PHYSICAL REVIEW D **91,** 024017 (2015)

Gravitational spin Hamiltonians from the *S* matrix

Varun Vaidya*

Department of Physics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, USA
(Received 8 November 2014; published 12 January 2015)

We utilize generalized unitarity and recursion relations combined with effective field theory techniques to compute spin-dependent interaction terms for an inspiralling binary system in the post-Newtonian (PN) approximation. Using these methods offers great computational advantage over traditional techniques involving Feynman diagrams, especially at higher orders in the PN expansion. As a specific example, we reproduce the spin-orbit (up to 2.5PN order) and the leading-order S^2 (2PN) Hamiltonian for a binary system with one of the massive objects having nonzero spin using the S-matrix elements of elementary particles. For the same system, we also obtain the S^3 (3.5PN) spin Hamiltonian for an arbitrary massive object, which was until now known only for a black hole. Furthermore, we derive the missing S^4 Hamiltonian at leading order (4PN), again for an arbitrary massive object and establish that the minimal coupling of an elementary particle to gravity automatically captures the physics of a spinning black hole. Finally, the Kerr metric is obtained as a series in G_N by comparing the action of a test particle in the vicinity of a spinning black hole to the derived potential.

DOI: 10.1103/PhysRevD.91.024017 PACS numbers: 04.25.Nx

I. INTRODUCTION

To observe gravitational waves, one needs very sensitive detectors due to the tiny cross section of the waves with matter. There are several ground-based detectors like VIRGO and LIGO [1,2] which have a good chance of detecting gravitational waves in the next few years. For the data analysis of such a signal, if and when it is discovered, it is necessary to have a theoretical template of the signal that is expected from inspiralling binary sources. While it is not possible to get an exact analytical solution for such a system in all regimes of its evolution, we can use approximate methods to get highly accurate analytical results in the slow-motion and wide-separation phase. The Hamiltonians for spinning and nonspinning objects in the post-Newtonian (PN) approximation known to date were neatly listed in Ref. [3]. These interactions have been derived using different formalisms: the Arnowitt-Deser-Misner formalism [4–7] which computes the Hamiltonians, nonrelativistic general relativity (NRGR) [8–11] which obtains the result in the form of a Lagrangian, and also the formalism used in Refs. [12–14] which used an nonrelativistic gravitational (NRG) effective field theory (EFT) to derive Hamiltonians.

In this paper we extend the method introduced in Ref. [15] to spinning sources, via effective field theory techniques using recent advances in S-matrix calculations in particle physics. A similar approach was used in a recent paper to compute quantum gravity effects [16]. We forgo Einstein's point of view of treating gravity as a manifestation of space-time geometry and instead treat all effects of gravity as the propagation of a massless spin-2 particle on a flat background. Classical spinning objects are treated as local sources of gravitons and the modes which give rise to the classical potential between such objects are factorized from the radiative modes in an EFT (see Ref. [17] for review). For example, the technique of NRGR relies on the explicit separation of scales relevant to the problem: the size of the objects r_s, the size of the orbit r and natural radiation wavelength r/v. Here the relative velocity $v \ll c$. Finite-sized effects are treated by including new terms in the wordline action which are needed to regularize the theory. This usually involves terms obeying the correct symmetry constructed using the Riemann tensor and the velocity v. The accuracy in the PN expansion can be improved by adding higher-dimensional operators. The coefficients of these operators are obtained by matching onto the full underlying theory which is general relativity. While doing calculations in such an EFT, Feynman diagrams will show up at the tree and loop level as perturbative techniques to iteratively solve for the Green's function of the full theory.

In our technique, we compute the interaction Hamiltonians as the nonrelativistic limit of on-shell scattering amplitudes. Modern methods of computation have dramatically reduced the effort involved in calculating loop amplitudes. Most of these involve the recursive use of on-shell amplitudes, which means that only the on-shell propagating modes of a field are used in any calculation. This technique automatically gets rid of the need for a gauge choice, thus eliminating the huge amount of redundancy involved in traditional Feynman diagrams.

The most useful of these for our purposes is the Britto-Cachazo-Feng-Witten (BCFW) recursion relation [18] and generalized unitarity methods [19,20]. These methods are traditionally applied for calculating on-shell S-matrix

*vpvaidya@andrew.cmu.edu

elements, but we are primarily interested in calculating the off-shell potential between spinning classical objects. The scattering amplitude is matched onto an EFT in which the graviton is essentially integrated out. This leads to a well-defined and IR-finite classical potential. The calculated potential is a series in the relative velocity for a virialized orbit $v^2 \sim \frac{Gm}{r}$ and the spin. Both quantities count as 1PN in the PN power-counting parameter.

This method has been applied for nonspinning objects in Ref. [15]. We extend this to the case of a binary system with one spinning component and demonstrate the use of this technique for calculating the spin-orbit Hamiltonian to 2.5 PN order. We also present Hamiltonians for S^2, S^3 and S^4 terms at leading order for an arbitrary spinning object and show that a minimal coupling to gravity gives the interaction terms for a black hole. The Kerr metric is then derived as a series in the PN power-counting parameter by expanding out the action of a test particle moving around a spinning black hole.

II. SPIN-DEPENDENT HAMILTONIANS FOR COMPACT BINARY SYSTEMS

In the calculations that follow, we obtain spin-dependent Hamiltonians from on-shell scattering amplitudes of a massive scalar particle with other massive particles with nonzero spin. From the addition rules of angular momentum, it is clear that the scattering of a scalar with a particle of spin j will generate terms of $2j, 2j-1, \ldots 0$ power in spin when we match onto an EFT. For example, the scattering with a spin-$1/2$ particle will produce terms which are spin independent and linear in spin. Since we are interested in terms up to the fourth order in the PN expansion, we need to go up to $S = 2$, to generate the S^4 piece. While it is true that all the relevant pieces that we need can be obtained by considering the scattering with a $S = 2$ particle, in order to obtain terms that are higher order in G, it is computationally efficient to consider the scattering of the smallest-spin particle that can give us the required result. To that end, we first consider the scattering of the scalar with a massive spin-$1/2$ particle to generate the spin-orbit Hamiltonian up to 2.5 PN order.

