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Thermodynamics of black holes in massive gravity

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We present a class of charged black hole solutions in an $(n+2)$-dimensional massive gravity with a negative cosmological constant, and study the thermodynamics and phase structure of the black hole solutions in both the grand canonical and canonical ensembles. The black hole horizon can have a positive, zero, or negative constant curvature characterized by the constant k. By using the Hamiltonian approach, we obtain conserved charges of the solutions and find that the black hole entropy still obeys the area formula and the gravitational field equation at the black hole horizon can be cast into a form similar to the first law of black hole thermodynamics. In the grand canonical ensemble, we find that the thermodynamics and phase structure depend on the combination $k-\mu^2/4+c_2m^2$ in the four-dimensional case, where μ is the chemical potential and c_2m^2 is the coefficient of the second term in the potential associated with the graviton mass. When it is positive, the Hawking-Page phase transition can happen; when as it is negative, the black hole is always thermodynamically stable with a positive capacity. In the canonical ensemble, the combination turns out to be $k+c_2m^2$ in the four-dimensional case. When it is positive, a first-order phase transition can happen between small and large black holes if the charge is less than its critical value. In the higher-dimensional $[(n+2)\geq 5]$ case, even when the charge is absent, the small/large black hole phase transition can also appear, and the coefficients for the third (c_3m^2) and/or fourth (c_4m^2) terms in the potential associated with the graviton mass in massive gravity can play the same role as that of the charge in the four-dimensional case.

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I. INTRODUCTION

In 1983, Hawking and Page [1] found that there is a phase transition between a Schwarzschild anti–de Sitter (AdS) black hole and a thermal gas in AdS space. In the literature, this phase transition is called the Hawking-Page (HP) phase transition. However, a similar phase transition does not exist for black holes in asymptotically flat or de Sitter spacetimes. The main reason for the existence of the Hawking-Page phase transition is as follows. In AdS space, there exists a minimal temperature below which there is a stable thermal gas solution but no black hole solution. However, above the minimal temperature, there are two black hole solutions with the same temperature: the black hole with the larger horizon radius is thermodynamically stable with a positive heat capacity, and the black hole with the smaller horizon radius is thermodynamically unstable with a negative heat capacity, behaving like a Schwarzschild black hole in an asymptotically flat spacetime. Thus beyond the minimal temperature, the thermal gas in AdS space will collapse to form the stable large black hole. This is just the Hawking-Page phase transition. Due to the AdS/CFT correspondence [2–4] [which says that a quantum gravity in AdS space is dual to a conformal field theory (CFT) living on the boundary of the AdS space], the Hawking-Page phase transition received another interpretation in the dual CFT side: the confinement/deconfinement phase transition of the dual gauge field theory [5].

Another remarkable difference between a black hole in AdS space and its counterpart in flat or de Sitter space is that the black holes in AdS space could have a Ricci flat or hyperbolic horizon, in addition to the sphere horizon. These black holes are usually called topological black holes in the literature [6–12] . It is quite interesting to note that the Hawking-Page phase transition does not happen for the AdS black holes with a Ricci flat or hyperbolic horizon [13].

Einstein's general relativity is a relativistic theory of gravity where the graviton is massless. A natural question is whether one can build a self-consistent gravity theory if the graviton is massive. It turns out that this is not a trivial

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matter. In Refs. [14–16], a class of nonlinear massive gravity theories were proposed, in which the ghost field is absent [17,18]. In this class of massive gravity, the energy-momentum is no longer conserved due to the breakdown of diffeomorphism invariance. Recently, Vegh [19] found a nontrivial black hole solution with a Ricci flat horizon in four-dimensional massive gravity with a negative cosmological constant [20]. He then found that the mass of the graviton can play the same role as the lattice in the holographic conductor model: the conductivity generally exhibits a Drude peak which approaches a delta function in the massless gravity limit. Some holographic consequences of the effect of the graviton mass in massive gravity were investigated in Refs. [21–25]. The propose of this paper is to generalize Vegh's black hole solution and to study the corresponding thermodynamical properties and phase structure of the black hole solutions.

The organization of this paper is as follows. In Sec. II we present the exact charged black hole solutions with any horizon topology in an $(n+2)$-dimensional massive gravity. In Sec. III we obtain thermodynamical quantities associated with the solution, and show that they obey the first law of black hole thermodynamics and the black hole entropy satisfies the area formula as in general relativity (GR). In particular—although the massive gravity is not diffeomorphism invariant—the equation of motion of the gravitational field at a black hole horizon can be cast into a form similar to the first law of black hole thermodynamics. In Sec. IV we study phase structure of a four-dimensional black hole in both the grand canonical and canonical ensembles. In Sec. V we discuss the case of a five-dimensional neutral black hole and show that there exists a first-order phase transition between small/large black holes, although in this case the electric charge is absent. We end the paper with conclusions in Sec. VI.[1]

II. THE BLACK HOLE SOLUTION

Let us consider the following action for an $(n+2)$-dimensional massive gravity with a Maxwell field and a negative cosmological constant [19]:

$$S=\frac{1}{2\kappa^2}\int d^{n+2}x\sqrt{-g}\left[R+\frac{n(n+1)}{l^2}-\frac{1}{4}F^2+m^2\sum_i^4 c_i\mathcal{U}_i(g,f)\right], \tag{1}$$

where f is a fixed symmetric tensor (usually called the reference metric), c_i are constants,[2] and $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the $(n+2)\times(n+2)$ matrix $\mathcal{K}^\mu{}_\nu \equiv \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$,

$$\begin{aligned}
\mathcal{U}_1 &= [\mathcal{K}],\\
\mathcal{U}_2 &= [\mathcal{K}]^2-[\mathcal{K}^2],\\
\mathcal{U}_3 &= [\mathcal{K}]^3-3[\mathcal{K}][\mathcal{K}^2]+2[\mathcal{K}^3],\\
\mathcal{U}_4 &= [\mathcal{K}]^4-6[\mathcal{K}^2][\mathcal{K}]^2+8[\mathcal{K}^3][\mathcal{K}]+3[\mathcal{K}^2]^2-6[\mathcal{K}^4].
\end{aligned} \tag{2}$$

The square root in $\mathcal{K}$ means $(\sqrt{A})^\mu{}_\nu(\sqrt{A})^\nu{}_\lambda = A^\mu{}_\lambda$ and $[\mathcal{K}]=\mathcal{K}^\mu{}_\mu$. The equations of motion turn out to be

$$\begin{aligned}
&R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\frac{n(n+1)}{2l^2}g_{\mu\nu}-\frac{1}{2}\left(F_{\mu\sigma}F_\nu{}^\sigma-\frac{1}{4}g_{\mu\nu}F^2\right)\\
&\quad+m^2\chi_{\mu\nu}=0,\\
&\quad\nabla_\mu F^{\mu\nu}=0,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\chi_{\mu\nu} = &-\frac{c_1}{2}(\mathcal{U}_1 g_{\mu\nu}-\mathcal{K}_{\mu\nu})-\frac{c_2}{2}(\mathcal{U}_2 g_{\mu\nu}-2\mathcal{U}_1\mathcal{K}_{\mu\nu}+2\mathcal{K}^2_{\mu\nu})\\
&-\frac{c_3}{2}(\mathcal{U}_3 g_{\mu\nu}-3\mathcal{U}_2\mathcal{K}_{\mu\nu}+6\mathcal{U}_1\mathcal{K}^2_{\mu\nu}-6\mathcal{K}^3_{\mu\nu})\\
&-\frac{c_4}{2}(\mathcal{U}_4 g_{\mu\nu}-4\mathcal{U}_3\mathcal{K}_{\mu\nu}+12\mathcal{U}_2\mathcal{K}^2_{\mu\nu}\\
&-24\mathcal{U}_1\mathcal{K}^3_{\mu\nu}+24\mathcal{K}^4_{\mu\nu}).
\end{aligned} \tag{4}$$

