Initial value formulation of dynamical Chern-Simons gravity

Térence Delsate,[1,*] David Hilditch,[2,†] and Helvi Witek[3,‡]

[1]*Theoretical and Mathematical Physics Department, University of Mons, 7000 Mons, Belgium*
[2]*Theoretical Physics Institute, University of Jena, 07743 Jena, Germany*
[3]*Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

(Received 5 August 2014; published 21 January 2015)

We derive an initial value formulation for dynamical Chern-Simons gravity, a modification of general relativity involving parity-violating higher derivative terms. We investigate the structure of the resulting system of partial differential equations thinking about linearization around arbitrary backgrounds. This type of consideration is necessary if we are to establish well-posedness of the Cauchy problem. Treating the field equations as an effective field theory we find that weak necessary conditions for hyperbolicity are satisfied. For the full field equations we find that there are states from which subsequent evolution is not determined. Generically the evolution system closes, but is not hyperbolic in any sense that requires a first order pseudodifferential reduction. In a cursory mode analysis we find that the equations of motion contain terms that may cause ill-posedness of the initial value problem.

DOI: 10.1103/PhysRevD.91.024027 PACS numbers: 02.30.Jr, 04.25.-g, 04.50.Kd, 04.60.Cf

I. INTRODUCTION

General relativity (GR) is the most successful theory of gravity to date, and has passed all experimental tests with flying colors. However, these tests, such as observations of pulsar binaries or observations inside the Solar System, are restricted to the range in which low order post-Newtonian calculations accurately describe the dynamics [1–4]. Bearing in mind the extrapolation of GR over many orders of magnitude, and the issues in wedding gravity with quantum physics, it would not be surprising if modifications to GR in the high curvature regime were discovered. Identifying *how* the field equations might be modified is however open to debate. One class of modifications is motivated by string theory in the low energy limit. Specifically, compactifications of 10-dimensional heterotic string theory to four spacetime dimensions yield modifications of the Einstein-Hilbert action involving higher derivative terms of the metric [5–7]. The gravity sector of the action including quadratic terms in the curvature is [5,6,8,9]

$$S = \int d^4x\sqrt{-g}\bigg[\kappa R + f_1(\theta)R^2 + f_2(\theta)R_{ab}R^{ab} + f_3(\theta)R_{abcd}R^{abcd} + f_4(\theta)^*RR - \frac{b_{\mathrm{CS}}}{2}(\nabla_a\theta\nabla^a\theta + 2V(\theta)) + \mathcal{L}_{\mathrm{m}}\bigg], \tag{1}$$

where the first term relates to GR with the gravitational coupling constant κ, $\mathcal{L}_{\mathrm{m}}$ denotes the Lagrangian for ordinary matter, θ is a dynamical scalar field, and $f_i(\theta)$ are functions specifying the coupling of the higher derivative contributions.

With the specific choice $f_1(\theta) = a_{\mathrm{GB}}\exp(-2\theta)$, $f_2(\theta) = -4f_1(\theta)$, $f_3(\theta) = f_1(\theta)$, $f_4(\theta) = 0$ we obtain the well-known dilaton-Gauss-Bonnet modification with the coupling a_{GB} [10]. Its parity-violating counterpart includes the Pontryagin density ${}^*RR = {}^*R^{abcd}R_{bacd} = -\frac{1}{2}\epsilon^{cd}{}_{ef}R^{abef}R_{abcd}$ with an axionic-type coupling to the scalar field θ, i.e. $f_1(\theta) = f_2(\theta) = f_3(\theta) = 0$, $f_4(\theta) = \frac{a_{\mathrm{CS}}}{4}\theta$, and is called dynamical Chern-Simons (dCS) theory [6,11]. If the kinematic term is discarded, the resulting model is called nondynamical Chern-Simons theory. If the scalar field is constant the corrections are topological in four dimensions and the equations of motion reduce to those of GR.

Some solutions of GR are inherited by the dCS model. Specifically, even parity spacetimes, such as the Schwarzschild solution, have vanishing Pontryagin density and are unaffected by the dCS modification. In contrast, the Kerr black hole (BH) is parity odd and therefore not a solution in dCS gravity. No complete solution for a rotating BH in dCS theory is known, but see Refs. [12–19] for perturbative calculations. Exploring dynamical BH solutions provides the possibility to explore gravity in the strong-field regime. In this context modifications to GR may become important. For example, studies of extreme-mass-ratio inspirals in dCS gravity revealed an additional polarization of gravitational waves (GWs) [20–24]. Furthermore rotating spacetimes are deformed in comparison to GR and may cause deviations in the GW signals. These "smoking-gun" effects may be observable with future space-based GW detectors along the lines of the eLISA mission [25,26] or, in the case of solar-mass BH

*terence.delsate@umons.ac.be
†david.hilditch@uni-jena.de
‡h.witek@damtp.cam.ac.uk

binaries, with existing or upcoming ground-based GW detectors such as the advanced LIGO/VIRGO detector network [27–31] or the KAGRA detector [32,33]. GW astronomy might furthermore yield more stringent bounds on the dCS coupling parameter [9], which so far has been constrained by table-top experiments [13] and observations of frame-dragging effects in the Solar System [34] to be $\sqrt{|a_{\rm CS}|} \lesssim 10^8 km$. Recently, it has been suggested that observations of highly spinning, solar-mass BHs could be employed to improve this bound to $\sqrt{|a_{\rm CS}|} \lesssim \mathcal{O}(10) km$ [19].

Investigating dCS gravity for comparable-mass binary systems, the most promising sources for ground-based GW detectors, is still outstanding in the high curvature regime (see Ref. [35] for a study in the PN approximation)—missing is a formulation which could be treated by standard numerical relativity techniques [36]. It was foreseen in Ref. [37] that the higher derivative equations might make such a formulation problematic. Given the ease in prescribing modifications to GR compatible with observational bounds it is natural to ask, what other tests could we subject the modified theory to? An obvious option is to look for logical inconsistencies, or for contradictions with some physical principle that we hold dear. In the present work we follow this tack. A fundamental question for any field theory is whether it has a locally well-posed initial value problem. We might furthermore insist on causality, or finite propagation speeds of information. While the linear stability of specific solutions has been studied [15,18,37–39] (see Refs. [40] and [41] for similar studies in Gauss-Bonnet and in Lovelock theories), it is not known whether or not dCS gravity makes sense as a time evolution system.

Therefore, we perform a $3+1$ decomposition of the dCS field equations along the lines of the Arnowitt-Deser-Misner (ADM)-York split [42–44] and begin studying the structure of the resulting partial differential equation (PDE) system. Guided by the similarities between the PDE structure of GR and Maxwell's theory we first investigate the properties of Maxwell's equations modified by the Chern-Simons term coupled to an axion field, which at first sight appears to be the electromagnetic analogue of dCS gravity. In contrast to Maxwell-Chern-Simons theory, we find that dCS gravity cannot be written in first order form, a necessary condition in many definitions of hyperbolicity (see for example [45]). Thus the attempted analysis fails: dCS gravity does not satisfy these definitions, even so far as GR does before fixing the gauge. Thus the naive expectation would be that even if the field equations admit a well-posed initial value problem, signals may travel arbitrarily fast. In any relativistic theory, however, physical signals, to be contrasted with gauge, should propagate at finite speeds.

dCS gravity is not normally viewed as a complete theory, but rather as an *effective* theory, emerging as a higher derivative modification to GR in string theory, loop quantum gravity [46–48], cosmological inflation [49] or particle physics [50]. The effective theory is a reasonable model when its solutions are a good approximation to those of the full theory. An approach in the literature is either to reduce the order of the highest derivative assuming a small coupling or to treat the effective field equations order by order in the coupling parameter. The resulting PDEs can be reduced to first order, thus fulfilling this very weak requirement to have a chance to be hyperbolic.

The paper is structured as follows. In Sec. II, we review relevant aspects of PDE theory. In Sec. III we discuss the Chern-Simons modification to the Maxwell equations. Subsequently, in Sec. IV we present the full dCS field equations in $3+1$ decomposed form. In Sec. V we discuss how some of the problems we encounter can be avoided when the model is treated as an effective theory. Finally, in Sec. VI, we conclude. We use geometrized units $G=1=c$ throughout. Early lower letters $a, b, \ldots \in 0, \ldots, 3$ denote spacetime indices; middle lower letters $i, j, \ldots \in 1, 2, 3$ denote spatial indices.

II. HYPERBOLIC AND PARABOLIC PDEs

Because the structure and properties of time evolution PDEs play a central role in the present paper we begin with a brief discussion highlighting the difference between hyperbolic and parabolic PDEs.

Any reasonable physical model should result in PDE problems that are well posed. Roughly speaking, well-posedness is the existence of a unique solution which depends continuously on given initial data. In a relativistic context we additionally insist on finite propagation speeds for physical fields, as opposed to gauge, given arbitrary data. Hyperbolic PDEs are characterized by this property. Formal definitions of hyperbolicity are given for first order systems in terms of algebraic properties of the coefficients of the derivatives [51,52]. Hyperbolicity of higher order derivative systems is defined by considering properties of fully first order (pseudo)differential reductions [45,53–55]. Therefore a necessary condition for the application of these definitions is the existence of a first order reduction of the PDE system in question. In the remainder of the paper we will refer to this definition of hyperbolicity without further comment. Consider the linear, constant coefficient, first order in time, second order in space (FT2S) system,

$$\begin{aligned} \partial_t u &= (A^u{}_u)^i \partial_i u + (A^u{}_v) v + S_u, \\ \partial_t v &= (A^v{}_u)^{ij} \partial_i \partial_j u + (A^v{}_v)^i \partial_i v + S_v. \end{aligned} \tag{2}$$

This system can be reduced to first differential order by introducing the variables $d_i = \partial_i u$, and then appropriately adding the constraint $c_i = d_i - \partial_i u$ to the resulting equations. In general, we will call equations with this shape "FTNS," which stands for *first order in time*, *Nth order in space*. Specifically, the dCS equations of motion (EoMs) contain third derivatives of the metric, so we might like to end up with a FT3S PDE system,

$$\begin{aligned}\partial_t u &= (A^u{}_u)^i\partial_i u + (A^u{}_v)v + S^u,\\ \partial_t v &= (A^v{}_u)^{ij}\partial_i\partial_j u + (A^v{}_v)^i\partial_i v + (A^v{}_w)w + S^v,\\ \partial_t w &= (A^w{}_u)^{ijk}\partial_i\partial_j\partial_k u + (A^w{}_v)^{ij}\partial_i\partial_j v\\ &\quad + (A^w{}_w)^i\partial_i w + S^w,\end{aligned} \tag{3}$$

which is easily seen [55] to be the natural generalization of Eq. (2).

The archetypal hyperbolic PDE is the wave equation, which can be written with a first order in time reduction as

$$\partial_t\Phi(t,x) = \Pi(t,x), \qquad \partial_t\Pi(t,x) = \partial_x^2\Phi(t,x). \tag{4}$$

The fundamental solution of the wave equation, that is the response to a Dirac delta function placed at the origin initially, is

$$\Phi(t,x) = \frac{1}{4}\Theta(t)[\mathrm{Sign}(t+x) + \mathrm{Sign}(t-x)], \tag{5}$$

where Θ is the Heaviside function. For every $t > 0$ the solution to the wave equation Φ has a compact support in x. Plotting the fundamental solution to the wave equation shows that the evolving pulse remains at all times inside the future null cone of the initial pulse. Contrast this with the heat equation,

$$\partial_t\Phi(t,x) = \partial_x^2\Phi(t,x). \tag{6}$$

Introducing here a reduction variable $d_x = \partial_x\Phi$ does not reduce the PDE to first order because the equation generates terms like $\partial_x^2 d_x$. The fundamental solution in this case is given by

$$\Phi(t,x) = \frac{1}{2\sqrt{\pi t}}\Theta(t)e^{-\frac{x^2}{4t}}. \tag{7}$$

For every $t > 0$ the solution to the heat equation has an infinite support. This means that a point impulse propagates instantaneously everywhere once $t > 0$ [56]. Similar statements can be made about Schrödinger-like equations. This "causality violation" property is present in other parabolic PDEs and is not permissible in relativistic physics where we have a natural speed limit.

The discussion so far is relevant for linear PDEs. When facing nonlinear problems we must linearize the equations about an arbitrary solution, and apply the linear theory. For certain types of equations, such as hyperbolic or parabolic, and if certain smoothness conditions are satisfied [51] then well-posedness of the linear problem guarantees *local in time* well-posedness of the nonlinear problem. It is possible that the local classification changes over the domain, the Tricomi equation $\partial_x^2 u = x\partial_y^2 u$ being the standard example of this behavior. We will also see that it is possible that, in some region, the PDE does not fall into any of the standard classes. In this case a more *ad hoc* analysis may be all that is available.