For all the amplitudes that we calculate, we will need the three-point interaction term of the scalar particle with a graviton. Assuming a minimal coupling to gravity gives us

$$M(p_3, p_4, m_b) = \frac{\kappa}{2}[p_{3\nu}p_{4\mu} + p_{3\mu}p_{4\nu} - \eta_{\mu\nu}(p_3 \cdot p_4 + m_b^2)] \quad (1)$$

where p_3 and p_4 are incoming momenta of the scalars with mass m_b and $\kappa = \sqrt{32\pi G_N}$.

We can also add a gauge-invariant operator $R\phi^2$, but this does not affect the classical result. For calculating the loop amplitude, we will also need the on-shell three-point amplitude in the spinor-helicity formalism (for a review see Refs. [21,22]). where we use 3 and 4 in place of p_3 and p_4, respectively, while using the spinor-helicity notation

$$iM(3, 4, 5^+) = \frac{\kappa}{2}\frac{\langle r\not{3}5]^2}{\langle r5\rangle^2}. \quad (2)$$

Here, r is any lightlike vector not proportional to the positive-helicity graviton momentum 5. The amplitude for the negative-helicity graviton is obtained by interchanging the angle and square brackets.

For future use, we give the four-point scalar graviton amplitude constructed using the BCFW recursion relation. This involves complexifying the momentum of two external massless particles while still maintaining momentum conservation. To apply this method, in principle, we need the theory to be BCFW constructible. This requires that the amplitude which is now a function of the complex variable z, should satisfy the condition $\lim_{z\to\infty} M(z)/z = 0$. However, in our case this condition can be relaxed, since the terms that are not captured by the recursion do not contribute to the classical potential. Also, we only need the four-point amplitude with opposite helicities for the gravitons [15]:

$$M(3, 4, 5^-, 6^+) = \frac{\kappa^2}{4}\frac{\langle 5\not{3}6]^4}{(5+6)^2}\left[\frac{1}{(5+3)^2 - m_b^2} + \frac{1}{(5+4)^2 - m_b^2}\right]. \quad (3)$$

A. Spin orbit

To begin, we consider the scattering of a scalar with a massive spin-$1/2$ fermion. For tree-level scattering, we will use the usual Feynman rules. As before, a minimal coupling to gravity gives us the interaction of fermions with gravity,

$$iM(p_1, p_2, m_a) = \frac{-i\kappa}{2}\left[(p_1 + p_2)_\nu\gamma_\mu + (p_1 + p_2)_\mu\gamma_\nu - \eta_{\mu\nu}\left(\frac{1}{2}(\not{p}_1 + \not{p}_2) - m_a\right)\right] \quad (4)$$

where p_1 is the incoming and p_2 is the outgoing momentum of the fermion with mass m_a. On the other hand for loop calculations, generalized unitarity methods become invaluable and to use them we need the on-shell three-point amplitude

$$M(1, 2, 5^+) = \frac{\kappa}{2}\bar{u}(2)\gamma_\mu u(1)\frac{\langle r\gamma^\mu 5]\langle r\not{1}5]}{\langle r5\rangle^2}. \quad (5)$$

The expression for the graviton with negative helicity is similar but with the angles interchanged with square brackets. Using this seed we can use the BCFW relation to construct the four-point amplitudes:

$$M(1,2,5^-,6^+) = \frac{\kappa^2}{4}\bar{u}(2)\gamma_\mu u(1)\frac{\langle 5\gamma^\mu 6]\langle 5 1\!\!/ 6]^3}{[(5+6)^2]^2} \times \left[\frac{1}{(1+5)^2 - m_a^2} + \frac{1}{(1+6)^2 - m_a^2}\right]. \quad (6)$$

For calculating tree-level scattering amplitudes, we use the graviton propagator in the harmonic or Feynman gauge. For all our calculations, we choose the incoming and outgoing scalar particles with rest mass m_b to have momenta p_3 and p_4, respectively. The particles with nonzero spin with mass m_a have momenta p_1 and p_2. In the center-of-mass frame

$$p_1^\mu = (E_1, \vec{p} + \vec{q}/2), \qquad p_2^\mu = (E_2, \vec{p} - \vec{q}/2),$$
$$p_3^\mu = (E_3, -\vec{p} - \vec{q}/2), \qquad p_4^\mu = (E_4, -\vec{p} + \vec{q}/2).$$

The nonrelativistic limit of this amplitude has been obtained in Ref. [23]. In order to calculate the spin-orbit piece up to 2.5 PN, we need to expand out the spin-independent piece to 1PN order, where we have kept only the classical contributions:

$$M = \frac{4\pi G m_a m_b}{\vec{q}^2}\left\{\chi_f^{a\dagger}\chi_i^a\left(1 + \frac{\vec{p}^2}{2m_a^2 m_b^2}(3m_a^2 + 3m_b^2 + 8m_a m_b)\right) + \frac{i\vec{S}\cdot(\vec{p}\times\vec{q})}{m_a^2 m_b}\left(\frac{4m_a + 3m_b}{2} + \frac{\vec{p}^2}{8m_a^2 m_b}[8m_a m_b - 5m_b^2 + 18m_a^2]\right)\right\} \quad (7)$$

where χ_f^a and χ_i^a are the spinors for the initial and final states of the fermion in the rest frame and $S^i = \chi_f^{a\dagger}\frac{\sigma^i}{2}\chi_i^a$ is the spin vector.

