalous" correction to the relation between velocity and momentum. Up to now the parameter τ was arbitrary. The action is reparametrization invariant so τ does not have to be the proper time. If we change the parametrization $\tau \to t$ in such a manner that the vielbein component $\dot x_0$ becomes unity, then the remaining $\dot x_i$, $i = 1, \ldots, d-1$, are the components of the velocity "3"-vector in the local Lorentz frame $\mathbf{e}_a$. The first component of (79) now becomes

$$
1 = \frac{1}{m^2}\left\{(\dot x \cdot k)E + S_{0c}\frac{Dk^c}{Dt}\right\}, \qquad (81)
$$

or, rearranging,

$$(\dot{x}\cdot k) = \frac{m^2 + \dot{k}^a S_{a0}}{E}, \tag{82}$$

where

$$\dot{k}^a \stackrel{\text{def}}{=} \frac{Dk^a}{Dt} = \frac{dk^a}{dt} + \omega^{ab}{}_c k_b \dot{x}^c. \tag{83}$$

Again use i and j for space indices, observe that $k^0 = E = \sqrt{m^2 + \sum_{i=1}^3 k^i k^i}$ gives

$$\dot{k}^0 = \frac{\partial k^0}{\partial k^j}\dot{k}^j = \frac{k^j}{E}\dot{k}^j = -\frac{k_j}{E}\dot{k}^j, \tag{84}$$

and make use of the skew symmetry in a, b of the spin connection $\omega^{ab}{}_\mu$. We find that

$$\dot{x}_i = \frac{k_i}{E} + \frac{1}{m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)\frac{Dk^j}{Dt} \tag{85}$$

Equation (85) has a familiar structure! It looks just like the anomalous velocity equation (2) with

$$\Omega_{ij} \to \frac{1}{m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right). \tag{86}$$

Furthermore, the associated two-form

$$\frac{1}{2}\Omega_{ij}dk^i \wedge dk^j = \frac{1}{2m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)dk^i \wedge dk^j \tag{87}$$

looks very much like our matrix-valued covariant Berry-connection curvature tensor

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)_{\alpha\beta} dk^i \wedge dk^j, \tag{88}$$

which in three dimensions is

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2}\frac{1}{\gamma}\left\{\frac{1}{2}\left(\boldsymbol{\sigma} + \frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)}\right)_{\alpha\beta}\right\}\cdot(d\mathbf{k}\times d\mathbf{k}). \tag{89}$$

The quantity in braces is the lab-frame spin of a particle with polarization $\mathbf{s} = \boldsymbol{\sigma}/2$. It is therefore natural to identify the classical spin angular momentum S_{ab} with expectation value

$$\bar{\psi}\frac{i}{4}[\gamma_a, \gamma_b]\psi = C^{*\alpha}(S_{ab})_{\alpha\beta}C^\beta \tag{90}$$

of the matrix-valued connection evaluated in the WKB state $\psi = u_\alpha C^\alpha$. Were we to quantize by integrating over λ in a path integral, we would expect S_{ab} to correspond to the operator $(S_{ab})_{\alpha\beta}$ that acts in the spin-polarization Hilbert space.

C. Return to the Berry connection

Our classical action (60) leads to dynamical evolution of the elements λ of the noncompact Lorentz group $SO(1, d-1)$. In the previous work [7,13,14] the phase-space was parametrized by $\mathbf{x}$, $\mathbf{k}$, and elements of a *compact* rotation group. We can connect the apparently distinct formalisms by a simple reparametrization of our degrees of freedom. We factorize each element λ as

$$\lambda = \lambda_k \sigma, \tag{91}$$

where λ_k is a chosen k-dependent Lorentz transformation that takes us from the reference $\mathbf{e}_0$ to momentum k, and σ lies in the little group of $\mathbf{e}_0$. For massive particles this little group is $SO(d-1)$. The two one-forms composing the action (60) now become

$$\Omega_1 = k_\mu dx^\mu \tag{92}$$

and

$$\begin{aligned}\Omega_2 &= \mathrm{tr}\left\{\Sigma\lambda^{-1}\left(d - \frac{1}{2}\sigma_{ab}\omega^{ab}\right)\lambda\right\}\\ &= \mathrm{tr}\left\{\Sigma\sigma^{-1}\left(d + (\lambda_k^{-1}d\lambda_k) - \frac{1}{2}(\lambda_k^{-1}\sigma_{ab}\lambda_k)\omega^{ab}\right)\sigma\right\}.\end{aligned} \tag{93}$$

and the action $S[x, \lambda]$ becomes $S[x, k, \sigma]$. As Σ lies in the Lie algebra of little group, the trace operation projects the Lorentz Lie-algebra element $\lambda_k^{-1}d\lambda_k$ into the Lie algebra of the little-group. The projected element $P\lambda_k^{-1}d\lambda_k P \equiv -i\mathfrak{a}_i dk^i$ is essentially the non-Abelian Berry connection that produces parallel transport on the little group in the formalism of [7,13,14]. A gauge transformation on this Berry connection is a change of choice $\lambda_k \to \lambda_k\sigma_k$ for some k-dependent element σ_k of the little group. It is "essentially" the same connection rather than "precisely" the same because we have $\lambda_k^{-1}d\lambda_k$ rather than $\lambda_k^\dagger d\lambda_k$. The present parallel transport is therefore the nonunitary covariant connection that gives rise to Thomas precession. In [7,13,14] we are considering massless particles, and the Berry connection provides unitary parallel transport on the group $SO(d-2)$. Connecting this massless case to our present formalism requires a more detailed consideration that we supply in the next section.

IV. MASSLESS PARTICLES

When our particles are massless the situation becomes rather more complicated. Even in the free case—no gravity, no electromagnetic field, and hence $\dot{k}^a = 0$—the Mathisson-Papapetrou-Dixon angular momentum equation

$$\frac{dS_{ab}}{d\tau} + \dot{x}_a k_b - k_a \dot{x}_b = 0 \tag{94}$$

supplemented by the Tulczyjew-Dixon condition $S_{ab}k^b = 0$ fails to have a unique solution. Suppose that $k^2 = 0$ and S_{ab} satisfies $S_{ab}k^b = 0$, then

$$\tilde{S}_{ab} = S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p \tag{95}$$

still satisfies $\tilde{S}_{ab}k^b = 0$. Further, if S_{ab} and x_a satisfy (94) and we set

$$\tilde{x}_a = x_a + S_{pa}\Theta^p, \tag{96}$$

then $\tilde{S}_{ab}$, $\tilde{x}_a$ are also a solution of (94) for any time-dependent $\Theta^p(\tau)$. This multiplicity of solutions is related to the absence of a well-defined center of mass, and to the corresponding difficulty of defining a covariant spin angular-momentum tensor for massless particles.

