

Gauss-Bonnet boson stars with a single Killing vectorLaura J. Henderson,^{1,*} Robert B. Mann,^{1,†} and Sean Stotyn^{2,‡}¹*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*²*Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1*

(Received 11 April 2014; published 6 January 2015)

We construct asymptotically anti-de Sitter boson stars in Einstein-Gauss-Bonnet gravity coupled to a $\frac{D-1}{2}$ -tuple of complex massless scalar fields both perturbatively and numerically in $D = 5, 7, 9, 11$ dimensions. These solutions possess just a single helical Killing symmetry due to the choice of scalar fields. The energy density at the center of the star characterizes the solutions, and for each choice of the Gauss-Bonnet coupling α we obtain a one parameter family of solutions. All solutions respect the first law of thermodynamics, in the numerical case to within 1 part in 10^6 . We describe the dependence of the angular velocity, mass, and angular momentum of the boson stars on α and on the dimensionality. For $D \geq 7$ these quantities reach maximum values and then decrease to eventually approach finite values as the central energy density tends to infinity. In the limit of diverging central energy density, the Kretschmann invariant at the center of the boson star also diverges. This is in contrast to the $D = 5$ case, where the Kretschmann invariant diverges at a finite value of the central energy density.

DOI: [10.1103/PhysRevD.91.024009](https://doi.org/10.1103/PhysRevD.91.024009)

PACS numbers: 04.50.-h

I. INTRODUCTION

The discovery of the Higgs boson has given a clear indication of the existence of at least one scalar field in nature. In addition to the fact that scalar fields are pervasive in many versions of quantum gravity, this has generated renewed interest in obtaining solutions to quantum-inspired effective theories of gravitation coupled to scalar fields.

Prominent amongst such solutions are boson stars: these are smooth, horizonless geometries composed of self-gravitating and (possibly) self-interacting bosonic matter [1,2]. Unlike ordinary stars and planets, they typically do not have a sharp edge (though solutions with such edges have recently been found [3]) but instead are bundles of field energy that decay at large distances from their cores. Their astrophysical relevance is not clear at this point in time but they may be possible dark matter halo candidates [4]. In an excited state they typically produce a more physically realistic, flatter rotation curve than in the ground state. However, such excited states decay to the ground state unless they are in rather specific mixed states [5]. Furthermore, boson stars can provide dark alternatives to astrophysical black hole candidates, which could potentially be discerned by gravitational wave astronomy [6,7].

From a theoretical perspective, asymptotically anti-de Sitter (AdS) boson stars may play an important role in holographic gauge theories through the AdS/CFT correspondence [8]. All such solutions are zero-temperature objects without horizons and, as such, they describe finite

energy excitations above the vacuum state. If AdS boundary conditions are present, their (nonlinear) stability is of key import, as it determines whether or not the corresponding state in the holographic dual CFT thermalizes and on what time scale. While recent work indicates that asymptotically AdS space-times suffer from a gravitational turbulent instability [9–12], which might lead one to expect that AdS boson stars are nonlinearly unstable to black hole formation, it has also been shown that there exists a wide range of initial data that are immune to this instability [13]. Furthermore, the turbulent instability of global AdS has been shown to be due to a high level of symmetry [14]: the normal mode frequencies are all integer multiples of the AdS frequency, leading to a large number of resonances responsible for the nonlinear instability. These results suggest that on the gauge theory side there is a family of strongly coupled CFT states that do not thermalize in finite time.

A natural question to ask is how generic this feature of the initial data considered in Ref. [13] is. In that respect, one of the immediate ongoing tasks is to map out the territory of boson star solutions. Quite a wide range of solutions has been obtained for various forms of scalar matter. These include a complex doublet of massive [15,16] and massless [17,18] scalar fields, self-interacting scalar fields [19,20], scalars with gauge charges [21,22], scalars in space-times with de Sitter [23,24], flat [19,25], or anti-de Sitter [15,26] boundary conditions, solutions in $(2 + 1)$ dimensions [15,27,28], and rotating doublets [18] and multiplets [17,29] of scalars.

In this paper we obtain both perturbative and numerical solutions for asymptotically AdS boson stars in $D = 5, 7, 9, 11$ dimensions in Einstein-Gauss-Bonnet (EGB)

*lhender@uwaterloo.ca

†rbmann@sciborg.uwaterloo.ca

‡sstotyn@phas.ubc.ca

gravity in the special case that the boson stars only possess a single Killing vector (SKV). EGB gravity is a special case of Lovelock gravity [30], a class of higher curvature theories of gravity that are distinguished by having their field equations not containing terms with more than two derivatives of the metric. They are of interest in quantum gravity insofar as one expects the Einstein action to be an effective gravitational action, valid for small curvature (or low energies), that will be modified by higher-curvature terms. A given contribution to the action in D dimensions consists of a product of the Euler density $\mathcal{L}^{(k)}$ of a $2k$ dimensional manifold and an arbitrary constant $\hat{\alpha}_{(k)}$ and so contributes to the equations of motion only if $D > 2k$. The simplest case beyond Einstein gravity is EGB gravity for which $k = 2$, implying $D \geq 5$.

EGB gravity is an incredibly rich theory with entirely new branches of solutions arising that have no counterpart in Einstein gravity. Most notably, for a given Gauss-Bonnet coupling constant α , there are two distinct vacua of the theory, assuming all matter sources present are appropriately bounded. In the absence of a cosmological constant, one of the vacua is the expected Minkowski vacuum while the other has an effective negative cosmological constant and is hence the AdS vacuum.¹ In Ref. [31] the authors considered the stability of linear gravitational perturbations about both vacua and concluded that the Minkowski vacuum is stable while the AdS vacuum exhibits ghostlike perturbations and is hence unstable.

When a negative cosmological constant is present, as in the present paper, there are still two distinct vacua, both of which are now AdS. However, one of these continuously deforms to the AdS vacuum of Einstein gravity as $\alpha \rightarrow 0$ while the other does not. In analogy with the analysis of [31] one expects the former to be stable and the latter to be unstable. Interestingly, there is a critical value, α_{cr} , where the two vacua degenerate and the corresponding EGB theory belongs to a class of quadratic curvature theories that have different properties than those of the standard EGB type [32]. Locally maximally symmetric solutions admit only one fixed radius of curvature, and while they admit spherically symmetric black hole solutions, these solutions do not obey the falloff boundary conditions that solutions in $\alpha \neq \alpha_{\text{cr}}$ theories require. Indeed, this class of theories admits Banados-Teitelboim-Zanelli (BTZ)-like solutions, in which there is a mass gap between the zero mass black hole and AdS space-time. While we shall not consider these theories here, analysis of boson star solutions in this case would be interesting, and require an approach similar to that recently employed for $D = 3$ rotating boson stars [29].

In the present paper, for any given value of the Gauss-Bonnet coupling constant $\alpha \neq \alpha_{\text{cr}}$ we obtain a one parameter family of solutions parametrized by the central

energy density of the boson star. For $D > 5$, in the limit that this central energy density tends to infinity, the mass, angular momentum, and angular velocity all limit to finite values, while the Kretschmann scalar at the center of the boson star diverges. We find that as the space-time dimension increases, the maximum value reached by these quantities decreases, and for $D > 7$ it also decreases with increasing values of α . For $D = 5$ we find markedly different behavior: the Kretschmann scalar at the center diverges for a finite value of the central energy density, with this critical value decreasing as α increases. All of the solutions constructed in this paper for $\alpha < \alpha_{\text{cr}}$, both perturbative and numerical, limit to the Einstein case [17,27] as $\alpha \rightarrow 0$; we do not consider alternate possible branches of solutions that do not have this limit,² though such branches may indeed exist with the addition of the Gauss-Bonnet term in the gravitational action.

Numerically we are unable to explore values of α in the range $0.99\alpha_{\text{cr}} < \alpha < \alpha_{\text{cr}}$ when $D \geq 7$ and $0.90\alpha_{\text{cr}} < \alpha < \alpha_{\text{cr}}$ when $D = 5$ due to numerical complications and the increase in computer time required to find solutions in this range. However it is possible to examine boson star solutions for $\alpha > \alpha_{\text{cr}}$, which correspond to boson stars immersed in the unstable AdS vacuum. Although we find that such solutions have positive Arnowitt-Deser-Misner (ADM) mass, we further find that such solutions are ghostlike as expected. We shall briefly comment on these solutions near the end of the paper.

The remainder of this paper is structured as follows: in Sec. II we present the metric and scalar field *Ansätze* and obtain the constraint equations and ordinary differential equations (ODEs) that must be solved. We describe the boundary conditions and the basic physical properties of the boson stars in Sec. III. We then analytically construct perturbative solutions in Sec. IV. In Sec. V we describe the numerical methods used to construct the full nonperturbative solutions, which are presented in Sec. VI along with a discussion about their salient features. In Sec. VII we discuss a preliminary investigation of the region of parameter space corresponding to the unstable AdS vacuum. Finally, in Sec. VIII we provide some concluding remarks.

II. SETUP

We begin with $D = n + 2$ dimensional Einstein-Gauss-Bonnet gravity with negative cosmological constant minimally coupled to an $\frac{n+1}{2}$ -tuple of complex massless scalar fields

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} (R - 2\Lambda + \alpha L_{GB} - 2|\nabla\vec{\Pi}|^2) \quad (2.1)$$

¹This assumes $\alpha > 0$ as is required by string theory, thereby eliminating the possibility of a de Sitter vacuum.

²This is controlled by requiring AdS boundary conditions. If these boundary conditions are relaxed, exotic branches may emerge but this is beyond the scope of the present paper.

where

$$L_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \quad (2.2)$$

and where Λ is the cosmological constant. The equations of motion resulting from this action are

$$G_{ab} + \Lambda g_{ab} - 2\alpha \left(-R_{acde}R_b{}^{cde} + 2R_{acbd}R^{cd} + 2R_{ac}R_b^c - RR_{ab} + \frac{1}{4}g_{ab}L_{GB} \right) = T_{ab} \quad (2.3)$$

$$\nabla^2 \vec{\Pi} = 0 \quad (2.4)$$

where

$$T_{ab} = (\partial_a \vec{\Pi}^* \partial_b \vec{\Pi} + \partial_a \vec{\Pi} \partial_b \vec{\Pi}^*) - g_{ab} (\partial_c \vec{\Pi} \partial^c \vec{\Pi}^*) \quad (2.5)$$

is the stress-energy tensor of the scalar fields. We will only be interested in solutions that asymptote to AdS and we thus require global AdS to be a solution to Eq. (2.3) with $\vec{\Pi} = 0$. This requirement fixes the bare cosmological constant Λ to

$$\Lambda = \frac{n(n+1)((n-1)(n-2)\alpha - \ell^2)}{2\ell^4}, \quad (2.6)$$

where ℓ is the effective AdS length.

We shall employ the following *Ansätze* for the metric and scalar fields [17]:

$$ds^2 = -f(r)g(r)dt^2 + \frac{dr^2}{f(r)} + r^2(h(r)(d\chi + A_i dx^i - \Omega(r)dt)^2 + g_{ij}dx^i dx^j) \quad (2.7)$$

$$\Pi_i = \Pi(r)e^{-i\omega t} z_i, \quad i = 1 \dots \frac{n+1}{2} \quad (2.8)$$

where z_i are complex coordinates such that $\sum_i dz_i d\bar{z}_i$ is the metric of a unit n -sphere. It is straightforward to show using the choice

$$z_i = \begin{cases} e^{i(\chi + \phi_i)} \cos \theta_i \prod_{j < i} \sin \theta_j, & i = 1 \dots \frac{n-1}{2} \\ e^{i\chi} \prod_{j=1}^{\frac{n-1}{2}} \sin \theta_j, & i = \frac{n+1}{2} \end{cases} \quad (2.9)$$

that $\sum_i dz_i d\bar{z}_i = (d\chi + A_i dx^i)^2 + g_{ij} dx^i dx^j$ is the Hopf fibration of the unit n -sphere, where

$$A_i dx^i = \sum_{i=1}^{\frac{n-1}{2}} \cos^2 \theta_i \left[\prod_{j < i} \sin^2 \theta_j \right] d\phi_i \quad (2.10)$$

and g_{ij} is the metric on a unit complex projective space $\mathbb{C}\mathbb{P}^{\frac{n-1}{2}}$. In these coordinates, χ and the ϕ_i all have period 2π

while the θ_i take values in the range $[0, \frac{\pi}{2}]$. This construction only works in odd dimensions, so we shall restrict our attention to $n = 3, 5, 7, 9$ (though our results can be extended to any odd value of $n \geq 3$).

The $n = 3$ form of this construction was first used in Ref. [33] to construct boson star solutions in 5-dimensional Einstein gravity coupled to a doublet of self-interacting scalars. The construction above is crucial in obtaining rotating boson star solutions in any odd dimension. The scalar fields can be regarded as coordinates on $\mathbb{C}\mathbb{P}^{\frac{n-1}{2}}$, with $\vec{\Pi}$ tracing out a round n -sphere (for any value of r) with a time-varying but otherwise constant phase. Constant r surfaces in the metric (2.7) correspond to squashed rotating n -spheres. The first term in the stress-energy tensor T_{ab} is the pull-back of the round metric on the n -sphere and the second term is proportional to g_{ab} . Hence T_{ab} has the same symmetries as the metric (2.7).

Note that while the matter stress tensor has the same symmetries as the metric, the scalar fields themselves do not. The scalar field multiplet (2.8) is only invariant under the combination

$$\kappa = \partial_t + \omega \partial_\chi, \quad (2.11)$$

whereas the metric (2.7) is indeed invariant under $\partial_t, \partial_\chi$ as well as the rotations of $\mathbb{C}\mathbb{P}^{\frac{n-1}{2}}$. Any solution with nontrivial scalar field content will only be invariant under the single Killing vector field κ given by (2.11).

The equations of motion yield a system of five coupled second-order ODEs. These are rather cumbersome to write down and so we have relegated them to Appendix A.

III. BOUNDARY CONDITIONS AND PHYSICAL CHARGES

In this section, we write down the boundary conditions that boson stars in EGB gravity must satisfy; we shall employ the same boundary conditions as in the Einstein case [29], both at the origin and asymptotically.