To extract the effective potential we match this result onto a nonrelativistic EFT in which the graviton is integrated out,

$$V_{si}(\vec{p},\vec{q})\psi^\dagger_{\vec{p}-\vec{q}/2}\psi_{\vec{p}+\vec{q}/2}\phi^\dagger_{-\vec{p}+\vec{q}/2}\phi_{-\vec{p}-\vec{q}/2} + V^j_{so}(\vec{p},\vec{q})S^j\phi^\dagger_{-\vec{p}+\vec{q}/2}\phi_{-\vec{p}-\vec{q}/2} \quad (8)$$

where V_{si} is the spin-independent piece and $V^j_{so}S^j$ is the spin-orbit piece of the potential. To get the complete spin orbit term at 2.5PN order, we need to consider the scattering amplitude at one loop. Using generalized unitarity methods we can construct the one-loop amplitude by sewing together the four-point amplitudes for the scalar and fermion as shown in Fig. 1:

$$M(1,2,3,4) = \int\frac{d^4 l}{(2\pi)^4}\frac{iM(1,2,l^-,-l'^+)iM(3,4,-l^+,l'^-) + (+\leftrightarrow -)}{l^2 l'^2}. \quad (9)$$

The basic idea is to simplify the numerator of the integrand by treating the gravitons (l, l') to be on-shell in four-dimensional space. After the simplification we will have a decomposition into standard scalar integrals. Using this we can accurately obtain the coefficients of those scalar integrals which contain *all* the cut propagators. In this case we are going for a t-channel cut which involves a cut on the two massless graviton propagators. The only scalar integrals which give a classical contribution are those given by the triangle diagram with exactly one massive propagator. This means that the t-channel cut is sufficient to calculate all the coefficients we need.

Moreover since we are using dimensional regularization, the loop integral in l is in d dimensions. But the reduction is much simpler in four dimensions and it is justified in this case since the errors produced are rational terms (polynomials) in the transfer momentum q which do not affect the long-range classical result. As before, we consider the nonrelativistic limit with a normalization factor $1/\sqrt{2E_1 2E_2 2E_3 2E_4}$ to give

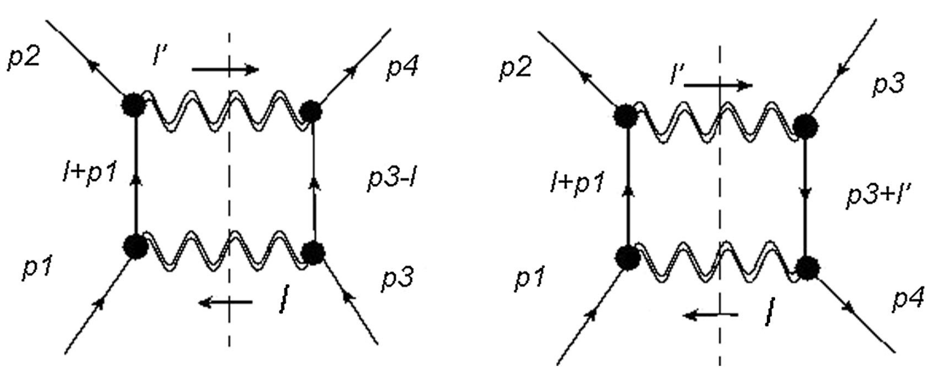

FIG. 1. Fusing two tree-level on-shell four-point amplitudes.

$$M^{(2)} = \frac{G^2\pi^2}{q}\left\{m_a m_b \chi_f^{a\dagger}\chi_i^a(6m_a+6m_b) + i\vec{S}\cdot(\vec{p}\times\vec{q}) \times\left[\frac{20m_a^3+9m_b^3+53m_a^2m_b+41m_am_b^2}{2m_a(m_a+m_b)} + \frac{m_a^2m_b^2(3m_b+4m_a)}{(m_a+m_b)p_0^2}\right]\right\} \tag{10}$$

where $p_0^2 = \vec{p}^2 + \vec{q}^2/4$. This result has a singular behavior in the limit $p_0 \to 0$. To define a well-behaved potential, we match onto our EFT. This requires us to subtract out the iterated tree-level potential from the loop scattering amplitude. To obtain the potential in position space, we Fourier transform the resulting coefficients of our EFT with respect to the transfer momentum vector $\vec{q}$. We now match this nonrelativistic EFT onto a point-particle Hamiltonian by treating r as the conjugate position variable to the canonical momentum p. This choice of a coordinate system is a specific one and hence makes the Hamiltonian gauge dependent:

$$\begin{aligned} H = {}& \frac{\vec{p}^2}{2m_a} + \frac{\vec{p}^2}{2m_b} - \frac{\vec{p}^4}{8m_a^3} - \frac{\vec{p}^4}{8m_b^3} - \frac{Gm_am_b}{r}\left\{1 + \frac{\vec{p}^2}{2m_a^2m_b^2}(3m_a^2+3m_b^2+8m_am_b)\right. \\ & \left. + \frac{\vec{S}\cdot(\vec{p}\times\vec{r})}{r^2m_a^2m_b}\left[\frac{4m_a+3m_b}{2} + \frac{\vec{p}^2}{8m_a^2m_b}(8m_am_b - 5m_b^2 + 18m_a^2)\right]\right\} \\ & + \frac{G^2\vec{S}\cdot(\vec{p}\times\vec{r})}{2r^4m_a(m_a+m_b)}(12m_a^3+10m_b^3+45m_b^2m_a+41m_am_b^2) \\ & + \frac{G^2}{2r^2}m_am_b(m_a+m_b)\left[1 + \frac{m_am_b}{(m_a+m_b)^2}\right]. \end{aligned} \tag{11}$$

We are working in a frame in which the momentum ($\vec{p}$) is directed transverse to $\vec{r}$ and we do not have a $\vec{p}\cdot\vec{r}$ term. In order to compare our result with the existing literature, we choose a different coordinate system to express our result. This amounts to a canonical transformation of the Hamiltonian. The most general form of the generator to implement this transformation is

$$g = a_1\frac{G(m_a+m_b)(\vec{p}\cdot\vec{r})}{r} + a_2\frac{G\vec{S}\cdot(\vec{p}\times\vec{r})(\vec{p}\cdot\vec{r})}{r^3}. \tag{12}$$

This generates a correction to the Hamiltonian $\{g, H\}$. The choice of constants $a_1 = \frac{m_am_b}{2(m_a+m_b)^2}$ and $a_2 = \frac{2m_a+m_b}{4m_a(m_a+m_b)}$ gives the result

$$\begin{aligned} H_{\rm int} = {}& -\frac{Gm_am_b}{r}\left(1 + \frac{\vec{p}^2}{2m_a^2m_b^2}(3m_a^2+3m_b^2+7m_am_b) + \frac{(\vec{p}\cdot\vec{r})^2}{2m_am_b}\right) + \frac{G^2}{2r^2}m_am_b(m_a+m_b) \\ & - \frac{G\vec{S}\cdot(\vec{p}\times\vec{r})}{r^3m_a}\left[\frac{4m_a+3m_b}{2} + \frac{\vec{p}^2}{8m_a^2m_b}(6m_am_b-5m_b^2+14m_a^2) + \frac{(\vec{p}\cdot\vec{r})^2}{4r^2m_a^2m_b}(6m_a+3m_b)\right] \\ & + \frac{G^2\vec{S}\cdot(\vec{p}\times\vec{r})}{2r^4m_a(m_a+m_b)}(12m_a^3+10m_b^3+38m_bm_a^2+36m_am_b^2). \end{aligned} \tag{13}$$

In this case, the spin-independent result agrees with the Einstein-Infeld-Hoffmann (EIH) potential. The spin-dependent piece agrees with the result obtained by Damour *et al.* [4] in the center-of-mass frame.