That there is going to be problem in the massless case is signalled by the factors of $1/m^2$ in our Berry curvature tensors. Indeed we expect a problem defining the spin angular-momentum tensor itself: when ψ is a Dirac spinor of definite chirality, the tensor $S_{ab} = i\bar{\psi}[\gamma_a, \gamma_b]\psi/4$ is identically zero. To understand the spin of massless particles, we need to appreciate Wigner's observation [45] that the little group for massless particles is the Euclidean group $\mathrm{SE}(d-2)$, and not the naïvely expected $\mathrm{SO}(d-2)$.

For massless particles in d-dimensional Minkowski space we can take the reference-momentum einbein to be the null-vector

$$N^a = \underbrace{(1, 0, \ldots, 0, 1)}_{d}. \tag{97}$$

The Lie algebra of the little group of N^a consists of the σ_{ab} with $0 < a, b, < d-1$ that generate $\mathrm{SO}(d-2)$, together with

$$\pi_a \stackrel{\text{def}}{=} N^b \sigma_{ba} = \sigma_{0a} + \sigma_{(d-1)a}, \qquad 0 < a < d-1. \tag{98}$$

Indeed, we can check that

$$[\pi_a, N^b\gamma_b] = 0, \qquad 0 < a < d. \tag{99}$$

From

$$[\sigma_{ij}, \sigma_{mn}] = \eta_{jm}\sigma_{in} - \eta_{im}\sigma_{jn} - \eta_{jn}\sigma_{im} + \eta_{in}\sigma_{jm} \tag{100}$$

we find that

$$[\pi_a, \pi_b] = 0, \qquad [\sigma_{ab}, \pi_c] = \eta_{bc}\pi_a - \eta_{ac}\pi_b. \tag{101}$$

The π_a therefore behave like translations, and together with the rotations generate the Euclidean group $\mathrm{SE}(d-2)$. Wigner argues in [45] that the quantum states of all known particles must be unaffected by these "translations."

For example, consider the $3+1$ massless Dirac equation. For $N^a = (1, 0, 0, 1)$ we have

$$\pi_1 = -\frac{1}{2}\begin{bmatrix} i\sigma_2 & \sigma_1 \\ \sigma_1 & i\sigma_2 \end{bmatrix}, \qquad \pi_2 = \frac{1}{2}\begin{bmatrix} i\sigma_1 & -\sigma_2 \\ -\sigma_2 & i\sigma_1 \end{bmatrix}, \tag{102}$$

and both these "translation" operators act as zero on the relevant positive energy, positive and negaive chirality, states

$$u_+(N) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_-(N) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{103}$$

We can obtain a general null-momentum $k^a = (|\mathbf{k}|, \mathbf{k}) = e^s(1, \mathbf{n})$ by applying to N^a a rapidity-s boost parallel to the $\mathbf{e}_3$ direction, and then a rotation that takes $\mathbf{e}_3$ to the unit vector $\mathbf{n}$. In the Dirac representation, this procedure is implemented by

$$\lambda_k = \exp\{-i\phi\Sigma_3\}\exp\{-i\theta\Sigma_2\}\exp\{sK_3\}, \tag{104}$$

where θ and ϕ are the polar angles of the direction of the 3-momentum $\mathbf{k}$, and

$$\Sigma_i = \frac{1}{2}\begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}, \qquad K_i = \frac{1}{2}\begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \tag{105}$$

are respectively the rotation and boost generators. The resulting covariantly-normalized spinor positive chirality spinor is $u_+(\mathbf{k}) = \lambda_k u_+(N)$ is

$$u_+(\mathbf{k}) = e^{s/2}\begin{pmatrix} \chi \\ \chi \end{pmatrix}, \tag{106}$$

where

$$\chi(\mathbf{k}) = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}. \tag{107}$$

The Dirac-equation eigenstates are therefore safely indifferent to any Wigner translations in $\lambda_k \to \lambda_k\sigma$.

The same is not true of the classical angular momentum tensor $S_{ab} = \mathrm{tr}\{\Sigma\lambda^{-1}\sigma_{ab}\lambda\}$. If we replace

$$\lambda \to \lambda\exp\left(\sum_{i=1}^{d-2}\theta^i\pi_i\right) \tag{108}$$

then we have a transformation

$$\delta_\Theta\colon\ S_{ab} \to S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p \tag{109}$$

where $\Theta^p = \Lambda^p{}_i\theta^i$ and $k^a = \Lambda^a{}_b N^b$. Thus S_{ab} is affected by the unphysical Wigner translations in the same manner as in (95). The Wigner-translation operation differs from that in (95), however, in that the parameter Θ^p in (95) is arbitrary but the parameter in (109) must satisfy $\Theta^p k_p = 0$. This constraint follows from the relation $\Theta^p = \Lambda^p_i\theta^i$, and is necessary for two successive translations with parameters Θ_1^p and Θ_2^p to be equivalent to one with parameters $\Theta_1^p + \Theta_2^p$. In particular, a transformation that is allowed by (95) but not by (109) is given by $\Theta_0^p \equiv (-E^{-1}, 0, \ldots 0)$. It takes

$$\delta_{\Theta_0}\colon\ S_{ab} \to \left(S_{ab} - \frac{k_a}{E}S_{0b} - S_{a0}\frac{k_b}{E}\right).$$

In other words it takes the spin tensor and projects it to Pauli-Lubansky tensor. Any subsequent Wigner translation leaves the Pauli-Lubansky tensor invariant. This tensor therefore captures the physically significant part of the spin angular momentum.

A Wigner translation, when combined with the translation $x_a \to x_a + S_{pa}\Theta^p$, leaves the free action invariant even for time-dependent $\Theta^p(\tau)$. The Wigner translation group must therefore be regarded as a *gauge invariance* [46]. The gauge group is slightly larger than just the Wigner translations because the action on x_a is not Abelian. Again requiring $\Theta^p k_p = 0$, we find that

$$[\delta_{\Theta_2}, \delta_{\Theta_1}]x_a = 2\Theta_1^p\Theta_2^q S_{pq}k_a \tag{110}$$

This means that translations $x_a \to x_a + \epsilon k_a$ must also be included in the gauge group of the free action [46].

Being gauge variant, the position of x_a of the particle is not an observable. This seems like a disaster for any mathematical model that claims to describe the motion of a particle. All is not lost, however. What has happened is that a massless particle has no rest frame and therefore no observer-independent center of mass. As explained in Appendix D, it still has well-defined mass *centroids*, but the location of these centroids depends on the reference frame of the observer.