We are now looking for a static black hole solution with the metric ansatz

$$\begin{aligned}
&ds^2=-N^2(r)f(r)dt^2+f^{-1}(r)dr^2+r^2h_{ij}dx^idx^j,\\
&i,j=1,2,3,\cdots,n,
\end{aligned} \tag{5}$$

where $h_{ij}dx^idx^j$ is the line element for an Einstein space with constant curvature $n(n-1)k$. Without loss of generality, one may take $k=1$, 0, or -1, corresponding to a sphere, Ricci flat, or hyperbolic horizon for the black hole, respectively. Following and generalizing the ansatz in Ref. [19], we take the following reference metric:

$$f_{\mu\nu}=\mathrm{diag}(0,0,c_0^2h_{ij}), \tag{6}$$

where c_0 is a positive constant. With the reference metric (6), we have

[1]While this work was being prepared, Ref. [26] appeared in the archive; the authors investigated the Hawking-Page phase transition in a four-dimensional neutral black hole in a special class of massive gravity (with the c_2m^2 term in this paper) and found that the Hawking-Page phase transition exists even when the temperature goes down to zero.

[2]For a self-consistent massive gravity theory, all these coefficients might be required to be negative if $m^2>0$. However, in this paper we do not impose this limit, since in AdS space the fluctuations of some fields with negative squared masses could still be stable if the squared mass obeys the corresponding Breitenlohner-Freedman bounds.

$$\begin{aligned}\mathcal{U}_1 &= nc_0/r,\\ \mathcal{U}_2 &= n(n-1)c_0^2/r^2,\\ \mathcal{U}_3 &= n(n-1)(n-2)c_0^3/r^3,\\ \mathcal{U}_4 &= n(n-1)(n-2)(n-3)c_0^4/r^4.\end{aligned} \tag{7}$$

We see that in the four-dimensional case with $n = 2$, one has identically $\mathcal{U}_3 = \mathcal{U}_4 = 0$, while $\mathcal{U}_4 = 0$ in the five-dimensional case with $n = 3$.

For the metric (5), the Hamiltonian action for the gravitational part turns out to be

$$I_{\mathrm{Grav}} = -\frac{nV_n}{2\kappa^2}\int dtdrNU', \tag{8}$$

where a prime denotes the derivative with respect to r and

$$U = r^{n-1}(k-f) + \frac{r^{n+1}}{l^2} + m^2\left(\frac{c_0c_1}{n}r^n + c_0^2c_2r^{n-1} + (n-1)c_0^3c_3r^{n-2} + (n-1)(n-2)c_0^4c_4r^{n-3}\right),$$

where V_n is the volume of the space spanned by the coordinates x_i. On the other hand, when considering a static charged solution it is easy to see that the Hamiltonian action for the Maxwell field part is of the form

$$I_{\mathrm{Max}} = \frac{V_n}{2\kappa^2}\int dtdr\left[\frac{N}{2r^n}p^2 + Vp'\right], \tag{9}$$

where p is the conjugate momentum of A_r and $V = -A_t$. Combining Eqs. (8) and (9), we have the total Hamiltonian action of the system,

$$I_{\mathrm{total}} = -\frac{nV_n}{2\kappa^2}\int dtdr\left[N\left(U' - \frac{1}{2nr^n}p^2\right) - \frac{1}{n}Vp'\right]. \tag{10}$$

Varying the action with respect to N, U, V, and p, respectively, we obtain the equations of motion,

$$U' - \frac{1}{2nr^n}p^2 = 0,\qquad N' = 0,\qquad p' = 0,\qquad V' = \frac{N}{r^n}p. \tag{11}$$

Integrating the above equations, we have

$$N(r) = N_0,\qquad p = q,\qquad V = \mu - \frac{N_0}{(n-1)r^{n-1}}q,\qquad U(r) = m_0 - \frac{1}{2n(n-1)r^{n-1}}q^2, \tag{12}$$

where N_0, q, μ, and m_0 are all constants. Without loss of generality, one can set $N_0 = 1$ by rescaling the coordinate t. To require a vanishing static electric potential at a black hole horizon r_+, the chemical potential at infinity is

$$\mu = \frac{1}{(n-1)r_+^{n-1}}q. \tag{13}$$

As a result, the metric function $f(r)$ is

$$f(r) = k + \frac{r^2}{l^2} - \frac{m_0}{r^{n-1}} + \frac{q^2}{2n(n-1)r^{2(n-1)}} + \frac{c_0c_1m^2}{n}r + c_0^2c_2m^2 + \frac{(n-1)c_0^3c_3m^2}{r} + \frac{(n-1)(n-2)c_0^4c_4m^2}{r^2} \tag{14}$$

expressed in terms of the mass parameter m_0 and electric charge q, or

$$f(r) = k + \frac{r^2}{l^2} - \frac{m_0}{r^{n-1}} + \frac{(n-1)\mu^2r_+^{2(n-1)}}{2nr^{2(n-1)}} + \frac{c_0c_1m^2}{n}r + c_0^2c_2m^2 + \frac{(n-1)c_0^3c_3m^2}{r} + \frac{(n-1)(n-2)c_0^4c_4m^2}{r^2} \tag{15}$$

expressed in terms of the mass parameter m_0 and the chemical potential. We see that in the four-dimensional case, (taking $c_3 = c_4 = 0$ and $k = 0$) one recovers the solution found in Ref. [19].[3]

III. THERMODYNAMICAL QUANTITIES AND THE FIRST LAW

In this section we calculate the conserved charges associated with the black hole solution found in the previous section. Note that the action density (10) can be written as

$$I = -\frac{nV_n(t_2-t_1)}{2\kappa^2}\int dr\left[N\left(U' - \frac{1}{2nr^n}p^2\right) - \frac{1}{n}Vp'\right] + B, \tag{16}$$

where B is a surface term, which should be chosen so that the action has an extremum under a variation of the fields

[3]Note that here the coordinate r is different from the one in Ref. [19]: $r_{\mathrm{here}} = l/r_{\mathrm{there}}$.

with appropriate boundary conditions. One requires that the fields approach the classical solutions at infinity. Varying the action, one finds the boundary term

$$\delta B = (t_2 - t_1)(N_0 \delta M - \mu \delta Q) + B_0. \tag{17}$$

The boundary term B is the conserved charge associated with the "improper gauge transformation" produced by time evolution. The constant B_0 is determined by some physical consideration, e.g., the mass vanishes when a black hole horizon goes to zero. Here M and N_0 are a conjugate pair, and μ and Q are another conjugate pair. According to the Hamiltonian approach, we have the mass M and charge Q as

$$M = \frac{nV_n}{2\kappa^2} m_0,$$
$$Q = \frac{V_n}{2\kappa^2} q = \frac{(n-1)r_+^{n-1} V_n}{2\kappa^2} \mu. \tag{18}$$