III. CHERN-SIMONS ELECTROMAGNETISM

A. Action and field equations

Bearing in mind the similarities between the PDE structure of GR and electromagnetism it is instructive to investigate the hyperbolicity properties of Maxwell's equations modified by a Chern-Simons term, which we will call Chern-Simons electromagnetism, before turning to the gravity case.

The action consists of an axionic deformation of the standard electromagnetic action [57]. The corresponding Lagrangian density is given by

$$\mathcal{L}_{\mathrm{CSE}} = -\frac{1}{4}F^{ab}F_{ab} - \frac{\lambda}{2}\psi^*F_{ab}F^{ab} - \frac{1}{2}\nabla^a\psi\nabla_a\psi - V(\psi), \tag{8}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ is the field strength, ${}^*F_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}$ is its dual, A^a is the $U(1)$ gauge field and λ denotes the coupling to the scalar field ψ. The term ${}^*F_{ab}F^{ab}$ imposes the parity violation and can be interpreted as the analogue to the Pontryagin density in the gravity case. The resulting EoMs are

$$\begin{aligned}\nabla^a\nabla_a\psi &= \frac{1}{2}\lambda{}^*F_{ab}F^{ab} + V'(\psi),\\ \nabla_b F^{ab} &= -2\lambda{}^*F^{ab}\nabla_b\psi, \qquad \nabla_b{}^*F^{ab} = 0.\end{aligned} \tag{9}$$

Note that the last relation is satisfied trivially when expressing it in terms of the vector potential and we only keep it for completeness. Already at this level, we observe that this system of PDEs can be made strongly hyperbolic because in the appropriate gauge it consists of a set of decoupled wave equations for the scalar and gauge fields, respectively, with some lower order source terms. In a PDE analysis language, this system is said to be minimally coupled.

B. 3 + 1 decomposition

In this section, we show explicitly that Eqs. (9) are indeed minimally coupled and rewrite them as a FT2S system. Therefore, we foliate a 4-dimensional spacetime manifold $\mathcal{M}$ into 3-dimensional spatial hypersurfaces Σ_t parametrized by the coordinate time t. We denote the spatial metric γ_{ij} and the unit timelike vector n^a which is orthogonal to the spatial slices and satisfies $n_a n^a = -1$. Furthermore, we introduce the projection operator $\gamma_a{}^b = \delta_a{}^b + n_a n^b$. Within this decomposition the spacetime line element is

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (10)$$

where α and β^i are the lapse function and shift vector.

To rewrite Eqs. (9) as a time evolution problem, we decompose the 4-vector potential A_a into its spatial part $\mathcal{A}_i$ and normal component Φ with a convenient normalization, according to

$$A_a = \mathcal{A}_a + \frac{\Phi}{\alpha} n_a, \qquad \mathcal{A}_i = \gamma^a{}_i A_a, \qquad \Phi = -\alpha n^a A_a. \quad (11)$$

Next, we introduce the electric and magnetic fields E_a and B_a given by the contractions of the Maxwell tensor and its dual with the unit timelike vector,

$$E_i = \gamma^a{}_i F_{ab} n^b, \qquad B_i = \gamma^a{}_i {}^*F_{ab} n^b. \quad (12)$$

The magnetic field is related to the spatial component of the vector potential via

$$B_i = \epsilon_i{}^{jk} D_j \mathcal{A}_k. \quad (13)$$

The magnetic field B_i is not treated as a dynamical variable itself. We employ it *purely as a shorthand* whenever economical.

Given these relations we can reexpress the Maxwell tensor in terms of the electric field and the spatial vector potential,

$$\begin{aligned} F_{ab} &= n_a E_b - n_b E_a + D_a \mathcal{A}_b - D_b \mathcal{A}_a, \\ {}^*F_{ab} &= n_a B_b - n_b B_a + \epsilon_{abc} E^c, \end{aligned} \quad (14)$$

where D_i denotes the 3-dimensional covariant derivative associated with the spatial metric γ_{ij} and the analogue of the Pontryagin density is ${}^*F^{ab}F_{ab} = -4E_i B^i$. We introduce the reduction variable Π_ψ,

$$\Pi_\psi \coloneqq -n^a \nabla_a \psi = -\frac{1}{\alpha}(\partial_t - \mathcal{L}_\beta)\psi. \quad (15)$$

Then, employing the $3+1$ decomposition, we obtain a set of time evolution equations,

$$\begin{aligned} \partial_t \psi &= -\alpha \Pi_\psi + \mathcal{L}_\beta \psi, \\ \partial_t \mathcal{A}_i &= -\alpha E_i - D_i \Phi + \mathcal{L}_\beta \mathcal{A}_i, \\ \partial_t \Pi_\psi &= \alpha[K\Pi_\psi + V'(\psi) - D^i D_i \psi + 2\lambda E_i B^i] \\ &\quad - D_i \psi D^i \alpha + \mathcal{L}_\beta \Pi_\psi, \\ \partial_t E^i &= D^j[\alpha(D^i \mathcal{A}_j - D_j \mathcal{A}^i)] + \alpha K E^i - 2\lambda\alpha \Pi_\psi B^i \\ &\quad - 2\lambda\alpha [E \times (D\psi)]^i + \mathcal{L}_\beta E^i, \end{aligned} \quad (16)$$

with the cross product defined by

$$[E \times (D\psi)]^i = \epsilon^{ijk} E_j D_k \psi. \quad (17)$$

For completeness we give also the time derivative of the magnetic field,

$$\partial_t B^i = \alpha K B^i - [D \times (\alpha E)]^i + \mathcal{L}_\beta B^i. \quad (18)$$

This equation is independent of any particular choice of model for the electromagnetic field, because it follows directly from $\nabla_b {}^*F^{ab} = 0$. Finally, the constraint equation reads

$$M = D^i E_i - 2\lambda B^i D_i \psi = 0. \quad (19)$$

We may formally compute the time derivative of the constraint, and find

$$\partial_t M = \alpha K M + \mathcal{L}_\beta M. \quad (20)$$

As expected the constraint subsystem is closed; i.e. if the constraint is satisfied initially, it is satisfied during the entire evolution. If we were going to analyze hyperbolicity of a particular formulation of the theory we would now adopt the free-evolution point of view, make a choice of gauge for the field Φ, expand the solution space with new constraints and couple them to the present system [58]. Instead we will focus simply on the structure of the equations.

We may view the model equations as telling us one constraint (20) and the three evolution equations for E^i. The remaining equations are differential identities following from the fact that F_{ab} is a closed 2-form. It is useful to keep this in mind in the gravitational case that follows.

Looking again at the Maxwell-Chern-Simons field equations expressed in a first order in time form, we see that the sets $(\mathcal{A}_i, E^i)$ and (ψ, Π_ψ) are minimally coupled. More precisely, the system (16) has the FT2S structure given in Eq. (2), where $\mathcal{A}_i, \psi$ are u-like variables and E^i, Π_ψ are v-like. In an appropriate formulation Φ will be a u-like variable. Furthermore the block of the principal symbol associated with the first pair $(\mathcal{A}_i, E^i)$ is identical to that of the pure Maxwell equations, and that of the second pair (ψ, Π_ψ) to that of the wave equation. In other words, after a suitable gauge choice, the full system can be rendered strongly hyperbolic according to a treatment identical to that for the Maxwell equations [59].

IV. CHERN-SIMONS GRAVITY

A. Motivation from string theory

Extensions of GR involving dCS modifications are motivated for example by the compactification of the bosonic part of 10-dimensional heterotic string theory to 4-dimensional $\mathcal{N} = 1$ supergravity [5]. The bosonic sector of this theory is given by

$$S_{10D} = \int d^{10}x\sqrt{-g_{10}}\left[R - \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{12}e^{-\phi}H_{abc}H^{abc} - \frac{1}{4}e^{-\frac{\phi}{2}}\mathrm{Tr}(F_{ab}F^{ab})\right], \tag{21}$$

where g_{10} is the 10-dimensional metric, F and H are 2- and 3-form field strengths, respectively, and ϕ is a scalar field.

It was shown that GR in even dimensions suffers from a gravitational anomaly [60], which can be cured by shifting the 3-form field strength with additional Chern-Simons terms, as shown by Green and Schwarz [61,62]:

$$H_3 = dB_2 - \frac{1}{4}(\Omega_3(A) - \Omega_3(\omega)), \tag{22}$$

where B_2 is a 2-form, A is the Yang-Mills 1-form and ω is the spin connection.

The terms involving Ω_3 are obtained from the Green-Schwarz prescription and are defined as

$$\Omega_3(A) = \mathrm{Tr}\left(dA \wedge A + \frac{2}{3}A \wedge A \wedge A\right). \tag{23}$$

Here, it is assumed that all moduli except for the axion are stabilized, and the resulting 4-dimensional action is that of dCS theory. See the review [6] for details and references.

For the 1-form A the Chern-Simons form (23) produces at most terms of the order $(\partial A)^2$ yielding an action that does not involve higher derivative terms. In Sec. III we saw that for electromagnetism, which has the same PDE structure as Yang-Mills, Chern-Simons-like terms coming from the anomaly canceling procedures are structurally fine. Thus one expects that, with a little work, dCS theory for a 1-form admits a well-posed initial value formulation. This picture changes dramatically if we consider the gravitational sector. Bearing in mind that the spin connection behaves as $\omega \sim \partial g_{10}$, it is evident that the Chern form (23) will introduce a term of the form $\partial^2 g_{10}$ leading to an action which contains higher derivatives of the metric. We will show in the remainder of this section that these higher derivative terms prevent dCS gravity from being hyperbolic.

B. Action and field equations

We now focus our attention on the case of Chern-Simons gravity coupled to a dynamical scalar field. We recover the corresponding action by setting $f_1(\theta) = f_2(\theta) = f_3(\theta) = 0$ and $f_4(\theta) = \frac{a_{\mathrm{CS}}}{4}\theta$ in Eq. (1) and note it here only for completeness [5,6]:

$$S = \int d^4x\sqrt{-g}\left[\kappa R + \frac{a_{\mathrm{CS}}}{4}\theta^* RR - \frac{b_{\mathrm{CS}}}{2}(\nabla^a\theta\nabla_a\theta + 2V(\theta)) + \mathcal{L}_{\mathrm{m}}\right], \tag{24}$$

where κ is the gravitational coupling, a_{CS} is the axionic coupling of the scalar field θ to the Pontryagin density ${}^*RR = -\frac{1}{2}\epsilon^{cd}{}_{ef}R^{abef}R_{abcd}$ and b_{CS} denotes the coupling to the kinetic term of the scalar field. One recovers GR minimally coupled to a scalar field if $a_{\mathrm{CS}} = 0$ and $b_{\mathrm{CS}} = 1$. From now on we will consider the absence of ordinary matter, i.e. $\mathcal{L}_{\mathrm{m}} = 0$, and vanishing scalar field potential $V(\theta) = 0$. If $V(\theta)$ contains no derivatives of the scalar field these assumptions will not change the outcome of the hyperbolicity analysis. The EoMs are

$$\begin{aligned} G_{ab} + \frac{a_{\mathrm{CS}}}{\kappa}C_{ab} &= \frac{b_{\mathrm{CS}}}{2\kappa}T^\theta_{ab}, \\ \Box\theta + \frac{a_{\mathrm{CS}}}{4b_{\mathrm{CS}}}{}^*RR &= 0, \end{aligned} \tag{25}$$

where G_{ab} is the Einstein tensor, T^θ_{ab} is the energy-momentum tensor related to the scalar field,

$$T^\theta_{ab} = \nabla_a\theta\nabla_b\theta - \frac{1}{2}g_{ab}\nabla^c\theta\nabla_c\theta, \tag{26}$$

and

$$C_{ab} = \nabla_c\theta\epsilon^{cd}{}_{e(a}\nabla^e R_{b)d} + \nabla^c\nabla^d\theta^* R_{d(ab)c} \tag{27}$$

is the C-tensor. Already at this stage it is evident that there is a distinction between the electromagnetic and gravitational Chern-Simons models, since the scalar is not minimally coupled to the parent field in the gravitational case.

In Refs. [37,63] it was observed that it is convenient to rewrite the C-tensor in terms of the Weyl tensor W_{abcd}, resulting in

$$C_{ab} = 2(\nabla^c\theta)\nabla^{d*}W_{d(ab)c} + (\nabla^c\nabla^d\theta)^* W_{d(ab)c} \tag{28}$$

where we have used the relation

$$\nabla_a W^{abcd} = \nabla^{[c}R^{d]b} + \frac{1}{6}g^{b[c}\nabla^{d]}R, \tag{29}$$

that follows from the Bianchi identities (see, e.g. Ref. [36] and references therein). The strength of this approach is the fact that contractions of the Weyl tensor with a timelike unit vector n^a define its electric and magnetic parts

$$E_{ij} = \gamma^a{}_i\gamma^b{}_j W_{acbd}n^c n^d, \qquad B_{ij} = \gamma^a{}_i\gamma^b{}_j * W_{acbd}n^c n^d, \tag{30}$$

in analogy with electromagnetism. Then, the Weyl tensor can be reconstructed from its electric and magnetic components [36]

$$W_{abcd} = 2(l_{a[c}E_{d]b} - l_{b[c}E_{d]a} - \epsilon^{e}{}_{ab}n_{[c}B_{d]e} - \epsilon^{e}{}_{cd}n_{[a}B_{b]e}), \tag{31}$$

where $l_{ab} = g_{ab} + 2n_a n_b$. The Pontryagin density can be written as ${}^*RR = -16E_{ij}B^{ij}$ [63].