In our massless action, we are still free to fix a gauge, and so pin down a position for the particle. A natural gauge choice is to factorize $\lambda = \lambda_k\sigma$ where σ is chosen to be an element of $\mathrm{SO}(d-2)$. In other words, we deliberately excluding the problematic Wigner translations from our action. Once we do this the free action becomes

$$\int (k_\mu dx^\mu - \mathrm{tr}\{\Sigma\sigma^{-1}(d + \lambda_k^{-1}d\lambda_k)\sigma\}), \tag{111}$$

and this is of the same form as the action in [7,13,14] where the internal spin degree of freedom lives only in the rotation part of the little group. For example, in 3 + 1 dimensions we write

$$\sigma = \exp\{i\Sigma_3\varphi\} \tag{112}$$

and

$$\lambda = \lambda_k\sigma = \exp\{-i\phi\Sigma_3\}\exp\{-i\theta\Sigma_2\}\exp\{sK_3\}\exp\{i\varphi\Sigma_3\}. \tag{113}$$

If we take take $\Sigma = J\Sigma_3/4$ then

$$\begin{aligned}\Omega_2 &= \frac{1}{4}J\mathrm{tr}\{\Sigma_3\lambda^{-1}d\lambda\}\\ &= \frac{1}{4}J\mathrm{tr}\{\Sigma_3\sigma^{-1}(d + \lambda_k^{-1}d\lambda_k)\sigma\}\\ &= iJ\mathrm{tr}(d\varphi - \cos\theta d\phi)\end{aligned}$$

The $d\varphi$ is total derivative and does not affect the equation of motion. The $iJ\cos\theta d\phi$ term is precisely the Berry phase for a spin J particle. Our action therefore reduces to that in [7].

In general dimensions the gauge fixed action gives the anomalous velocity of the lab-frame centroid in terms of Wigner-translation invariant Pauli-Lubanski tensor.

$$\dot{x}_i = \frac{k_i}{E} + \frac{1}{E^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)\dot{k}^j. \tag{114}$$

In the massless case the Pauli-Lubanski tensor not only has vanishing time components (as does the massive case) but is also perpendicular to the space components of the momentum. This condition is the higher-dimensional analogue of the spin being slaved to the momentum.

The gauge-fixing is frame-dependent, and consequently the action is no longer manifestly Lorentz covariant. For complete covariance we need to allow λ to be any Lorentz transformation matrix—not only one that omits the Wigner translations. When we make a Lorentz transformation, we must therefore make a corresponding gauge transformation so as to restore the noncovariant gauge choice in the new frame. The gauge transformation involves the spacetime-translation in (96), and this translation corresponds to the relocation of the lab-frame mass centroid defined in (D12). We can understand the shift by simple kinematics: consider a massless particle with lab-frame 4-momentum and spin vector

$$\begin{aligned}k &= (|k|, \mathbf{k}) \equiv (|k|, 0, 0, |k|),\\ \mathbf{S} &= |S|(0, 0, 1).\end{aligned} \tag{115}$$

From a frame moving with rapidity s along the x^1 axis we see these vectors as

$$\begin{aligned}k' &= (|k|\cosh s, -|k|\sinh s, 0, |k|)\\ \mathbf{S}' &= |S|(-\tanh s, 0, \mathrm{sech} s),\end{aligned} \tag{116}$$

and so observe nonzero x^1 component of spin $S'^1 = -|S|\tanh s$. Since the x^1 component of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is unchanged by the boost, there must be a compensating orbital angular momentum component $L^1 = |S|\tanh s$. This can only arise from a sideways shift δx^2 of the particle's trajectory such that $L^1 = k^3 \delta x^2$. Thus we expect a transverse shift

$$\delta x^2 = \frac{|S|}{|k|}\tanh s. \tag{117}$$

It is straightforward, if a little tedious, to verify analytically that the energy centroid of a circularly-polarized gaussian light beam experiences a sideways displacement of exactly this amount when observed from a moving frame [47]. This shift is precisely the unusual Lorentz transformation uncovered in [16]. It is not just a mathematical artifact: the energy-centroid is where a photon detector at rest in this reference frame would locate the beam. For more discussions of the effect of rotations and boosts on light beams see [48,49].

If the change of beam direction under a small s boost is compensated for by a small rotation, the net effect is a small Wigner translation of the beam. A sequence of such combined boosts and rotations can translate the beam through an arbitrary amount. Consequently, being a physical shift, the Wigner translation gauge invariance is necessarily violated by beam stops and interactions. For example, if $\dot{k}_a$ is nonzero we find that the angular momentum conservation equation changes into

$$\begin{aligned}\frac{d\tilde{S}_{ab}}{d\tau} + \dot{\tilde{x}}_a k_b - k_a \dot{\tilde{x}}_b &= (\dot{k}_a S_{pb} - \dot{k}_b S_{pa})\Theta^p \\ &= ((x_a - \tilde{x}_a)\dot{k}_b - \dot{k}_a (x_b - \tilde{x}_b)).\end{aligned} \tag{118}$$

What has happened is that, with a nonzero net force, the external torque depends on the point about which moments are taken. The nonzero right-hand side of (118) is the torque about the new particle location $\tilde{x}_a$ due to the force acting at the old particle position x_a.

Once we are no longer allowed to make gauge transformations, the gauge-fixed theory and manifestly covariant theory are no longer exactly equivalent. As a consequence exact Lorentz invariance has been lost in the gauge-fixed theory. This may seem unsatisfactory, but it is to be expected. There are two related reasons. Firstly the proof cited in Appendix D, that the angular momentum of an extended body defined by (D2) is actually a Lorentz tensor depends crucially on there being no external force on the body. When we make a Lorentz transformation, the time-slice integral samples different epochs in the body's history, and the history-dependent momentum acquisition spoils the tensor property. This fact necessarily causes a problem for any point-particle approximation to the extended body unless the body is very compact and the external force small. The force being small is also a necessary condition for the validity of the adiabatic approximation which is a prerequisite for Berry transport. The adiabatic approximation also depends on the difference in energy between the $\pm|E|$ states being large compared to the inverse timescale of the change of the states. The point-particle actions used in [7,13,14] are therfore only applicable to particles with a large $E = |\mathbf{k}|$—but a Lorentz transformation can take a large $E = |\mathbf{k}|$ particle to one with arbitrarily small E. Therefore only we expect Lorentz invariance only under suitably "small" transformations.

V. DISCUSSION

We have seen that for massive particles Berry-phase-containing equations of motion such as (1) and (2) can be the three-dimensional reduction of a manifestly covariant equation of motion for the particle's center of mass. The same is not true for massless particles. In the absence of a rest frame in which to define the center of mass, the best we can do is derive an equation of motion for the lab-frame energy centroid of the particle, and when the massless particle is spinning the position of this energy centroid is observer dependent.