The black hole horizon is determined by $f(r)|_{r=r_+} = 0$. Thus the mass M can be expressed in terms of the horizon radius r_+,

$$M = \frac{nV_n r_+^{n-1}}{2\kappa^2} \left(k + \frac{r_+^2}{l^2} + \frac{q^2}{2n(n-1) r_+^{2(n-1)}} + \frac{c_0 c_1 m^2}{n} r_+ + c_0^2 c_2 m^2 + \frac{(n-1)c_0^3 c_3 m^2}{r_+} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r_+^2} \right). \tag{19}$$

The Hawking temperature of the black hole can be easily obtained as

$$T = \frac{1}{4\pi r_+} \left((n-1)k + (n+1)\frac{r_+^2}{l^2} - \frac{q^2}{2n r_+^{2(n-1)}} + c_1 c_0 m^2 r_+ + (n-1)c_2 c_0^2 m^2 + \frac{(n-1)(n-2)c_3 c_0^3 m^2}{r_+} + \frac{(n-1)(n-2)(n-3)c_4 c_0^4 m^2}{r_+^2} \right) \tag{20}$$

by requiring the Euclidean time ($\tau = it$) to have a period $\beta = 4\pi / f'(r)|_{r=r_+}$, so that the potential conical singularity at the black hole horizon is moved and the period just gives the inverse Hawking temperature of the black hole. It is easy to see that the black hole entropy obeys the area formula, which gives

$$S = \frac{2\pi V_n}{\kappa^2} r_+^n. \tag{21}$$

We then obtain the following first law of black hole thermodynamics:

$$dM = TdS + \mu dQ. \tag{22}$$

Now let us notice an interesting property of the gravitational field equations at a black hole horizon. The first equation in Eq. (11) can be written as

$$(n-1)r^{n-2}(k-f) - r^{n-1}f' + (n+1)\frac{r^n}{l^2} + m^2(c_0 c_1 r^{n-1} + (n-1)c_0^2 c_2 r^{n-2}$$
$$(n-1)(n-2)c_0^3 c_3 r^{n-3} + (n-1)(n-2)(n-3)c_0^4 c_4 r^{n-4}) = \frac{1}{2nr^n} q^2, \tag{23}$$

On the black hole horizon, one has $f(r)|_{r=r_+} = 0$, and the black hole temperature

$$T = \frac{1}{4\pi} f'|_{r=r_+}. \tag{24}$$

By multiplying both sides of Eq. (23) by $nV_n/(2\kappa^2)dr_+$, we see that it can be rewritten as

$$dM - TdS - \mu dQ = 0, \tag{25}$$

where S, M, and Q are the entropy in Eq. (21), and the mass and charge in Eq. (18), respectively. Thus we have shown that the Hamiltonian constraint—or rather, the $t-t$ component of the gravitational field equations—on the horizon can be cast into a form similar to the first law of black hole thermodynamics (like in GR), although the massive gravity manifestly breaks the diffeomorphism invariance (like in Horava-Lifshitz gravity). For the latter, we have also shown that the equation of motion of the gravitational field at the black hole horizon can be written as a form of the first law of black hole thermodynamics [27].

IV. PHASE STRUCTURE OF FOUR-DIMENSIONAL BLACK HOLES

In this section we focus on the four-dimensional case. In this case, we have $\mathcal{U}_3 = \mathcal{U}_4 = 0$ ($c_3 = c_4 = 0$) and the metric function $f(r)$ becomes

$$f(r) = k + \frac{r^2}{l^2} - \frac{m_0}{r} + \frac{q^2}{4r^2} + \frac{c_1 m^2}{2} r + c_2 m^2, \tag{26}$$

where without loss of generality we have set $c_0 = 1$. Note that the vacuum solution with $m_0 = q = 0$ is

$$f_0(r) = k + \frac{r^2}{l^2} + \frac{c_1 m^2}{2} r + c_2 m^2, \tag{27}$$

which is not an AdS space unless $m^2 = 0$. Note that the black hole mass (18) is defined with respect to the vacuum solution.

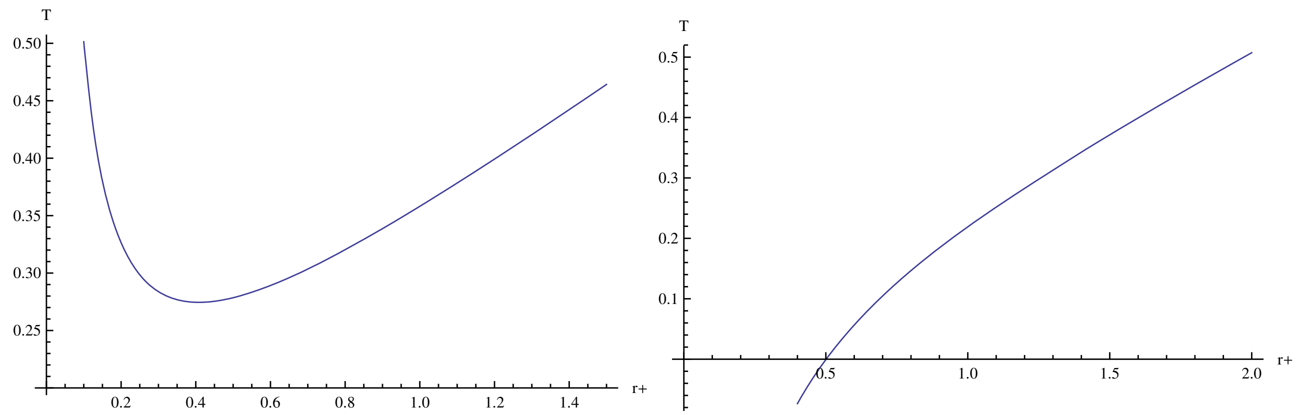


FIG. 1 (color online). The temperature T of the black hole with $k=-1$ versus horizon radius r_+. Here we take $l=c_1m^2=1$. Left: $\tilde{\mu}^2=-6$. Right: $\tilde{\mu}^2=1$.

A. Grand canonical ensemble

In a grand canonical ensemble with a fixed chemical potential μ associated with the charge Q, the Gibbs free energy is[4]

$$\begin{aligned} G_4 &= M - TS - \mu Q, \\ &= \frac{V_2 r_+}{2\kappa^2}\left(k - \frac{r_+^2}{l^2} + c_2 m^2 - \frac{1}{4}\mu^2\right). \end{aligned} \tag{28}$$

The Hawking temperature can be written as

$$T_4 = \frac{1}{4\pi r_+}\left(k + 3\frac{r_+^2}{l^2} - \frac{1}{4}\mu^2 + c_1 m^2 r_+ + c_2 m^2\right), \tag{29}$$

and the heat capacity with a fixed chemical potential is given by

$$C_\mu = T\left(\frac{dS}{dT}\right)\bigg|_\mu = \frac{(4\pi)^2 V_2 T r_+^3}{\kappa^2(-k + 3\frac{r_+^2}{l^2} + \frac{\mu^2}{4} - c_2 m^2)}. \tag{30}$$

From these thermodynamical quantities, we see that in this grand canonical ensemble, the term $\mu^2 - 4c_2m^2$ behaves as an effective chemical potential $\tilde{\mu}^2 = \mu^2 - 4c_2m^2$ for a charged black hole in GR. We now discuss the cases of $k=-1$, 0, and 1, respectively.