B. Spin quadrupole

The S^2 piece of the amplitude can be obtained by scattering a scalar with a massive spin-1 particle. We begin with the Proca action for a massive particle of spin 1,

$$S=\int d^4x\left[-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}+\frac{1}{2}m^2\phi^\mu\phi_\mu\right]. \tag{14}$$

We consider a minimal coupling to gravity to determine the interaction. In this paper, we are interested only in the leading-order S^2 piece, and hence a tree-level scattering amplitude is sufficient. This means that the massive particles are always on-shell and we can use equations of motion to simplify the stress energy tensor:

$$T^{\mu\nu}=\frac{\kappa}{2}[\partial^\mu\phi_\alpha\partial^\nu\phi^\alpha+\partial_\alpha\phi^\mu\partial^\alpha\phi^\nu-\partial^\mu\phi^\alpha\partial_\alpha\phi^\nu-\partial^\nu\phi^\alpha\partial_\alpha\phi^\mu-m^2\phi^\mu\phi^\nu]+\frac{\kappa}{4}\eta^{\mu\nu}[-\partial_\alpha\phi_\beta\partial^\alpha\phi^\beta+\partial_\alpha\phi_\beta\partial^\beta\phi^\alpha+m^2\phi_\alpha\phi^\alpha]. \tag{15}$$

Scattering this off a scalar with mass m_b gives us the following scattering amplitude in the center-of-mass frame:

$$iM=-\frac{4\pi Gm_am_b}{\vec{q}^2}\{\epsilon^*(p_2)\cdot\epsilon(p_1)+\hat{p}_1\cdot\epsilon^*(p_2)\hat{p}_2\cdot\epsilon(p_1)+2[\hat{p}_2\cdot\epsilon(p_1)\hat{p}_3\cdot\epsilon^*(p_2)+\hat{p}_1\cdot\epsilon^*(p_2)\hat{p}_3\cdot\epsilon(p_1)-\hat{p}_2\cdot\epsilon(p_1)\hat{p}_1\cdot\epsilon^*(p_2)]\} \tag{16}$$

where $\hat{p}_1=p_1/m_a,\hat{p}_2=p_2/m_a,\hat{p}_3=p_3/m_b,\hat{p}_4=p_4/m_b$. We now consider the nonrelativistic limit of this amplitude using the following approximations:

$$\begin{aligned}\epsilon^*(p_2)\cdot\epsilon(p_1)&\approx-\hat{\epsilon}_1\cdot\hat{\epsilon}_2^*-\frac{1}{2m_a^2}\vec{q}\cdot\hat{\epsilon}_1\,\vec{q}\cdot\hat{\epsilon}_2^*-\frac{1}{2m_a^2}(q^ip^j-p^iq^j)\hat{\epsilon}_1{}^i\hat{\epsilon}_2^{*j},\quad \hat{p}_2\cdot\epsilon(p_1)\hat{p}_3\cdot\epsilon^*(p_2)+\hat{p}_1\cdot\epsilon^*(p_2)\hat{p}_3\cdot\epsilon(p_1)\\&\approx-\frac{1}{m_a^2}\vec{q}\cdot\hat{\epsilon}_1\,\vec{q}\cdot\hat{\epsilon}_2^*-\left(\frac{1}{m_a^2}+\frac{1}{m_am_b}\right)(q^ip^j-p^iq^j)\hat{\epsilon}_1{}^i\hat{\epsilon}_2^{*j},\\ \hat{p}_2\cdot\epsilon(p_1)\hat{p}_1\cdot\epsilon^*(p_2)&\approx-\frac{1}{m_a^2}\vec{q}\cdot\hat{\epsilon}_1\,\vec{q}\cdot\hat{\epsilon}_2^*\end{aligned} \tag{17}$$

where $\hat{\epsilon}_i$ is the polarization tensor of the spin-1 particle with momentum p_i in the rest frame. This reduces the amplitude to the following compact form:

$$M\simeq\frac{4\pi Gm_am_b}{\vec{q}^2}\left[\hat{\epsilon}_1{}^i\hat{\epsilon}_2^{*i}-\frac{1}{m_a^2}q^iq^j\hat{\epsilon}_1{}^i\hat{\epsilon}_2^{*j}+\left(\frac{3m_b+4m_a}{m_a^2m_b}\right)q^ip^j(\hat{\epsilon}_1{}^i\hat{\epsilon}_2^{*j}-\hat{\epsilon}_2^{*i}\hat{\epsilon}_1{}^j)\right]. \tag{18}$$

The effective potential between the two objects in position space is

$$V(\vec{p},\vec{r})=Gm_am_b\left[-\frac{1}{r}\hat{\epsilon}_1{}^i\hat{\epsilon}_2{}^{*i}-\frac{1}{m_a^2}\left(\frac{3r^ir^j}{r^5}-\frac{\delta^{ij}}{r^3}\right)\hat{\epsilon}_1{}^i\hat{\epsilon}_2{}^{*j}+i\left(\frac{3m_b+4m_a}{m_a^2m_b}\right)\frac{r^i}{r^3}p^j(\hat{\epsilon}_1{}^i\hat{\epsilon}_2{}^j-\hat{\epsilon}_2{}^{*i}\hat{\epsilon}_1{}^j)\right].$$