We can understand physically why the spin angular-momentum plays a central role in the anomalous velocity. A spinning object of mass m and acted on by a force $\mathbf{F}$ possesses a *hidden momentum* [50,51] of

$$\mathbf{P}^{\text{spin}}_{\text{hidden}} \approx -\frac{\mathbf{S}\times\mathbf{F}}{mc^2}. \tag{119}$$

(We have restored the factors of c to emphasize that this is a relativistic effect.) Therefore the total momentum of the body is given by

$$\mathbf{P}_{\text{tot}} \approx m\dot{\mathbf{x}} - \frac{\mathbf{S}\times\mathbf{F}}{mc^2}. \tag{120}$$

Identifying $\mathbf{P}_{\text{tot}}$ with $\mathbf{k}$, and $\mathbf{F}$ with $\dot{\mathbf{k}}$, gives us

$$\dot{\mathbf{x}} \approx \frac{\mathbf{k}}{m} + \frac{\mathbf{S}\times\dot{\mathbf{k}}}{m^2c^2}. \tag{121}$$

Now, at low speed, and taking into account that $\dot{x}_a = -\dot{x}^a$ and $k_a = -k^a$, our anomalous velocity equation (85) reduces to

$$\dot{\mathbf{x}} = \frac{\mathbf{k}}{m} + \frac{\mathbf{S}\times\dot{\mathbf{k}}}{m^2c^2}, \tag{122}$$

so anomalous velocity is precisely accounted for by the hidden momentum.

We have as yet explored only the effects of gravity, and have not included electromagnetic forces. In particular we have ignored the consequences of any intrinsic magnetic moment possessed by the particle. We imposed this restriction because a nonzero gyromagnetic ratio g requires us to modify the mass-shell condition to

$$|\mathbf{k}|^2 = m^2 + \frac{1}{2} e g S_{\mu\nu} F^{\mu\nu} \tag{123}$$

[38,44], and accepting this modification would obscure some of the geometric effects that we wished to display. However, in order to compare with other approaches to Dirac particle dynamics, we must now allow for a nonzero moment.

The first effect of a magnetic moment is to alter the relation between energy and momentum. For a massive particle at rest in three space dimensions, the modified mass-shell condition leads to

$$E = m\sqrt{1 - \frac{eg}{m^2}\mathbf{S}\cdot\mathbf{B}} = m - \frac{eg}{2m}\mathbf{S}\cdot\mathbf{B} + O(|\mathbf{B}|^2). \tag{124}$$

This is the usual energy shift due to a magnetic moment $\boldsymbol{\mu} = eg\mathbf{S}/2m$. In general dimensions, there is a similar shift but the vector $\boldsymbol{\mu}$ must be replaced by a skew symmetric tensor

$$\mathcal{M}_{ab} = \frac{ge}{2m} S_{ab}. \tag{125}$$

For a massless particle, we first need to replace the gauge dependent $S_{\mu\nu}$ by the Gauge invariant Pauli-Lubanski tensor whose $\mu, \nu = 0$ components vanish. Then we find

$$\begin{aligned} E(\mathbf{k}) &\sim |\mathbf{k}| - \frac{eg}{2|k|}\mathbf{S}\cdot\mathbf{B} + O(1/|\mathbf{k}|^2) \\ &= |\mathbf{k}| - \frac{e}{2|\mathbf{k}|}\hat{\mathbf{k}}\cdot\mathbf{B} + O(1/|\mathbf{k}|^2), \end{aligned} \tag{126}$$

where the second inequality applies to the $g = 2$, $S = \frac{1}{2}$ Dirac particle. This is the modified energy-momentum relation found in [16,52].

In the presence of a field induced torque, the angular-momentum conservation equation becomes [53]

$$\frac{D}{D\tau} S_{ab} + \dot{x}_a k_b - k_a \dot{x}_b = F_a{}^c \mathcal{M}_{cb} - \mathcal{M}_{ac} F^c{}_b, \tag{127}$$

and if we approximate (by ignoring terms higher in $|S|$)

$$\dot{k}_b = eF_{bc}\dot{x}^c \sim eF_{bc}k^c/m, \tag{128}$$

then we find that our anomalous-velocity equation is replaced by

$$\dot{x}_i = \frac{k_i}{E} + \left(1 - \frac{g}{2}\right)\frac{1}{m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)\dot{k}^j. \tag{129}$$

Again we can understand this *via* a hidden momentum. An accelerating magnetic dipole possesses a hidden momentum of

$$\mathbf{P}^{\mathrm{EM}}_{\mathrm{hidden}} = \frac{\boldsymbol{\mu}\times\mathbf{E}}{c^2} \tag{130}$$

[54,55], so

$$\mathbf{P}_{\mathrm{tot}} = m\dot{\mathbf{x}} + \frac{\boldsymbol{\mu}\times\mathbf{E}}{c^2} - \frac{\mathbf{S}\times\mathbf{F}}{mc^2}. \tag{131}$$

Once we set $\mu = ge/2m$ and $e\mathbf{E} = \dot{\mathbf{k}}$, we find that

$$\mathbf{k} = m\dot{\mathbf{x}} + \left(\frac{g}{2} - 1\right)\frac{1}{mc^2}\mathbf{S}\times\dot{\mathbf{k}}, \tag{132}$$

or

$$\dot{\mathbf{x}} = \frac{\mathbf{k}}{m} + \left(1 - \frac{g}{2}\right)\frac{1}{m^2c^2}\mathbf{S}\times\dot{\mathbf{k}}, \tag{133}$$

which is again consistent with the Mathisson-Papapetrou-Dixon equation modified to include the effect of the magnetic moment. That there is no anomalous-velocity correction when $g = 2$ is also a conclusion in [56].

The result in (133) does *not* coincide with the wave packet calculation in [19]. In [19] the anomalous velocity is entirely accounted for by the electromagnetic hidden momentum. There is no sign of the spin hidden momentum that is intimately connected with our Thomas precession curvature. This discrepancy is presumably due to the position "$\mathbf{x}$" in the massive Mathisson-Papapetrou-Dixon equations being the center of mass extracted from moments of the energy momentum tensor. The position "$\mathbf{x}$" in [19] is the center of charge or probability density of the wave packet. Since a Lorentz-boosted magnetic moment acquires an electric-dipole moment, the charge center will move away from the mass-centroid in a velocity-dependent manner and this momentum-dependent shift will also contribute to $\dot{\mathbf{x}}$. Whether this shift completely explains the discrepancy requires further study.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant No. NSF DMR 13-06011. In the course of this work M. S. has exchanged many useful emails with Peter Horváthy and Christian Duval. In particular we thank them for sharing with us early drafts of [57], which covers many of the topics in the present paper—but from a different yet complementary perspective. M. S. would also like to the Konstantin Bliokh for sanity-preserving discussions about the observer dependence of the location of light rays, and for drawing our attention to [49]. We also thank the authors of [16] for sending us an early version of their work.