1. The case of $k=-1$

In this case, if $c_2 \le 0$, we see that $\tilde{\mu}^2$ is always positive, and the Gibbs free energy is always negative,[5] while the Gibbs free energy changes its sign at

$$r_{gc} = l\sqrt{-\tilde{\mu}^2/4 - 1}, \tag{31}$$

if $c_2 > 0$, so that $-\tilde{\mu}^2 > 4$. Namely, when $r_+ < r_{gc}$, one has $G_4 > 0$, while $G_4 < 0$ when $r_+ > r_{gc}$. In other words, the Hawking-Page phase transition does not exist if $c_2 < 0$, while it does if $4c_2m^2 - \mu^2 > 4$. The Hawking-Page phase transition temperature is given by

$$T_{\rm HP} = \frac{1}{4\pi r_{gc}}\left(2\frac{r_{gc}^2}{l^2} + c_1 m^2 r_{gc}\right). \tag{32}$$

To have a positive $T_{\rm HP}$, we see that c_1m^2 has to satisfy $c_1m^2 > -2r_{gc}/l^2$.

The local thermodynamical stability of the black hole is determined by the heat capacity (30). We see that C_μ is always positive if $c_2 \le 0$; however, if $c_2 > 0$ so that $-\tilde{\mu}^2 > 4$, one has $C_\mu < 0$ when $r_+ < r_{\mu c}$, and $C_\mu > 0$ when $r_+ > r_{\mu c}$, which diverges at

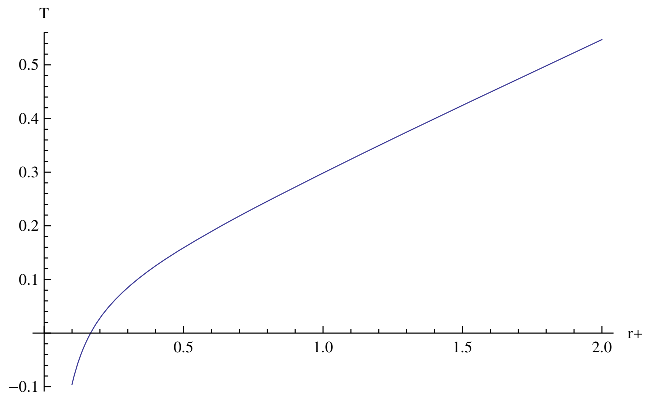
$$r_{\mu c} = l\sqrt{(-\tilde{\mu}^2/4 - 1)/3} = r_{gc}/\sqrt{3}. \tag{33}$$

The divergence point corresponds to the minimal temperature (see the left plot of Fig. 1). We see that in this case, the temperature behavior of the black hole with a hyperbolic horizon is qualitatively the same as that of a Schwarzschild AdS black hole in GR: there exists a minimal temperature, and the smaller black holes with $r_+ < r_{\mu c}$ are unstable with a negative heat capacity, while the larger black holes with $r_+ > r_{\mu c}$ are stable with a positive heat capacity.

In addition, we see from the temperature (29) that there exists a minimal horizon radius r_m,

$$r_m = \frac{c_1 m^2 l^2}{6}\left[-1 \pm \sqrt{1 + \frac{12}{l^2 c_1^2 m^4}\left(1 + \frac{1}{4}\tilde{\mu}^2\right)}\right], \tag{34}$$

[4]Note that the Gibbs free energy (28) is the same as the Euclidean action difference of the black hole and the corresponding vacuum solution multiplied by the black hole temperature.
[5]In this paper we always assume $m^2 > 0$.



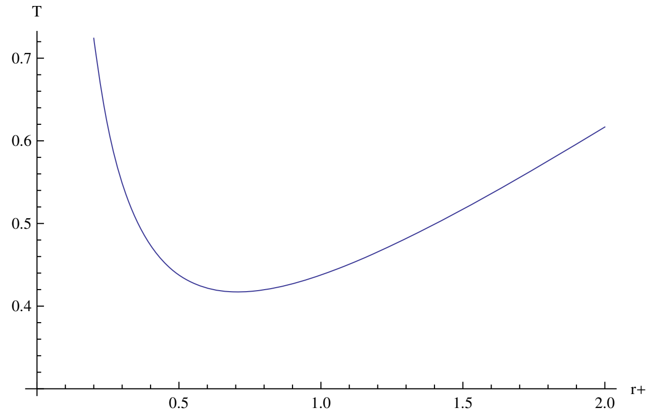


FIG. 2 (color online). The temperature T of the black hole with $k=0$ versus horizon radius r_+. Here we take $l = c_1 m^2 = 1$. Left: $\tilde{\mu}^2 = 1$. Right: $\tilde{\mu}^2 = -6$.

where the Hawking temperature vanishes; it corresponds to an extremal black hole (see the right plot of Fig. 1). Note that as $c_1 \geq 0$, one requires $1 + \tilde{\mu}^2/4 \geq 0$ in order to have a real root $r_m \geq 0$ [in this case, one takes "+" in Eq. (34)]. On the other hand, if $c_1 < 0$, the condition $1 + \tilde{\mu}^2/4 + l^2 c_1^2 m^4/12 \geq 0$ has to be satisfied [in this case, one takes "−" in Eq. (34)]. Furthermore, if $c_1 = 0$, the minimal horizon radius is $r_m = l\sqrt{(\tilde{\mu}^2/4 + 1)/3}$.

As a summary we find that if both $c_1 < 0$ and $c_2 < 0$, the situation is qualitatively the same as in GR: the black hole is not only globally thermodynamically stable with $G_4 < 0$, but it is also locally thermodynamically stable with $C_\mu > 0$, i.e., the minimal horizon radius (34) exists. However, if $c_2 > 0$ so that $c_2 m^2 > \mu^2/4$, the thermodynamical behavior of the black hole is qualitatively the same as the Schwarzschild AdS black hole: there does exist a minimal temperature and the Hawking-Page phase transition happens, although the black hole discussed here has a hyperbolical horizon.

2. The case of $k = 0$

In this case, the Gibbs free energy is always negative if $c_2 \leq 0$. But when $c_2 > 0$ so that $-\tilde{\mu}^2 > 0$, the Gibbs free energy is positive for small horizon radius $r_+ < r_{gc}$ and negative for $r_+ > r_{gc}$, and it changes its sign at

$$r_{gc} = l\sqrt{-\tilde{\mu}^2/4}. \tag{35}$$

Namely, if $c_2 > 0$, the Hawking-Page phase transition can happen at $r_+ = r_{gc}$.

The extremal black hole exists with a horizon radius

$$r_m = \frac{c_1 m^2 l^2}{6}\left(-1 \pm \sqrt{1 + \frac{3}{l^2 c_1^2 m^4}\tilde{\mu}^2}\right). \tag{36}$$

If $c_1 \geq 0$, we take "+" in the above equation and require $\tilde{\mu}^2 > 0$; if $c_1 < 0$, we take "−" and require $1 + 3\tilde{\mu}^2/(l^2 c_1^2 m^4) > 0$; and if $c_1 = 0$, the minimal horizon radius is $r_m = l\sqrt{\tilde{\mu}^2/12}$ (see the left plot of Fig. 2).