A. Boundary conditions at the origin

Since a boson star geometry must be smooth and horizonless, all metric functions must be regular at the origin. Furthermore, surfaces of constant t in the vicinity of the origin are described by round n -spheres (with r the proper radial distance). Multiplying (A5) by r^2 , we note that Π must vanish at the origin in order to yield consistent equations of motion. Thus, the boundary conditions at the center of the boson star take the form

$$\begin{aligned} f|_{r \rightarrow 0} &= 1 + \mathcal{O}(r^2), & g|_{r \rightarrow 0} &= g(0) + \mathcal{O}(r), \\ h|_{r \rightarrow 0} &= 1 + \mathcal{O}(r^2), & \Omega|_{r \rightarrow 0} &= \Omega(0) + \mathcal{O}(r), \\ \Pi|_{r \rightarrow 0} &= q_0 \frac{r}{\ell} + \mathcal{O}(r^2), \end{aligned} \quad (3.1)$$

for all n , where q_0 is a dimensionless parameter such that the energy density of the scalar field, T^{00} , at the origin (hereafter called the central energy density) is proportional to q_0^2 . In fact for a given value of α , q_0 uniquely parametrizes the one-parameter family of boson star solutions in each dimension. Formally, it is defined by $q_0 \equiv \ell \Pi'(0)$.

B. Asymptotic boundary conditions

In order to simplify the asymptotic boundary conditions, we first make note of a residual gauge freedom. It is straightforward to show that the transformation

$$\chi = \tilde{\chi} + \lambda t, \quad \Omega(r) = \tilde{\Omega}(r) + \lambda, \quad \omega = \tilde{\omega} + \lambda, \quad (3.2)$$

for some arbitrary constant λ , leaves both the metric (2.7) and scalar field (2.8) unchanged. We find it convenient in our numerical analysis to set $\lambda = \omega$ so that we can set $\tilde{\omega} = 0$: in this frame, the coordinates are rigidly rotating asymptotically so that $\tilde{\Omega}(r) \rightarrow -\omega$ as $r \rightarrow \infty$. In what follows, we use $\tilde{\chi}$, $\tilde{\Omega}(r)$, and ω but we drop the tildes for notational convenience.

As $r \rightarrow \infty$, the boson stars asymptote to AdS with corrections for mass and angular momentum. This determines the metric functions up to constants C_f , C_h , and C_Ω . The boundary condition for the scalar field is set by requiring Π to be normalizable. Explicitly we obtain

$$\begin{aligned} f|_{r \rightarrow \infty} &= \frac{r^2}{\ell^2} + 1 + \frac{C_f \ell^{n-1}}{r^{n-1}} + \mathcal{O}(r^{-n}), \\ g|_{r \rightarrow \infty} &= 1 - \frac{C_h \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \\ h|_{r \rightarrow \infty} &= 1 + \frac{C_h \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \\ \Omega|_{r \rightarrow \infty} &= -\omega + \frac{C_\Omega \ell^n}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \\ \Pi|_{r \rightarrow \infty} &= \frac{\epsilon \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \end{aligned} \quad (3.3)$$

where ϵ is a dimensionless measure of the amplitude of the scalar field at infinity. The constants C_f , C_h , and C_Ω are determined from solving the equations; note that the leading order corrections to $g(r)$ and $h(r)$ are not independent. In the next section we shall solve the equations perturbatively, and will find (as in the Einstein case [17]) that ϵ uniquely parametrizes the boson star solutions. Nonperturbatively the situation is different, as we shall explore in Sec. VI.

At this point, it is important to note that we are using ℓ to define the asymptotic behavior of $f(r)$, and as long as $\ell^2 > 0$, the space-time will be asymptotically AdS. By defining the AdS length in this way, the relation (2.6) for the cosmological constant follows from the field equations. Note that Eq. (2.6) can also be written as a quadratic equation for ℓ^2 , which has solutions

$$\ell^2 = -\frac{n(n+1)}{4\Lambda} \left(1 \pm \sqrt{1 + \frac{8\Lambda\alpha(n-1)(n-2)}{n(n+1)}} \right), \quad (3.4)$$

where the $+$ sign reduces to the Einstein relation, $\Lambda = -\frac{n(n+1)}{2\ell^2}$, in the limit $\alpha \rightarrow 0$ while the $-$ sign implies $\ell^2 \rightarrow 0$ as $\alpha \rightarrow 0$. The $-$ sign thus corresponds to the unstable AdS vacuum of EGB theory while the $+$ sign corresponds to the stable AdS vacuum, in analogy with the work of Ref. [31]. Note that there is a critical value of α in terms of Λ where ℓ^2 becomes degenerate:

$$\alpha_{\text{cr}} = -\frac{n(n+1)}{8\Lambda(n-1)(n-2)}, \quad (3.5)$$

where $\Lambda < 0$ to ensure $\alpha > 0$. For this value of α , the EGB theory has a unique AdS vacuum and becomes part of a class of theories dubbed Lovelock unique vacuum (LUV) theories, as first studied in [32], and has very different properties from theories with $\alpha \neq \alpha_{\text{cr}}$.

The convention used here differs from another common convention [32,34], which sets

$$\Lambda = -\frac{n(n+1)}{2L^2} \quad (3.6)$$

(where L is the would-be AdS length in the absence of the Gauss-Bonnet term) and leads to an upper bound on α ,

$$\alpha < \frac{L^2}{4(n-1)(n-2)}, \quad (3.7)$$

in order to satisfy AdS boundary conditions [34]. In terms of the convention used here, this inequality can be expressed as

$$(2(n-1)(n-2)\alpha - \ell^2)^2 > 0, \quad (3.8)$$

which immediately implies

$$\alpha \neq \frac{\ell^2}{2(n-1)(n-2)} \equiv \alpha_{\text{cr}}. \quad (3.9)$$

Note that this is the same α_{cr} found in Eq. (3.5) that makes the EGB theory a LUV theory [32]. This critical value of α also naturally appears in the perturbative boson star solutions, as will be seen shortly.

C. Physical charges

The SKV boson stars are invariant under the single Killing field (2.11). However since the scalar fields vanish at infinity with sufficient falloff, both ∂_t and ∂_χ are each asymptotic Killing fields (the metric alone being invariant

under them for the full solution). Consequently conserved charges can be defined where ∂_t and ∂_χ are readily associated with a conserved energy and angular momentum respectively. We use the definitions proposed in Ref. [35], given by

$$M = \frac{(n+1)\pi^{\frac{n-1}{2}}\ell^{n-1}}{16\binom{n+1}{2}!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) ((n+1)C_h - nC_f), \quad (3.10)$$

$$J = \frac{(n+1)^2\pi^{\frac{n-1}{2}}\ell^n}{16\binom{n+1}{2}!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) C_\Omega \quad (3.11)$$

where C_f , C_h , and C_Ω are the constants appearing in the asymptotic boundary conditions (3.3). This prescription emerges from a semiclassical treatment whereby the actions are calculated using the boundary counterterms found in [35], and agrees with the Ashtekar/Das definition [36,37] of mass in the $\alpha \rightarrow 0$ limit. The above definitions ensure that the first law of thermodynamics is satisfied for black holes in EGB theory, at least in the spherically symmetric case where an explicit solution is known.

The existence of these asymptotic Killing symmetries also guarantees that the boson stars satisfy the first law of thermodynamics. They have vanishing temperature and entropy, so the first law takes the form

$$dM = \omega dJ. \quad (3.12)$$

This relation provides a useful and important numerical tool: by requiring the solutions to respect the first law (3.12) to at least one part in 10^6 , we obtain a primary cross-check on the validity of the numerical methods used.

IV. PERTURBATIVE SOLUTIONS

In this section, we consider perturbative solutions to the Einstein-Gauss-Bonnet equations that satisfy the boundary conditions of Sec. III. These solutions are horizonless and are constructed as perturbations around AdS; they generalize those found previously for Einstein gravity in [17]. We take the scalar field condensate parameter, ϵ , as the perturbative expansion parameter and we give results up to order ϵ^6 for the space-time dimensions of interest in string theory, namely $D = 5, 7, 9, 11$. As a perturbative construction, these results will only be valid for small energies and angular momenta. They will provide another useful check on the numeric solutions we present later in the paper.

A. Perturbative boson star

Since we require global AdS to be a solution when the scalar field amplitude vanishes, we employ the expansion

$$\begin{aligned} F(r, \epsilon) &= \sum_{i=0}^m \tilde{F}_{2i}(r) \epsilon^{2i} \\ \Pi(r, \epsilon) &= \sum_{i=0}^m \tilde{\Pi}_{2i+1}(r) \epsilon^{2i+1} \\ \omega(\epsilon) &= \sum_{i=0}^m \tilde{\omega}_{2i} \epsilon^{2i} \end{aligned} \quad (4.1)$$

where $F \in \{f, g, h, \Omega\}$ is shorthand for each of the metric functions in (2.7). Note that the scalar fields are expanded in odd powers of ϵ whereas the metric functions are expanded in even powers. The perturbative solution to the equations is thus obtained by starting with global AdS—the zeroth terms in the metric function expansions above.

The scalar equation (A5) is then solved in this background, yielding $\tilde{\Pi}_1(r)$. The full set of equations of motion (A1)–(A6) are then satisfied up to order ϵ . The stress-energy for these scalars is then of order ϵ^2 , which induces source corrections to the gravitational fields $\tilde{F}_2(r)$ (the $i = 1$ terms in the metric functions). These in turn backreact on the scalar fields, and the next-order solution to (A5) yields $\tilde{\Pi}_3(r)$, thereby satisfying the equations of motion up to order ϵ^3 .

This bootstrapping procedure can be carried out to arbitrary order in ϵ . The angular frequency must also be expanded in even powers of ϵ because the scalar fields backreact on the metric, inducing nontrivial frame-dragging effects, which in turn affect the rotation of the scalar fields. These corrections to ω are found by imposing the boundary conditions.

Global AdS is given by

$$f_0 = 1 + \frac{r^2}{\ell^2}, \quad g_0 = 1, \quad h_0 = 1, \quad \Omega_0 = 0. \quad (4.2)$$

The most general massless scalar field solution to (A5) in this background that is consistent with the asymptotic boundary conditions (3.3) is

$$\begin{aligned} \Pi_1(r) &= \frac{r\ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}} \\ &\times {}_2F_1 \left[\frac{n+2 - \omega\ell}{2}, \frac{n+2 + \omega\ell}{2}; \frac{n+3}{2}; \frac{\ell^2}{r^2 + \ell^2} \right] \end{aligned} \quad (4.3)$$

where ${}_2F_1$ is the hypergeometric function. Note that the Gauss-Bonnet parameter α does not enter at this order. In order to satisfy the boundary conditions at the origin (3.1) we must require

$$\omega\ell = n + 2 + 2k, \quad k = 0, 1, 2, \dots \quad (4.4)$$

where the non-negative integer, k , describes the various possible radial modes of the scalar field. While any radial profile for the scalars is possible, it must be built up from a

linear combination of the radial modes that necessitate the inclusion of multiple frequency parameters, ω_k . This is inconsistent with the existence of the Killing vector field (2.11). These higher modes represent excited states, whereas the $k=0$ mode is the ground state. In what follows, we only consider the ground state, in which case (4.3) simplifies to

$$\Pi_1(r) = \frac{r\ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}}. \quad (4.5)$$

Next, we insert (4.5) and the expansion (4.1) into the equations of motion (A1)–(A6), expand in ϵ and solve at order ϵ^2 . Using the boundary conditions to fix the two constants of integration that emerge, the order ϵ^2 corrections to the metric functions, $\bar{F}_2(r)$, are then inserted into the equation of motion for $\Pi(r)$ to find its order ϵ^3 correction. This procedure, carried out up to but not including order ϵ^6 , yields

$$\begin{aligned} f(r) = 1 + \frac{r^2}{\ell^2} - \frac{r^2\ell^{n+1}f_{n;2}}{(r^2 + \ell^2)^{n+1}(\ell^2 - 2(n-1)(n-2)\alpha)}\epsilon^2 \\ - \frac{r^2\ell^{n+3}f_{n;4}}{(r^2 + \ell^2)^{2n+3}(\ell^2 - 2(n-1)(n-2)\alpha)^3}\epsilon^4 \\ + \mathcal{O}(\epsilon^6) \end{aligned} \quad (4.6)$$

$$\begin{aligned} g(r) = 1 - \frac{2\ell^{2n+4}((n+1)r^2 + \ell^2)}{n(r^2 + \ell^2)^{n+2}(\ell^2 - 2(n-1)(n-2)\alpha)}\epsilon^2 \\ - \frac{\ell^{n+5}g_{n;4}}{(r^2 + \ell^2)^{2n+4}(\ell^2 - 2(n-1)(n-2)\alpha)^3}\epsilon^4 \\ + \mathcal{O}(\epsilon^6) \end{aligned} \quad (4.7)$$

$$\begin{aligned} h(r) = 1 + \frac{r^2\ell^{n+5}h_{n;4}}{(r^2 + \ell^2)^{2n+3}(\ell^2 - 2(n-1)(n-2)\alpha)^3}\epsilon^4 \\ + \mathcal{O}(\epsilon^6) \end{aligned} \quad (4.8)$$

$$\begin{aligned} \Omega(r) = \frac{\ell^{n+2}\Omega_{n;2}}{(r^2 + \ell^2)^{n+1}(\ell^2 - 2(n-1)(n-2)\alpha)}\epsilon^2 \\ + \frac{\ell^{n+4}\Omega_{n;4}}{(r^2 + \ell^2)^{2n+3}(\ell^2 - 2(n-1)(n-2)\alpha)^3}\epsilon^4 \\ + \mathcal{O}(\epsilon^6) \end{aligned} \quad (4.9)$$

$$\begin{aligned} \Pi(r) = \frac{r\ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}}\epsilon \\ + \frac{r\ell^{n+5}\Pi_{n;3}}{(r^2 + \ell^2)^{\frac{3n+4}{2}}(\ell^2 - 2(n-1)(n-2)\alpha)}\epsilon^3 \\ + \frac{r\ell^{n+7}\Pi_{n;5}}{(r^2 + \ell^2)^{\frac{5(n+2)}{2}}(\ell^2 - 2(n-1)(n-2)\alpha)^3}\epsilon^5 \\ + \mathcal{O}(\epsilon^7) \end{aligned} \quad (4.10)$$

where the fields $\{f_{n;s}, g_{n;s}, h_{n;s}, \Omega_{n;s}, \Pi_{n;s}\}$ are simple polynomials in r ; in this notation s labels the order in ϵ and $n = D - 2$ labels the space-time dimension.³ These fields are cataloged in Appendix B for $n = 3, 5, 7, 9$ up to order ϵ^6 , with the associated energies and angular momenta in Appendix C. It is straightforward to check that these latter quantities obey the first law (3.12) to order ϵ^6 .