APPENDIX A: SPINORS, POLARIZATION AND SPIN

In three dimensions we may take the Dirac gamma matrices to be

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \gamma^a = \begin{bmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{bmatrix},$$
$$\alpha^i = -\gamma_0\gamma_i = \gamma^0\gamma^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}. \tag{A1}$$

The eigenspinors with Lorentz-covariant normalization $\bar{u}_\alpha u_\beta = -\bar{v}_\alpha v_\beta = \delta_{\alpha\beta}$, $\bar{u}_\alpha v_\beta = \bar{v}_\alpha u_\beta = 0$ can be taken to be

$$u_\alpha(\mathbf{k}) = \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} (E+m)\chi_\alpha \\ (\boldsymbol{\sigma}\cdot\mathbf{k})\chi_\alpha \end{bmatrix},$$
$$v_\alpha(\mathbf{k}) = \frac{1}{\sqrt{2m(|E|+m)}} \begin{bmatrix} (\boldsymbol{\sigma}\cdot\mathbf{k})\chi_\alpha \\ (|E|+m)\chi_\alpha \end{bmatrix}, \tag{A2}$$

where χ_α are the unit two-spinors $\chi_1 = (1,0)^T$ and $\chi_2 = (0,1)^T$. The label α on u_α is therefore that of the spin in the rest frame of the particle, where

$$u_\alpha \to \begin{bmatrix} \chi_\alpha \\ 0 \end{bmatrix}. \tag{A3}$$

The spin in the particles's rest frame is usually called the "polarization," and is a more transparent quantity to work with than the lab-frame spin [30].

Define the spin generators

$$\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{A4}$$

and assemble the spatial parts into a spin three-vector $\Sigma = (\Sigma_{23}, \Sigma_{31}, \Sigma_{12})$ where

$$\Sigma = \frac{1}{2}\begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix}. \tag{A5}$$

We can now evalaute

$$\begin{aligned} u_\alpha^\dagger \Sigma u_\beta &= \frac{1}{2}\chi_\alpha^\dagger\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\sigma})}{(1+\gamma)}\right)\chi_\beta \\ &= \frac{1}{2}\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\sigma})}{(1+\gamma)}\right)_{\alpha\beta} \\ &= \frac{1}{2}\left(\boldsymbol{\sigma} + \frac{\gamma^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{\sigma})}{(1+\gamma)}\right)_{\alpha\beta}, \end{aligned} \tag{A6}$$

where $\boldsymbol{\beta} = \mathbf{v} = \mathbf{k}/E$ is the 3-velocity, and $\gamma = (1-|\boldsymbol{\beta}|^2)^{-1/2}$.

The physical meaning of the combination of $\boldsymbol{\sigma}$'s in parentheses in (A6) can be understood by defining a spin four-vector $(S^0, \mathbf{S})$ that takes the value $(0, \mathbf{s})$ particle's rest frame. Then, by performing a Lorentz transformation, we find that the corresponding lab-frame components are given by

$$\begin{aligned} \mathbf{S} &= \mathbf{s} + \frac{\gamma^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{s})}{(1+\gamma)} \\ S^0 &= \gamma\boldsymbol{\beta}\cdot\mathbf{s} = \boldsymbol{\beta}\cdot\mathbf{S}. \end{aligned} \tag{A7}$$

We see that $u_\alpha^\dagger \Sigma u_\beta$ coincides with the Lorentz transform of the matrix elements of the operator that measures the polarization $\mathbf{s}$.

Alternatively, we can define the Pauli-Lubansky spin four-vector operator

$$\mathfrak{S}^\kappa = \frac{1}{2}\epsilon^{\kappa\mu\nu\lambda}\Sigma_{\mu\nu}\left(\frac{k_\lambda}{m}\right), \qquad \epsilon^{1230} = 1, \tag{A8}$$

that reduces to $(0, \Sigma)$ in the particle's rest frame where $k^\mu = (m, \mathbf{0})$. Its three-space components are

$$\mathfrak{S}^i = \frac{1}{2}\gamma\epsilon^{ijk}\left(\Sigma_{jk} - \frac{k_i}{E}\Sigma_{0k} - \Sigma_{j0}\frac{k_k}{E}\right), \tag{A9}$$

and the time component is

$$\mathfrak{S}^0 = \gamma(\boldsymbol{\beta}\cdot\Sigma) = \boldsymbol{\beta}\cdot\mathfrak{S}. \tag{A10}$$

Because the matrix elements $\bar{u}_\alpha\Sigma_{\mu\nu}u_\beta$ transform as a tensor, and the matrix elements of the space components Σ_i and $\mathfrak{S}_i$ coincide in the particle's rest frame, we must have that

$$\frac{1}{\gamma}u_\alpha^\dagger\Sigma_i u_\beta = \bar{u}_\alpha\left\{\frac{1}{2}\epsilon^{ijk}\left(\Sigma_{jk} - \frac{k_i}{E}\Sigma_{0k} - \Sigma_{j0}\frac{k_k}{E}\right)\right\}u_\beta, \tag{A11}$$

as can be confirmed by direct calculation. The left-hand side of (A11) comprises the matrix elements of the 3-spin operator in a plane-wave beam normalized to one particle per unit volume in the lab frame. A physical interpretation of the right-hand side is provided in Appendix D.

Although the Pauli-Lubansky four-vector can only be defined in $3+1$ dimensions, the identity

$$\frac{1}{\gamma}u_\alpha^\dagger\Sigma_{ij}u_\beta = \bar{u}_\alpha\left(\Sigma_{ij} - \frac{k_i}{E}\Sigma_{0j} - \Sigma_{i0}\frac{k_j}{E}\right)u_\beta, \tag{A12}$$

is true in all dimensions. Equation (A12) follows from setting $\lambda = 0$, $\mu = i$, $\nu = j$ in the covariant identity

$$\frac{1}{2}\bar{u}_\alpha\{\gamma_\lambda, \Sigma_{\mu\nu}\}u_\beta = \frac{1}{m}\bar{u}_\alpha(k_\lambda\Sigma_{\mu\nu} - k_\mu\Sigma_{\lambda\nu} - \Sigma_{\mu\lambda}k_\nu)u_\beta. \tag{A13}$$

In turn, Eq. (A13) holds because the right- and left-hand sides are both totally antisymmetric tensors, whose components coincide in the rest frame of the particle. With the λ

index raised, the left-hand side of (A13) comprises the matrix elements of the spin-current tensor $S^{\lambda}{}_{\mu\nu}$.

APPENDIX B: GENERAL PROPERTIES OF DIRAC SPINORS AND MATRIX ELEMENTS

We collect some properties of the solutions $u_\alpha(k)$, $v_\alpha(k)$ to the plane-wave Dirac equation in any space-time dimension. The positive energy eigenvectors u_α, $\bar{u}_\alpha \equiv u_\alpha^\dagger \gamma_0$ satisfy

$$\begin{aligned}(\gamma_\mu k^\mu - m)u_\alpha &= 0,\\ \bar{u}_\alpha(\gamma_\mu k^\mu - m) &= 0.\end{aligned} \tag{B1}$$

The corresponding negative energy eigenvectors $v_\alpha(k)$, $\bar{v}_\alpha \equiv v_\alpha^\dagger \gamma_0$, obey

$$\begin{aligned}(\gamma_\mu k^\mu + m)v_\alpha &= 0,\\ \bar{v}_\alpha(\gamma_\mu k^\mu + m) &= 0.\end{aligned} \tag{B2}$$