If $\mu < 2m\sqrt{c_2}$, there exists a minimal temperature at

$$r_{\mu c} = l\sqrt{(c_2 m^2 - \mu^2/4)/3} = r_{gc}/\sqrt{3}. \tag{37}$$

When $r_+ < r_{\mu c}$ the black hole has a negative heat capacity, and it is positive when $r_+ > r_{\mu c}$. The heat capacity diverges at $r_+ = r_{\mu c}$ (see the right plot of Fig. 2).

As a summary, if $c_1 < 0$ and $c_2 < 0$, the situation is qualitatively the same as in GR: the black hole is not only globally stable with $G_4 < 0$, but also locally stable with $C_\mu > 0$, i.e., the extremal black hole with the minimal horizon radius (36) exists. However, if $c_2 > 0$ so that $-\tilde{\mu}^2 > 0$, once again the situation qualitatively behaves like the Schwarzschild AdS black hole, although now the black hole discussed here has a Ricci flat horizon.

3. The case of $k = 1$

In this case, we can see from Eq. (28) that the Gibbs free energy changes its sign at

$$r_{gc} = l\sqrt{1 - \tilde{\mu}^2/4}, \tag{38}$$

which means that the Hawking-Page phase transition happens at $r_+ = r_{gc}$ [26] if $1 - \tilde{\mu}^2/4 > 0$. The Gibbs free energy is positive for small black holes with $r_+ < r_{gc}$ and negative for large black holes with $r_+ > r_{gc}$. Of course, if $1 - \tilde{\mu}^2/4 < 0$, the Hawking-Page phase transition does not appear.

From Eq. (30) we see that if $1 - \tilde{\mu}^2/4 > 0$, there exists a minimal temperature at

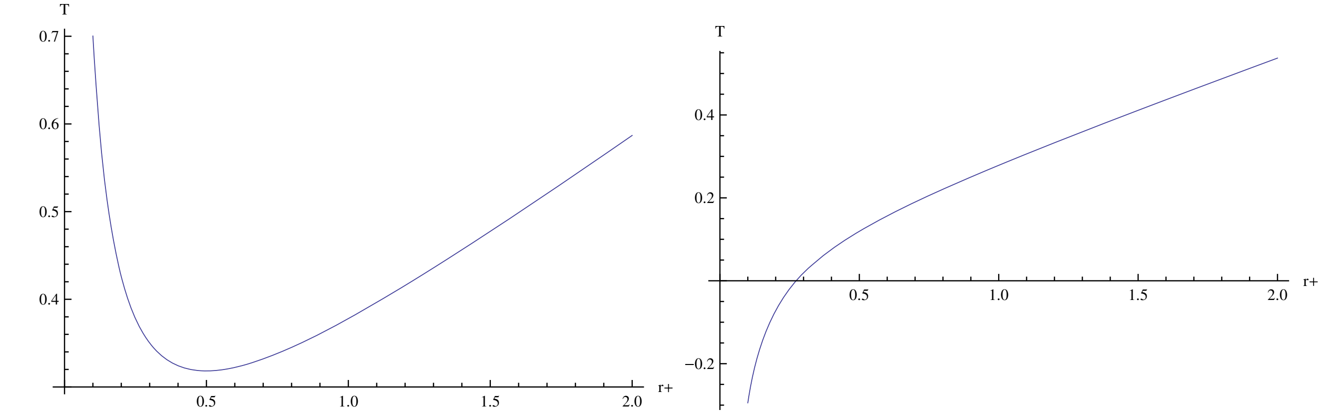


FIG. 3 (color online). The temperature T of the black hole with $k = 1$ versus horizon radius r_+. Here we take $l = c_1 m^2 = 1$. Left: $\tilde{\mu}^2 = 1$. Right: $\tilde{\mu}^2 = 6$.

$$r_{\mu c} = l\sqrt{(1 - \tilde{\mu}^2/4)/3} = r_{gc}/\sqrt{3}. \tag{39}$$

When $r_+ < r_{\mu c}$ the heat capacity is negative, and it is positive when $r_+ > r_{\mu c}$. The heat capacity diverges at $r_+ = r_{\mu c}$ (see the left plot of Fig. 3).

On the other hand, if $1 - \tilde{\mu}^2/4 < 0$ the extremal back hole solution with a vanishing Hawking temperature exists and its horizon radius is given by

$$r_m = \frac{l^2 c_1 m^2}{6}\left[-1 \pm \sqrt{1 - \frac{12}{l^2 c_1^2 m^4}\left(1 - \frac{1}{4}\tilde{\mu}^2\right)}\right]. \tag{40}$$

We see that if $c_1 > 0$, we take the "+" sign in Eq. (40); if $c_1 < 0$, we take the "−" sign; and if $c_1 = 0$, we have $r_m = l\sqrt{\tilde{\mu}^2/4 - 1}$ (see the right plot of Fig. 3).

Thus we see that when both $c_1 < 0$ and $c_2 < 0$, the situation is qualitatively the same as the case of a Schwarzschild AdS black hole in GR: the black hole is locally unstable for small black holes with $r_+ < r_{\mu c}$ and it is stable for large black holes with $r_+ > r_{\mu c}$, and thus the Hawking-Page phase transition happens at $r_+ = r_{gc}$. The Hawking-Page phase transition does not exist if the effective potential $\tilde{\mu}^2/4 > 1$.

To summarize the three cases, we find that the thermodynamics and phase structures of the black holes crucially depend on the combination $k - \mu^2/4 + c_2 m^2$. When it is positive, there is a minimal temperature: the black hole with the smaller (larger) horizon is thermodynamically unstable (stable), and the Hawking-Page phase transition can happen for any topological horizon ($k = 1, 0, -1$). This is quite different from the case in GR, where the Hawking-Page phase transition can only appear when $k = 1$. When the combination is negative, the black hole is always thermodynamically stable and no phase transition can happen.

B. Canonical ensemble

In a canonical ensemble with a fixed charge Q, the Helmholtz free energy is[6]

$$F_4 = M - TS = \frac{V_2 r_+}{2\kappa^2}\left(k - \frac{r_+^2}{l^2} + c_2 m^2 + \frac{3q^2}{4r_+^2}\right), \tag{41}$$

the associated heat capacity is given by

$$C_Q = T\left(\frac{dS}{dT}\right)\bigg|_Q = \frac{(4\pi)^2 V_2 T r_+^3}{\kappa^2(-k + 3\frac{r_+^2}{l^2} + \frac{3q^2}{4r_+^2} - c_2 m^2)}, \tag{42}$$

and the black hole temperature can be expressed in terms of the horizon radius and charge as

$$T_4 = \frac{1}{4\pi r_+}\left(k + 3\frac{r_+^2}{l^2} - \frac{1}{4}\frac{q^2}{r_+^2} + c_1 m^2 r_+ + c_2 m^2\right). \tag{43}$$

We see from these thermodynamical quantities that we can combine k and $c_2 m^2$ to create an effective horizon curvature, $\tilde{k} = k + c_2 m^2$.