The perturbative solutions all reduce to those of the Einstein case [17] in the limit $\alpha \rightarrow 0$. Perturbative boson star solutions obeying the boundary conditions (3.3) do not exist for $\alpha = \alpha_{\text{cr}}$ in any dimension.

V. NUMERICAL CONSTRUCTION

We numerically construct boson star solutions by approximating the metric and scalar field functions $\{f(r), g(r), h(r), \Omega(r), \Pi(r)\}$ as Chebyshev polynomials and using a relaxation method on a Chebyshev grid. A detailed review of this procedure can be found in [38].

Before the Chebyshev grid can be constructed, we must first compactify the infinite domain $r \in [0, \infty)$ to the finite domain $y \in [0, 1]$, which is done via the simple coordinate transformation

$$y = \frac{r^2}{r^2 + \ell^2}. \quad (5.1)$$

The grid then consists of the set of points $y_k = (\cos(k\pi/N) + 1)/2$, $k = 0, \dots, N$, which are the $N + 1$ extrema of the N th-order Chebyshev polynomial, $T_N(2y - 1)$.

Next, we define auxiliary functions $\{q_f, q_g, q_h, q_\Omega, q_\Pi\}$, which are analytic over the entire domain $y \in [0, 1]$, by taking into account the boundary conditions (3.1) and (3.3):

$$f(y) = \frac{y}{1-y} + 1 + (1-y)^{\frac{n-1}{2}}q_f(y), \quad (5.2)$$

$$g(y) = 1 + (1-y)^{\frac{n+1}{2}}q_g(y), \quad (5.3)$$

$$h(y) = 1 + (1-y)^{\frac{n+1}{2}}q_h(y), \quad (5.4)$$

$$\Omega(y) = \frac{q_\Omega(y)}{\ell}, \quad (5.5)$$

$$\Pi(y) = \sqrt{y}(1-y)^{\frac{n+1}{2}}q_\Pi(y). \quad (5.6)$$

The remaining discussion will use the coordinate y , with a prime $'$ denoting a derivative with respect to y .

The boundary conditions for the q functions are found by first converting the field equations (A6)–(A3) into functions of y , then Taylor expanding them around the two end points $y_0 = 0, 1$ and requiring that they vanish order by order. The lowest-order terms give nontrivial relationships

³The notation is the same as that of Ref. [17], except that an additional “0” index (relevant for perturbative black hole solutions) is dropped.

between the q functions and their first derivatives. At $y_0 = 1$ we obtain

$$\begin{aligned} q'_f(1) + \frac{n-1}{2} q_f(1) + q_h(1) &= 0, \\ q_g(1) + q_h(1) &= 0, \\ q'_\Omega(1) &= 0, \end{aligned} \quad (5.7)$$

$$\begin{aligned} q'_h(1) + \frac{n+1}{2} q_h(1) &= 0, \\ q'_\Pi(1) + \frac{1}{2(n+3)} ((n+2)^2 - q_\Omega(1)^2) q_\Pi(1) &= 0. \end{aligned} \quad (5.8)$$

At $y_0 = 0$ we find

$$\begin{aligned} q_f(0) &= 0, \\ q_h(0) &= 0, \\ q_\Pi(0) - q_0 &= 0, \end{aligned} \quad (5.9)$$

$$\begin{aligned} &\frac{2\ell^2 q_0^2 q_\Omega(0)}{(n+3)(2\alpha(n-1)((n-2)(q'_f(0)+1) + 3q'_h(0)) - \ell^2)} \\ &+ q'_\Omega(0) = 0, \end{aligned} \quad (5.10)$$

$$\begin{aligned} & q'_g(0) + \frac{1}{2\alpha(n-2)(n-1)(n(q'_f(0)+1) + 3q'_h(0)) - \ell^2 n} \\ & \times \left(q'_f(0) \left(\alpha(n-2)(n-1)(3(n-1)(q_g(0)+1)q'_h(0) + n(n+1)) - \frac{1}{2}\ell^2 n(n+1)(q_g(0)+1) \right) \right. \\ & + \frac{1}{2}\alpha(n-2)(n-1)n(n+1)(q_g(0)+1)(q'_f(0))^2 - \ell^2(n-1)q_0^2(q_g(0)+1) \\ & + (n-1)q'_h(0) \left(\alpha(n-2)(3(n-1) - 6q_g(0)) - \frac{3}{2}\ell^2(q_g(0)+1) \right) \\ & \left. + \frac{15}{2}\alpha(n-3)(n-1)(q_g(0)+1)(q'_h(0))^2 + n(n+1)q_g(0) \left(\frac{\ell^2}{2} - \alpha(n-2)(n-1) \right) \right) = 0, \end{aligned} \quad (5.11)$$

where $q'_f(0)$ is given by

$$\begin{aligned} q'_f(0) &= -\frac{1}{2\alpha(n-2)(n-1)n} [(-24\alpha^2(n-2)(n-1)^2(n+3)q'_h(0)^2 + \ell^4 n^2 - 4\alpha\ell^2(n-2)(n-1)n^2 \\ & + 8\alpha\ell^2(n-2)(n-1)nq_0^2 + 4\alpha^2(n-2)^2(n-1)^2 n^2)^{1/2} + 6\alpha(n-2)(n-1)q'_h(0) \\ & + \ell^2(-n) + 2\alpha(n-2)(n-1)n]. \end{aligned} \quad (5.12)$$

We generate solutions to the equations of motion by approximating each of the q functions as an order N Chebyshev expansion, using the method described in [38]. These expansions are substituted into the equations of motion (A1)–(A6) at the $N-1$ interior grid points and into the boundary conditions (5.9) and (5.7) at the two boundary points. In so doing, the numerical integration is reduced to a set of $5(N+1)$ algebraic equations in the spectral coefficients of the Chebyshev expansions. These equations are then linearized with respect to each spectral coefficient and a standard Newton-Raphson method is used to solve the resulting system of linear equations, with convergence defined as a change in the spectral coefficients less than 10^{-30} between iterations.

A one parameter family of solutions, parametrized by q_0 , is built up using a step size of $\Delta q_0 = 0.01$; along this chain of discrete solutions, the previous numeric solution is used as the seed for the next one. The initial seed we use to start the procedure has $q_0 = 0.01$ and is

given by the perturbative solution (4.6)–(4.10), where the numeric parameter q_0 is related to the perturbative parameter ϵ through the equality

$$q_0 = \lim_{r \rightarrow 0} \frac{(r^2 + \ell^2)^{\frac{n+2}{2}}}{r\ell^{n+1}} \Pi(r),$$

and $\Pi(r)$ is taken to be the perturbative solution of Eq. (4.10).

To ensure the numeric solutions are physical, we demand that they obey the first law of thermodynamics (3.12) to one part in 10^6 . When this limit is no longer satisfied, this is taken as an indication that the Chebyshev grid is too coarse; at this juncture we increase N , refining the grid, and continue the procedure. This provides a useful cross-check on the validity of the solutions, but not the only one. Due to the exponential convergence of the Chebyshev approximation (for example, see Ref. [38]), the truncation error in the approximations of the q functions can be easily estimated via

$$|\text{error}| \leq \sum_{j=N+1}^{\infty} C_j \sim \int_N^{\infty} C_0 e^{-kj} dj = \frac{C_0 e^{-kN}}{k} \quad (5.13)$$

for some real number k , characteristic of the Chebyshev expansion. Here we have used the property that $-1 \leq T_j(x) \leq 1$ and that $C_j \approx C_0 e^{-kj}$. Using the % error $\approx \frac{C_N}{C_0}$ to quantify the truncation error, we impose that it must be less than 10^{-7} for all solutions.

The numerical integration method requires that the equations of motion be satisfied only on the grid points. This does not demand that they be solved everywhere. We find that, in general, the largest interpolation error occurs at the midpoint between grid points, i.e. the zeros of the N th Chebyshev polynomial. As long as the grid is dense enough to satisfy the constraint on the first law, this error is less than 10^{-5} .

Finally, we note that the critical value of α imposes constraints on the efficiency of our numerical work: in seeking to obtain solutions for α less than but close to α_{cr} , we find that significantly larger resolution is required,

necessitating a corresponding increase in the number of grid points and thus the computational processing time; above a certain value of α this becomes intractable. We have not been able to obtain solutions for $0.99\alpha_{\text{cr}} < \alpha < \alpha_{\text{cr}}$ in any dimension.

VI. RESULTS

We find it convenient to write the expressions for the physical quantities of the boson star in terms of the boundary values of the q functions and their derivatives. For example, the thermodynamic quantities are given by [35]

$$M = \frac{\pi^{\frac{n-1}{2}} \ell^{n-1}}{8 \left(\frac{n-1}{2}\right)!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) ((n+1)q_h(1) - nq_f(1)), \quad (6.1)$$

$$J = \frac{(-1)^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} \ell^n}{4 \left(\left(\frac{n-1}{2}\right)!\right)^2} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) q_{\Omega}^{(\frac{n+1}{2})}(1), \quad (6.2)$$

$$\omega = -q_{\Omega}(1) \quad (6.3)$$

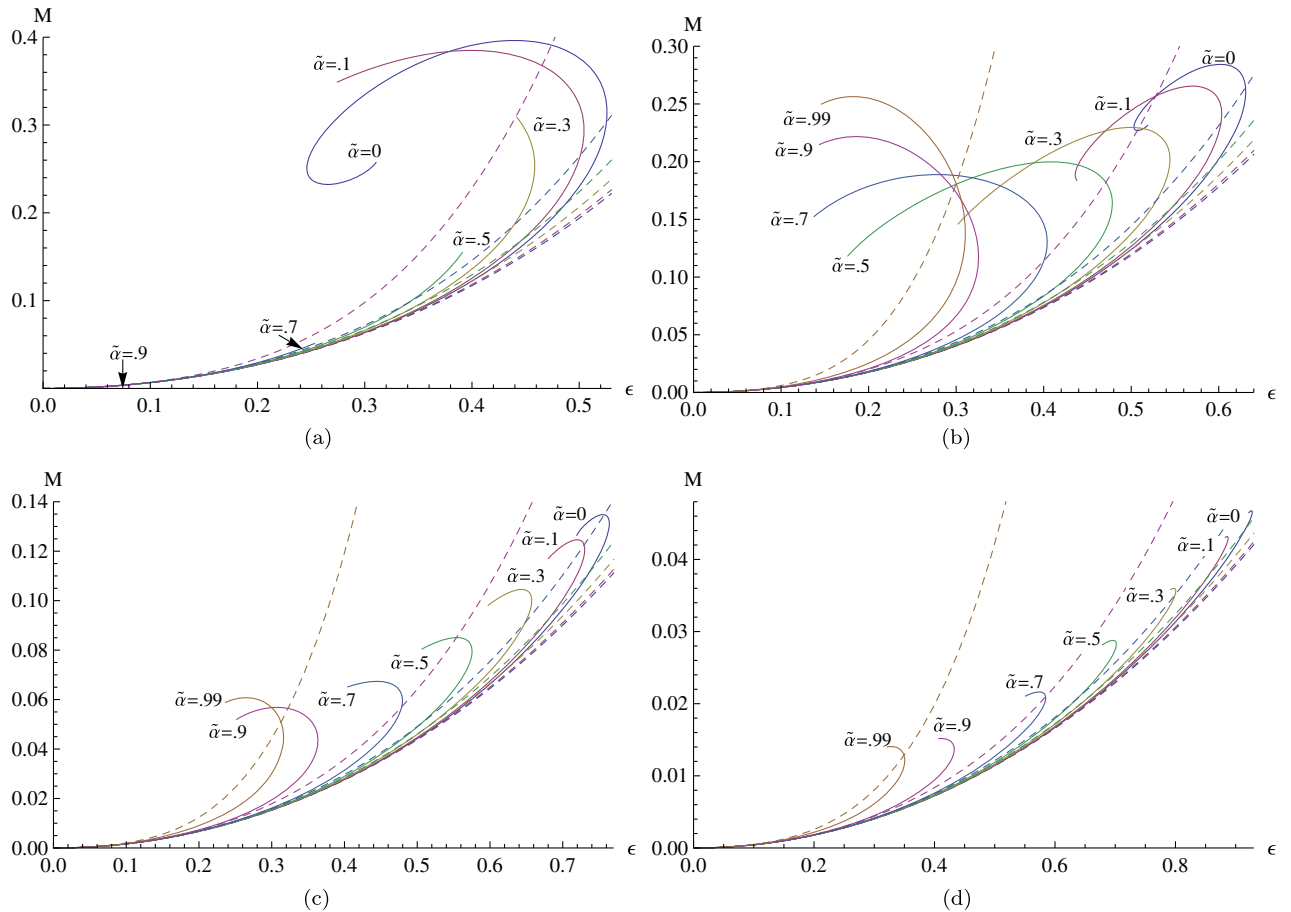


FIG. 1 (color online). The boson star mass plotted against the perturbative parameter ϵ in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. In the nonperturbative regime, ϵ no longer uniquely parametrizes the boson star family. The spirals tighten with increasing dimension and the truncated behavior for $D=5$ is associated with diverging curvature at finite q_0 . The dashed lines correspond to the perturbative solutions.