In both cases the momenta lie on the positive energy mass shell. $k^2 = m^2$, $k^0 > 0$. We impose the Lorentz covariant normalization $\bar{u}_\alpha u_\beta = -\bar{v}_\alpha v_\beta = \delta_{\alpha\beta}$, $\bar{u}_\alpha v_\beta = \bar{v}_\alpha u_\beta = 0$. The completeness relation is therefore

$$\begin{aligned}\mathbb{I} &= u_\alpha \bar{u}_\alpha - v_\alpha \bar{v}_\alpha\\ &= \Lambda_+ + \Lambda_-,\end{aligned} \tag{B3}$$

where the projection operators are

$$\begin{aligned}\Lambda_+ &= \frac{1}{2m}(m + \gamma_\mu k^\mu) = u_\alpha \bar{u}_\alpha\\ \Lambda_- &= \frac{1}{2m}(m - \gamma_\mu k^\mu) = -v_\alpha \bar{v}_\alpha.\end{aligned} \tag{B4}$$

By varying the equation $\bar{u}_\alpha(\gamma_\mu k^\mu - m)u_\beta = 0$ and making use of the normalization conditions, we find the 4-current matrix elements

$$\bar{u}_\alpha \gamma^\mu u_\beta = \bar{v}_\alpha \gamma^\mu v_\beta = \delta_{\alpha\beta} k^\mu / m \equiv \delta_{\alpha\beta} V^\mu. \tag{B5}$$

Here $V^\mu = \gamma(1, \boldsymbol{\beta})$ is the 4-velocity derived from the group velocity $\boldsymbol{\beta} = \partial k^0 / \partial \mathbf{k}$.

Similarly, by varying (B1), (B2), we find that

$$\begin{aligned}\bar{v}_\alpha \delta u_\beta &= \frac{1}{2m}\delta k^\mu (\bar{v}_\alpha \gamma_\mu u_\beta),\\ \delta \bar{u}_\alpha v_\beta &= \frac{1}{2m}\delta k^\mu (\bar{u}_\alpha \gamma_\mu v_\beta).\end{aligned} \tag{B6}$$

We must keep k^μ on the mass-shell; consequently the δk^μ are not independent.

Now consider the covariant Berry connection $\mathfrak{a}_{\alpha\beta} = i\bar{u}_\alpha du_\beta$. From $\bar{u}_\alpha du_\beta + d\bar{u}_\alpha u_\beta = 0$, (B6), and the completeness relation, we find that the corresponding curvature is given by

$$\begin{aligned}\mathfrak{F}_{\alpha\beta} &\stackrel{\text{def}}{=} (d\mathfrak{a} - i\mathfrak{a}^2)_{\alpha\beta}\\ &= (id\bar{u}_\alpha du_\beta - d\bar{u}_\alpha u_\gamma \bar{u}_\gamma du_\beta)\\ &= i(d\bar{u}_\alpha u_\gamma \bar{u}_\gamma du_\beta - d\bar{u}_\alpha v_\gamma \bar{v}_\gamma du_\beta - d\bar{u}_\alpha u_\gamma \bar{u}_\gamma du_\beta)\\ &= -i(d\bar{u}_\alpha v_\gamma)(\bar{v}_\gamma du_\beta)\\ &= -i(\bar{u}_\alpha \gamma_\mu v_\gamma)(\bar{v}_\gamma \gamma_\nu u_\beta) dk^\mu \wedge dk^\nu / 4m^2\\ &= i(\bar{u}_\alpha \gamma_\mu \gamma_\nu u_\beta) dk^\mu \wedge dk^\nu / 4m^2\\ &= i(\bar{u}_\alpha [\gamma_\mu, \gamma_\nu] u_\beta) dk^\mu \wedge dk^\nu / 8m^2\\ &= \frac{1}{2m^2}(S_{\mu\nu})_{\alpha\beta} dk^\mu \wedge dk^\nu.\end{aligned} \tag{B7}$$

Another set of covariant matrix elements are the

$$(S_{\mu\nu})_{\alpha\beta} = \bar{u}_\alpha \Sigma_{\mu,\nu} u_\beta. \tag{B8}$$

that occur in (A11), (A12) and (A13). They play the role of the components of a covariant angular momentum tensor. We find directly from the Dirac equation, that they obey

$$k^\mu (S_{\mu\nu})_{\alpha\beta} = 0. \tag{B9}$$

APPENDIX C: COVARIANT BERRY TRANSPORT IS FERMI-WALKER TRANSPORT

Here we show that if we expand $\psi = u_\beta C^\beta$ and $\bar{\psi} = C^{*\alpha} \bar{u}_\alpha$ then the covariant parallel transport of the coefficients C^α leads to the angular momentum tensor

$$S^{\mu\nu} \equiv \frac{i}{4} \bar{\psi} \Sigma^{\mu\nu} \psi \tag{C1}$$

being Fermi-Walker transported along the trajectory.

Berry transport of the C^α means that

$$\delta C^\alpha = -(\bar{u}_\alpha \delta u_\beta) C^\beta. \tag{C2}$$

The states $u_\alpha(k)$ themselves change with k^μ so that

$$\delta u_\beta = u_\alpha \bar{u}_\alpha \delta u_\beta - v_\alpha \bar{v}_\alpha \delta u_\beta. \tag{C3}$$

Putting these two results together we have

$$\delta(u_\beta C^\beta) = -v_\alpha (\bar{v}_\alpha \delta u_\beta) C^\beta, \tag{C4}$$

and

$$\delta(C^{*\alpha} \bar{u}_\alpha) = -C^{*\alpha} (\delta \bar{u}_\alpha v_\beta) \bar{v}_\beta. \tag{C5}$$

We now use the formulas (B6) for $(\delta \bar{u}_\alpha v_\beta)$ and $(\bar{v}_\alpha \delta u_\beta)$ to find

$$
\begin{aligned}
4i\delta S^{\mu\nu} &= C^{*\alpha}\delta\bar{u}_\alpha v_\rho \bar{v}_\rho[\gamma^\mu,\gamma^\nu]\psi - \bar{\psi}[\gamma^\mu,\gamma^\nu]v_\sigma\bar{v}_\sigma\delta u_\beta C^\beta,\\
&= \frac{\delta k^\lambda}{2m}(\bar{\psi}\gamma_\lambda v_\rho\bar{v}_\rho[\gamma^\mu,\gamma^\nu]\psi - \bar{\psi}[\gamma^\mu,\gamma^\nu]v_\sigma\bar{v}_\sigma\gamma_\lambda\psi),\\
&= +\frac{\delta k^\lambda}{2m}(\bar{\psi}\gamma_\lambda v_\rho\bar{v}_\rho\gamma^\mu u_\sigma\bar{u}_\sigma\gamma^\nu\psi - (\mu\leftrightarrow\nu))\\
&\quad -\frac{\delta k^\lambda}{2m}(\bar{\psi}\gamma_\lambda v_\rho\bar{v}_\rho\gamma^\mu v_\sigma\bar{v}_\sigma\gamma^\nu\psi - (\mu\leftrightarrow\nu))\\
&\quad +\frac{\delta k^\lambda}{2m}(\bar{\psi}\gamma^\mu u_\rho\bar{u}_\rho\gamma^\nu v_\sigma\bar{v}_\sigma\gamma_\lambda\psi - (\mu\leftrightarrow\nu))\\
&\quad -\frac{\delta k^\lambda}{2m}(\bar{\psi}\gamma^\mu v_\rho\bar{v}_\rho\gamma^\nu v_\sigma\bar{v}_\sigma\gamma_\lambda\psi - (\mu\leftrightarrow\nu)). \qquad (C6)
\end{aligned}
$$