From Eq. (41) we see that the Helmholtz free energy is positive for small black holes with $r_+ < r_{fc}$, is negative for large black holes with $r_+ > r_{fc}$, and changes sign at $r_+ = r_{fc}$, where

$$r_{fc}^2 = \frac{l^2}{2}\left(\tilde{k} + \sqrt{\tilde{k}^2 + \frac{3q^2}{l^2}}\right). \tag{44}$$

[6]If we define an extremal black hole with charge q as the reference background [28], then the Helmholtz free energy should be changed to $\tilde{F}_4 = F_4 - M_{\rm ext}$, where $M_{\rm ext}$ is the mass of the extremal black hole with vanishing Hawking temperature. The conclusions are not qualitatively different if one takes F_4 instead of $\tilde{F}_4$, and therefore we consider F_4 here for simplicity.

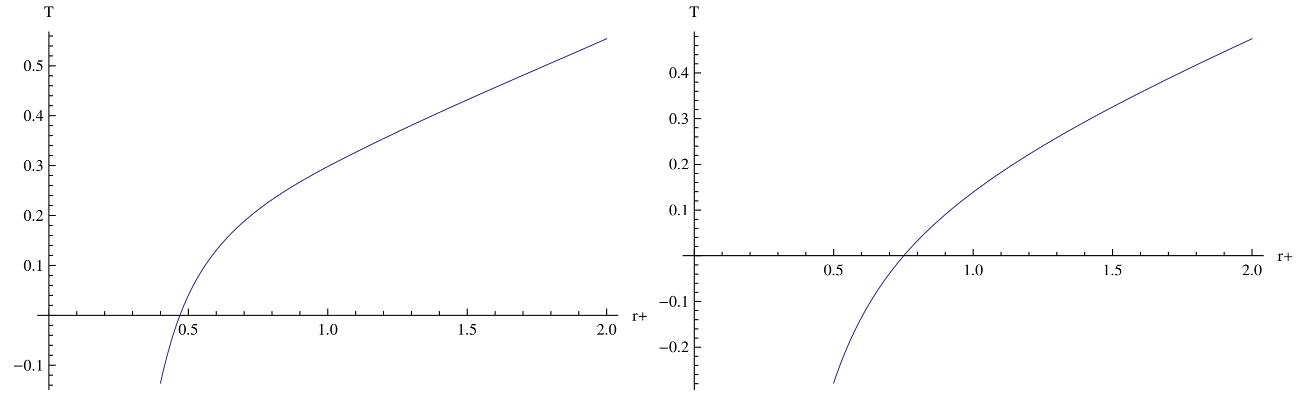


FIG. 4 (color online). The temperature T of the black hole with $q = 1$ versus horizon radius r_+. Here we take $l = c_1 m^2 = 1$. Left: $\tilde{k} = 0$. Right: $\tilde{k} = -2$.

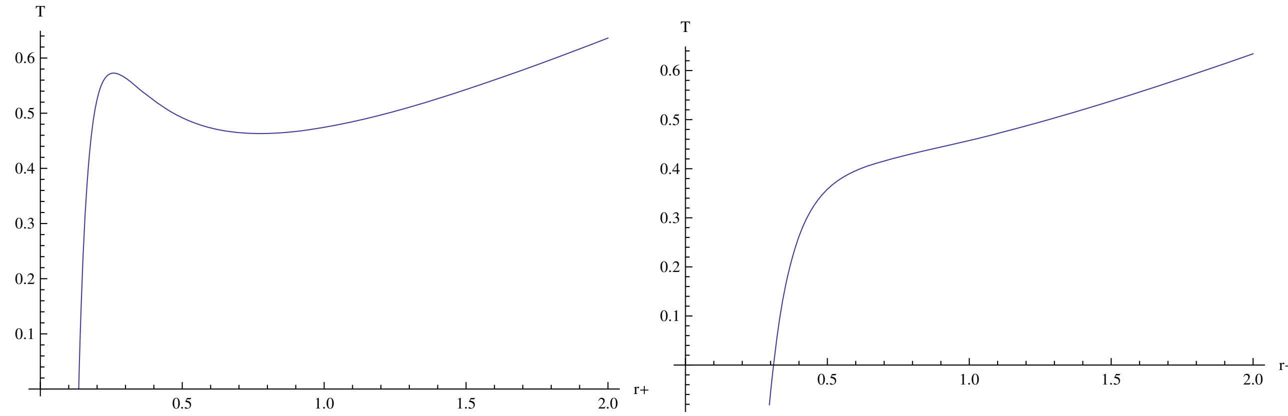


FIG. 5 (color online). The temperature T of the black hole with $\tilde{k} = 2$ versus horizon radius r_+. Here we take $l = c_1 m^2 = 1$. Left: $q = 0.4$ ($< q_{\rm crit}$). Right: $q = 1$ ($> q_{\rm crit}$).

On the other hand, we see from the heat capacity (42) that if $\tilde{k} \leq 0$, the heat capacity is always positive. However, if $\tilde{k} > 0$ the heat capacity is positive for small black holes with $r_+ < r_{qc-}$ and large black holes with $r_+ > r_{qc+}$, while it is negative for intermediate black holes with $r_{qc-} < r_+ < r_{qc+}$. The heat capacity diverges at

$$r_{qc\pm}^2 = \frac{l^2}{6}\left(\tilde{k} \pm \sqrt{\tilde{k}^2 - \frac{9q^2}{l^2}}\right) \tag{45}$$

if the charge is less than the critical one, $q_{\rm crit}^2 = l^2\tilde{k}^2/9$. Of course, if the charge is larger than the critical one, the black hole is always stable with positive heat capacity. These properties can be seen from the behavior of the Hawking temperature (see Figs. 4 and 5). Note that these properties are independent of the sign of c_1, as in the grand canonical ensemble.

Note that the horizon radius for an extremal black hole with vanishing Hawking temperature is determined by the equation

$$\tilde{k} + 3\frac{r_m^2}{l^2} - \frac{1}{4}\frac{q^2}{r_m^2} + c_1 m^2 r_m = 0, \tag{46}$$

which has a simple root

$$r_m^2 = \frac{l^2}{6}\left(-\tilde{k} + \sqrt{\tilde{k}^2 + \frac{3q^2}{l^2}}\right) \tag{47}$$

if $c_1 = 0$. Let us recall here that the minimal horizon radius always exists for any $\tilde{k}$.

In Fig. 6 we plot the Helmholtz free energy of the black holes with $\tilde{k} = 2$ with respect to temperature in the cases of $q < q_{\rm crit}$ and $q > q_{\rm crit}$, respectively. In the left plot,[7] we see that a typical first-order phase transition signature (a swallow tail) appears when $q < q_{\rm crit}$, and it disappears when $q > q_{\rm crit}$. Thus we see that as in GR, if $\tilde{k} > 0$, there

[7]Note that if one plots the free energy $\tilde{F}_4 = F_4 - M_{\rm ext}$, the free energy $\tilde{F}_4$ has the same shape as F_4. The only difference is that the whole free-energy curve $\tilde{F}_4$ will move below the horizontal axis, i.e., $\tilde{F}_4$ is always negative.