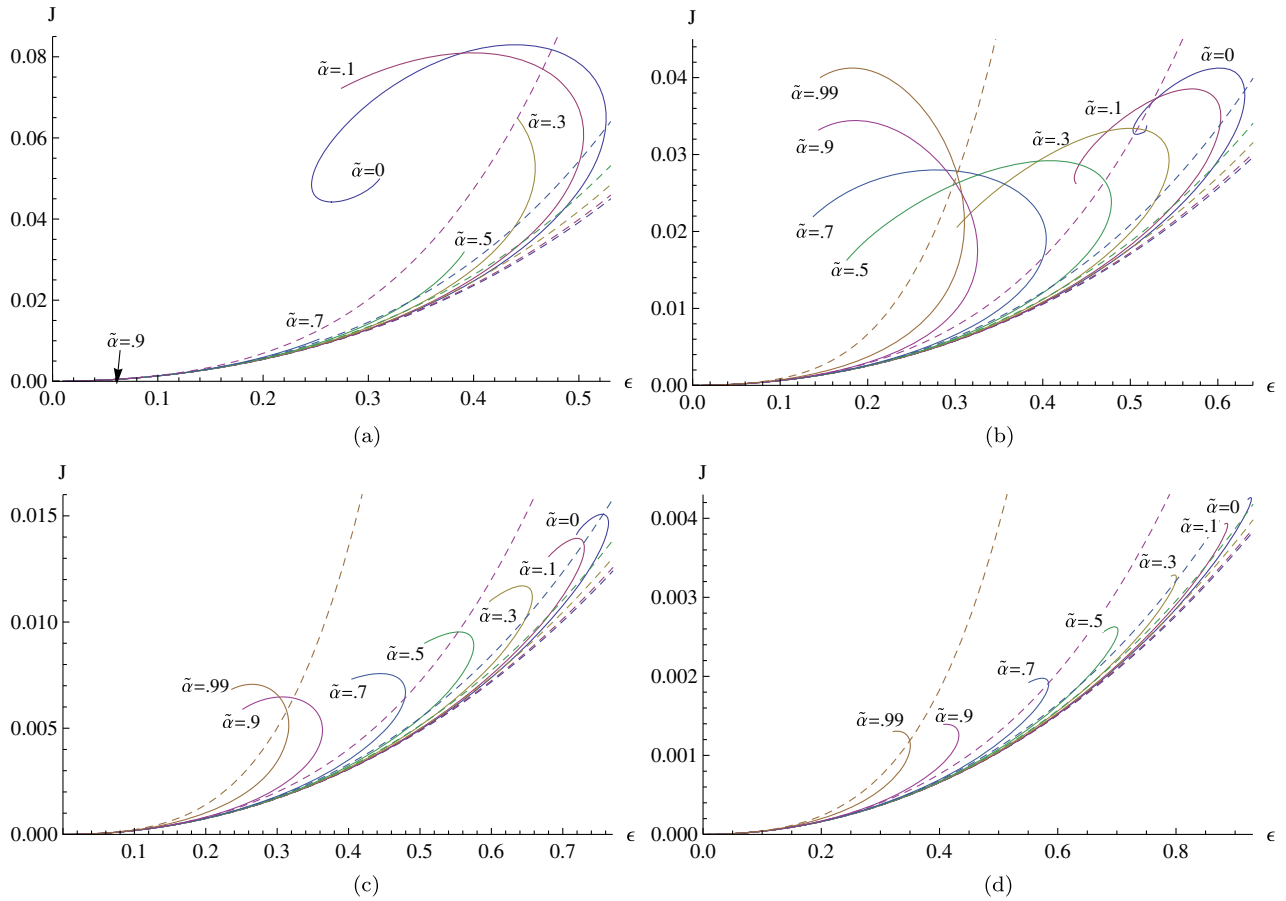


FIG. 2 (color online). The boson star angular momentum plotted against the perturbative parameter ϵ in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. In the nonperturbative regime, ϵ no longer uniquely parametrizes the boson star family. As with the mass, the spirals tighten with increasing dimension and the truncated behavior for $D = 5$ is associated with diverging curvature at finite q_0 . The dashed lines correspond to the perturbative solutions.

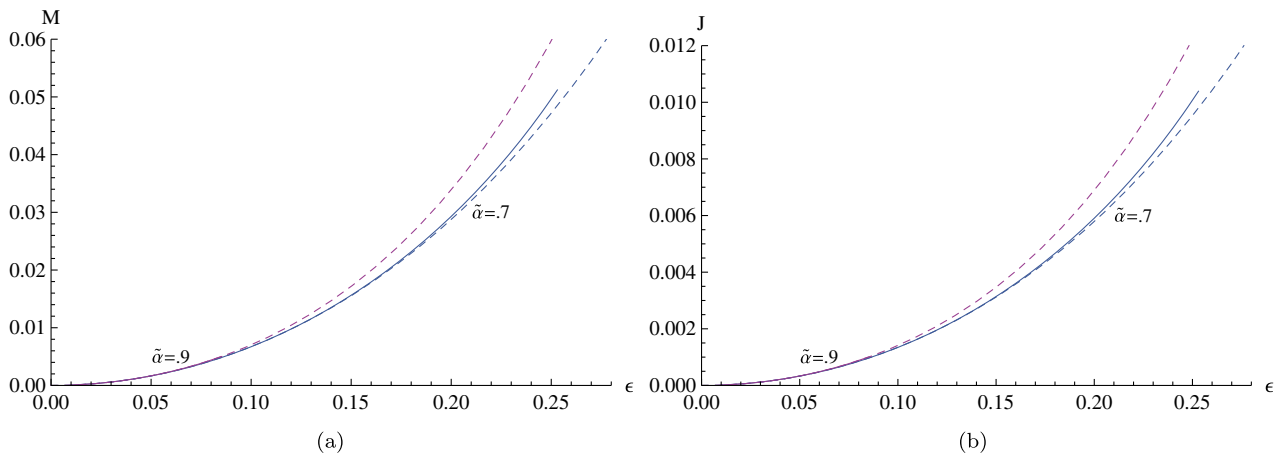


FIG. 3 (color online). Detail plots of mass (a) and angular momentum (b) against the perturbative parameter ϵ in 5 dimensions for $\tilde{\alpha} = \frac{7}{10}$ and $\tilde{\alpha} = \frac{9}{10}$. Perturbative results are plotted as dashed lines. The truncation, which is associated with diverging curvature, occurs at smaller values of ϵ as α increases.

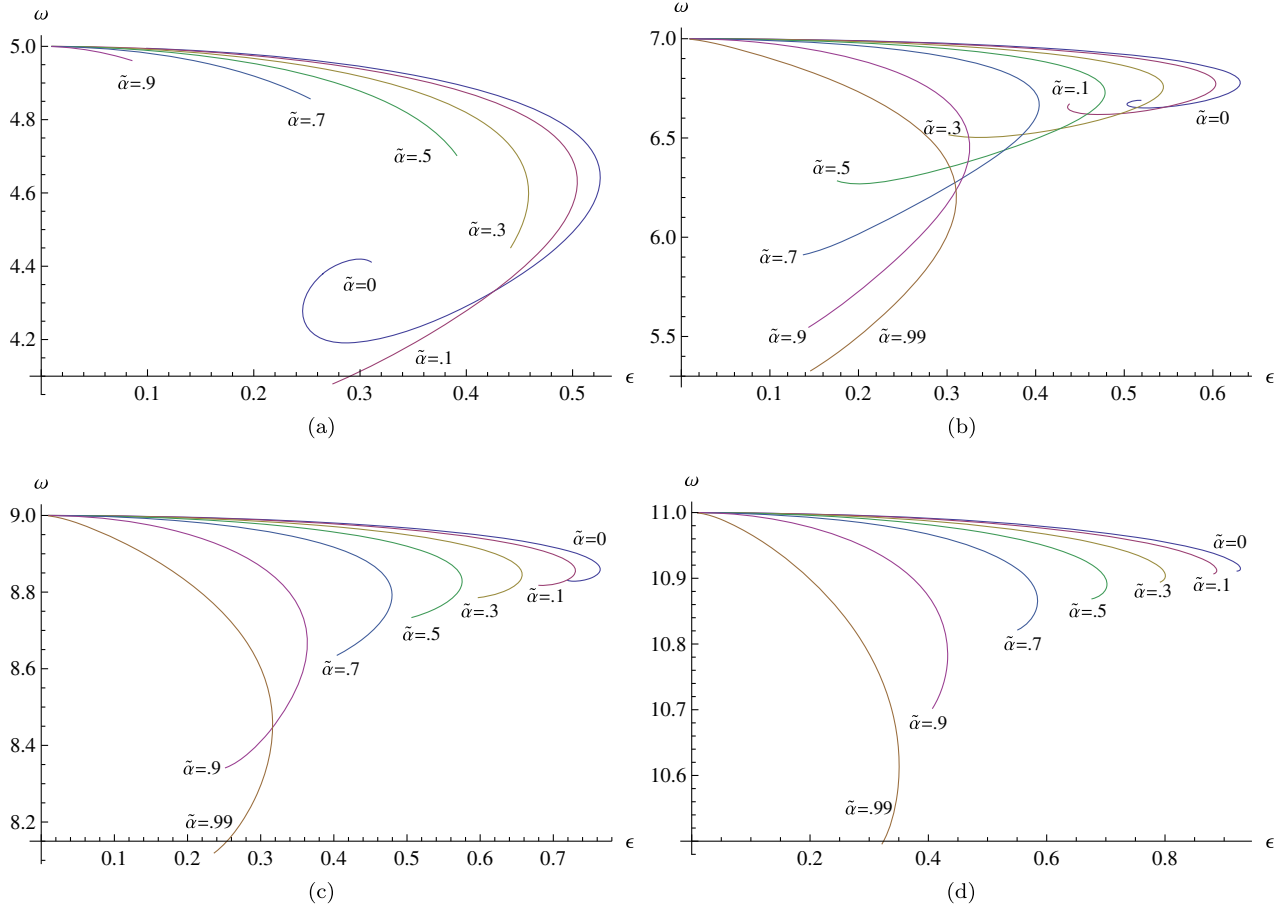


FIG. 4 (color online). The angular frequency ω plotted against the perturbative parameter ϵ in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$.

which can be easily calculated using (3.10) and (5.2)–(5.6). The Kretschmann scalar $K = R_{abcd}R^{abcd}$ at the center of the boson star is given by

$$\begin{aligned}
 K_n = & \frac{2(n+1)}{\ell^4} \left(\frac{2q'_f(0)(2q'_g(0) + 3(q_h(0) + 1)q'_h(0) + q_h(0) + n + 2)}{q_h(0) + 1} + (n+2)q'_f(0)^2 - \frac{2q'_g(0)((n-1)q_h(0) - 2)}{(q_h(0) + 1)^2} \right. \\
 & \left. + \frac{2q'_g(0)^2}{(q_h(0) + 1)^2} + 3(n+2)q'_h(0)^2 + \frac{(\frac{1}{2}(n-1)(n+1) + 1)q_h(0)^2}{(q_h(0) + 1)^2} + \frac{n+2}{(q_h(0) + 1)^2} + 6q'_h(0) + \frac{2q_h(0)}{(q_h(0) + 1)^2} \right), \quad (6.4)
 \end{aligned}$$

which further simplifies using the boundary conditions (5.9), yielding an expression in terms of just $q'_h(0)$. The resulting expressions are rather cumbersome, and so we have placed them in Appendix D.

We use these quantities to study the behavior of boson stars as a function of the Gauss-Bonnet parameter α for $n = 3, 5, 7, 9$. In order to easily make comparisons between the different dimensions, we have chosen to look at values of α in fractions of the critical value: $\tilde{\alpha} \equiv \frac{\alpha}{\alpha_{cr}} = (0, \frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10})$. The $\alpha = 0$ case yields results that are commensurate with those obtained for the Einstein case in

Ref. [29]. This provides a useful cross-check on our analysis, since the form of the field equations used here (A1)–(A6) differs significantly from the form in Ref. [29].

The most noteworthy feature to emerge from our study is the distinction between the $D = 5$ ($n = 3$) case and that of higher dimensions. We shall first discuss the $D > 5$ cases.

The M and J curves as functions of ϵ have the familiar spirallike behavior that was seen in the Einstein case [29], evident in Figs. 1 and 2, with better resolution given for the $D = 5$ values in Fig. 3. As $\tilde{\alpha}$ increases, the spirals become larger. The spirals also tighten with increasing

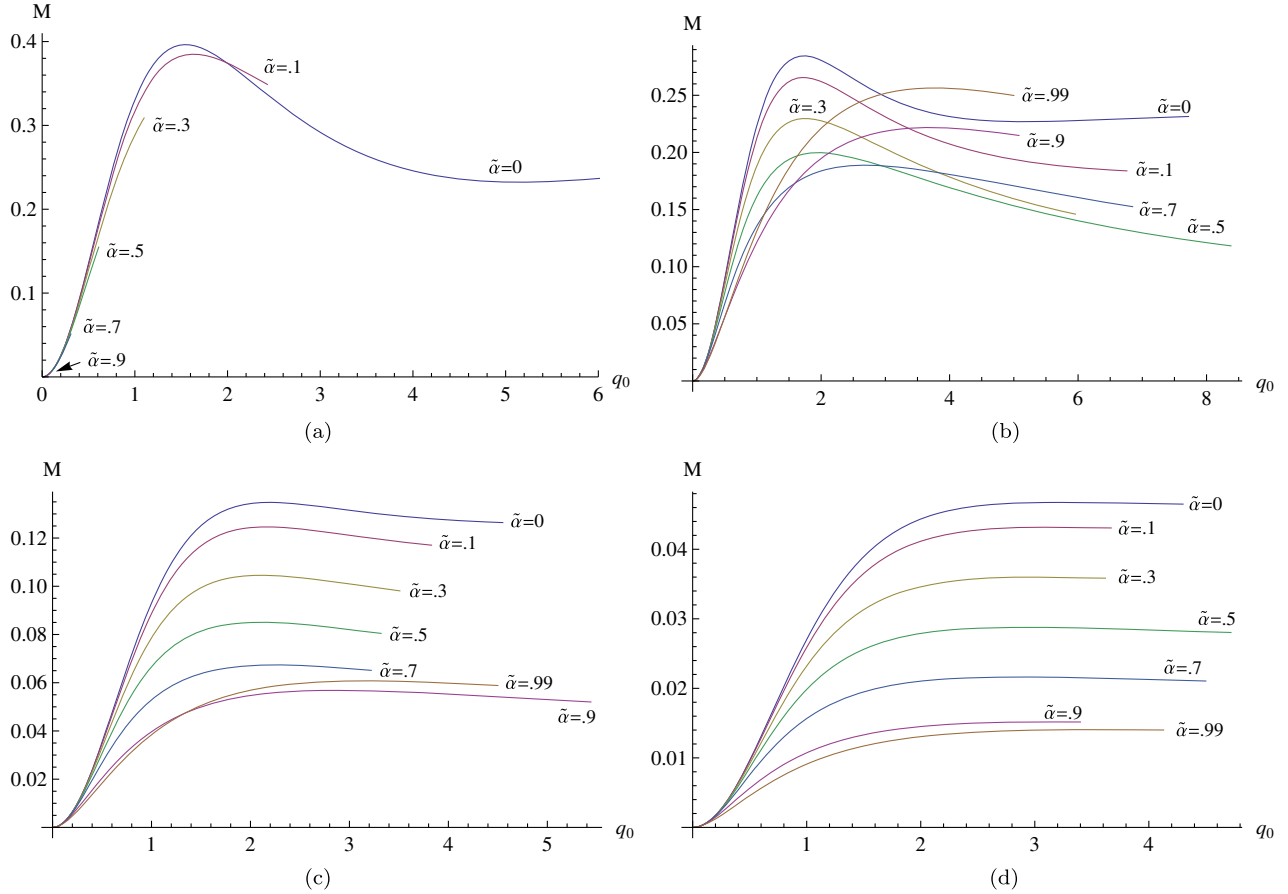


FIG. 5 (color online). The boson star mass plotted against the parameter q_0 in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. The maximum mass decreases nonlinearly with the space-time dimension. For $D = 5$ ($n = 3$), except for $\tilde{\alpha} = 0$, the curves all terminate at their end points, at which the Kretschmann scalar diverges.

dimension—they are barely visible for $D = 11$ ($n = 9$). Similar spiral-type behavior is also present for ω plotted as a function of ϵ , as seen in Fig. 4.