C. Formulation as Cauchy problem

We proceed in our analysis by rewriting dCS gravity as a Cauchy problem. For this we $3+1$ decompose the EoMs (25). Although this is conceptually straightforward it becomes rather involved due to the presence of covariant derivatives of the Ricci tensor (yielding third derivatives of the metric) in the C-tensor; cf. Eq. (27). The computation was carried out using the xTensor [64] package. For clarity we suppress some details of the derivation and refer the interested reader to the notebooks [65]. Some of the relations we derive are purely of geometrical origin and are, therefore, *independent* of the gravitational field equations. A second set of equations stems from the EoMs and so is model dependent. Specifically, the constraint equations all originate from various projections of the EoMs. Instead, the time evolution equations consist of both kinematical, i.e. geometric or model-independent, as well as dynamical, i.e. model-dependent, degrees of freedom. A similar decomposition was made elsewhere [66], but given in a less geometric language without employing E_{ij} and B_{ij}, which unfortunately gives the impression that the constraint equations depend on the coordinate gauge.

1. Structure equations and choice of variables

The foliation of spacetime into 3-dimensional spatial slices introduces the spatial metric γ_{ij} together with the extrinsic curvature,

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t - \mathcal{L}_\beta)\gamma_{ij}. \tag{32}$$

The spacetime coordinates are described by the lapse function α and shift vector β^i. The line element in terms of $3+1$ variables is given in Eq. (10). We introduce the reduction variable to the scalar field θ,

$$\Pi = -n^a\nabla_a\theta = -\frac{1}{\alpha}(\partial_t - \mathcal{L}_\beta)\theta. \tag{33}$$

It proves useful to split rank-2 tensors into their trace and trace-free parts. Specifically, we split the extrinsic curvature K_{ij} and spatial Ricci tensor $\mathcal{R}_{ij}$ according to

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \qquad \mathcal{R}_{ij} = \mathcal{R}^{\mathrm{TF}}_{ij} + \frac{1}{3}\gamma_{ij}\mathcal{R}. \tag{34}$$

The electric and magnetic parts of the Weyl tensor also enter the equations of motion. In $3+1$ language their definitions give

$$\begin{aligned} E_{ij} &= \frac{1}{2}\mathcal{R}^{\mathrm{TF}}_{ij} + \frac{1}{2\alpha}[D_iD_j\alpha]^{\mathrm{TF}} + \frac{1}{2}\mathcal{L}_n A_{ij} \\ &\quad + \frac{1}{6}KA_{ij} + \frac{1}{3}A^{kl}A_{kl}\gamma_{ij}, \\ B_{ij} &= (D\times A)_{ij} \equiv \epsilon_{(i|}{}^{kl}D_kA_{l|j)}. \end{aligned} \tag{35}$$

Both quantities are already trace free. While E_{ij} joins the state vector of dCS gravity as a dynamical variable the magnetic part will be employed purely as a shorthand. The magnetic part satisfies the geometric identities

$$\begin{aligned} \partial_t B_{ij} &= \alpha[[D\times(2E - E^{\mathrm{GR}})]_{ij} - 3A^k{}_{(i}B_{j)k} \\ &\quad - 2\epsilon_{(i}{}^{kl}E_{j)k}D_l\ln\alpha - \epsilon_i{}^{kl}B_{km}A_{ln}\epsilon_j{}^{mn} \\ &\quad + \frac{1}{3}KB_{ij} + \frac{1}{2}\epsilon_{(i}{}^{kl}A_{j)k}M^{\mathrm{GR}}_l] + \mathcal{L}_\beta B_{ij}, \\ D^jB_{ij} &= \epsilon_i{}^{jk}A^l{}_kE^{\mathrm{GR}}_{jl} + \frac{1}{2}\epsilon_i{}^{jk}D_jM^{\mathrm{GR}}_k, \end{aligned} \tag{36}$$

which follow from the projections of the Bianchi identities. Here, we use the shorthand,

$$E^{\mathrm{GR}}_{ij} = \mathcal{R}^{\mathrm{TF}}_{ij} - A^k{}_iA_{jk} + \frac{1}{3}\gamma_{ij}A^{kl}A_{kl} + \frac{1}{3}KA_{ij}, \tag{37}$$

for the expression that follows for the electric part of the Weyl tensor in *vacuum* GR, and

$$M^{\mathrm{GR}}_i = D^jA_{ij} - \frac{2}{3}D_iK, \tag{38}$$

for the expression that appears in the vacuum momentum constraint in GR. In what follows we will also use the expression for the vacuum Hamiltonian constraint in GR,

$$H^{\mathrm{GR}} = \mathcal{R} - A_{ij}A^{ij} + \frac{2}{3}K^2. \tag{39}$$

To summarize, at this stage the independent, dynamical variables are taken to be $(\gamma_{ij}, \theta, A_{ij}, K, E_{ij}, \Pi)$, while the remaining quantities are used as shorthand notation.

2. Auxiliary variables

For the sake of simplifying the expressions, we define

$$X_{ij} = E_{ij} - E^{\mathrm{GR}}_{ij}, \qquad \mathcal{O}_{ij}{}^{kl} = \gamma_{(i}{}^{(k}\epsilon_{j)}{}^{l)m}D_m\theta, \tag{40}$$

and the auxiliary tensor

$$\tilde{X}_{ij} = \mathcal{O}_{ij}{}^{kl}X_{kl}, \tag{41}$$

which will turn out to be an important object. We use the same notation to denote the operator $\mathcal{O}$ acting on other symmetric tensors. This operator is not invertible, which can be checked by explicitly computing its determinant in a particular basis. It is also not nilpotent (see Appendix A), which is an important property for our purposes. In the model-dependent EoMs we will find a coupling to the gradient of the scalar field θ. As long as this gradient is nonvanishing it is useful to introduce a unit normal vector s^i parallel to $D^i\theta$ and, in particular, we define

$$s_i = L^{-1} D_i\theta, \qquad L^2 = D_i\theta D^i\theta. \tag{42}$$

Furthermore, we introduce the 2-metric q_{ij} of the hypersurface orthogonal to s^i, which defines a projection operator, and the corresponding antisymmetric tensor

$$q_{ij} = \gamma_{ij} - s_i s_j, \qquad \epsilon_{jk} = \epsilon_{ijk} s^i. \tag{43}$$

Then, the operator $\mathcal{O}$ can be expressed as

$$L^{-1}\mathcal{O}_{ij}{}^{kl} = q_{(i}{}^{(l}\epsilon_{j)}{}^{k)} + s_{(i}s^{(l}\epsilon_{j)}{}^{k)}. \tag{44}$$

Using the projector, X_{ij} can be written as

$$X_{ij} = X_{ss}\left[s_i s_j - \frac{1}{2}q_{ij}\right] + 2X_{sA}q^A{}_{(i}s_{j)} + X^{\mathrm{TF}}_{AB}[q^A{}_{(i}q^B{}_{j)} - \frac{1}{2}q_{ij}q^{AB}], \tag{45}$$

with

$$X^{\mathrm{TF}}_{AB} = \left(q^i{}_{(A}q^j{}_{B)} - \frac{1}{2}q_{AB}q^{ij}\right)X_{ij},$$
$$X_{sA} = q_A{}^k s^l X_{kl}, \qquad X_{ss} = s^i s^j X_{ij}, \tag{46}$$

where we use uppercase latin indices to denote components that have been projected onto the 2-surface and where indices s refer to quantities contracted with the normal vector s^i. Then,

$$L^{-1}\tilde{X}_{ij} = X^{\mathrm{TF}}_{AC}\epsilon_B{}^C q^A{}_{(i}q^B{}_{j)} + X_{sA}\epsilon_B{}^A q^B{}_{(i}s_{j)}. \tag{47}$$

The consequence of the operator $\mathcal{O}$ being noninvertible is that it is not possible to solve for all of the components of X_{ij} given $\tilde{X}_{ij}$. Indeed, we can *only* solve for X_{sA} and the projected trace-free part X^{TF}_{AB}:

$$X^{\mathrm{TF}}_{AB} = L^{-1}\tilde{X}_{ij}q^i{}_C q^j{}_{(A}\epsilon^C{}_{B)},$$
$$X_{sA} = 2L^{-1}\tilde{X}_{si}q^i{}_B\epsilon^B{}_A. \tag{48}$$

In conclusion, four of the five components of X_{ij} can be expressed in terms of $\tilde{X}_{ij}$. This is in big part—but not exclusively—the main problem that arises in dCS gravity, as we will see in the following. For convenience we will express all equations in terms of 3-dimensional, spatial variables, resorting to the $2+1$ split only where it is necessary for clarity.

3. Constraint equations

The $3+1$ split of the EoMs (25) along the lines of the ADM-York decomposition [36,42,43] yields a set of constraint and time evolution equations. As in GR we obtain scalar $\mathcal{H}$ and vector constraints $\mathcal{M}_i$ considering various projections of the EoMs. Contracting the tensorial Eq. (25) twice with the normal vector n^a yields the scalar constraint,

$$\mathcal{H} = H^{\mathrm{GR}} - \frac{b_{\mathrm{CS}}}{2\kappa}(\Pi^2 + D^i\theta D_i\theta) - \frac{2a_{\mathrm{CS}}}{\kappa}[2A^{ij}\tilde{X}_{ij} - B^{ij}(\Pi A_{ij} - D_iD_j\theta) + (D\times M^{\mathrm{GR}})_i D^i\theta]. \tag{49}$$

We obtain the vector constraint of the dCS gravity model by considering the mixed projection of Eq. (25). The computation gives

$$\mathcal{M}_i = M^{\mathrm{GR}}_i - \frac{b_{\mathrm{CS}}}{2\kappa}\Pi D_i\theta + \frac{a_{\mathrm{CS}}}{\kappa}\epsilon_i{}^{jk}\left(D^l\theta D_j X_{kl} - \frac{1}{2}D_j\theta D^l(3E^{\mathrm{GR}}_{kl} + 4X_{kl}) + A^l{}_j E^{\mathrm{GR}}_{kl}\Pi + (E^{\mathrm{GR}}_{jl} + X_{jl})D^l D_k\theta - \frac{1}{2}D_j M^{\mathrm{GR}}_k\Pi + \frac{1}{2}A^l{}_j\left(M^{\mathrm{GR}}_k D_l\theta - \frac{1}{2}M^{\mathrm{GR}}_l D_k\theta\right)\right) + \frac{a_{\mathrm{CS}}}{\kappa}\left(\frac{1}{2}D^j\theta(3A^k{}_i B_{jk} - A^k{}_j B_{ik}) + B_{ij}\left(\frac{1}{3}KD^j\theta - D^j\Pi\right)\right). \tag{50}$$

These are the model-dependent constraints associated with spatial diffeomorphism invariance and the freedom in the foliation. It is not clear how standard methods for constructing solutions to the constraints in GR could be modified to deal with these constraints when θ and Π are nonvanishing.