In order to match this amplitude onto the EFT, we need to consider the relevant operators that will appear in our EFT Lagrangian. In the rest frame of the particles, the only nontrivial vector operator that is available is spin. This implies that any tensor constructed using the polarization vectors has to map onto some linear combination of corresponding tensors constructed using the spin vector and other invariant tensors. We can define the spin operators using the following identities:

$$\begin{aligned}&\hat{\epsilon}_1{}^i\hat{\epsilon}_2{}^{*j}-\hat{\epsilon}_2{}^{*i}\hat{\epsilon}_1{}^j=\frac{i}{2}\epsilon^{ijm}\langle s=1,m_2|S^m|s=1,m_1\rangle,\\&\frac{3}{2}(\hat{\epsilon}_1{}^i\hat{\epsilon}_2{}^{*j}+\hat{\epsilon}_2{}^{*i}\hat{\epsilon}_1{}^j)-\delta^{ij}\hat{\epsilon}_1{}^k\hat{\epsilon}_2{}^{*k}\\&\quad=-\left\langle s=1,m_2\left|\frac{3}{2}(S^iS^j+S^jS^i)-\vec{S}^2\delta^{ij}\right|s=1,m_1\right\rangle.\end{aligned} \tag{19}$$

Here m_1 and m_2 are the z components of the spin in the initial and final states for the massive spin-1 particle. Apart from the minimal coupling, we can add other gauge-invariant operators to the Proca Lagrangian. It turns out that the only relevant operator that we can add which has a nontrivial effect on the classical result is

$$L_{\text{int}}=\frac{C_1}{8}R_{\mu\nu\alpha\beta}G^{\mu\nu}G^{\alpha\beta}. \tag{20}$$

This additional piece leaves the Newtonian and spin-orbit terms unchanged, but alters the spin-quadrupole term giving us the final result

$$V(\vec{p},\vec{r})=Gm_am_b\left[-\frac{1}{r}+\left(C_1+\frac{1}{2m_a^2}\right)\frac{1}{r^3}\left(\frac{3(\vec{S}\cdot\vec{r})^2}{r^2}-\vec{S}^2\right)+\left(\frac{3m_b+4m_a}{2m_a^2m_br^3}\right)\vec{S}\cdot(\vec{r}\times\vec{p})\right].$$

By comparing this with the existing literature [24] we see that this is the result for a black hole when $C_1 = 0$. This indicates that minimal coupling to gravity corresponds to a black hole structure. This also demonstrates the universal form of the spin-orbit term for the interaction between any two classical objects. The arbitrary coefficient C_1 allows us to account for any other massive classical object (e.g. a neutron star). In order to determine this coefficient, we can do a matching procedure using any other spin-dependent observable related to the star. For example, Ref. [25] used an EFT to model any star as a point source with finite-size effects encoded into effective operators. This is essentially an expansion in multipolar degrees of freedom. The dynamics of these multipoles can be obtained by matching the gravitational field of the actual star with that of the effective point source.

C. Spin octupole

To derive the spin-octupole Hamiltonian at leading order, we need to consider the scattering of a spin-2 particle. We begin with the Fierz-Pauli action for a massive elementary particle with spin 2 [26],

$$S = \int d^4x \left[-\frac{1}{2}\partial_\lambda \phi_{\mu\nu}\partial^\lambda \phi^{\mu\nu} + \partial_\mu \phi_{\nu\lambda}\partial^\nu \phi^{\mu\lambda} - \partial_\mu \phi^{\mu\nu}\partial_\nu \phi + \frac{1}{2}\partial_\lambda \phi \partial^\lambda \phi - \frac{1}{2}m^2(\phi_{\mu\nu}\phi^{\mu\nu} - \phi^2) \right]$$

where $\phi = \phi^\mu_\mu$ is the trace over the spin-2 tensor.

The equations of motion from this free-field Lagrangian give a symmetric traceless rank-2 tensor which restricts the number of on-shell modes to five:

$$\partial_\mu \phi^{\mu\nu} = 0, \quad \phi = 0, \quad (\partial^2 + m^2)\phi^{\mu\nu} = 0. \tag{21}$$

We consider a minimal coupling to gravity

$$S = \int d^4x \sqrt{|g|} \left[-\frac{1}{2}\nabla_\lambda \phi_{\mu\nu}\nabla^\lambda \phi^{\mu\nu} + \nabla_\mu \phi_{\nu\lambda}\nabla^\nu \phi^{\mu\lambda} - \nabla_\mu \phi^{\mu\nu}\nabla_\nu \phi + \frac{1}{2}\nabla_\lambda \phi \nabla^\lambda \phi - \frac{1}{2}m^2(\phi_{\mu\nu}\phi^{\mu\nu} - \phi^2) \right].$$

This gives us a symmetric and conserved stress-energy tensor which we again simplify using the equations of motion,

$$\begin{aligned} T^{\gamma\delta} = &-\partial^\gamma \phi_{\nu\lambda}\partial^\nu \phi^{\delta\lambda} - \partial^\delta \phi_{\nu\lambda}\partial^\nu \phi^{\gamma\lambda} + \partial_\mu \phi^{\nu\delta}\partial_\nu \phi^{\mu\gamma} \\ &+ \frac{1}{2}\partial^\gamma \phi_{\nu\lambda}\partial^\delta \phi^{\lambda\nu} + \partial_\mu \phi^{\nu\gamma}\partial^\mu \phi^\delta_\nu - \partial_\mu \partial_\nu \phi^{\gamma\delta}\phi^{\mu\nu} \\ &- m^2 \phi^\gamma_\mu \phi^{\mu\delta} + \frac{1}{2}\eta^{\gamma\delta}\left[-\frac{1}{2}\partial_\lambda \phi_{\mu\nu}\partial^\lambda \phi^{\mu\nu} + \partial_\mu \phi_{\nu\lambda}\partial^\nu \phi^{\mu\lambda} + \frac{1}{2}m^2 \phi_{\mu\nu}\phi^{\mu\nu} \right]. \end{aligned} \tag{22}$$