Plotting both M , J and ω as functions of q_0 , we see from Figs. 5, 6 and 8 (with detail for $D = 5$ given in Fig. 7) that for the $n = 5, 7, 9$ cases the behavior of the Einstein case [29] is still present. For all dimensions, the mass and angular momentum of the solutions have a shallower initial slope and, except for $D = 7$, reach a smaller maximum with increasing α .

We find notably different behavior for the $n = 3$ ($D = 5$) case. A perusal of Figs. 1 and 2 indicates that even for $\tilde{\alpha} = 0.1$ the spiral behavior is eliminated. The curves terminate at finite values of (M, J) , with our code

breaking down at critical values of q_0 . This behavior is seen in more striking terms in Figs. 5, 6 and 7: as $\tilde{\alpha}$ increases, the curves terminate at such small values of q_0 that for $\tilde{\alpha} \geq 0.3$ the curves no longer decrease after the maxima are reached.

The underlying reason for this behavior can be traced to the Kretschmann scalar, as delineated in Fig. 9 (with detail given in Fig. 10). For all values of n we find numerically that the Kretschmann scalar initially decreases to a minimum and then becomes a strictly increasing function of q_0 . However for $n = 5, 7, 9$ we find that it remains finite for finite values of q_0 , whereas for $n = 3$ it diverges at a critical value q_h^* of $q_h'(0)$, where

$$\begin{aligned}
 q_h^* &= \pm \frac{1}{2(n-1)\alpha} \sqrt{\frac{4n^2(n-1)^2(n-2)^2\alpha^2 + n^2\ell^4 + 4n(n-1)(n-2)\alpha\ell^2(2q_0^2 - n)}{6(n+3)(n-2)}} \\
 &= -\frac{\sqrt{144\alpha^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2}}{24\alpha} \quad \text{for } n = 3 \quad \text{and} \quad \alpha < \alpha_{\text{cr}}.
 \end{aligned} \tag{6.5}$$

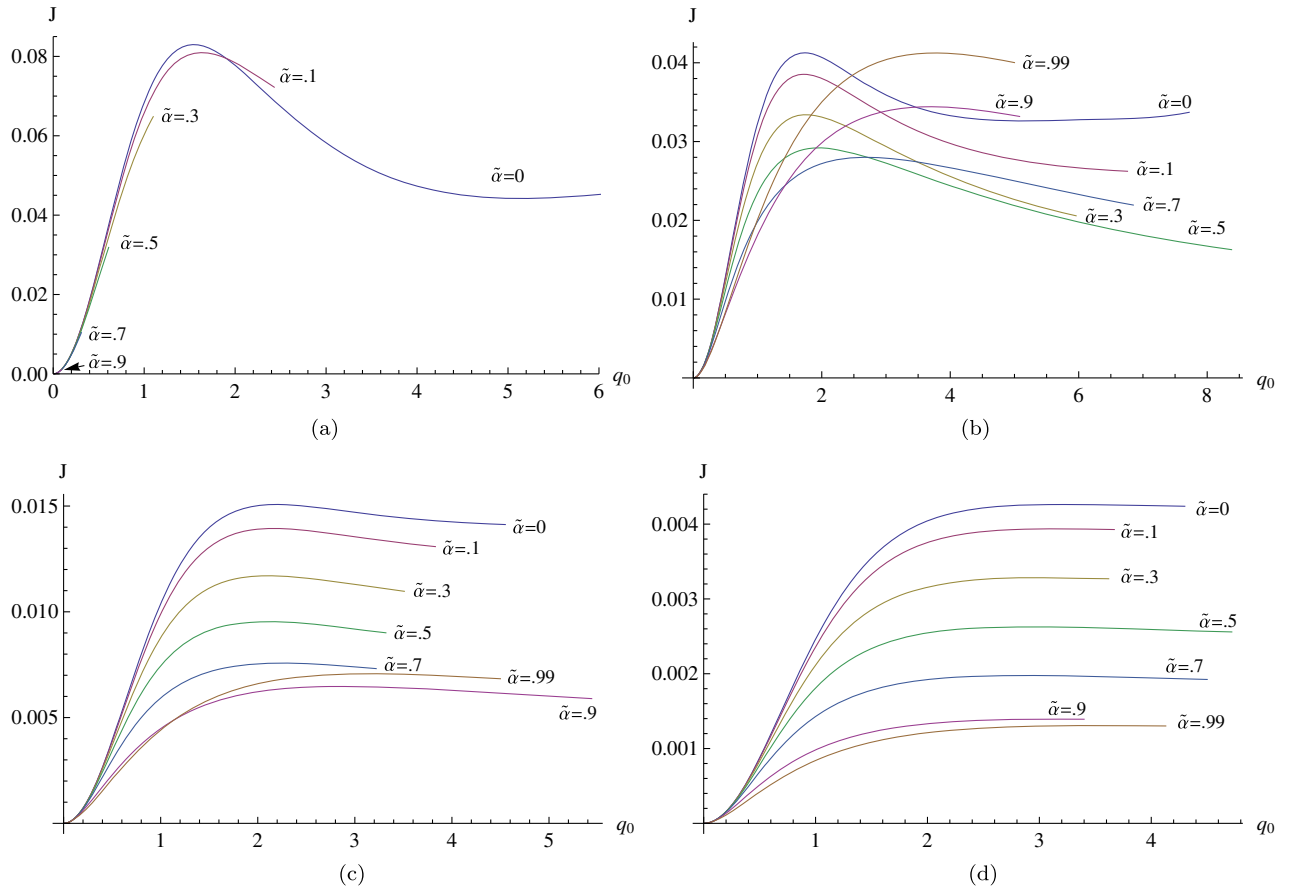


FIG. 6 (color online). The boson star angular momentum plotted against the parameter q_0 in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. As with the mass, the maximum angular momentum decreases nonlinearly with the space-time dimension. For $D = 5$ ($n = 3$), except for $\tilde{\alpha} = 0$, the curves all terminate at their end points, at which the Kretschmann scalar diverges.

The sign of q_h^* is chosen so that it matches the sign of $q_h'(0)$, which is negative for all space-time dimensions when $\alpha < \alpha_{\text{cr}}$. Although there is a critical value q_h^* in any dimension, we find numerically that as q_0 increases $q_h'(0)$ departs further from this critical value, implying that the Kretschmann scalar remains finite for all finite q_0 . This pattern can be seen in Figs. 11(c) and 11(d). However, for $n = 3$ we find that $q_h'(0) \rightarrow q_h^*$ at finite q_0 , signaling a divergence in the Kretschmann scalar. Furthermore, this critical value of q_0 decreases as $\tilde{\alpha}$ increases; to estimate the critical value of q_0 , in Figs. 11(a) and 11(b) we plot $q_h'(0) - q_h^*$ and employ a ninth-order polynomial fit to interpolate the curve. The specific values are shown in Table I.

This behavior is present neither in the Einstein case nor in the perturbative solutions. It suggests the existence of a critical central energy density in $D = 5$ at or before which the behavior of the boson star must radically change. This situation is reminiscent of the numerical boson star solution obtained in $D = 3$ [28] insofar as the mass, angular momentum, and angular velocity all approach finite terminal values at a critical value of the central energy density,

whereas in higher dimensions the central energy density is unbounded and the corresponding physical quantities approach finite limiting values. However numerical evidence for the $D = 3$ case indicates the formation of an extremal BTZ black hole at the critical value, with vanishing scalar field in the exterior, whereas in the present case we do not find the scalar field to vanish and we find no indication of the formation of a horizon. Nevertheless, since the curvature at the center is so high, we expect that for some q_0 sufficiently close to $q_{0\text{critical}}$ that a finite size perturbation will eventually concentrate enough energy at the center so as to form a horizon. We therefore expect that these solutions will become dynamically unstable to hairy black hole formation for some $q_0 < q_{0\text{critical}}$.

In $D = 3$ the Ricci scalar is the highest-curvature term that can appear in that dimension; in $D = 5$ it is the Gauss-Bonnet term that fulfills this criterion. It is tempting to conjecture that such critical energies appear in all odd dimensions in which the gravitational theory includes its highest-possible curvature term. While this would be challenging to numerically check, it is conceivably feasible for the $D = 7$ third-order Lovelock and $D = 9$ fourth-order Lovelock cases.

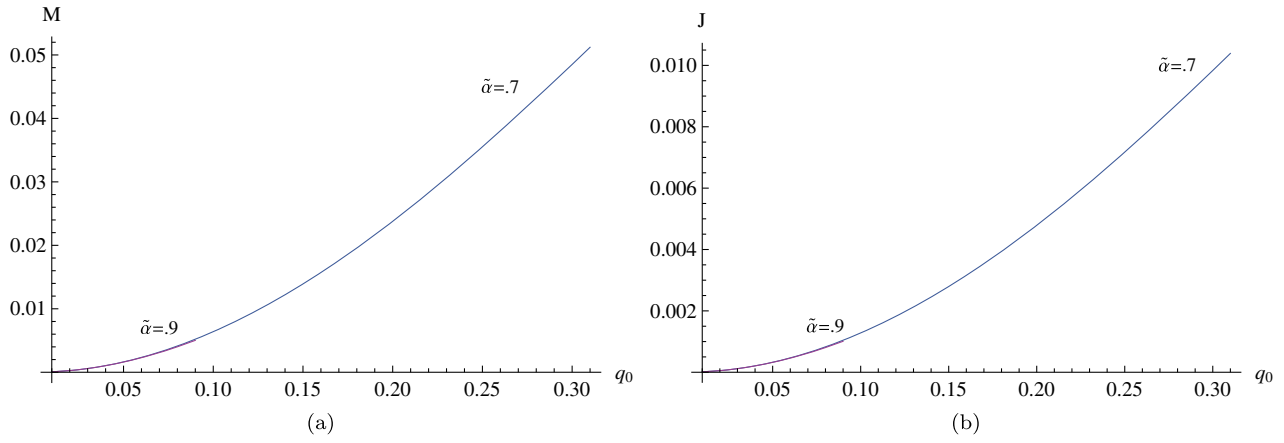


FIG. 7 (color online). Detail plots of (a) mass and (b) angular momentum against the perturbative parameter q_0 in 5 dimensions for various values of $\tilde{\alpha}$. The truncation, which is a result of the divergent curvature, occurs at smaller values of q_0 for as α increases.

To conclude our analysis, in Fig. 12 we plot M vs J for each dimension and the values of $\tilde{\alpha}$ indicated. The most striking feature is that for each dimension, all of the solutions lie on almost the same line in the energy vs

angular momentum graph regardless of the value of $\tilde{\alpha}$, at least for sufficiently small values of (M, J) . For a given value of $\tilde{\alpha}$ these curves “turn back,” making a zigzag pattern familiar from the Einstein case [29], albeit much tighter.

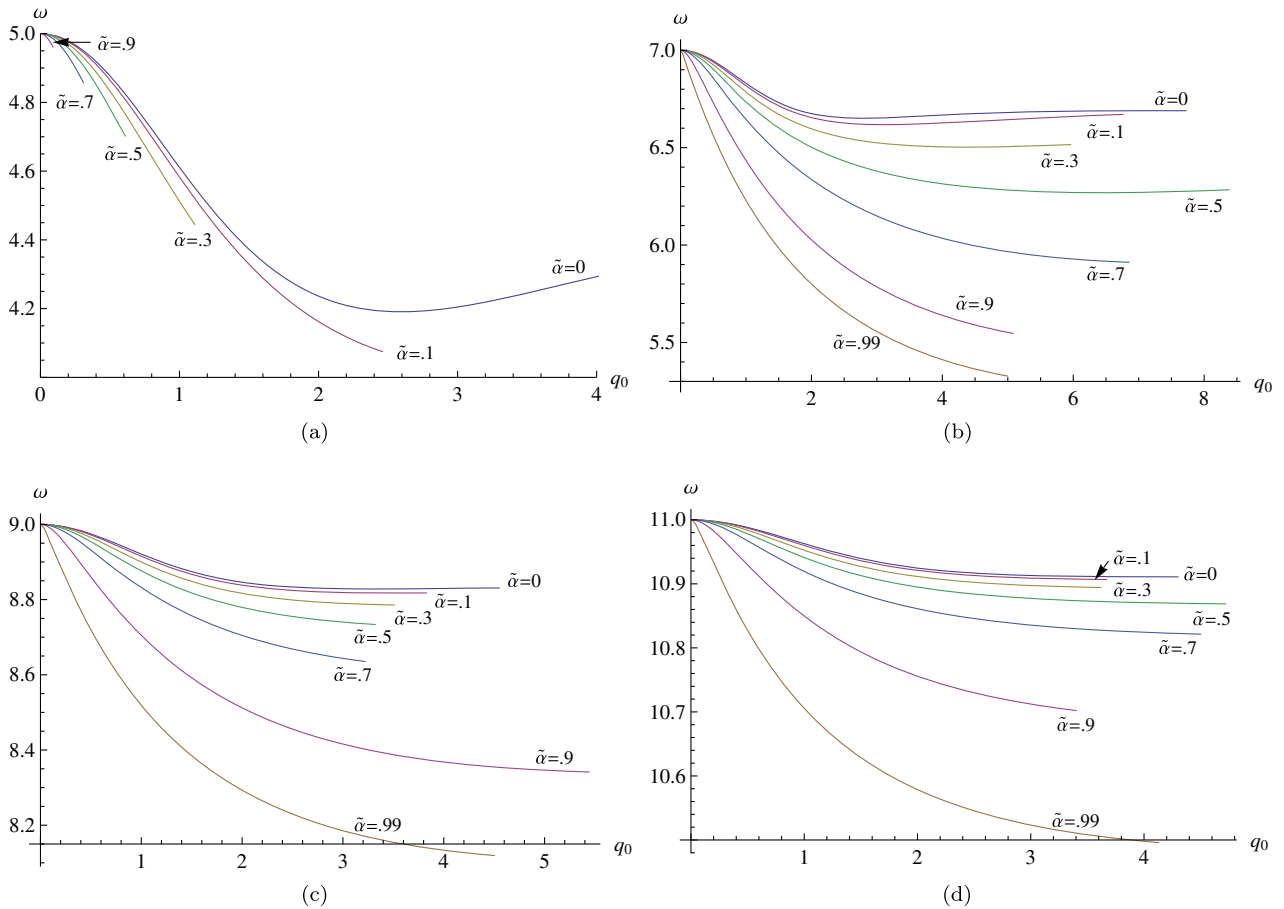


FIG. 8 (color online). The angular velocity ω plotted against the perturbative parameter q_0 in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. The termination of the curves in $D = 5$ is due to the divergence of the Kretschmann scalar at the end points, except for $\tilde{\alpha} = 0$ which is immune to this divergence.