We can simplify by using the current matrix elements to get

$$
\begin{aligned}
\delta S^{\mu\nu} &= \frac{i}{4m^2}(k^\nu\delta k^\lambda\bar{\psi}[\gamma^\mu,\gamma_\lambda]\psi + k^\mu\delta k^\lambda\bar{\psi}[\gamma_\lambda,\gamma^\nu]\psi)\\
&= \frac{1}{m^2}(k^\nu\delta k^\lambda S^\mu{}_\lambda + k^\mu\delta k^\lambda S_\lambda{}^\mu) \qquad (C7)
\end{aligned}
$$

Thus we have found that

$$
\frac{\partial S^{\mu\nu}}{\partial\tau} + \frac{1}{m^2}\left(k^\nu\frac{\partial k^\lambda}{\partial\tau}S^\mu{}_\lambda + k^\mu\frac{\partial k^\lambda}{\partial\tau}S_\lambda{}^\nu\right) = 0. \qquad (C8)
$$

At the level of WKB, where we see no anomalous velocity, we have $V^\mu = k^\mu/m$. Consequently (C8) is

$$
\frac{\partial S^{\mu\nu}}{\partial\tau} + V^\nu\frac{\partial V^\lambda}{\partial\tau}S^\mu{}_\lambda + V^\mu\frac{\partial V^\lambda}{\partial\tau}S_\lambda{}^\nu = 0. \qquad (C9)
$$

Given that $V_\mu S^{\mu\nu} = (k_\mu/m)S^{\mu\nu} = 0$, Eq. (C9) is the statement that $S^{\mu\nu}$ is being Fermi-Walker transported.

APPENDIX D: CENTROIDS AND THE CENTER OF MASS

We review some standard material on centroids and centers of mass of extended bodies that should apply to wave-packets of Dirac particles. We work in flat space and suppose there are no external forces. Our extended body therefore possesses a conserved and compactly supported symmetric energy-momentum tensor

$$
\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\nu\mu}. \qquad (D1)
$$

Let $x^\mu_{\rm A}$ be a space-time event, Σ a spacelike surface, and define the angular momentum of the body about $x_{\rm A}$ by

$$
M^{\mu\nu}_{\rm A} = \int_\Sigma \{(x^\mu - x^\mu_{\rm A})T^{\nu\gamma} - (x^\nu - x^\nu_{\rm A})T^{\mu\gamma}\}d\Sigma_\gamma \qquad (D2)
$$

then ([58] page 161) $M^{\mu\nu}_{\rm A}$ is a tensor, and independent of the choice of Σ.

We now choose a lab frame and, with i, j running over space indices only, we define the energy and three-momentum of the body to be

$$
E = \int_{x^0=t} T^{00}d^3x, \qquad p^i = \int_{x^0=t} T^{i0}d^3x, \qquad (D3)
$$

respectively. We also define the *mass-centroid* $X^i_{\rm L}$ in the lab frame by

$$
\left\{\int_{x^0=t} T^{00}d^3x\right\}X^i_{\rm L} = \int_{x^0=t} x^iT^{00}d^3x. \qquad (D4)
$$

Now

$$
\partial_t\int_{x^0=t} T^{00}d^3x = \int_{x^0=t}\partial_0T^{00}d^3x = -\int_{x^0=t}\partial_jT^{j0}d^3x = 0, \qquad (D5)
$$

and

$$
\begin{aligned}
\partial_t\int_{x^0=t} x^iT^{00}d^3x &= \int_{x^0=t} x^i\partial_0T^{00}d^3x = -\int_{x^0=t} x^i\partial_jT^{j0}d^3x\\
&= \int_{x^0=t}\delta^i_jT^{j0}d^3x = p^i. \qquad (D6)
\end{aligned}
$$

So, differentiating its definition with respect to t, we read off that the ordinary 3-velocity of the centroid is

$$
\dot{\mathbf{X}}_{\rm L} = \mathbf{p}/E. \qquad (D7)
$$

Now take $x^\mu_{\rm A}$ to be point in the $x^0 = t$ surface. Then

$$
\begin{aligned}
M^{i0}_{\rm A} &= \int_{x^0=t}\{(x^i - x^i_{\rm A})T^{00} - (x^0 - x^0_{\rm A})T^{i0}\}d^3x\\
&= (X^i_{\rm L} - x^i_{\rm A})E.
\end{aligned}
$$

(The second term on the right in the first line is zero because $x^0 - x^0_{\rm A}$ is zero everywhere in the integral.) Thus M^{i0}_A is zero when A is the centroid in the lab frame. If we replace the lab frame with an inertial frame having 4-velocity V^μ we have that $M^{\mu\nu}_{\rm A}V_\nu = 0$ if and only if A is the mass centroid in that frame.

Define the *center of mass* $X^i_{\rm CM}$ to be the mass-centroid in the frame where $p^i = 0$, and the *intrinsic angular momentum* $S^{\mu\nu}$ to be the angular momentum about the center of mass. Thus $S^{\mu\nu}p_\nu = 0$ and we automatically have the Tulczyjew-Dixon condition.

Now, looking back at the definition of angular momentum, we see that if we change reference points we have

$$
M^{\mu\nu}_{\rm A} + x^\mu_{\rm A}p^\nu - x^\nu_{\rm A}p^\mu = M^{\mu\nu}_{\rm B} + x^\mu_{\rm B}p^\nu - x^\nu_{\rm B}p^\mu. \qquad (D8)
$$

Let us take $\mathbf{x}_{\rm A} = \mathbf{X}_{\rm CM}$ and $\mathbf{x}_{\rm B} = \mathbf{X}_{\rm L}$ to be the centroid in the lab frame. Then

$$S_{\rm A}^{\mu\nu} + X_{\rm CM}^{\mu} p^{\nu} - X_{\rm CM}^{\nu} p^{\mu} = M_{\rm L}^{\mu\nu} + X_{\rm L}^{\mu} p^{\nu} - X_{\rm L}^{\nu} p^{\mu}, \quad \text{(D9)}$$

and

$$S^{\mu\nu} + (X_{\rm CM}^{\mu} - X_{\rm L}^{\mu}) p^{\nu} - (x_{\rm CM}^{\nu} - X_{\rm L}^{\nu}) p^{\mu} = M_{\rm L}^{\mu\nu}. \quad \text{(D10)}$$

The lab-frame centroid condition gives us $M_{\rm L}^{i0} = 0$, and we have $(X_{\rm CM}^{0} - X_{\rm L}^{0}) = 0$, so

$$S^{0\nu} - (X_{\rm CM}^{\nu} - X_{\rm L}^{\nu}) E = 0. \quad \text{(D11)}$$

We write this as

$$(X_{\rm CM}^{\nu} - X_{\rm L}^{\nu}) = \frac{S^{0\nu}}{E}, \qquad \left(= \frac{1}{E^2} S^{\nu i} p_i \right) \quad \text{(D12)}$$

and find

$$M_{\rm L}^{\mu\nu} = \left(S^{\mu\nu} - S^{\mu 0} \frac{p^{\nu}}{E} - \frac{p^{\mu}}{E} S^{0\nu} \right). \quad \text{(D13)}$$

Thus we have a physical interpretation of the Pauli-Lubansky spin-tensor components that appears many times in this paper. It is the intrinsic angular momentum about the lab-frame centroid.