We now consider the leading-order elastic scattering amplitude between a massive spin-2 particle and a massive scalar,

$$\begin{aligned} M = \frac{4\pi G m_a m_b}{\vec{q}^2} \{ &\epsilon(p_1)^{\mu\nu}\epsilon^*(p_2)_{\mu\nu} - 4\epsilon(p_1)^{\alpha\beta}\epsilon^*(p_2)_{\beta\nu} \\ &\times (\hat{p}_{2\alpha}\hat{p}_3{}^\nu + \hat{p}_{3\alpha}\hat{p}_1{}^\nu) + 2\epsilon(p_1)^{\alpha\beta}\epsilon^*(p_2)^{\mu\nu} \\ &\times (2\hat{p}_{2\alpha}\hat{p}_{3\beta}\hat{p}_{1\mu}\hat{p}_{3\nu} + \hat{p}_{3\alpha}\hat{p}_{3\beta}\hat{p}_{1\mu}\hat{p}_{1\nu} \\ &+ \hat{p}_{2\alpha}\hat{p}_{2\beta}\hat{p}_{3\mu}\hat{p}_{3\nu}) \}. \end{aligned} \tag{23}$$

In order to extract the effective potential, we take the nonrelativistic limit of this amplitude. This can be done using the following approximations:

$$\begin{aligned} \epsilon(p_1)^{\mu\nu}\epsilon^*(p_2)_{\mu\nu} \simeq \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*ki} &+ \left[\frac{1}{m_a^2}q^i q^j - \frac{1}{m_1^2}(q^i p^j - p^i q^j) \right] \\ &\times \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*kj} + \left[\frac{1}{2m_a^4}q^i q^j (p^k q^l - p^l q^k) \right. \\ &\left. + \frac{1}{4m_a^4}q^i q^j q^k q^l \right] \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*jl} \end{aligned} \tag{24}$$

where repeated indices on the right-hand side are summed over and are all spatial. $\hat{\epsilon_1}^{ij}, \hat{\epsilon_2}^{*kl}$ are the polarization tensors in the rest frame,

$$\begin{aligned} \epsilon(p_1)^{\alpha\beta}\epsilon^*(p_2)_{\beta\nu}(\hat{p}_{2\alpha}\hat{p}_3{}^\nu + \hat{p}_{3\alpha}\hat{p}_1{}^\nu) \simeq &\left[\frac{1}{m_a^2}q^i q^j - \left(\frac{1}{m_a^2} + \frac{1}{m_a m_b} \right)(q^i p^j - p^i q^j) \right] \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*kj} \\ &+ \left[\left(\frac{1}{m_a^4} + \frac{1}{2m_a^3 m_b} \right) q^i q^j (p^k q^l - p^l q^k) + \frac{1}{2m_a^4}q^i q^j q^k q^l \right] \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*jl}, \end{aligned} \tag{25}$$

$$\begin{aligned} &\epsilon(p_1)^{\alpha\beta}\epsilon^*(p_2)^{\mu\nu}(2\hat{p}_{2\alpha}\hat{p}_{3\beta}\hat{p}_{1\mu}\hat{p}_{3\nu} + \hat{p}_{3\alpha}\hat{p}_{3\beta}\hat{p}_{1\mu}\hat{p}_{1\nu} + \hat{p}_{2\alpha}\hat{p}_{2\beta}\hat{p}_{3\mu}\hat{p}_{3\nu}) \\ &\simeq \left[2\left(\frac{1}{m_a^4} + \frac{1}{m_a m_b} \right) q^i q^j (p^k q^l - p^l q^k) + \frac{1}{m_a^4}q^i q^j q^k q^l \right] \hat{\epsilon_1}^{ik}\hat{\epsilon_2}^{*jl}. \end{aligned} \tag{26}$$

This reduces the amplitude to the following compact form:

$$M \simeq \frac{4\pi G m_a m_b}{\vec{q}^2}\left\{\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*ki} - \frac{3}{m_a^2} q^i q^j \hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} + \left(\frac{3m_b + 4m_a}{m_a^2 m_b}\right) q^i p^j (\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} - \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{kj}) \right.$$
$$\left. + \left(\frac{1}{2m_a^4} + \frac{2}{m_a^3 m_b}\right) q^i q^j p^k q^l (\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2^{*jl} - \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{jl}) + \frac{1}{4m_a^4} q^i q^j q^k q^l \hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*jl}\right\} \tag{27}$$

which in turn gives us the potential

$$V(\vec{p},\vec{r}) = G m_a m_b \left\{ -\frac{1}{r}\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*ki} - \frac{3}{m_a^2}\left(\frac{3r^i r^j}{r^5} - \frac{\delta^{ij}}{r^3}\right)\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} + i\left(\frac{3m_b + 4m_a}{m_a^2 m_b}\right)\frac{r^i}{r^3} p^j (\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} - \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{kj}) \right.$$
$$+ 3i\left(\frac{1}{2m_a^4} + \frac{2}{m_a^3 m_b}\right) p^i \left(\frac{\delta^{kl} r^j}{r^5} + \frac{\delta^{jl} r^k}{r^5} + \frac{\delta^{kj} r^l}{r^5} - 5\frac{r^k r^j r^l}{r^7}\right)(\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*jl} - \hat{\epsilon}_2{}^{*il}\hat{\epsilon}_1{}^{jk}) - \hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*jl}\frac{3}{4m_a^4}$$
$$\left. \times \left\{\frac{\delta^{ij}\delta^{kl}}{r^5} + \frac{\delta^{ik}\delta^{jl}}{r^5} + \frac{\delta^{il}\delta^{jk}}{r^5} - \frac{5}{r^7}(r^i r^j \delta^{kl} + r^i r^k \delta^{jl} + r^j r^l \delta^{ik} + r^j r^k \delta^{il} + r^j r^l \delta^{kj} + r^k r^l \delta^{ij}) + 35\frac{r^i r^j r^k r^l}{r^9}\right\}\right\}. \tag{28}$$

As for the case of spin 1, we now match onto the spin operators. The easiest way to do this for the case of spin 2 is to match the coefficients of irreducible tensor structures,

$$\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} - \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{kj} = \frac{-i}{2}\epsilon^{ijm}\langle s=2, m_2|S^m|s=2, m_1\rangle,$$
$$\frac{3}{2}(\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*kj} + \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{kj}) - \delta^{ij}\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*ki} = -\frac{1}{6}\left\langle s=2, m_2\left|\frac{3}{2}(S^i S^j + S^j S^i) - \vec{S}^2\delta^{ij}\right|s=2, m_1\right\rangle.$$