4. Evolution equations

We now turn to the derivation of the time evolution equations. Their geometric subset provides the kinematic degrees of freedom describing the evolution of the 3-metric γ_{ij}, the scalar field θ and the trace-free part of the extrinsic curvature A_{ij}. They come from the definitions of the time reduction variables, Eqs. (32) and (33), and of the Weyl tensor, Eq. (31), yielding

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha\left(A_{ij} + \frac{1}{3}\gamma_{ij}K\right) + \mathcal{L}_\beta \gamma_{ij}, \\
\partial_t \theta &= -\alpha \Pi + \mathcal{L}_\beta \theta, \\
\partial_t A_{ij} &= -[D_i D_j \alpha]^{\mathrm{TF}} + \alpha\Big(2X_{ij} + E^{\mathrm{GR}}_{ij} \\
&\quad - A^k_i A_{jk} - \frac{1}{3}\gamma_{ij}A^{kl}A_{kl}\Big) + \mathcal{L}_\beta A_{ij},
\end{aligned} \tag{51}
$$

where E^{GR}_{ij} is given in Eq. (37). The previous expressions have been derived solely from *geometric* relations. The model-dependent, dynamic degrees of freedom enter through the EoMs yielding evolution equations for the time reduction variable of the scalar field, the trace of the extrinsic curvature and the electric part of the Weyl tensor. They encode information about the considered theory of gravity including GR as well as higher derivative modifications. Because we will employ the well-known relations in GR as abbreviations in the following, let them serve as an example. We recover the field equations of GR (minimally coupled to a scalar field) if we set $a_{\mathrm{CS}} = 0$ and $b_{\mathrm{CS}} = 1$ in Eqs. (25). We find

$$
\begin{aligned}
X_{ij} &= -\frac{1}{4\kappa}[D_i\theta D_j\theta]^{\mathrm{TF}}, \\
\partial_t K &= \alpha(\mathcal{R} + K^2) - D^i D_i \alpha - \frac{\alpha}{2\kappa}D^i\theta D_i\theta + \mathcal{L}_\beta K, \\
\partial_t \Pi &= -D^i\alpha D_i\theta - \alpha(D^i D_i\theta - K\Pi) + \mathcal{L}_\beta \Pi.
\end{aligned} \tag{52}
$$

Together with an appropriate choice of gauge conditions for the lapse function and shift vector these relations close the PDE system. We recognize that GR essentially results in four constraints, evolution equations for K and Π and five algebraic relations for the electric part of the Weyl tensor in terms of other $3+1$ quantities. Due to the presence of higher derivative terms this is no longer true in dCS gravity. Using Eqs. (25), we are still able to find evolution equations for the time reduction variable of the scalar field and the trace of the extrinsic curvature, which are

$$
\begin{aligned}
\partial_t \Pi &= -D^i\alpha D_i\theta - \alpha(D^i D_i\theta - K\Pi) \\
&\quad + 4\alpha\frac{a_{\mathrm{CS}}}{b_{\mathrm{CS}}}B^{ij}(X_{ij} + E^{\mathrm{GR}}_{ij}) + \mathcal{L}_\beta \Pi, \\
\partial_t K &= -D^i D_i\alpha + \alpha[H^{\mathrm{GR}} + A_{ij}A^{ij} + \frac{1}{3}K^2] + \mathcal{L}_\beta K \\
&\quad - \alpha a_{\mathrm{CS}}[2A^{ij}\tilde{X}_{ij} + (D\times M^{\mathrm{GR}})_i D^i\theta \\
&\quad - B^{ij}(A_{ij}\Pi - D_i D_j\theta)] - \alpha\frac{b_{\mathrm{CS}}}{2\kappa}D_i\theta D^i\theta.
\end{aligned} \tag{53}
$$

The relation involving the electric part of the Weyl tensor can be derived as the trace-free contribution of the spatial projection of Eq. (25). This computation yields a lengthy equation for $\partial_t \tilde{X}_{ij}$ of the form

$$
\begin{aligned}
\partial_t \tilde{X}_{ij} &\simeq -\frac{\alpha\Pi}{L}\Big(D_s\tilde{X}_{ij} - 2D_{(i}\tilde{X}_{j)s} - s_{(i}D^k\tilde{X}_{j)k} \\
&\quad - \frac{3L}{2}\epsilon_{(i}{}^k s_{j)}D_k X_{ss} + 3s_{(i|}D_s\tilde{X}_{|j)s} \\
&\quad + \gamma_{ij}D^k\tilde{X}_{sk}\Big) + \mathcal{L}_\beta \tilde{X}_{ij}
\end{aligned} \tag{54}
$$

where we present only terms corresponding to the highest spatial derivatives of the metric. The full equation is presented in Appendix B.

5. Closing the system

As we discussed in the beginning of this section, knowledge of $\tilde{X}_{ij}$ is not sufficient to close the PDE system, since it does not yield X_{ss}. It is however possible to derive an algebraic equation for X_{ss} by projecting the trace-free part of the spatial projection of the field equations (25) along the gradient of the scalar field, leading to

$$
\begin{aligned}
\left(1 - \frac{3a^2_{\mathrm{CS}}}{\kappa b_{\mathrm{CS}}}B^2_{ss}\right)X_{ss} - \frac{2a_{\mathrm{CS}}}{L\kappa}\Pi D^i\tilde{X}_{is} &= -\frac{b_{\mathrm{CS}}}{6\kappa}L^2 + \frac{a_{\mathrm{CS}}}{3\kappa}\Pi A^{ij}B_{ij} + \frac{2a_{\mathrm{CS}}}{3\kappa}\Pi K B_{ss} - \frac{a_{\mathrm{CS}}}{\kappa}\Pi A^i{}_s B_{is} + \frac{2a^2_{\mathrm{CS}}}{\kappa b_{\mathrm{CS}}}B^{ij}E^{\mathrm{GR}}_{ij}B_{ss} \\
&\quad + \frac{a_{\mathrm{CS}}}{\kappa}(B_s{}^i D_s D_i\theta - B_{ss}D^i D_i\theta) - \frac{a_{\mathrm{CS}}}{3\kappa}B^{ij}D_i D_j\theta + \frac{2a^2_{\mathrm{CS}}}{L\kappa b_{\mathrm{CS}}}B_{ss}B^{\mathrm{TF}}_{AB}q^{AC}\epsilon^{DB}\tilde{X}^{\mathrm{TF}}_{CD} \\
&\quad - \frac{8a^2_{\mathrm{CS}}}{L\kappa b_{\mathrm{CS}}}B_{ss}B_{sA}\epsilon^{AB}\tilde{X}_{sB} + \frac{2a_{\mathrm{CS}}}{\kappa}A^i{}_s\tilde{E}^{\mathrm{GR}}_{is} - \frac{2a_{\mathrm{CS}}}{L\kappa}D^i\Pi\tilde{E}^{\mathrm{GR}}_{is} + \frac{a_{\mathrm{CS}}}{6\kappa}(D\times M^{\mathrm{GR}})_i D^i\theta \\
&\quad + \frac{a_{\mathrm{CS}}}{L\kappa}\Pi\tilde{A}^i{}_s M^{\mathrm{GR}}_i + \frac{a_{\mathrm{CS}}}{3\kappa}A^{ij}\tilde{X}_{ij} - \frac{2a_{\mathrm{CS}}}{L\kappa}D^i\Pi\tilde{X}_{is} + \frac{2a_{\mathrm{CS}}}{\kappa}A^i{}_s\tilde{X}_{is} - \frac{3a_{\mathrm{CS}}}{2L^2\kappa}\Pi\tilde{X}^{ij}D_i D_j\theta.
\end{aligned} \tag{55}
$$

Besides X_{ss} this expression involves quantities for which the time evolution equations are known and thus this relation closes the evolution equations, provided that we are in the generic situation where $3a^2_{\mathrm{CS}}B^2_{ss} \neq \kappa b_{\mathrm{CS}}$ and $D^i\theta \neq 0$. Once the generic case is examined we ought to treat the special case in which this equation cannot be inverted, and the case in which s^i is not well defined.

6. Structure of the field equations

Let us begin by assuming that we are in the generic situation, and consider the shape of the resulting equations. In short, the system does not have a FTNS structure. The term breaking the structure is the second on the left-hand side of Eq. (55). Since this term is present in the vector constraint (50), one might try eliminating it by adding multiples of the constraint. To avoid any suspense: it is easily checked that a FTNS structure is *not* recovered with this strategy. Indeed, the constraint additionally contains terms of the form $D_i M_j^{\mathrm{GR}}$, involving higher derivatives of the metric, which do not cancel. In order to see explicitly the structural problem, let us keep only terms containing the highest spatial derivative in Eq. (55) and plug these into Eq. (54). We observe the following terms spoiling the FTNS structure

$$\partial_t \tilde{X}_{ij} \sim \Pi s_{(i} \epsilon_{j)}{}^k D_k X_{ss}. \tag{56}$$

These look diffusive and, indeed, with $X_{ss} \sim \Pi D^i \tilde{X}_{si}$, the highest derivative terms acting on $\tilde{X}_{ij}$ are given by

$$\partial_t \tilde{X}_{si} \sim \Pi \epsilon_i{}^j D_j D_k \tilde{X}_s^k, \tag{57}$$

where we have projected Eq. (54) along s^i to show exactly the problematic term.

This second order combination does not necessarily vanish, and cannot be replaced by lower derivatives using the constraints without introducing different higher derivative terms. Note that the simplifying procedure involves pushing s^i inside two derivatives, producing third order derivatives of θ, but these terms are consistent with the FTNS structure. The heart of the problem lies in the fact that the operator $\mathcal{O}$ is not invertible. This leads to one degree of time derivative less in one of the equations, without affecting the number of spatial derivatives. Repeated application of the operator $\mathcal{O}$ does not allow closure of the evolution system in a different way because the operator $\mathcal{O}$ is not nilpotent, as we have shown in Appendix A. This means that in the generic case the system is not consistent with the first requirement of defining a hyperbolic system.

7. Special case I

Let us now turn our attention to the special case for which the coefficient in front of X_{ss} in Eq. (55) vanishes, i.e. $3a_{\mathrm{CS}}^2 B_{ss}^2 = b_{\mathrm{CS}}\kappa$. In this case, it is not possible to recover X_{ss} which implies that we cannot determine all components of X_{ij} and the PDE system is not even closed.

8. Special case II

Now consider the case in which the gradient of the scalar field vanishes, implying that the spatial normal vector $s^i \sim D^i\theta = 0$ is not defined. Let us focus on the full field equation (B2), which is the crucial relation that needs to be solved to close the system. Taking $D_i\theta = 0$ yields

$$\begin{aligned}\partial_t \tilde{X}_{ij} = \alpha\Bigg(&\Pi(D\times X)_{ij} - \frac{\kappa}{a_{\mathrm{CS}}} X_{ij} - \Pi[A^k{}_{(i}B_{j)k}]^{\mathrm{TF}} \\ &+ \frac{2}{3}K\Pi B_{ij} + 2\frac{a_{\mathrm{CS}}}{b_{\mathrm{CS}}} B_{ij}B^{kl}(E_{kl}^{\mathrm{GR}} + X_{kl}) \\ &+ \Pi\epsilon_{(i|}{}^{kl}\left(\frac{1}{2}A_{|j)k}M_l^{\mathrm{GR}} - X_{|j)k}D_l \ln\alpha\right) \\ &+ \epsilon_{(i|}{}^{kl}E_{|j)k}^{\mathrm{GR}} D_l\Pi\Bigg) + \mathcal{L}_\beta \tilde{X}_{ij}.\end{aligned} \tag{58}$$

Although this equation provides a prescription for the time evolution of $\tilde{X}_{ij}$, this tensor vanishes and we have no means to recover X_{ij} which is required to close the PDE system. Instead we could regard Eq. (58) (with the left-hand side vanishing) as the differential relation for X_{ij} only. Albeit this equation appears to be somewhat simpler than in the generic case, we have not found a solution to this differential equation or a way to use it for prescribing a time evolution equation for X_{ij}. Note, that we cannot resort to the $2+1$ decomposition that we employed in the generic case, because $D_i\theta = 0$. Thus, we have not been able to close the PDE system in the case that the scalar field gradient vanishes.

9. Summary of the initial value problem

To set up the Cauchy problem for dCS gravity such that neither of the special cases above occurs initially, one requires initial data for the evolution variables $\gamma_{ij}, A_{ij}, K, \tilde{X}_{ij}$ in the gravity sector and for the fields θ, Π in the scalar sector with the additional conditions that $D_i\theta \neq 0$ and $3a_{\mathrm{CS}}^2 B_{ss}^2 \neq \kappa b_{\mathrm{CS}}$ everywhere. These variables must satisfy the constraints (49)–(50). They evolve according to (51), (53) and (54). Note that the variable $\tilde{X}_{ij}$ could be replaced in the state vector by the electric part of the Weyl tensor E_{ij} according to (40) using (45) and (55). While doing so makes the resulting expressions more cumbersome, it does not affect the basic structure of the system.

D. Well-posedness discussion

It was previously shown in Refs. [37] and [39] that upon linearization around a Schwarzschild or slowly rotating BH background, respectively, the dCS field equations admit superluminal mode solutions, which are damped away. Studies of dCS gravity in the background of a Kerr BH revealed that the scalar field diverges on the inner horizon [18]. As described in Sec. II this type of analysis is not strong enough to draw conclusions about well-posedness of the initial value problem, where we are required to consider an arbitrary background. We also presented the structure that a FT3S system of PDEs must have in order to have a chance to be strongly hyperbolic. We have seen in the previous section that dCS gravity does not have this form. This is most evident by combining Eqs. (54) and (55), in which the operator $\mathcal{O}$ plays the key role.