[1] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[2] R. Loganayagam, arXiv:1106.0277
[3] R. Loganayagam and P. Surówka, J. High Energy Phys. 04 (2013) 097.
[4] N. Banerjee, S. Dutta, S. Jain, R. Loganayagam, and T. Sharma, J. High Energy Phys. 03 (2013) 048.
[5] S. Dubovsky, L. Hui, and A. Nicolis, Phys. Rev. D **89**, 045016 (2014).
[6] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Phys. Rev. D **85**, 085029 (2012).
[7] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. **109**, 162001 (2012).
[8] F. D. M Haldane, Phys. Rev. Lett. **93**, 206602 (2004).
[9] D. T. Son and N. Yamamoto, Phys. Rev. D **87**, 085016 (2013).
[10] J-W. Chen, S. Pu, Q. Wang, and X-N. Wang, Phys. Rev. Lett. **110**, 262301 (2013).
[11] D. Xiao, J. Shi, and Q. Niu, Phys. Rev. Lett. **95**, 137204 (2005).
[12] C. Duval, Z. Horváth, P. A. Horváthy, L. Martina, and P. C. Stichel, Mod. Phys. Lett. B **20**, 373 (2006).
[13] M. Stone and V. Dwivedi, Phys. Rev. D **88**, 045012 (2013).
[14] V. Dwivedi and M. Stone, J. Phys. A **47**, 025401 (2014).
[15] D. T. Son and N. Yamamoto, Phys. Rev. Lett. **109**, 181602 (2012); arXiv:1203.2697.
[16] J-Y. Chen, D. T. Son, M. A. Stephanov, H-U. Yee, and Y. Yin, Phys. Rev. Lett. **113**, 182302 (2014).
[17] K. Yu. Bliokh, Europhys. Lett. **72**, 7(2005).
[18] A Bérard and H. Mohrbach, Phys. Lett. A **352**, 190 (2006).
[19] M-C. Chang and Q. Niu, J. Phys. Condens. Matter **20**, 193202 (2008).
[20] M. Mathisson, Acta Phys. Pol. **6**, 163 (1937); Gen. Relativ. Gravit. **42**, 1011 (2010).
[21] A. Papapetrou, Proc. R. Soc. A **209**, 248 (1951).
[22] W. G. Dixon, Proc. R. Soc. A **314**, 499 (1970).
[23] E. N. Adams and E. I. Blount, J. Phys. Chem. Solids **10**, 286 (1959); E. I. Blount, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, New York, 1962), Vol. 13, p. 305.
[24] G. Sundaram and Q. Niu, Phys. Rev. B **59**, 14915 (1999).
[25] Y. D. Chong, Phys. Rev. B **81**, 052303 (2010).
[26] P. A. Horváthy, Phys. Lett. A **359**, 705 (2006);
[27] We can include the force due to a magnetic moment in an inhomogeneous field by including a term $M^{\mu\nu} F_{\mu\nu}/2$ into the Hamiltonian, where $M^{\mu\nu}$ is the magnetic moment tensor.
[28] W. Pauli, Helv. Phys. Acta **5**, 179 (1932).
[29] S. I. Rubinov and J. B. Keller, Phys. Rev. **131**, 2789 (1963).
[30] M. Rosen, Il Nuovo Cimento **XXXIII**, 1667 (1964).
[31] J. Bolte and S. Keppler, Ann. Phys. (N.Y.) **274**, 125 (1999).
[32] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).
[33] R. G. Littlejohn and W. G. Flynn, Phys. Rev. A **44**, 5239 (1991).
[34] J. A. Rhodes and M. D. Semon, Am. J. Phys. **72**, 943 (2004).
[35] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), 3rd ed.
[36] H. Mathur, Phys. Rev. Lett. **67**, 3325 (1991).
[37] L. H. Thomas, Nature (London) **117**, 514 (1926).
[38] H. P. Künzle, J. Math. Phys. **13**, 739 (1972).
[39] A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, Phys. Lett. B **89**, 199 (1980).
[40] B. S. Skagerstam and A. Stern, Phys. Scr. **24**, 493 (1981).
[41] C. Duval and P. A. Horváthy, Ann. Phys. (N.Y.) **142**, 10 (1982).
[42] W. M. Tulczyjew, Acta Phys. Pol. **18**, 393 (1959).
[43] F. A. E. Pirani, Acta Phys. Pol. **15**, 389 (1956); Gen. Relativ. Gravit. **41**, 1215 (2009).
[44] C. Duval, Annales de l'Institut Henri Poincaré, Phys. Theor. **25**, 345 (1976).
[45] E. Wigner, Ann. Math. **40**, 149 (1939).
[46] M. V. Atre, A. P. Balachandran, and T. R. Govindarajan, Int. J. Mod. Phys. A **02**, 453 (1987).
[47] M. Stone, V. Dwivedi, and T. Zhou (to be published).
[48] A. Aiello, N. Lindlein, C. Marquardt, and G. Leuchs, Phys. Rev. Lett. **103**, 100401 (2009).
[49] K. Y. Bliokh and F. Nori, Phys. Rev. Lett. **108**, 120403 (2012).
[50] S. E. Gralla, A. I. Harte, and R. M. Wald, Phys. Rev. D **81**, 104012 (2010).
[51] L. F. Costa, C. Herdeiro, J. Natário, and M. Zilhão, Phys. Rev. D **85**, 024001 (2012).

[52] C. Manuel and J. M. Torres-Rincon, arXiv:1404.6409.
[53] J. M. Souriau, Annales de l'Institute Henri Poincaré A **20**, 315 (1974).
[54] S. Coleman and J. H. Van Vleck, Phys. Rev. **171**, 1370 (1968).
[55] L. Vaidman, Am. J. Phys. **58**, 978 (1990).
[56] C. Duval and P. Horváthy, Phys. Lett. B **594**, 402 (2004).
[57] C. Duval and P. Horváthy, arxiv:1406.0718.
[58] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).