The identities for S^3 and S^4 operators are more involved due to the multitude of nonequivalent structures that are possible:

$$\{2\delta^{jk}(\hat{\epsilon}_1{}^{hi}\hat{\epsilon}_2{}^{*lh} - \hat{\epsilon}_2{}^{*hi}\hat{\epsilon}_1{}^{lh}) - 5(\hat{\epsilon}_1{}^{ij}\hat{\epsilon}_2{}^{*kl} - \hat{\epsilon}_2{}^{*ik}\hat{\epsilon}_1{}^{jl})\} + (j\leftrightarrow l) + (k\leftrightarrow l)$$
$$= \frac{1}{18}\langle s=2, m_2|i\{\delta^{jk}[3\epsilon^{ilm}S^m\vec{S}^2 - S^i\epsilon^{alm}S^m S^a - \epsilon^{alm}S^m S^a S^i]$$
$$-\frac{5}{2}[\epsilon^{ilm}S^m S^j S^k + S^l\epsilon^{ijm}S^m S^k + S^l S^j\epsilon^{ikm}S^m + \epsilon^{ikm}S^m S^l S^j + S^k\epsilon^{ilm}S^m S^j + S^k S^l\epsilon^{ijm}S^m]\}$$
$$+ (j\leftrightarrow l) + (k\leftrightarrow l)|s=2, m_1\rangle, \tag{29}$$

$$\{\hat{\epsilon}_1{}^{hm}\hat{\epsilon}_2{}^{*hm}(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk}) - 5(\hat{\epsilon}_1{}^{ih}\hat{\epsilon}_2{}^{*jh}\delta^{kl} + \hat{\epsilon}_1{}^{hk}\hat{\epsilon}_2{}^{*hl}\delta^{il} + \hat{\epsilon}_1{}^{ih}\hat{\epsilon}_2{}^{*jh}\delta^{kj} + \hat{\epsilon}_1{}^{hk}\hat{\epsilon}_2{}^{*hl}\delta^{ij}) + 35\hat{\epsilon}_1{}^{ik}\hat{\epsilon}_2{}^{*jl}\}$$
$$+ \text{all permutations of } i,j,k,l = \frac{1}{6}\langle s=2, m_2|\{(\vec{S}^2\vec{S}^2\delta^{ij}\delta^{kl} + \vec{S}^a\vec{S}^b\vec{S}^a\vec{S}^b\delta^{ik}\delta^{jl} + \vec{S}^2\vec{S}^2\delta^{il}\delta^{jk})$$
$$-5(\vec{S}^2 S^i S^j\delta^{kl} + S^i S^a S^k S^a\delta^{jl} + S^a S^j S^a S^l\delta^{ik} + S^a S^j S^k S^a\delta^{il} + \vec{S}^2 S^i S^l\delta^{kj} + \vec{S}^2 S^k S^l\delta^{ij}) + 35 S^i S^j S^k S^l\}$$
$$+ \text{all permutations of } i,j,k,l|s=2, m_1\rangle. \tag{30}$$

We can also add three relevant gauge-invariant operators:

$$L_{\text{int}} = \frac{C_1}{8m_a^2}R_{\mu\nu\alpha\beta}U^{\mu\nu\gamma}U_\gamma^{\alpha\beta} + C_2 R_{\alpha\beta\gamma\rho}(\phi^{\alpha\gamma}\phi^{\beta\rho} - \phi^{\beta\gamma}\phi^{\alpha\delta}) + \frac{C_3}{2m_a^2}R_{\mu\nu\alpha\beta}\partial^\mu\phi^{\rho\alpha}\partial_\rho\phi^{\nu\beta} \tag{31}$$

where $U^{\mu\nu\gamma} = \partial^\mu\phi^{\nu\gamma} - \partial^\nu\phi^{\mu\gamma}$. These additional pieces leave the Newtonian and spin-orbit terms unchanged, but they alter the spin-quadrupole and -octupole terms giving us the result

$$V(\vec{p},\vec{r}) = Gm_a m_b \Bigg\{ -\frac{1}{r} + \left[\left(C_1 + \frac{1}{2} + \frac{C_2}{3} \right) \frac{1}{m_a^2 r^3} \right] \left(\frac{3(\vec{S}\cdot\vec{r})^2}{r^2} - \vec{S}^2 \right) + \left(\frac{3m_b + 4m_a}{2m_a^2 m_b r^3} \right) \vec{S}\cdot(\vec{r}\times\vec{p}) + \frac{1}{2r^4} \left\{ (C_3 + 4C_1) \left(\frac{1}{m_a^2} + \frac{1}{m_a m_b} \right) + \left(\frac{1}{2m_a^4} + \frac{2}{m_a^3 m_b} \right) \right\} \vec{S}\cdot(\vec{r}\times\vec{p}) \left(\vec{S}^2 - 5\frac{(\vec{S}\cdot\vec{r})^2}{r^2} \right) - \left[\frac{C_1}{m_a^4 r^5} + \frac{4C_2+1}{8m_a^4 r^5} \right] \left(3(\vec{S}^2)^2 - 30\frac{\vec{S}^2(\vec{S}\cdot\vec{r})^2}{r^2} + 35\frac{(\vec{S}\cdot\vec{r})^2}{r^4} \right) \Bigg\}. \quad (32)$$

This gives us the result for the missing H_{S^4} Hamiltonian for a compact star,

$$H_{S^4} = -\left[\frac{C_1}{m_a^4 r^5} + \frac{4C_2+1}{8m_a^4 r^5} \right] \times \left(3(\vec{S}^2)^2 - 30\frac{\vec{S}^2(\vec{S}\cdot\vec{r})^2}{r^2} + 35\frac{(\vec{S}\cdot\vec{r})^2}{r^4} \right). \quad (33)$$

As before, we recover the universal form of the spin-orbit piece. Since we have three additional operators for the spin-2 case, we get arbitrary coefficients for the S^2, S^3 and S^4 pieces. The limit for the black hole is obtained for $C_1 = C_2 = C_3 = 0$ [27], which again demonstrates that a minimal coupling to the graviton implies a black hole. The Wilson coefficients for these operators can be obtained from a matching procedure with any other spin-dependent observable. The result for the S^3 Hamiltonian was derived in the limit of a black hole [24,27]. An attempt to derive the quartic spin Hamiltonian for a black hole was made in Ref. [27] but was found to be inconsistent with the results of Ref. [28] which computed the binding energy of a test particle in the extreme mass ratio in a circular orbit with the spin of the massive star aligned perpendicular to the orbit. In this limit, the Hamiltonian above reduces to

$$H_{S^4} = -\frac{3Gm_b}{8m_a^3 r^5}(\vec{S}^2)^2. \quad (34)$$

A comparison with the result in Ref. [28] gives a match for the binding energy.