1. Model for the structure of the dCS equations

For illustration consider the model equation:

$$O\partial_t\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} v \\ u \end{pmatrix}' = 0, \tag{59}$$

where O is the noninvertible and non-nilpotent operator,

$$O = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{60}$$

for definiteness. This is the situation present in dCS gravity where we should think of Ou as $\tilde{X}_{ij}$ and Ov as X_{ss}. The first equation is a differential equation for u while the second is an algebraic relation for v. Plugging the solution for v into the equation for u leads to

$$\dot{u} + u - u'' = 0. \tag{61}$$

This is the heat equation plus a nonprincipal term, which does not have a FTNS structure. Of course, in the dCS equations the specific form of the resulting PDE is not as simple as this, but also does not admit a first order reduction.

2. Gravitational wave degrees of freedom

Constructing hyperbolic formulations of systems with gauge freedom is more subtle than for simpler examples like the wave equation [58]. Therefore one might object to the nonexistence of a first order reduction by suggesting that the problem is related to a poor gauge choice. One approach might be to try and ape the construction of the generalized harmonic formulation of GR, but taking θ as the time coordinate. Since the equation of motion for θ contains $\square\theta$, and the choice simultaneously removes the troublesome $D_i\theta$ terms, this approach initially seems promising. Unfortunately the "gauge source function" would then behave as $E_{ij}B^{ij}$, and these terms would again take a nonhyperbolic character. In fact, since gravitational waves can be thought of as the propagating part of the Weyl tensor and the electric part of the Weyl tensor is $E_{ij} = E^{\mathrm{GR}}_{ij} + X_{ij}$ we have shown that the lack of hyperbolicity in the dCS field equations, in the generic case of $D_i \neq 0$ and keeping the background *arbitrary*, occurs *precisely* in the GW degrees of freedom.

3. Classification of dCS gravity

The dCS field equations admit a set of constraint and evolution PDEs. After appropriate manipulation, their analogue in GR corresponds to a set of elliptic- and hyperbolic-type PDEs, respectively. In contrast, for dCS gravity we have seen that the PDEs encoding the propagation of the Weyl tensor are not hyperbolic. How may we classify them? Since we have higher spatial derivatives the first guess is to check for parabolicity, or perhaps a mixed hyperbolic-parabolic structure. But since the higher derivatives do not appear in the form $a^{ij}\partial_i\partial_j u$ with a^{ij} positive definite, this possibility is also to be discarded, as is that of a mixed hyperbolic-Schrödinger class. A further possibility is that the equations may form a mixed hyperbolic-elliptic system as found for example in Ref. [67]. In such a system some variables appear without evolution equations, instead satisfying elliptic equations. The evolution equations (up to the expected freedom in the choice of lapse and shift) in dCS gravity, however, are generically complete, and so not of this type. Since the equations seem not to lie in any particular PDE class, there is no definitive theory of well-posedness available to fall back on for the initial value problem (IVP). Therefore we present here a preliminary calculation to demonstrate what type of behavior the present higher derivative terms may cause, leaving a detailed study for future work.

4. Cursory mode analysis

Equation (54) prescribes the evolution of 4 degrees of freedom, 2 each in $\tilde{X}_{si}$ and $\tilde{X}_{AB}$. Consider the second and first derivatives terms in these equations, by Fourier transforming the spatial dependence according to $X(t, x^i) = X(t)\exp(i\omega\hat{\omega}_i x^i)$, where $\hat{\omega}_i$ is a unit vector, that we choose to be orthogonal to s^i. Let us further define $\hat{v}^i$ such that $\hat{v}$ is orthogonal to both s^i and $\hat{\omega}_i$. The state vector is $(\tilde{X}_{s\hat{\omega}}, \tilde{X}_{\hat{\omega}\hat{\omega}}, \tilde{X}_{s\hat{v}}, \tilde{X}_{\hat{\omega}\hat{v}})$, and X_{ss} is to be replaced using its equation. Keeping only the highest derivative terms, we get

$$\partial_t\begin{pmatrix} \tilde{X}_{s\hat{\omega}} \\ \tilde{X}_{\hat{\omega}\hat{\omega}} \\ \tilde{X}_{s\hat{v}} \\ \tilde{X}_{\hat{\omega}\hat{v}} \end{pmatrix} = -i\omega\frac{\Pi}{L}\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ C_1\omega & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{X}_{s\hat{\omega}} \\ \tilde{X}_{\hat{\omega}\hat{\omega}} \\ \tilde{X}_{s\hat{v}} \\ \tilde{X}_{\hat{\omega}\hat{v}} \end{pmatrix}, \tag{62}$$

where

$$C_1 = -\frac{3ia_{\mathrm{CS}}\Pi}{L(1 - 3\frac{a_{\mathrm{CS}}}{\kappa b_{\mathrm{CS}}}B_{ss}^2)}. \tag{63}$$

Generically $C_1 \neq 0$ since otherwise either $a_{\mathrm{CS}} = 0$, implying that the dCS modification disappears, or $\Pi = 0$. If the latter condition holds everywhere this is not dynamical dCS gravity.

This equation is obtained under the simplifying assumptions that B_{ij}, s^i, L, Π are constant. This approximation is justified by the fact that PDE analysis implies freezing coefficients and treating them as independent. For consistency, we should have kept $D_i B_{jk}$ terms and $D_i M_j^{\mathrm{GR}}$, but our aim here is just to point out the effect that higher derivative terms are likely to have in this type of analysis. Except for the additional factor of ω inside the matrix in Eq. (62), the matrix looks like the principal symbol of a weakly hyperbolic PDE. Computing the general solution one finds frequency-dependent growth like $\omega^2 t$, so the problem is ill posed.

5. A final model

Now consider a model problem indicating the implausibility of obtaining well-posedness of the IVP for the full dCS system. Take

$$\partial_t u = a\partial_x u + b\partial_x v,$$
$$\partial_t v = c\partial_x^2 u + d\partial_x v, \tag{64}$$

with a, b, c and d real constants. Fourier transforming in space we have an ordinary differential equation (ODE) with

$$\partial_t \tilde{U} = M\tilde{U}, \tag{65}$$

where $\tilde{U} = (\tilde{u}, \tilde{v})^T$. Assume that c is nonzero; otherwise we are in the standard first order case. For brevity we also assume that $\omega > 0$. The symbol is

$$M = \begin{pmatrix} a & b \\ i\omega c & d \end{pmatrix} i\omega. \tag{66}$$

If $b \neq 0$ the eigenvalues of the symbol are

$$\lambda_\pm = \frac{1}{2} i\omega\left[(a+d) \pm \sqrt{(a-d)^2 - 4i\omega bc}\right], \tag{67}$$

which results in mode solutions that propagate with arbitrarily fast group velocity and, worse, blow up in a frequency-dependent exponential manner. Thus, the IVP is ill posed. Assume next that $b = 0$; then the eigenvalues of the symbol are $i\omega a$ and $i\omega d$. If furthermore $a \neq d$ the symbol can be diagonalized by the similarity matrix,

$$S = \begin{pmatrix} 1 & 0 \\ \frac{c}{a-d} i\omega & 1 \end{pmatrix}. \tag{68}$$

But as $\omega \to \infty$, we find that $|S|$ diverges, which prevents application of the Kreiss matrix theorem (see Theorem 2.4.1 in [51]) to build estimates on solutions; the PDE is once again ill posed. Finally consider the case that $a = d$. This is closest to what we obtained for dCS gravity in Eq. (62). In this case the symbol is not diagonalizable. We again find that the system is ill posed, although only with growth like $\omega^2 t$. Although this seems the mildest ill-posedness, note that in the presence of lower order terms this growth becomes as rapid as before. In summary, the model problem (64) always has an ill-posed IVP. Naturally one should not use this sketch to draw conclusions about the full dCS theory, but in the absence of a simple model of the same structure with a well-posed IVP, there seems little reason to be optimistic.

V. CHERN-SIMONS GRAVITY AS AN EFFECTIVE THEORY

It has been argued that dCS gravity can be treated as an effective field theory [5–7,9] in which the coupling constant is treated as a parameter in a perturbative expansion around GR. We explore the effective field theory approach in two steps. In Sec. V A we start with a "reduced-order model" suggested in Ref. [9], in which the effective EoMs have at most second derivatives of the metric but the dynamical variables are left arbitrary. In Sec. V B we follow the more common approach (see e.g. Refs. [13,17–19]) and perform an order-by-order reduction, in which both the EoMs and dynamical variables are expanded in terms of the dCS coupling parameter.

A. Reduced-order model

Under the small-coupling assumption we can remove the higher derivative terms in the EoMs (25) that prevent dCS gravity, when regarded as a "full" theory, from having a FTNS shape. We accomplish this order reduction by substituting the trace-reversed form of Eq. (25) into the C-tensor (27) and keeping only terms up to $\mathcal{O}(a_{\mathrm{CS}})$. With this treatment the higher derivative terms are replaced by derivatives of the C-tensor thus becoming a contribution of order $\mathcal{O}(a_{\mathrm{CS}}^2)$ which we discard. This procedure yields modified EoMs

$$G_{ab} + \frac{a_{\mathrm{CS}}}{\kappa} C_{ab}^{(2)} - \frac{b_{\mathrm{CS}}}{2\kappa} T_{ab}^{\theta} = 0,$$
$$\Box\theta + \frac{a_{\mathrm{CS}}}{4b_{\mathrm{CS}}} {}^*RR = 0, \tag{69}$$

where the energy-momentum tensor T_{ab}^{θ} is given by Eq. (26) and

$$C_{ab}^{(2)} = (\nabla^c \nabla^d \theta)^* W_{d(ab)c} \tag{70}$$

denotes the second term of the C-tensor given in Eq. (28). In order to analyze the PDE structure of the order-reduced equations of motion (69) we perform a spacetime split and formulate them as a first order in time PDE system. In analogy to Sec. IV we employ the electromagnetic decomposition of the Weyl tensor (30) and, in particular, we will again use the tensor $X_{ij} = E_{ij} - E_{ij}^{\mathrm{GR}}$ instead of the electric part E_{ij} itself.

The kinematic evolution equations which result from geometry, i.e. those for the 3-metric γ_{ij}, scalar field θ and trace-free part A_{ij} of the extrinsic curvature, remain unaltered and are given by Eqs. (51). On the other hand, the dynamic degrees of freedom given in the EoMs determine the constraints and the evolution equations for the momentum Π of the scalar field, the trace K of the extrinsic curvature and a relation for X_{ij}. Considering the order-reduced EoMs (69) we find the scalar and vector constraints

$$\begin{aligned}\mathcal{H} &= H^{\mathrm{GR}} - \frac{b_{\mathrm{CS}}}{2\kappa}(\Pi^2 + D^i\theta D_i\theta) \\ &\quad + \frac{2a_{\mathrm{CS}}}{\kappa}B^{ij}(\Pi A_{ij} - D_iD_j\theta), \\ \mathcal{M}_i &= M_i^{\mathrm{GR}} - \frac{b_{\mathrm{CS}}}{2\kappa}\Pi D_i\theta \\ &\quad + \frac{a_{\mathrm{CS}}}{\kappa}B_{ij}\left(A^{jk}D_k\theta + \frac{1}{3}KD^j\theta - D^j\Pi\right) \\ &\quad + \frac{a_{\mathrm{CS}}}{\kappa}\epsilon_i{}^{jk}(\Pi A^l{}_j - D^lD_j\theta)(E_{kl}^{\mathrm{GR}} + X_{kl}), \qquad (71)\end{aligned}$$

where H^{GR} and M_i^{GR} are the Hamiltonian and momentum constraints for vacuum GR given by Eqs. (39) and (38). The time evolution of the scalar field momentum and the trace of the extrinsic curvature are prescribed by

$$\begin{aligned}\partial_t\Pi &= -D^i\theta D_i\alpha - \alpha(D^iD_i\theta - K\Pi) \\ &\quad + 4\alpha\frac{a_{\mathrm{CS}}}{b_{\mathrm{CS}}}B^{ij}(X_{ij} + E_{ij}^{\mathrm{GR}}) + \mathcal{L}_\beta\Pi, \\ \partial_tK &= -D^iD_i\alpha + \alpha\left(H^{\mathrm{GR}} + A_{ij}A^{ij} + \frac{K^2}{3}\right) + \mathcal{L}_\beta K \\ &\quad + \alpha\frac{a_{\mathrm{CS}}}{\kappa}B^{ij}(\Pi A_{ij} - D_iD_j\theta) - \alpha\frac{b_{\mathrm{CS}}}{2\kappa}D^i\theta D_i\theta. \qquad (72)\end{aligned}$$

The final piece of information comes from the (trace-free part of the) spatial projection of the EoMs. In contrast to the full theory the order-reduced model (69) provides a relation *algebraic* in X_{ij}, as in GR. Keeping only terms up to $\mathcal{O}(a_{\mathrm{CS}})$ yields

$$\begin{aligned}X_{ij} &= \frac{a_{\mathrm{CS}}}{\kappa}\bigg(\frac{2}{3}K\Pi B_{ij} - \Pi[B_{k(i}A^k{}_{j)}]^{\mathrm{TF}} \\ &\quad + [B_{k(i}D_{j)}D^k\theta]^{\mathrm{TF}} - B_{ij}D^kD_k\theta \\ &\quad + \epsilon_{(i|}{}^{kl}E_{|j)k}^{\mathrm{GR}}\left(D_l\Pi - \frac{1}{3}KD_l\theta - A_{lm}D^m\theta\right)\bigg) \\ &\quad + \frac{a_{\mathrm{CS}}b_{\mathrm{CS}}}{4\kappa^2}\epsilon_{(i|}{}^{kl}D_{|j)}\theta D_l\theta(D_k\Pi - A_{kn}D^n\theta) \\ &\quad - \frac{b_{\mathrm{CS}}}{4\kappa}[D_iD_j\theta]^{\mathrm{TF}}. \qquad (73)\end{aligned}$$

We have been able to eliminate all terms involving a coupling between X_{ij} and the gradient of the scalar field θ. In the case of small couplings to the dCS correction Eq. (73) closes the system of evolution equations for any value of the scalar field.