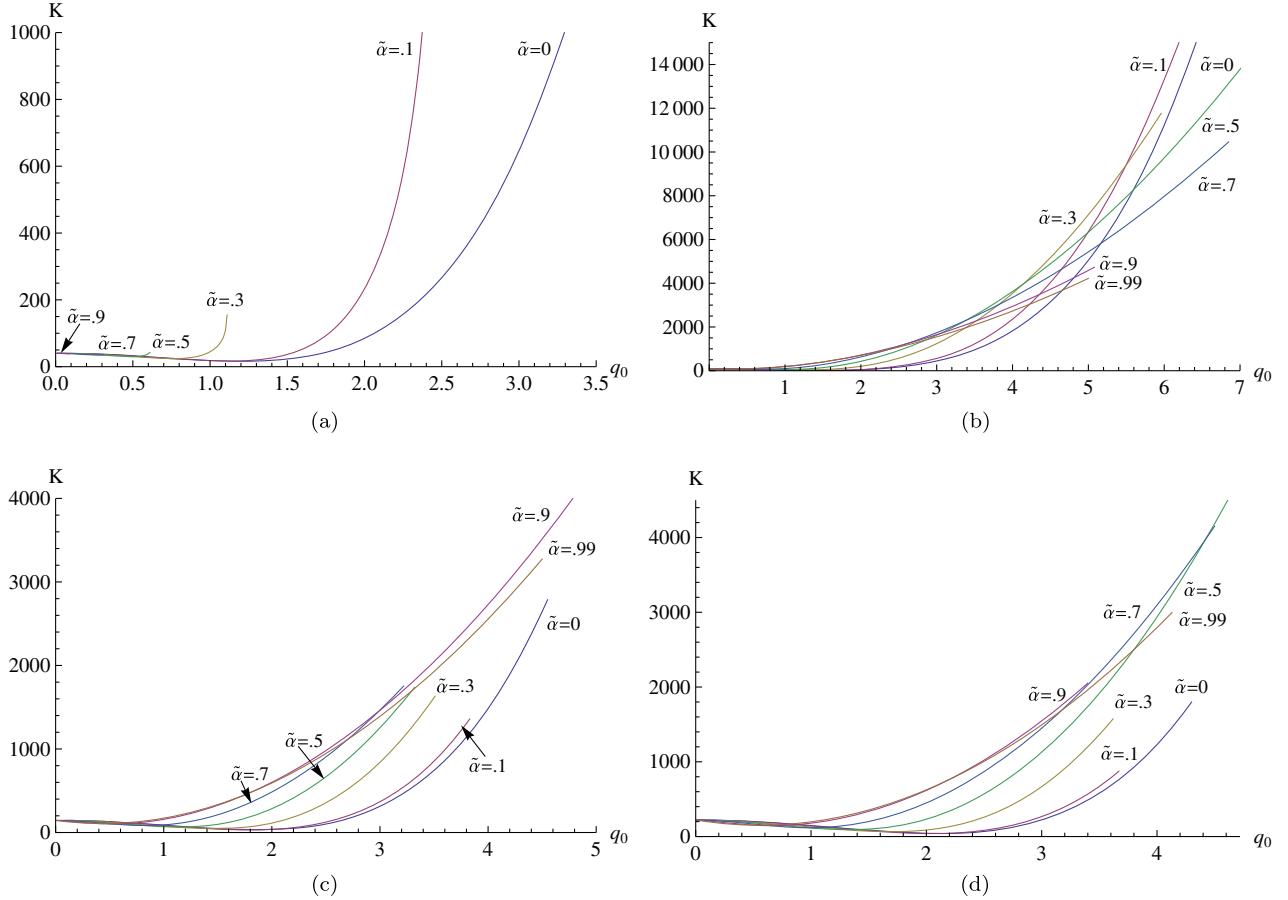


FIG. 9 (color online). The Kretschmann scalar $K = R^{abcd}R_{abcd}$ plotted against the perturbative parameter q_0 in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. The divergent behavior of K in $D = 5$ is drastically different than in higher dimensions for reasons explained in the text.

As $\tilde{\alpha}$ increases, the turnaround point for the zigzag pattern is closer to the beginning of the curve [at smaller values of (M, J)] for $D \neq 7$. As the dimensionality increases the effect of the zigzag is also suppressed. It is most pronounced in $D = 5$; for this dimension we found no numerical evidence for a zigzag pattern for $\alpha \geq 0.3\alpha_{\text{cr}}$ as the Kretschmann scalar diverges before this pattern can emerge.

VII. BEYOND α_{cr}

It is possible to numerically analyze solutions for which $\alpha > \alpha_{\text{cr}}$. While a full analysis is beyond the scope of this paper, we present some preliminary results. We find for all boson star solutions that the combinations of the quantities C_f , C_h and C_Ω in (3.10) become negative in this regime, seemingly indicating they have negative ADM mass. However the factor of $(1 - \alpha/\alpha_{\text{cr}})$ in (3.10) also becomes negative, ensuring that the overall signs of M and J do not change in going from $\alpha < \alpha_{\text{cr}}$ to $\alpha > \alpha_{\text{cr}}$.

As before, in order to easily make comparisons between different space-time dimensions, we choose a fixed value of $\tilde{\alpha} = \frac{6}{5}$. Boson star solutions in this parameter range are

qualitatively very different. Unlike for $\alpha < \alpha_{\text{cr}}$, the spiral behavior in the M and J curves as functions of ϵ is not present in any dimension but these quantities instead increase monotonically, as can be seen in Fig. 13. Similarly, as is seen in Fig. 14, when the M and J curves are plotted as functions of q_0 , they also increase monotonically, and plotting the natural log of M and J against q_0 , as shown in Fig. 15, reveals that the curves approach exponential growth as q_0 increases. At first glance it appears that the growth rate is universal for $D \geq 7$; however closer inspection reveals that the growth rate increases slightly with increasing space-time dimension. It is clear from these plots that the physical quantities will not approach a finite limiting value. We omit ω plotted as a function of ϵ since it also increases monotonically with increasing ϵ and lacks the characteristic spiral behavior seen for $\alpha < \alpha_{\text{cr}}$.

We again find a difference between boson star behavior in $D = 5$ and $D > 5$, namely that there seems to be a maximum value of q_0 beyond which the numerics break down. At first, when looking at Fig. 16, the Kretschmann scalar appears to remain finite for all values of q_0 ; however,

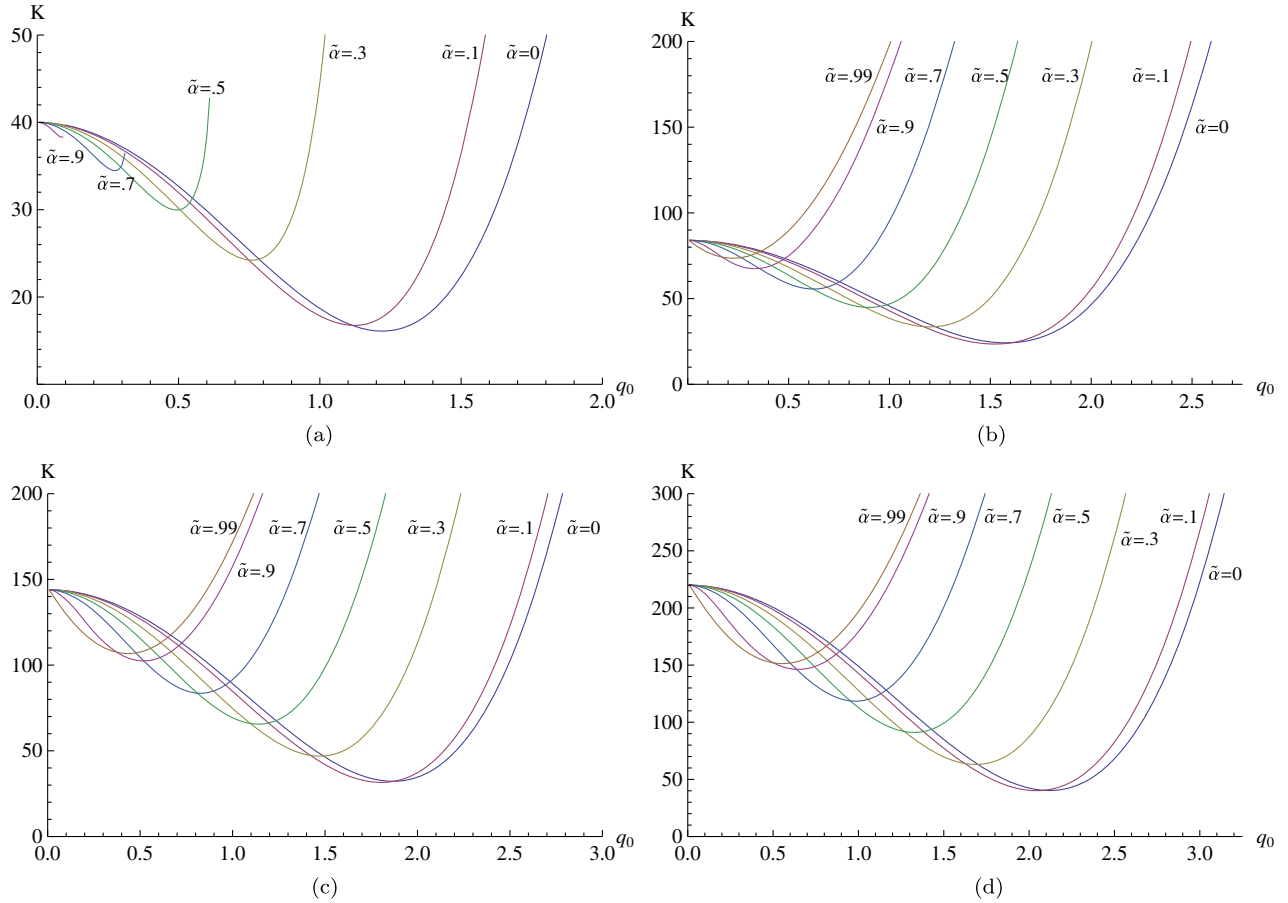


FIG. 10 (color online). Detail of the Kretschmann scalar K plotted against the perturbative parameter q_0 in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. Except for the scale on the K axis, dimensions $D = 7, 9, 11$ behave qualitatively the same as $\tilde{\alpha}$ increases, whereas $D = 5$ exhibits distinct behavior.

for $D = 5$ Fig. 17 reveals that $q_h^* - q_h(0)$ does indeed have a root at some finite $q_{0\text{critical}}$. We again use a ninth-order polynomial best-fit curve to estimate $q_{0\text{critical}}$ at 0.172. Similar to the $\alpha < \alpha_{\text{cr}}$ case for $D > 5$, we find the Kretschmann scalar does not diverge and the difference $q_h^* - q_h(0)$ is nowhere vanishing and monotonically increasing q_0 , which can be seen in Fig. 17(b).

Finally, in Fig. 18 we plot M vs J . Note that in each dimension the $\alpha > \alpha_{\text{cr}}$ solutions are nearly collinear with the $\alpha < \alpha_{\text{cr}}$ solutions. However, for $\alpha > \alpha_{\text{cr}}$, these lines do not zigzag and “turn back” at any value of q_0 due to the lack of spiral behavior in M and J . We choose $\tilde{\alpha} = 0.70$ to be representative of the $\alpha < \alpha_{\text{cr}}$ solutions, but any value could have been chosen as they all lie on the same curve (see Fig. 12). It appears that for a specified dimension, the M vs J curve is universal to all values of $\alpha \neq \alpha_{\text{cr}}$.

Our brief exploration of the $\alpha > \alpha_{\text{cr}}$ parameter space has shown that these boson star solutions have properties that are significantly different from $\alpha < \alpha_{\text{cr}}$ solutions. This regime is numerically difficult to explore since a large number of Chebyshev points are needed to find solutions for relatively small values of q_0 . The reason behind this need for increased resolution is not fully understood in the $D > 5$ case, as the

Kretschmann scalar remains relatively small compared with the $\alpha < \alpha_{\text{crit}}$ case. A full exploration of the reason behind these surprising results will be left for future work.

VIII. CONCLUSIONS

We have obtained asymptotically AdS rotating SKV boson star solutions in Einstein-Gauss-Bonnet gravity coupled to a multiplet of massless scalar fields in all odd space-time dimensions of interest in string theory, i.e. $D = 5, 7, 9, 11$, both perturbatively in powers of the scalar field amplitude and numerically. For the latter, our approach was to employ the same relaxation procedure on a Chebyshev grid used in the Einstein case [28,29]. In all space-time dimensions, the perturbative solutions match the numerical solutions for sufficiently small values of q_0 , as expected. Furthermore, each dimension exhibits a critical value of the Gauss-Bonnet coupling constant, α_{cr} , for which the solutions no longer have the asymptotic behavior in (3.3).