III. KERR METRIC

As another consistency check we can easily obtain the Kerr metric to leading power in G and up to fourth order in spin using the calculation done so far. We consider the worldline action of a probe particle in a Kerr background field,

$$S = -m_b \int dt \sqrt{g_{00} + g_{0i}v_i + g_{ij}v_{ij}}. \quad (35)$$

For the leading-order spin-dependent pieces, this gives us the result

$$g_{00} = 1 + 2Gm_a \left[-\frac{1}{r} + \frac{1}{2m_a^2 r^3} \left(\frac{3(\vec{S}\cdot\vec{r})^2}{r^2} - \vec{S}^2 \right) - \frac{1}{8m_a^4 r^5} \left(3(\vec{S}^2)^2 - 30\frac{\vec{S}^2(\vec{S}\cdot\vec{r})^2}{r^2} + 35\frac{(\vec{S}\cdot\vec{r})^4}{r^4} \right) \right]. \quad (36)$$

Comparing g_{00} with the corresponding result for the Kerr metric in harmonic coordinates [29] again confirms the S^4 Hamiltonian piece.

IV. SUMMARY AND OUTLOOK

We have used modern methods of amplitude computation combined with EFT techniques to obtain spin-dependent Hamiltonians for a binary inspiralling system in the PN approximation. The use of on-shell methods substantially reduces the effort of computing loop diagrams. We have also shown how the idea of treating gravity as spin-2 massless particle provides a natural way of obtaining higher-order spin corrections for arbitrary classical objects. The possible gauge-invariant interaction operators that we can write down, automatically account for any spinning classical objects including a black hole. Using a massive spin-2 particle scattering at tree level, we were able to obtain the S^3 interaction for an arbitrary object in terms of coefficients which depend on the specific equation of state for a star, which was until now known only for a black hole. We were also able to calculate in a simple manner the hitherto unknown S^4 Hamiltonian and show that three independent operators are needed to account for other stellar equations of state up to fourth order in the PN expansion. What is really interesting, is the universality of the interaction terms that appear as we move to particles of higher spin. Also, a curious fact is revealed that the minimal coupling of a massive elementary particle to gravity automatically accounts for any spin-dependent interactions of a black hole. In principle, all the spin-dependent Hamiltonians up to 4PN order can be obtained by considering loop corrections for the scattering of two spin-1 particles. The spin-2 particle scattering is required only at tree level.

ACKNOWLEDGMENTS

I thank Ira Rothstein for his help on several conceptual aspects of this project and for comments on this manuscript. This work is supported by DOE Grant No. DE-FG02-04ER41338 and FG02-06ER41449.

Note added.—As this paper was being finalized, another paper appeared [30] which also investigated the cubic and quartic spin Hamiltonians using the NRGR formalism which uses the traditional effective field theory approach.

[1] J. Aasi *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. D **87**, 022002 (2013).
[2] J. Abadie *et al.* (LIGO Scientific and Virgo Collaborations), Classical Quantum Gravity **27**, 173001 (2010).
[3] S. Hergt, J. Steinhoff, and G. Schafer, J. Phys. Conf. Ser. **484**, 012018 (2014).
[4] T. Damour, P. Jaranowski, and G. Schafer, Phys. Rev. D **77**, 064032 (2008).
[5] J. Steinhoff, G. Schafer, and S. Hergt, Phys. Rev. D **77**, 104018 (2008).
[6] J. Steinhoff, S. Hergt, and G. Schafer, Phys. Rev. D **77**, 081501 (2008).
[7] S. Hergt, J. Steinhoff, and G. Schafer, Classical Quantum Gravity **27**, 135007 (2010).
[8] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006).
[9] R. A. Porto, Classical Quantum Gravity **27**, 205001 (2010).
[10] R. A. Porto and I. Z. Rothstein, Phys. Rev. D **78**, 044013 (2008); **81**, 029905(E) (2010).
[11] R. A. Porto and I. Z. Rothstein, Phys. Rev. D **78**, 044012 (2008); **81**, 029904(E) (2010).
[12] M. Levi, Phys. Rev. D **85**, 064043 (2012).
[13] M. Levi, Phys. Rev. D **82**, 104004 (2010).
[14] M. Levi, Phys. Rev. D **82**, 064029 (2010).
[15] D. Neill and I. Z. Rothstein, Nucl. Phys. **B877**, 177 (2013).
[16] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, J. High Energy Phys. 02 (2014) 111.
[17] I. Z. Rothstein, arXiv:hep-ph/0308266.
[18] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B715**, 499 (2005).R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005).
[19] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[20] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[21] L. J. Dixon, in *Qcd & Beyond: Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics (Tasi '95), Boulder, Colorado, USA, 1995* (World Scientific, Singapore, 1996), p. 539.
[22] M. E. Peskin, arXiv:1101.2414.
[23] B. R. Holstein and A. Ross, arXiv:0802.0716.
[24] S. Hergt and G. Schaefer, Phys. Rev. D **77**, 104001 (2008).
[25] S. Chakrabarti, T. Delsate, and J. Steinhoff, Phys. Rev. D **88**, 084038 (2013).
[26] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).
[27] S. Hergt and G. Schaefer, Phys. Rev. D **78**, 124004 (2008).
[28] J. Steinhoff and D. Puetzfeld, Phys. Rev. D **86**, 044033 (2012).
[29] J. M. Aguirregabiria, L. Bel, J. Martin, A. Molina and E. Ruiz, Gen. Relativ. Gravit. **33**, 1809 (2001).
[30] M. Levi and J. Steinhoff, arXiv:1410.2601.