The relation (73) involves terms that have at most second spatial derivatives of the metric (given by terms $\sim E_{ij}^{\mathrm{GR}} \sim \mathcal{R}_{ij}^{\mathrm{TF}}$) and in the scalar field and at most first spatial derivatives of the extrinsic curvature [given by terms $\sim B_{ij} = (D \times A)_{ij}$]. This implies that the entire system of evolution PDEs of the order-reduced dCS model given by Eqs. (51), (72) and (73) has a FTNS structure (specifically FT2S), and has a chance to be hyperbolic. In contrast to the situation in GR, however, the coefficients entering the principal part do not depend only on the metric, but also connection terms, for example K and A_{ij}. In a full hyperbolicity analysis we must take these coefficients to have arbitrary values in the background solution, and then we expect that there will be situations in which the resulting linearized equations are not hyperbolic. Somehow these background solutions will have to be disallowed by the theory, if the IVP is to be well posed for all admissible initial data.

The computation presented in this section shows that, unlike the full field equations, the order-reduced dCS model admits a first order reduction. The order-reduced field equations take the form needed for the application of the Cauchy-Kowalevskya theorem, as applied to Lovelock gravity in Ref. [68]. The calculation thus supports the claim that a higher derivative gravity theory may be transformed into a hyperbolic system, by employing the order-reduction method for effective field theories (see, e.g. Ref. [9] and references therein). However, hyperbolicity of the resulting equations will depend crucially on the background solution under consideration.

The key assumption underlying this discussion is not only a small coupling, but also that the higher derivative terms in the series expansion modifying GR are at most of the same magnitude as the lower derivative terms, so that terms of order $\mathcal{O}(a_{\mathrm{CS}}^2)$ are negligible. This is in direct contradiction with the approximations made in the PDEs' analysis, in which the highest derivative terms are taken to dominate. Even given initial data satisfying the condition it is not clear whether the higher derivative terms will remain small in the course of the evolution, unless it is enforced explicitly by the numerical scheme.

B. Small-coupling expansion

In this section we treat dCS gravity as an "effective theory" that would be solved order by order in the perturbation parameter which is taken to be the coupling. Using a simple counting argument we will show that to *every order* in the dCS coupling, i.e. to every order in the perturbation, the EoMs can (i) be formulated as first order in time reduction of the theory, and (ii) have the structure of a hyperbolic PDE system.

Let us assume that the metric and the scalar field can be expanded according to

$$\theta = \sum_N c^N \theta^{(N)}, \qquad g_{ab} = \sum_N c^N g_{ab}^{(N)}, \tag{74}$$

where $c = a_{\rm CS}/\kappa$. Note that we chose to scale the coupling with κ, but it can equivalently be scaled with $b_{\rm CS}$. We stress that the expansion is made over the coupling parameter, which is formally different than a "regular" perturbative approach because the small parameter of the perturbative approach appears explicitly in the field equations. The approach we follow here is somehow similar to that in Refs. [18,19] where the rotating BH solution in dCS gravity is approximated with a perturbation in the coupling. Assuming that we know all the fields up to order c^{N-1} for a given value of N, the equations for the components of the metric and scalar field at order N are given by a linear perturbation around the background of the metric and scalar field truncated up to order c^{N-1}. The important point of the argument is that the basic properties of the PDE system are encoded in the principal symbol of the equations, which are unaffected by lower order terms in the coupling c. These lower order terms appear as sources for the equations at order c^N. In order to show explicitly this statement, let us first formally expand the d'Alembertian in power of c:

$$\Box\theta \approx \sum_{n=0}^{\infty}\sum_{i=0}^{n} c^n \Box_{[n-i]}\theta^{(i)}, \tag{75}$$

where $\Box_{[i]}$ is the d'Alembertian truncated at order i in the expansion. The d'Alembertian $\Box_N$ at order N is explicitly given by

$$\Box_N = -\frac{1}{2} g^{(N)}{}_c{}^c \Box_0 + \frac{1}{\sqrt{-g^0}}\partial_a\sqrt{-g^{(0)}}\left(g^{(N)ab} + \frac{1}{2} g^{(0)ab} g^{(N)}{}_c{}^c\right)\partial_b. \tag{76}$$

The strategy we use is to build a linear perturbation, say δg_{ab} around the order c^{N-1} background, say g_{ab}, and set $\delta g_{ab} = c^N g_{ab}^{(N)}$, $g_{ab} = \sum_{i=0}^{N-1} c^i g_{ab}^{(i)}$. Then, the terms contributing to c^N are all of the form $g^{(N)} g^{(0)}$. Along these lines, indices of $g_{ab}^{(N)}$ are raised and lowered with $g_{ab}^{(0)}$.

To order c^N, the fields to be solved for are $\theta^{(N)}$ and $g_{ij}^{(N)}$. The two terms involving these fields in Eq. (75) are $\Box_0\theta^{(N)}$ and $\Box_N\theta^{(0)}$, and their derivative structure is of the schematic form

$$\partial^2\theta^{(N)} + \partial g_{ab}^{(N)} = \text{Source terms}. \tag{77}$$

The same reasoning applies to the gravitational equations, where the dynamical part coming from the Einstein tensor has the structure

$$\partial^2 g_{ab}^{(N)} + \partial g_{ab}^{(N)} = \text{Source terms}. \tag{78}$$

Finally, the terms causing the pathologies of the nonlinear theory come from the C-tensor and are always associated with the coupling $c = a_{\rm CS}/\kappa$, reducing the overall order of such terms by 1. As a consequence, to a given order N, these pathological terms are always evaluated from solutions of order $N-1$, i.e. terms already known in an iterative scheme.

The whole argument works only if everything is well defined to order c^0; i.e. the background is a solution of GR minimally coupled to a scalar field. Then the argument can be applied iteratively. This is indeed the case, since to order c^0, the theory is only GR with a scalar field, which is known to pose no problem.

In summary, the equations at order N have the following structure:

$$\begin{aligned}
\Box_0\theta^{(N)} &= V'(\theta)|_{c=0}\theta^{(N)} + \text{l.o.t.},\\
\frac{1}{2}(\Delta^{\rm GR})^{(0)}_{ab}{}^{cd} g^{(N)}{}_{cd} &= \frac{b_{\rm CS}}{\kappa}\left(\nabla^{(0)}_{(a}\theta^{(N)}\nabla^{(0)}_{b)}\theta^{(0)} - \frac{1}{2}\nabla^{(0)}_c\theta^{(N)}\nabla^{(0)c}\theta^{(0)} g^{(0)}_{ab}\right) + \text{l.o.t.},
\end{aligned} \tag{79}$$

where $(\Delta^{\rm GR})^{(0)}$ is the operator governing perturbations around an arbitrary background in GR, evaluated with the order c^0 background, $\nabla^{(0)}_a$ is the covariant derivative compatible with $g^{(0)}$, and "l.o.t." denotes lower order terms.

As a consequence, the principal symbol at order c^N is schematically given by

$$\mathcal{P} = \begin{pmatrix} \Box_0 & 0 \\ 0 & (\Delta^{\rm GR})^{(0)} \end{pmatrix}, \tag{80}$$

where it is understood that $\mathcal{P}$ acts on v,

$$v = \begin{pmatrix} \theta^{(N)} \\ g^{(N)}_{ab} \end{pmatrix}. \tag{81}$$

In other words, in an effective approach where all the corrections in the coupling are computed order by order, the highest order operator decouples. This implies that the third order derivatives always appear only in the source terms and dCS gravity—when treated as an *effective theory*—can be formulated as a hyperbolic set of PDEs.

The zeroth order in the coupling trivially reduces to the Einstein equations:

$$\begin{aligned}
G_{ab}(g^{(0)}_{cd}) &= \frac{b_{\rm CS}}{2\kappa}\left(\nabla_a\theta^{(0)}\nabla_b\theta^{(0)} - \frac{1}{2}\nabla_c\theta^{(0)}\nabla^c\theta^{(0)} g^{(0)}_{ab} + V(\theta^{(0)}) g^{(0)}_{ab}\right),\\
\Box^{(0)}\theta^{(0)} &= V'(\theta^{(0)}).
\end{aligned} \tag{82}$$

The first order in c correction given by the dCS modification is then given by

$$\frac{1}{2}(\Delta^{\mathrm{GR}})^{(0)cd}_{ab}g^{(1)}_{cd} + C_{ab}(g^{(0)}_{cd},\theta^{(0)}) = \frac{b_{\mathrm{CS}}}{2\kappa}\Big(2\partial_{(a}\theta^{(1)}\partial_{b)}\theta^{(0)} - \partial_c\theta^{(1)}\partial^c\theta^{(0)}g^{(0)}_{ab} - \frac{1}{2}\partial_c\theta^{(0)}\partial^c\theta^{(0)}g^{(1)}_{ab} + V(\theta^{(0)})g^{(1)}_{ab} + V'(\theta^{(0)})\theta^{(1)}g^{(0)}_{ab}\Big),$$

$$\Box^{(0)}\theta^{(1)} + \frac{1}{2}\partial_a\theta^{(0)}\nabla^{(0)}g^{(1)c}_{c} = -\frac{\kappa}{4b_{\mathrm{CS}}}{}^*R^{(0)}{}_{abcd}R^{(0)abcd}, \tag{83}$$

where $\Box^{(0)}$ is the d'Alembertian constructed from $g^{(0)}$ only, and $C(g^{(0)},\theta^{(0)})_{ab}$ is the C-tensor evaluated with the zeroth order terms of the metric and scalar field.

Higher orders become cumbersome but the structure is the same: the terms causing troubles in the nonexpanded theory are now evaluated on lower order in the coupling. In conclusion, the dynamical Chern-Simons model, treated in this manner, can be made hyperbolic in the same way as GR minimally coupled to a scalar field.

VI. CONCLUSIONS

We have investigated the initial value formulation and PDE structure of dynamical Chern-Simons gravity. This modification of GR, motivated for example by string theory, loop-quantum gravity or cosmology, has recently attracted a lot of attention. Previous studies have been concerned with the construction of solutions to dCS gravity and their stability properties, but well-posedness of the initial value problem has remained outstanding. We have started filling this gap by deriving an initial value formulation of dCS theory and investigating its PDE structure.

We encountered a number of difficulties. First, in the generic situation when the spatial gradient of the Chern-Simons field is nonvanishing, if additionally $3a_{\mathrm{CS}}^2B_{ss}^2 = b_{\mathrm{CS}}\kappa$, the field equations do not close. This means that given suitable initial data for the variables $\gamma_{ij},\theta,A_{ij},K,\tilde{X}_{ij}$ and Π we cannot compute all components of the time derivative of the trace-free part of the extrinsic curvature because the electric part of the Weyl tensor $E_{ij} = E^{\mathrm{GR}}_{ij} + X_{ij}$ is not completely determined. Likewise when the scalar field gradient vanishes it seems impossible to obtain the electric part of the Weyl tensor, and again we cannot compute the time derivative of A_{ij}. To avoid either pathology one would have to demonstrate that these cases cannot occur.

Next, in the generic case that the spatial gradient of the scalar field is nonvanishing and $3a_{\mathrm{CS}}^2B_{ss}^2 \neq b_{\mathrm{CS}}\kappa$ we succeeded in formulating dCS gravity as an evolution problem. But we found that the higher derivative terms present in the dCS gravity EoMs have a different structure than CS electromagnetism. A crucial tool in investigating well-posedness of a hyperbolic PDE system (following for example Refs. [45,53,54]) is the use of a first order reduction. The dCS gravity EoMs do not admit such a reduction, and so are not hyperbolic in this sense. Therefore one would naively expect that even if the dCS IVP could be made well posed, signals could propagate arbitrarily fast. But, in fact, in a very rough mode analysis obtained by taking a subset of the full EoMs, we do not find unbounded speeds, but instead that the IVP admits frequency-dependent growth of solutions, and so is ill posed. The evolution PDEs of dCS gravity do not fall into any of the standard PDE classifications. To understand what problems the higher derivative terms might cause we looked systematically at a simple toy model with its structure inspired by dCS. The toy always has an ill-posed initial value problem regardless of how the various parameters present were chosen. It seems that further advances in PDE theory will be needed to make conclusive statements about the well-posedness of the IVP of dCS gravity, but the expectation gained from the analysis of simplified models is that it will be ill posed.