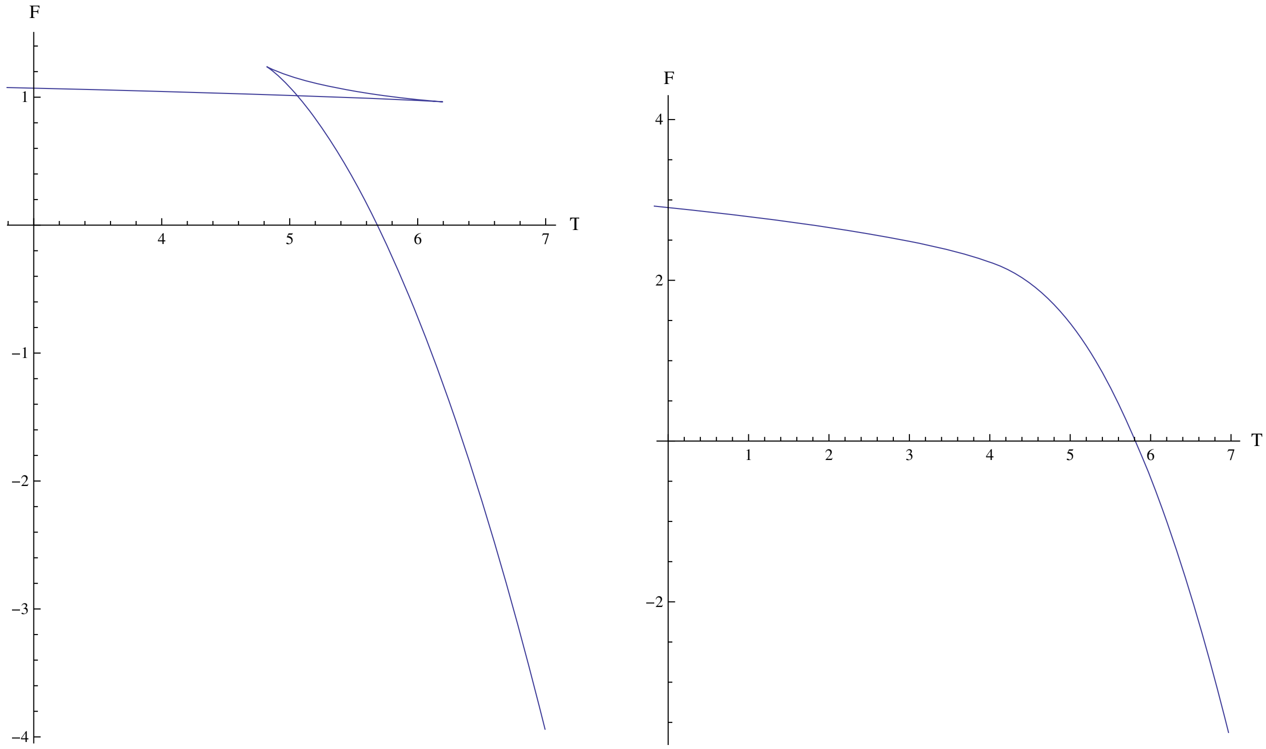


FIG. 6 (color online). The Helmholtz free energy of a black hole with $\tilde{k} = 2$ versus temperature T. Here we take $l = 1$, $c_1 m^2 = 0$. Left: $q = 0.4$ ($< q_{\rm crit}$). Right: $q = 1$ ($> q_{\rm crit}$).

exists a first-order phase transition between small and large black holes if the charge is less than the critical one. The phase transition disappears when the charge is larger than the critical one. The critical point appears when the charge is equal to the critical one. This phase transition behaves very much like that in the van der Waals system in the case of charged black holes in GR [28]. Note that such a phase transition does not exist when $\tilde{k} \le 0$.

As a summary, we see that in the canonical ensemble the thermodynamical behavior of the charged black holes in massive gravity is qualitatively the same as that of Reissner-Nordström AdS black holes with an effective horizon curvature $\tilde{k} = k + c_2 m^2$. If $\tilde{k} > 0$, there exists a small/large black hole phase transition if the charge is less than the critical one. This phase transition disappears when the charge is larger than the critical one.

V. THE PHASE TRANSITION IN HIGHER-DIMENSIONAL NEUTRAL BLACK HOLES

In a higher-dimensional case with $n > 2$, one has $c_3 \neq 0$ if $n \geq 3$, and $c_3 \neq 0$, $c_4 \neq 0$ if $n \geq 4$, in general. Note that the charge plays a crucial role in the existence of the small/large black hole phase transition in the four-dimensional case. In this section, we show that $c_3 m^2$ and $c_4 m^2$ terms can play the same role as the charge in higher-dimensional cases. Here we consider the case of five-dimensional neutral black holes. To clearly see the effect of the term $c_3 m^2$, we set $c_1 = c_2 = q = 0$ for simplicity in this section. Thus the Hawking temperature is given by

$$T_5 = \frac{1}{4\pi r_+}\left(2k + 4\frac{r_+^2}{l^2} + \frac{2c_3 m^2}{r_+}\right), \tag{48}$$

the mass of the black hole is

$$M_5 = \frac{3V_3 r_+^2}{2\kappa^2}\left(k + \frac{r_+^2}{l^2} + \frac{2c_3 m^2}{r_+}\right), \tag{49}$$

and the Helmholtz free energy is

$$F_5 = \frac{V_3 r_+^2}{2\kappa^2}\left(k - \frac{r_+^2}{l^2} + \frac{4c_3 m^2}{r_+}\right). \tag{50}$$

We see that when $c_3 > 0$, nothing special happens. This situation is qualitatively the same as the case of topological black holes in GR [13]: when $k = 1$, the Hawking-Page phase transition occurs, and black holes with a small horizon radius are thermodynamically unstable with a negative heat capacity and those with a large horizon radius are stable with a positive heat capacity; however, when $k = 0$ and -1, the Hawking-Page phase transition does not occur and the black holes are always thermodynamically stable with positive capacity.

When $c_3 < 0$, however, some interesting things appear. In Fig. 7, we show the behavior of the temperature of the

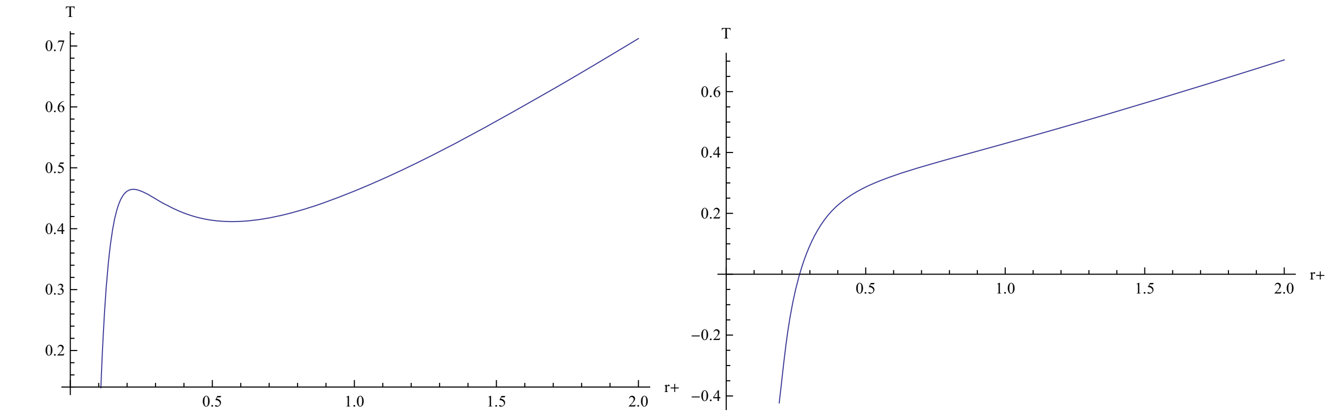


FIG. 7 (color online). The temperature T of a black hole with $k = 1$ versus horizon radius r_+. Here we take $l = 1$. Left: $c_3m^2 = -0.1$. Right: $c_3m^2 = -0.3$