The most striking result of the present work is the distinction between the $D = 5$ case and its higher dimensional counterparts. For $D > 5$, the physical quantities

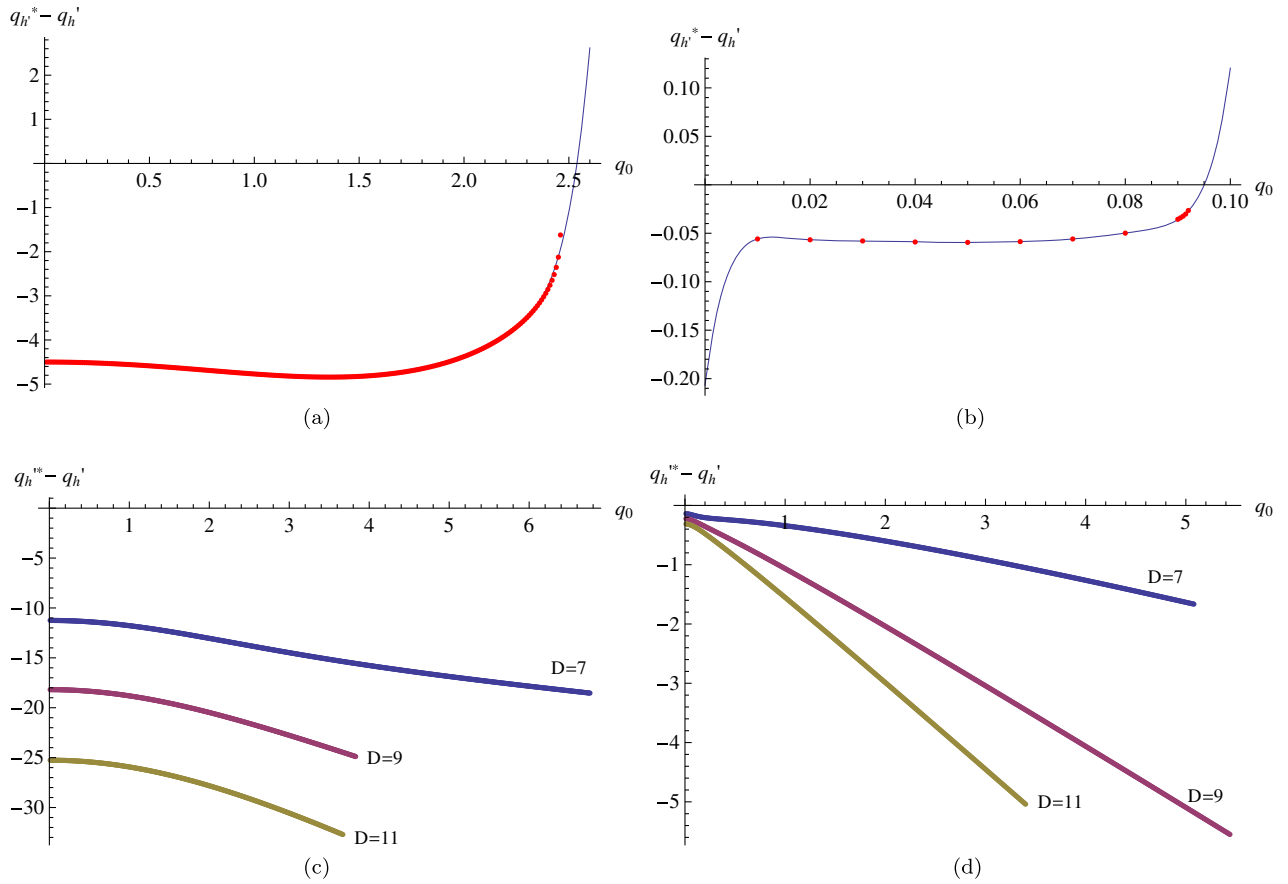


FIG. 11 (color online). Plots of $q_h^{**} - q_h'$ against the perturbative parameter q_0 in 5 dimensions for (a) $\tilde{\alpha} = 0.1$ and for (b) $\tilde{\alpha} = 0.9$ and in 7, 9, and 11 dimensions for (c) $\tilde{\alpha} = 0.1$ and for (d) $\tilde{\alpha} = 0.9$. The solid curves in panels (a), (b) are the best-fit ninth-order polynomials to the points used to obtain $q_{0\text{critical}}$.

(M, J, ω) reach maximum values and then decrease to eventually approach finite limiting values as q_0 increases without bound. However, for $D = 5$ and for values of α as small as $\alpha = 0.1\alpha_{\text{cr}}$, M, J , and ω each terminate at a finite critical value $q_{0\text{critical}}$, at which point the Kretschmann scalar diverges. This situation is redolent of the AdS boson star solutions recently obtained in $D = 3$ [28]. In that case the boson star mass, angular momentum, and angular velocity all monotonically increase with the central energy density up to a critical value, at which point the boson star branch of solutions smoothly connects with the extremal BTZ black

hole. In the present case for $D = 5$ there is no evidence that a black hole forms at the critical value of q_0 , although the diverging Kretschmann scalar indicates the formation of a singularity; this boson star branch is likely dynamically unstable to forming a black hole at some value $q_0 < q_{0\text{critical}}$.

Recall that demanding the field equations to have derivatives no higher than second order, the Ricci scalar is the highest-curvature term that can appear in 3 space-time dimensions, while the Gauss-Bonnet term is the highest-curvature term in 5 dimensions. We conjecture that critical values of q_0 will appear in all odd dimensions in which the gravitational theory includes its highest possible curvature term that maintains second-order field equations. It is conceivably feasible to numerically check this for the $D = 7$ third-order Lovelock and $D = 9$ fourth-order Lovelock cases. It would also be interesting to extend our work to that of massive scalar fields and scalar fields with potential terms to see what effect the scalar field mass has on the critical value of q_0 .

As with the Einstein case [9], one might expect asymptotically AdS space-times in Einstein-Gauss-Bonnet gravity to be nonperturbatively unstable to the formation

TABLE I. Estimate of the critical value of q_0 at which the Kretschmann scalar diverges in $D = 5$ for various values of $\tilde{\alpha}$ using the best fit polynomials in Fig. 11.

$\tilde{\alpha}$	$q_{0\text{critical}}$
0.1	2.54
0.3	1.15
0.5	0.648
0.7	0.329
0.9	0.0950

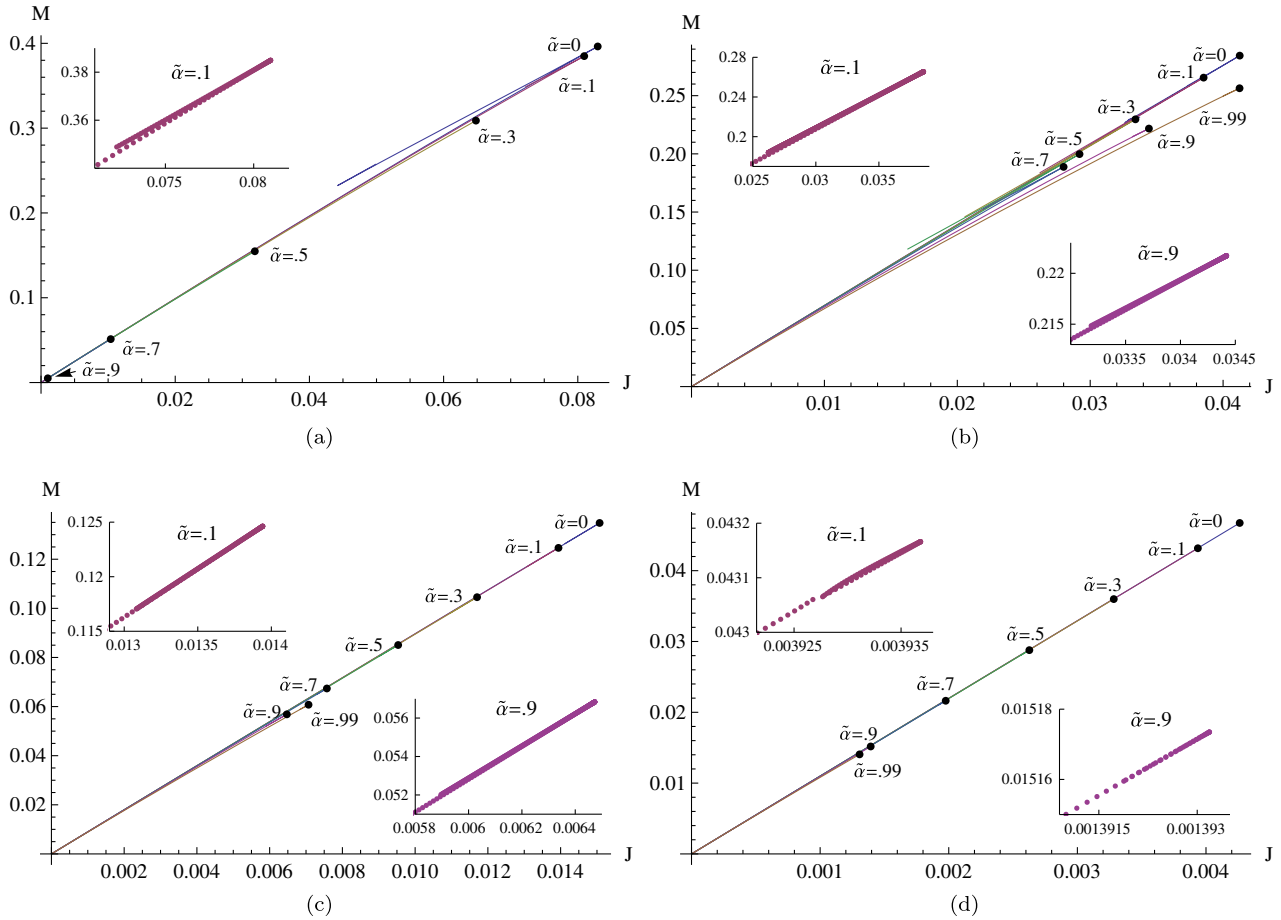


FIG. 12 (color online). M vs J in (a) 5, (b) 7, (c) 9, and (d) 11 dimensions for various values of $\tilde{\alpha}$. The black dots correspond to the point at which these curves turn back for a particular value of $\tilde{\alpha}$. The insets are close-ups of the zigzag patterns for $\tilde{\alpha} = 0.1$ (top left corner) and $\tilde{\alpha} = 0.9$ (bottom right corner) for each dimension.

of black holes. Heuristically, global AdS is nonperturbatively unstable to black hole formation because its reflecting boundary conditions imply that finite energy perturbations, given enough time, will come together in

sufficient concentration to form a black hole. Since Gauss-Bonnet gravity tends to increase gravitational attraction, we expect such an effect to be enhanced, though a proper study remains to be carried out. Indeed, since it has

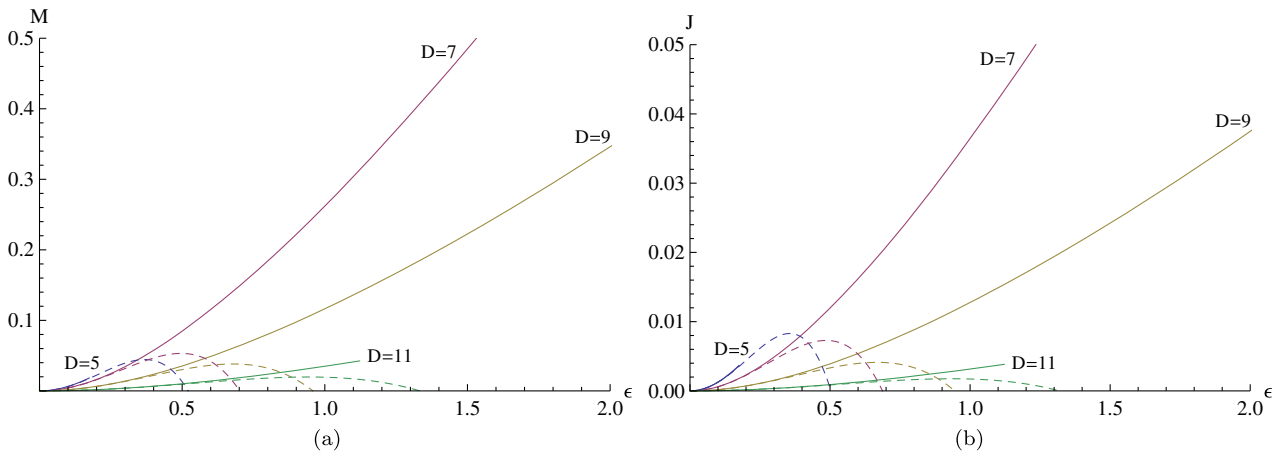


FIG. 13 (color online). (a) The boson star mass M and (b) angular momentum J plotted against the perturbative parameter ϵ for $\tilde{\alpha} = 1.20$ in $D = 5, 7, 9, 11$ ($n = 3, 5, 7, 9$). Perturbative results are plotted as dashed lines. Both M and J increase monotonically with increasing ϵ .

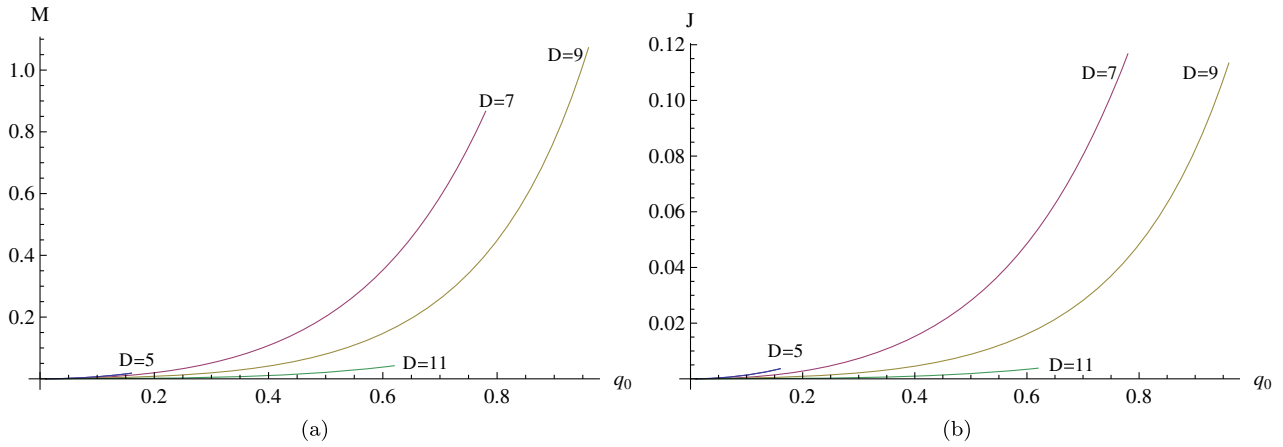


FIG. 14 (color online). (a) The boson star mass M and (b) angular momentum J plotted against the parameter q_0 for $\tilde{\alpha} = 1.20$ in $D = 5, 7, 9, 11$ ($n = 3, 5, 7, 9$). Both M and J increase monotonically with increasing q_0 .