Perhaps anticipating this result, it has been argued that dCS gravity should instead be viewed as an effective model resulting from a more fundamental theory. Taking on this viewpoint the dCS modifications are treated as the lowest order contribution in a series expansion around GR. We have order reduced the EoMs to eliminate the higher derivative terms yielding a systematically well-defined time evolution formulation. While a proper hyperbolicity analysis of the order-reduced PDE system is beyond the scope of this paper, we note that it is a FTNS system and can potentially be cast into a strongly hyperbolic problem. That said, this potential seems unlikely to be realized generically because, in contrast to GR, the resulting principal symbol contains multiple tensor fields. Somehow the field equations will have to disallow any "bad" combination of these fields. One might expect a similar situation in dilaton Gauss-Bonnet gravity.

We have taken the previous treatment, focusing on a series expansion only of the EoMs, a step further and considered perturbations of the metric and scalar field around an arbitrary background where the expansion parameter is given by the dCS coupling constant. This case closely resembles most previous studies involving dCS gravity. We have shown that in this order-by-order expansion the higher derivative contributions always only enter as lower order source terms to Einstein's equations.

Several further assumptions besides the coupling constant being small underlie this computation. To justify the small-coupling assumption it has been argued that the dCS modification itself can be interpreted as the lowest order contribution to a series expansion of the underlying theory which would take the form $\mathcal{L} \sim \sum_n a^n \mathcal{O}(R^{n+1})$. The effective field theory approximation, i.e. truncating the model at $\mathcal{O}(a)$, can only be valid if terms at different orders are at most comparable to each other, for which there is no guarantee. This assumption is particularly questionable in dynamical scenarios. Consider some solution to dCS gravity in the small-coupling limit, e.g. the approximate superposition of two Schwarzschild BHs, as the initial configuration. One could investigate the dynamical evolution of this system using the Cauchy formulation of the order-reduced model, which can be cast into a time evolution system. However, near the plunge of the two BHs higher curvature modifications may become important, possibly exceeding the energy cutoff, and the small-coupling approximation would break down. This suspicion is supported by a recent study exploring highly rotating BHs in dCS gravity [19], where it has been shown that the range of validity of the perturbative approach (considering only the dCS modification) shrinks with increasing BH spin.

Finally, thinking of the "more fundamental" theory as being string theory, it is tempting to relate the pathology in the effective theory, dCS gravity, to the origin of the modification to GR. Recall that the dCS term derives from an anomaly cancellation in the gravitational sector of the 10-dimensional heterotic string model. We argued that the anomaly cancellation procedure seems to have the same derivative structure as the effective dCS model. This suggests that a careful analysis of the anomaly canceled model should be carried out. One might worry about the procedure itself when the base field theory has a Lagrangian with the same structure as GR, though we will not enter this debate here.

ACKNOWLEDGMENTS

We thank V. Cardoso for carefully proofreading the manuscript. We thank V. Cardoso, T.-j. Chen, L. Gualtieri, E. Lim, T. Sotiriou, P. Spindel, L. Stein and N. Yunes for many useful discussions and feedback. H. W. acknowledges the kind hospitality of the Yukawa Institute for Theoretical Physics at Kyoto University during the YITP-T-14-1 workshop on "Holographic vistas on Gravity and Strings." This work was supported by the FCT– Portugal project No. CERN/FP/123593/2011, the FP7 ERC Starting Grant "The dynamics of black holes: testing the limits of Einstein's theory" Grant No. DyBHo-256667, the STFC GR Rolling Grant No. ST/L000636/1, the NRHEP 295189 FP7-PEOPLE-2011-IRSES Grant and the Action de Recherche Concertée (ARC) Contract No. AUWB-2010/15-UMONS-1. D. H. is supported in part by DFG grant SFB/Transregio 7 "Gravitational Wave Astronomy." H. W. acknowledges financial support provided under the *ERC-2011-StG 279363–HiDGR* ERC Starting Grant.

APPENDIX A: The case of $\mathcal{O}^n$

In this section, we show that repeated application of the operator $\mathcal{O}$ on itself never vanishes; in other words that the operator $\mathcal{O}$ is not nilpotent. If this were the case, we could close the system by defining a series of new fields of the form $X_n \coloneqq \mathcal{O}^n \mathcal{L}_n X$ and end up with an equation for $\mathcal{L}_n X_n$.

We use the notation introduced in Sec. IV. Recall that the operator $\mathcal{O}$ is written as

$$\mathcal{O}_{ij}{}^{kl} = Lq_{(i}{}^{(l}\epsilon_{j)}{}^{k)} + Ls_{(i}s^{(l}\epsilon_{j)}{}^{k)}. \tag{A1}$$

Recall the following useful relation:

$$\epsilon_{ik}\epsilon_{jl} = q_{ij}q_{kl} - q_{ik}q_{jl}. \tag{A2}$$

It is a crucial remark that the term involving a Lie derivative of the electric part of the Weyl tensor in Eq. (54) is precisely given by $\mathcal{O}_{ij}{}^{kl}E_{kl}$, giving support to the idea that a decomposition along the gradient of θ is relevant.

Repeated applications of $\mathcal{O}$ consist in contracting the last two indices of $\mathcal{O}$ with the first two indices of the next occurrence, e.g. $(\mathcal{O}^2)_{ij}{}^{mn} = \mathcal{O}_{ij}{}^{kl}\mathcal{O}_{kl}{}^{mn}$.

Straightforward algebra then shows that powers of $\mathcal{O}$ are given by

$$\begin{aligned}
(\mathcal{O}^{4n+2})_{ij}{}^{kl} &= \frac{1}{4^n}L^{4n+2}\left(\frac{1}{2}q_{ij}q^{kl} - q_{(i}^{(k}q_{j)}^{l)}\right), \quad n \in \mathbb{N},\\
(\mathcal{O}^{4n})_{ij}{}^{kl} &= \frac{1}{2^{2n-1}}L^{4n}q_{(i}^{(k}q_{j)}^{l)}, \quad n \in \mathbb{N}^*,\\
(\mathcal{O}^{2n+1})_{ij}{}^{kl} &= \frac{(-1)^n}{2^n}L^{2n+1}q_{(i}^{(k}\epsilon_{j)}^{l)}, \quad n \in \mathbb{N}^*,
\end{aligned} \tag{A3}$$

which completes the proof.

APPENDIX B: EVOLUTION EQUATION FOR $\tilde{X}_{ij}$

For completeness we present the entire time evolution equation for the dynamical variable $\tilde{X}_{ij}$ which contains the electric part of the Weyl tensor. We have presented its highest derivative terms in Eq. (54), highlighted in boldface in expression (B2) below. The trace-free part of the EoMs (25) fully projected onto the spatial slice is given by

$$-2X_{ij} + \frac{a_{\rm CS}}{\kappa}C^{\rm TF}_{ij} - \frac{b_{\rm CS}}{2\kappa}L^2 s_i s_j = 0, \quad \text{with} \quad C^{\rm TF}_{ij} = \left(\gamma^k{}_i\gamma^l{}_j - \frac{1}{3}\gamma_{ij}\gamma^{kl}\right)C_{kl}. \tag{B1}$$

Employing the notation $\mathcal{L}_n\tilde{X}_{ij} = \frac{1}{\alpha}(\partial_t - \mathcal{L}_\beta)\tilde{X}_{ij}$, the trace-free, spatial projection of the C-tensor is