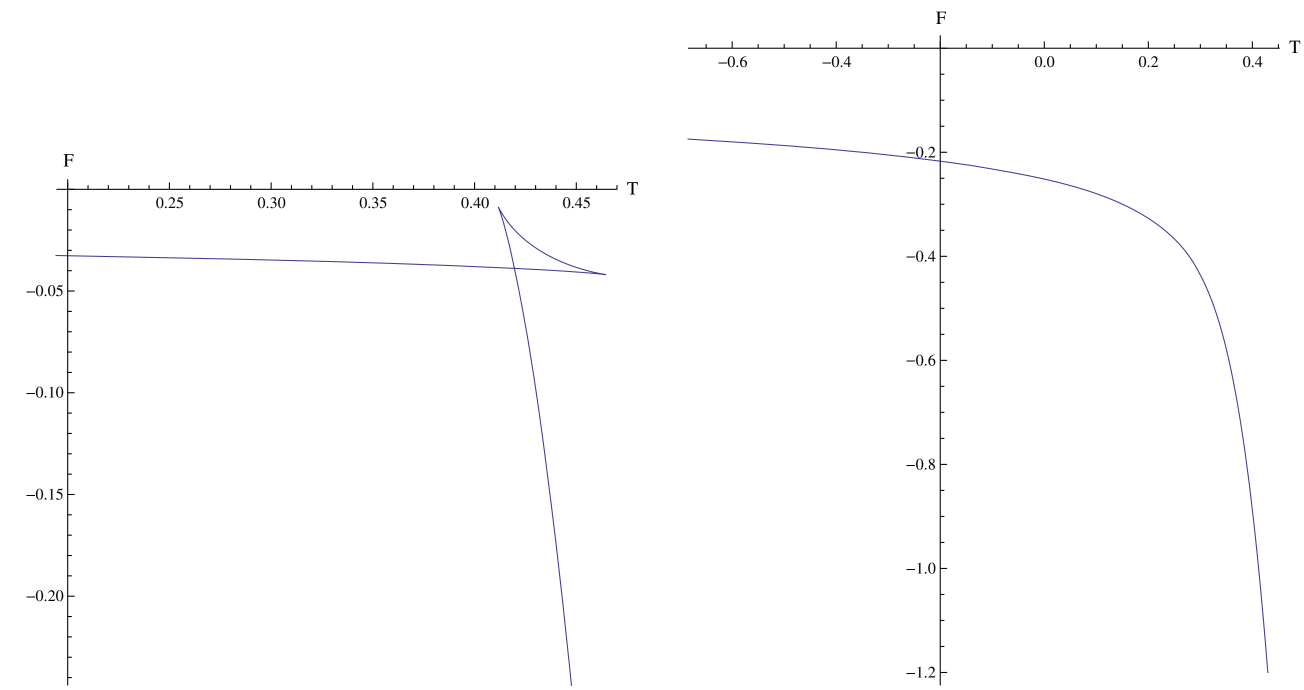


FIG. 8 (color online). The Helmholtz free energy of a black hole with $k = 1$ versus temperature T. Here we take $l = 1$. Left: $c_3m^2 = -0.1$. Right: $c_3m^2 = -0.3$.

black holes with $k = 1$ when $c_3 = -0.1$ and $c_3 = -0.3$, respectively, while the corresponding free energies are plotted in Fig. 8.

From Fig. 7 we see that the temperature behaviors are quite similar to the cases shown in Fig. 5, which implies that there exists a critical $c_{3\mathrm{crit}}$. When $c_3 < c_{3\mathrm{crit}}$ the black holes are always thermodynamically stable with a positive capacity, when $c_{3\mathrm{crit}} < c < 0$ there exist two stable branches with positive capacity (a small horizon branch and a large horizon branch), and for the intermediate horizon the black holes are thermodynamically unstable with negative capacity. The heat capacity is given by

$$C_5 = T\left(\frac{dS}{dT}\right) = \frac{6\pi^2 V_3 T_5 r_+^4}{\kappa^2(-k + \frac{2r_+^2}{l^2} - \frac{2c_3m^2}{r_+})}. \qquad (51)$$

The critical c_3 is given by $c_{3\mathrm{crit}} = -l/m^2\sqrt{54}$ for the case when $k = 1$. From the free-energy behavior shown in Fig. 8, we see that the first-order phase transition occurs when $c_{3\mathrm{crit}} < c_3 < 0$ and it disappears if $c_3 < c_{3\mathrm{crit}}$.

Note that the first-order phase transition discussed above occurs only in the case when $k = 1$. In addition, it is easy to see from the temperature behavior (20) that the term c_4m^2 in the six-dimensional case can play the same role as the term c_3m^2 in the five-dimensional case.

VI. CONCLUSIONS

In this paper we have presented a class of charged black hole solutions in an $(n+2)$-dimensional massive gravity with a negative cosmological constant, and studied the thermodynamics and phase structure of the black hole solutions in both the grand canonical and canonical ensembles. The black hole horizon can have a positive, zero, or negative constant curvature. In the massive gravity we have considered, there are four terms in the potential associated with the graviton mass. In the four-dimensional case, only two of the terms appear in the solution, while the other two terms may occur in higher-dimensional cases. By using the Hamiltonian approach, we have obtained conserved charges of the solutions and they satisfy the first law of black hole thermodynamics. It turns out that the entropy of the black hole still obeys the area formula as in GR, although the massive gravity is not diffemorphism invariant. In addition, we have shown that the gravitational field equations at the black hole horizon can be cast into a form similar to the first law of black hole thermodynamics.

In the four-dimensional case, the black hole thermodynamics and phase structure crucially depend on the horizon curvature of the black holes and the sign of c_2. In the grand canonical ensemble, the Hawking-Page phase transition happens for the case $k-\mu^2/4+c_2m^2 \geq 0$. Namely, it can appear only for the case $k=1$ if $c_2<0$, while for the $k=0$ or -1 case, the black holes are always thermodynamically stable with positive capacity. When $c_2>0$, however, we have found that the Hawking-Page phase transition can always appear for any horizon curvature if $k-\mu^2/4+c_2m^2 \geq 0$. In the canonical ensemble, when the charge of the black hole is less than its critical one, a small/large black hole first-order phase transition happens if $k+c_2m^2 \geq 0$. This phase transition behaves like the one in the van der Waals system [28].

For the higher-dimensional $(n+2 \geq 5)$ case, we have found that even when the charge of the black hole is absent, the small/large black hole phase transition can appear. The coefficients c_3m^2 and/or c_4m^2 can play the same role as the charge does in the four-dimensional case if c_3m^2 and/or c_4m^2 are negative. This is a remarkable result in massive gravity, which does not appear in GR.

Finally we mention here that the black hole solutions presented in this paper crucially depend on the choice of the reference metric (6). In general, if one can take the ansatz $f_{\mu\nu}=\mathrm{diag}(0,0,c_0^2F(r)h_{ij})$, where $F(r)$ is a continuous function of r, we can also obtain the exact solution of the theory once $F(r)$ is specified. In this sense, the choice of the reference metric is an important issue in this class of massive gravity.

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