$$
\begin{aligned}
C_{ij}^{\mathrm{TF}} = & -\mathcal{L}_n \tilde{X}_{ij} + \frac{3}{2}\epsilon_{(i}{}^{k} s_{j)} \Pi \mathbf{D_k X_{ss}} + \Pi\left(\frac{2}{3} K B_{ij} - [A^{k}{}_{(i} B_{j)k}]^{\mathrm{TF}}\right) - A^{k}{}_{(i}\tilde{X}_{j)k} - \frac{1}{3} K \tilde{X}_{ij} + A^{kl}\tilde{X}_{kl}\left(s_i s_j - \frac{2}{3}\gamma_{ij}\right) \\
& - 3 s_{(i} A_{j)}{}^{k} \tilde{X}_{sk} + \tilde{X}_{s(i} A_{j)s} - s_{(i}\tilde{X}_{j)k} A^{k}_{s} + 2 s_i s_j A^{s}_{k} \tilde{X}_{sk} - 3 A_{ss} s_{(i}\tilde{X}_{j)s} + \frac{3}{2}\Pi \epsilon_{(i}{}^{k} s_{j)} X_{ss} D_k \ln\alpha \\
& + \frac{3}{2}\Pi X_{ss}(\epsilon_{(i|}{}^{k} D_k s_{|j)} - \epsilon_{(i}{}^{k} s_{j)} D_s s_k + s_i s_j \epsilon^{kl} D_k s_l) + \epsilon_{(i}{}^{k} E^{\mathrm{GR}}_{j)k} D_s \Pi + \epsilon^{kl} s_{(i} E^{\mathrm{GR}}_{j)k} D_l \Pi - \epsilon_{(i}{}^{k} E^{\mathrm{GR}}_{j)s} D_k \Pi \\
& + \frac{1}{2}\Pi(\epsilon^{kl} s_{(i} A_{j)k} M^{\mathrm{GR}}_{l} - \epsilon_{(i}{}^{k} A_{j)s} M^{\mathrm{GR}}_{k} + \epsilon_{(i}{}^{k} A_{j)k} M^{\mathrm{GR}}_{s}) \\
& + L\bigg[-\frac{1}{4} s_{(i}\epsilon_{j)}{}^{k} D_s M^{\mathrm{GR}}_{k} + B_{k(i} D^{k} s_{j)} - B_{ij} D^{k} s_k - \frac{1}{3}\gamma_{ij} D^{k} B_{sk} - \frac{1}{2}\epsilon_{(i}{}^{k} A_{j)k}(X_{ss} + H^{\mathrm{GR}}) + \frac{3}{2} s_{(i}\epsilon_{j)}{}^{k} A_{sk} X_{ss} \\
& + \frac{1}{2}\epsilon_{(i}{}^{k} E^{\mathrm{GR}}_{j)l} A^{l}{}_{k} - \frac{1}{2}\epsilon_{(i}{}^{k} A_{j)}{}^{l} E^{\mathrm{GR}}_{kl} - \epsilon_{(i}{}^{k} E^{\mathrm{GR}}_{j)k}\left(A_{ss} + \frac{1}{3} K\right) + \epsilon_{(i}{}^{k} E^{\mathrm{GR}}_{j)s} A_{sk} + \frac{1}{2}\epsilon^{kl} A^{m}{}_{k} E^{\mathrm{GR}}_{lm}\left(\frac{1}{3}\gamma_{ij} - s_i s_j\right) \\
& + \frac{1}{2} s_{(i}\epsilon_{j)}{}^{k}(E^{\mathrm{GR}}_{kl} A^{l}_{s} - A^{l}{}_{k} E^{\mathrm{GR}}_{ls}) - \epsilon^{kl} s_{(i} E^{\mathrm{GR}}_{j)k} A_{sl} + \frac{1}{4} s_{(i}\epsilon_{j)}{}^{k} D_k M^{\mathrm{GR}}_{s} + \frac{1}{4}\epsilon_{(i}{}^{k} D_{j)} M^{\mathrm{GR}}_{k} + \frac{1}{4}\epsilon_{(i|}{}^{k} D_k M^{\mathrm{GR}}_{|j)} \\
& + \frac{1}{2}\epsilon_{(i}{}^{k} M^{\mathrm{GR}}_{j)} D_k \ln\alpha + \frac{1}{2}\epsilon_{(i|}{}^{k} M^{\mathrm{GR}}_{k} D_{|j)} \ln\alpha + \frac{1}{4}\epsilon^{kl} D_k M^{\mathrm{GR}}_{l}\left(s_i s_j + \frac{1}{3}\gamma_{ij}\right)\bigg] \\
& + \frac{\Pi}{L}\bigg[-\mathbf{D_s \tilde{X}_{ij}} + \frac{1}{2}\mathbf{D_{(i}\tilde{X}_{j)s}} + \mathbf{s_{(i} D^{k}\tilde{X}_{j)k}} - 3\mathbf{s_{(i|} D_s \tilde{X}_{|j)s}} - 2\gamma_{ij}\mathbf{D^{k}\tilde{X}_{sk}} + 3 s_i s_j D^{k}\tilde{X}_{sk} \\
& + \frac{1}{2}(\epsilon_{(i}{}^{k}\tilde{X}_{j)l} D_s \epsilon_{kl} + \tilde{X}^{l}{}_{k}\epsilon_{(i|}{}^{k} D_s \epsilon_{|j)l} - 3 s_{(i}\epsilon_{j)}{}^{l}\tilde{X}^{k}_{s} D_s \epsilon_{kl}) - \frac{7}{2}\tilde{X}_{s(i|} D_s s_{|j)} \\
& - (\tilde{X}_{ij} + 3 s_{(i}\tilde{X}_{j)s}) D_s \ln\alpha - \frac{1}{2}\tilde{X}_{ij} D_k s^{k} + \frac{1}{2}\tilde{X}_{k(i} D_{j)} s^{k} + \frac{1}{2}\gamma_{ij}\tilde{X}_{kl} D^{k} s^{l} - \frac{3}{2} s_i s_j \tilde{X}^{k}_{s}\epsilon^{lm} D_m \epsilon_{kl} \\
& + \frac{1}{2}\tilde{X}_{k(i} D^{k} s_{j)} - s_{(i}\tilde{X}_{j)s} D_k s^{k} + \frac{9}{2}\tilde{X}_{sk} s_{(i} D^{k} s_{j)} - 2\epsilon_{(i|}{}^{k}\tilde{X}^{l}_{s} D_k \epsilon_{|j)l} \\
& + \frac{1}{2}(s_{(i}\tilde{X}^{k}{}_{j)}\epsilon^{lm} D_l \epsilon_{km} - \epsilon_{k}{}^{m}\tilde{X}_{lm} s_{(i} D^{k}\epsilon_{j)}{}^{l}) + 2\tilde{X}_{s(i} D_{j)} \ln\alpha + s_{(i}\tilde{X}_{j)k} D^{k} \ln\alpha + (3 s_i s_j - 2\gamma_{ij})\tilde{X}_{sk} D^{k} \ln\alpha\bigg] \\
& + \frac{b_{\mathrm{CS}} L^{3}}{6\kappa}\epsilon_{(i}{}^{k} A_{j)k} + \frac{a_{\mathrm{CS}}}{b_{\mathrm{CS}}} B_{ij}\left[B^{kl} E^{\mathrm{GR}}_{kl} + 3 B_{ss} X_{ss} - \frac{2}{L}\epsilon^{kl}(B_{km}\tilde{X}^{m}{}_{l} + 3 B_{sk}\tilde{X}_{sl})\right] \\
& + \frac{a_{\mathrm{CS}} L}{3\kappa}\bigg[-\Pi \epsilon_{(i}{}^{k} A_{j)k} A^{lm} B_{lm} - s_{(i} A^{k}{}_{j)}\epsilon^{lm} A_{kl}\tilde{X}_{sm} + \epsilon_{(i}{}^{k} A_{j)l} A^{lm}\tilde{X}_{km} - 2 A_{k(i} A^{l}{}_{j)}\epsilon^{km}\tilde{X}_{lm} \\
& - \tilde{X}_{k(i} A^{l}{}_{j)}\epsilon^{km} A_{lm} + 3\epsilon_{(i}{}^{k} A^{l}{}_{j)}\tilde{X}_{sk} A_{sl} - \epsilon_{(i}{}^{k} A_{j)s} A^{l}_{s}\tilde{X}_{kl} + \epsilon^{kl}(2\tilde{X}_{sk} A_{l(i} A_{j)s} - A_{sk}\tilde{X}_{l(i} A_{j)s}) \\
& + s_{(i} A_{j)k} A_{sm}(\epsilon^{kl}\tilde{X}^{m}{}_{l} - \epsilon^{lm}\tilde{X}^{k}{}_{l}) - 3 A_{ss}(\epsilon_{(i}{}^{k} A_{j)s}\tilde{X}_{sk} - s_{(i} A_{j)k}\epsilon^{kl}\tilde{X}_{sl})\bigg] \\
& + \frac{a_{\mathrm{CS}} L^{2}}{3\kappa}\bigg[\epsilon_{(i}{}^{k} A_{j)k} D^{l} B_{sl} - A_{s(i} A^{k}{}_{j)} E^{\mathrm{GR}}_{sk} + s_{(i} A^{k}{}_{j)}(A^{l}{}_{k} E^{\mathrm{GR}}_{l} - A^{l}_{s} E^{\mathrm{GR}}_{kl}) + A_{k(i} E^{\mathrm{GR}}_{j)l} s^{k} A^{l}_{s} \\
& + A^{k}{}_{(i} A^{l}{}_{j)} E^{\mathrm{GR}}_{kl} - A_{k(i} E^{\mathrm{GR}}_{j)l} A^{kl} + \frac{1}{2} A_{s(i|} D_s M^{\mathrm{GR}}_{|j)} - \frac{1}{2} s_{(i} A^{k}{}_{j)} D_s M^{\mathrm{GR}}_{k} + \frac{1}{2} A^{k}{}_{(i} D_{j)} M^{\mathrm{GR}}_{k} - \frac{1}{2} A_{k(i} D^{k} M^{\mathrm{GR}}_{j)} \\
& + \frac{1}{2} s_{(i} A_{j)k} D^{k} M^{\mathrm{GR}}_{s} - \frac{1}{2} A_{s(i} D_{j)} M^{\mathrm{GR}}_{s}\bigg]. \qquad \text{(B2)}
\end{aligned}
$$

[1] C. M. Will, Living Rev. Relativity **17**, 4 (2014).
[2] J. R. Gair, M. Vallisneri, S. L. Larson, and J. G. Baker, Living Rev. Relativity **16**, 7 (2013).
[3] B. Sathyaprakash and B. Schutz, Living Rev. Relativity **12**, 2 (2009).
[4] J. Antoniadis *et al.*, Science **340**, 6131 (2013).
[5] J. Polchinski, *Superstring Theory and Beyond*, String Theory Vol. 2 (Cambridge University Press, Cambridge, UK, 1998).
[6] S. Alexander and N. Yunes, Phys. Rep. **480**, 1 (2009).
[7] M. Adak and T. Dereli, Eur. Phys. J. C **72**, 1979 (2012).
[8] N. Yunes and L. C. Stein, Phys. Rev. D **83**, 104002 (2011).
[9] N. Yunes and X. Siemens, Living Rev. Relativity **16**, 9 (2013).
[10] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[11] R. Jackiw and S. Pi, Phys. Rev. D **68**, 104012 (2003).
[12] N. Yunes and F. Pretorius, Phys. Rev. D **79**, 084043 (2009).
[13] K. Yagi, N. Yunes, and T. Tanaka, Phys. Rev. D **86**, 044037 (2012).
[14] N. Yunes and C. F. Sopuerta, Phys. Rev. D **77**, 064007 (2008).
[15] C. Molina, P. Pani, V. Cardoso, and L. Gualtieri, Phys. Rev. D **81**, 124021 (2010).
[16] T. Delsate, V. Cardoso, and P. Pani, J. High Energy Phys. 06 (2011) 055.
[17] P. Pani, C. F. Macedo, L. C. Crispino, and V. Cardoso, Phys. Rev. D **84**, 087501 (2011).
[18] K. Konno and R. Takahashi, Phys. Rev. D **90**, 064011 (2014).
[19] L. C. Stein, Phys. Rev. D **90**, 044061 (2014).
[20] J. Gair and N. Yunes, Phys. Rev. D **84**, 064016 (2011).
[21] C. F. Sopuerta and N. Yunes, Springer Proc. Phys. **137**, 281 (2011).
[22] C. F. Sopuerta and N. Yunes, Phys. Rev. D **80**, 064006 (2009).
[23] P. Canizares, J. R. Gair, and C. F. Sopuerta, Phys. Rev. D **86**, 044010 (2012).
[24] P. Pani, V. Cardoso, and L. Gualtieri, Phys. Rev. D **83**, 104048 (2011).
[25] P. Amaro-Seoane *et al.*, GW Notes **6**, 4 (2013)., arXiv:1201.3621
[26] P. A. Seoane *et al.* (eLISA Collaboration), arXiv:1305.5720.
[27] B. Abbott *et al.* (LIGO Scientific Collaboration), Rep. Prog. Phys. **72**, 076901 (2009).
[28] F. Acernese *et al.*, Classical Quantum Gravity **25**, 184001 (2008).
[29] J. Abadie *et al.* (LIGO Scientific Collaboration, Virgo Collaboration), Phys. Rev. D **83**, 122005 (2011).
[30] J. Aasi *et al.* (LIGO Scientific Collaboration, Virgo Collaboration), arXiv:1304.0670.
[31] Advanced LIGO website, https://www.advancedligo.mit.edu/.
[32] Y. Aso, Y. Michimura, K. Somiya, M. Ando, O. Miyakawa, T. Sekiguchi, D. Tatsumi, and H. Yamamoto, Phys. Rev. D **88**, 043007 (2013).
[33] K. Somiya (KAGRA Collaboration), Classical Quantum Gravity **29**, 124007 (2012).
[34] Y. Ali-Haimoud and Y. Chen, Phys. Rev. D **84**, 124033 (2011).
[35] K. Yagi, L. C. Stein, N. Yunes, and T. Tanaka, Phys. Rev. D **85**, 064022 (2012).
[36] M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, International Series of Monographs on Physics (Oxford Univ. Press, Oxford, 2008).
[37] D. Garfinkle, F. Pretorius, and N. Yunes, Phys. Rev. D **82**, 041501 (2010).
[38] V. Cardoso and L. Gualtieri, Phys. Rev. D **80**, 064008 (2009).
[39] D. Ayzenberg, K. Yagi, and N. Yunes, Phys. Rev. D **89**, 044023 (2014).
[40] K. Izumi, Phys. Rev. D **90**, 044037 (2014).
[41] H. Reall, N. Tanahashi, and B. Way, arXiv:1406.3379.
[42] R. L. Arnowitt, S. Deser, and C. W. Misner, Gen. Relat. Gravit. **40**, 1997 (2008).
[43] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University Press, Cambridge, 1979), pp. 83–126.
[44] E. Gourgoulhon, arXiv:gr-qc/0703035.
[45] G. Nagy, O. E. Ortiz, and O. A. Reula, Phys. Rev. D **70**, 044012 (2004).
[46] A. Ashtekar, A. P. Balachandran, and S. Jo, Int. J. Mod. Phys. A **04**, 1493 (1989).
[47] S. Mercuri and V. Taveras, Phys. Rev. D **80**, 104007 (2009).
[48] V. Taveras and N. Yunes, Phys. Rev. D **78**, 064070 (2008).
[49] J. Garcia-Bellido, M. Garcia-Perez, and A. Gonzalez-Arroyo, Phys. Rev. D **69**, 023504 (2004).
[50] T. Mariz, J. Nascimento, E. Passos, and R. Ribeiro, Phys. Rev. D **70**, 024014 (2004).
[51] H. O. Kreiss and J. Lorenz, *Initial-boundary Value Problems and the Navier-Stokes Equations* (Academic Press, New York, 1989).
[52] B. Gustafsson, H. O. Kreiss, and J. Oliger, *Time Dependent Problems and Difference Methods* (Wiley, New York, 1995).
[53] M. E. Taylor, *Pseudodifferential Operators* (Princeton University Press, Princeton, NJ, 1981).
[54] C. Gundlach and J. M. Martín-García, Classical Quantum Gravity **23**, S387 (2006).
[55] D. Hilditch and R. Richter., J. Hyperbol. Differ. Eq. (to be published).
[56] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics Vol. 19 (American Mathematical Society, Providence, Rhode Island, 1998).
[57] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, Phys. Rev. D **81**, 123530 (2010).
[58] D. Hilditch, Int. J. Mod. Phys. A **28**, 1340015 (2013).
[59] D. Hilditch and R. Richter, arXiv:1303.4783.
[60] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1984).
[61] M. B. Green, J. H. Schwarz, and E. Witten, *Introduction*, Superstring Theory Vol. 1 (Cambridge University Press, UK, 1987).
[62] M. B. Green, J. H. Schwarz, and E. Witten, *Loop Amplitudes, Anomalies and Phenomenology*, Superstring Theory Vol. 2 (Cambridge University Press, Cambridge, UK, 1987).
[63] D. Grumiller and N. Yunes, Phys. Rev. D **77**, 044015 (2008).
[64] xAct package for Mathematica, http://www.xAct.es.
[65] D. Hilditch homepage, https://www.tpi.uni-jena.de/tiki-index.php?page=Staff List.
[66] S. Wood, Ph.D. thesis, The Pennsylvania State University, The Graduate School, Department of Physics, 2009.
[67] L. Andersson and V. Moncrief, arXiv:gr-qc/0110111.
[68] Y. Choquet-Bruhat, J. Math. Phys. (N.Y.) **29**, 1891 (1988).