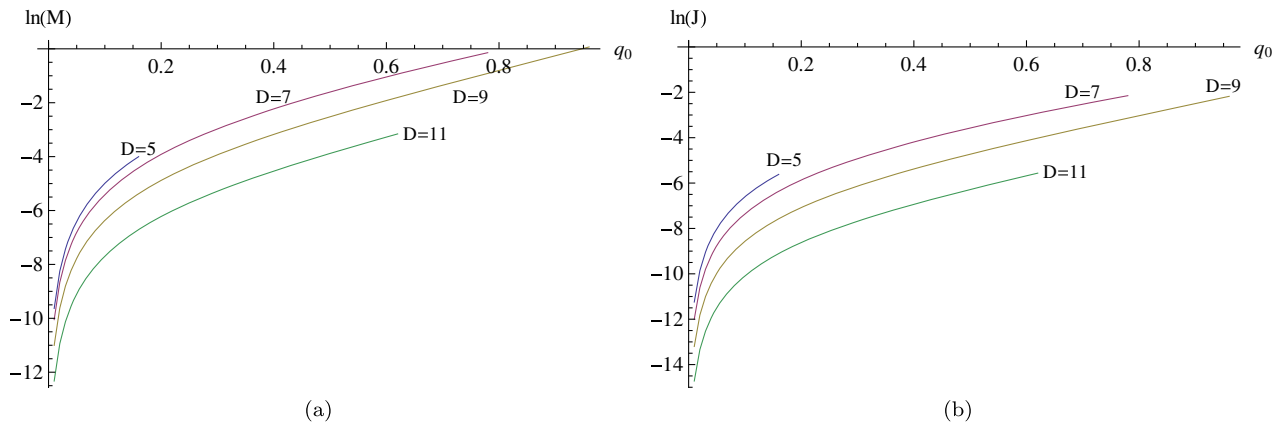


FIG. 15 (color online). (a) The natural log of the boson star mass M and (b) angular momentum J plotted against the parameter q_0 for $\tilde{\alpha} = 1.20$ in $D = 5, 7, 9, 11$ ($n = 3, 5, 7, 9$). Both M and J approach exponential growth with increasing q_0 .

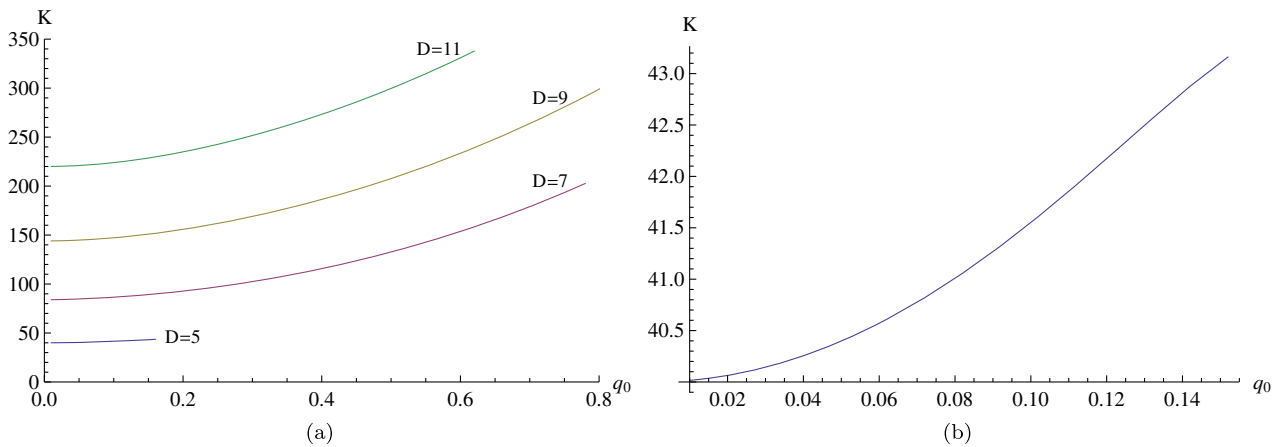


FIG. 16 (color online). The Kretschmann scalar K plotted against the parameter q_0 for $\tilde{\alpha} = 1.20$ in (a) $D = 5, 7, 9, 11$, ($n = 3, 5, 7, 9$) and (b) $D = 5$ ($n = 3$) to show detail.

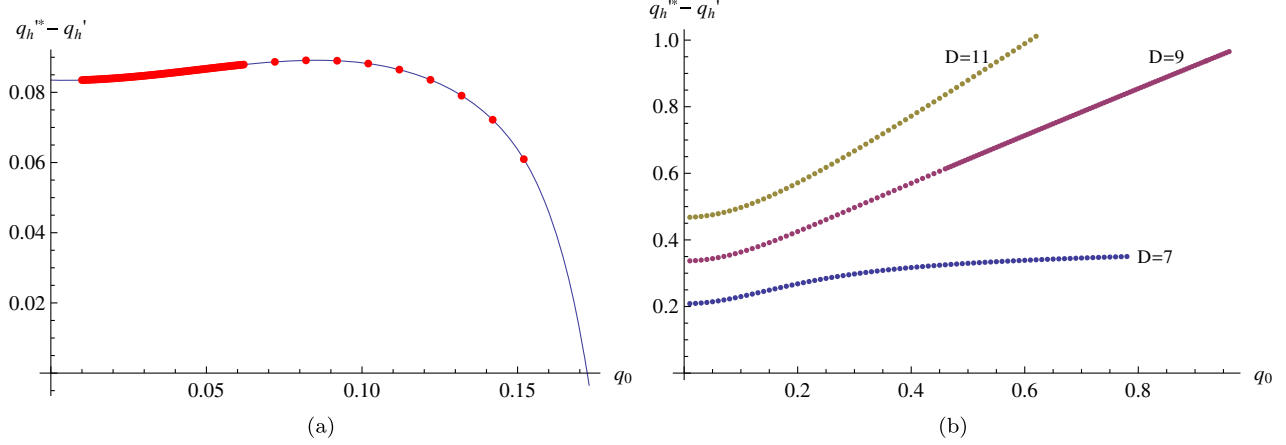


FIG. 17 (color online). Plots of $q_h^{ns} - q_h'$ against the perturbative parameter q_0 in (a) 5 and (b) 7, 9, and 11 dimensions for $\tilde{\alpha} = 1.20$. The solid curve in part (a) is the best-fit ninth-order polynomials to the points used to obtain $q_{0\text{critical}}$.

been shown that the AdS instability is really due to the high level of symmetry present in global AdS (the normal mode frequencies are all resonant with the AdS frequency [14]) we expect the boson star solutions constructed in this paper to be nonperturbatively stable, although verifying this explicitly is beyond the scope of the present work.

Further work includes a proper analysis of the $\alpha > \alpha_{\text{cr}}$ and $\alpha = \alpha_{\text{cr}}$ cases. We have given a brief discussion of the former case. It is straightforward to show that transverse traceless excitations of the metric about an AdS vacuum obey the equation

$$\left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) \left(\nabla^2 h_{ab} + \frac{2}{\ell^2} h_{ab}\right) = 8\pi G T_{ab} \quad (8.1)$$

indicating that for $\alpha > \alpha_{\text{cr}}$ such excitations are ghostlike [39]. This is congruent with the analysis of [31] in which the authors found ghostlike metric perturbations in the vacuum that does not limit to the Einstein vacuum in the absence of a Gauss-Bonnet term. Just as matter sources in

their analysis still carry positive energy, the boson stars we obtain here have positive mass, although the ghostlike excitations imply an instability of the vacuum. The $\alpha = \alpha_{\text{cr}}$ case requires a separate analysis due to the different boundary conditions that must be employed. It will be interesting to see what features solutions in this class have in common with the $D = 3$ case [29].

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada. We are grateful to R. Myers and K. Schleich for helpful discussions.

Note added.—Recently, we became aware of similar work in $D = 5$ [40], in which rotating EGB boson stars were studied for scalar fields in a potential. The zero-potential case corresponds to the one we study, though they evidently discuss only the $\alpha = 0$ case when rotation is present and the potential vanishes. For nonzero potential they also find in EGB gravity that rotating solutions exist up to some maximal value of the central energy density, though they claim the limiting solutions for this case are regular, in contrast to the massless case that we consider here.

APPENDIX A: FIELD EQUATIONS

Here we record the field equations we employ in obtaining both our perturbative and numerical solutions. While it is possible to rewrite these so that the second-derivative terms are isolated, this produces expressions even more cumbersome than those presented here, and so we do not carry out this step. There are five coupled second-order ODEs for the five metric and scalar field functions as follows:

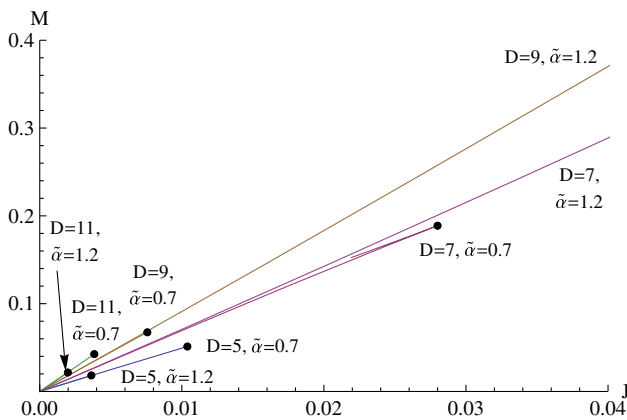


FIG. 18 (color online). M vs J for $\tilde{\alpha} = 0.70$ and $\tilde{\alpha} = 1.20$ in $D = 5, 7, 9, 11$ ($n = 3, 5, 7, 9$). The black dots correspond to the maximum (M, J) value reached on each curve.

$$\begin{aligned}
& r^2[2g(f^2gh(n-2)(n-1) - 2h\Pi^2r^2(\omega - \Omega)^2) \\
& + fhr(gr(2gf'' + 2fg'' + 4fg(\Pi')^2 - 3hr^2(\Omega')^2) + gf'(3rg' + 4g(n-1)) + 2fg(n-1)g' - fr(g')^2) \\
& + 2fg^2(h(3h(n-1) - n^2 + 2\Lambda r^2 + 1) + 2\Pi^2(h(n-1) - 1))] \\
& + 2\alpha fh(1-n)[g^2(n-3)(f^2(n-4)(n-2) + 2f(n-4)(3h-n-1) + 3h(5h-2(n+1)) + n^2-1) \\
& + r(gr(2(gf'' + fg''))(f(n-2) + 3h-n-1) - 3hr^2(\Omega')^2(f(n-2) + 5h-n-1)) \\
& + 2g^2(n-2)r(f')^2 + gf'(rg'(5f(n-2) + 9h-3(n+1)) + 4g(n-3)(f(n-2) + 3h-n-1)) \\
& + fr(g')^2(-f(n-2) - 3h+n+1) + 2fg(n-3)g'(f(n-2) + 3h-n-1)] = 0
\end{aligned} \tag{A1}$$

$$\begin{aligned}
& f^2hr^4\Omega(g')^2 + fg^2r^2(r(2hr(f\Omega'' - \Omega f'') + f\Omega'(3rh' + 2h(n+2))) - 4h(n-1)r\Omega f' \\
& - 2\Omega(h(n-1)(f(n-2) + 3h-n-1) + 2\Pi^2(h(n-1) + 1) + 2h\Lambda r^2) - 4fhr^2\Omega(\Pi')^2 + 8\Pi^2\omega) \\
& + ghr^3(r\Omega(-2f^2g'' + 3fhr^2(\Omega')^2 + 4\Pi^2(\omega - \Omega)^2) - fg'(3r\Omega f' + 2f(n-1)\Omega + fr\Omega')) \\
& + 2\alpha f(n-1)\{g^2h(n-3)\Omega(f^2(n-4)(n-2) + 2f(n-4)(3h-n-1) + 15h^2 - 6h(n+1) + n^2-1) \\
& + r[g(2hr((f-1)n-2f+3h-1)(g\Omega f'' + f\Omega g'' - fg\Omega'')) \\
& + (f(-n) + 2f-5h+n+1)(3fgrh'\Omega' + 3h^2r^3\Omega(\Omega')^2)) + 2g^2h(n-2)r\Omega(f')^2 \\
& + ghf'(r\Omega g'(5f(n-2) + 9h-3n-3) + 4g(n-3)\Omega((f-1)n-2f+3h-1) - 2fg(n-2)r\Omega') \\
& + 2fg^2hn\Omega'(f(-n) + 2f-3h+n+1) + fghg'(f(n-2) + 3h-n-1)(2(n-3)\Omega + r\Omega') \\
& + fhr\Omega(g')^2(-f(n-2) - 3h+n+1)]\} = 0
\end{aligned} \tag{A2}$$

$$\begin{aligned}
& r^2\{2gh(-2h^2\Pi^2(\omega - \Omega)^2\Omega^2r^4 + fgh(2(-\omega^2 - 2\Omega\omega + (h(n-1) + 2)\Omega^2)\Pi^2 \\
& + h(-n^2 + 3h(n-1) + 2r^2\Lambda + 1)\Omega^2)r^2 - f^3g^2h(n-1)n \\
& + f^2g(-2g(h(n-1) + 1)\Pi^2 + h^2(n-2)(n-1)r^2\Omega^2 + gh(n^2 - h(n-1) - 2r^2\Lambda - 1))) \\
& - fr[fr^3\Omega^2(g')^2h^3 - 2fgr^2\Omega g'((n-1)\Omega + r\Omega')h^3 \\
& + gf'(g(2h(fgn - 2h(n-1)r^2\Omega^2) + fgrh') - 3h^2r^3\Omega^2g')h \\
& + g(-f^2g^2r(h')^2 + 2fgh(3h\Omega\Omega'r^3 + fg(n+1))h' + hr(2f^2h''g^2 + 4fh(fg - hr^2\Omega^2)(\Pi')^2g \\
& + 4fh^2(n+2)r\Omega\Omega'g + h^2r^2(3hr^2\Omega^2 + fg)(\Omega')^2 - 2h^2r^2\Omega(g\Omega f'' + f\Omega g'' - 2fg\Omega'')))] \\
& + 2f(n-1)\alpha\{-2g^2h^3(n-2)\Omega^2(f')^2r^4 + fh^3(3h + f(n-2) - n-1)\Omega^2(g')^2r^4 \\
& - 2fgh^3(3h + f(n-2) - n-1)\Omega g'((n-3)\Omega + r\Omega')r^3 \\
& + ghf'[h^2(-9h - 5f(n-2) + 3n+3)\Omega^2g'r^3 + g(fg(3h + 3f(n-2) - n-1)rh' \\
& + 2h(2fh(n-2)\Omega\Omega'r^3 - 2h(n-3)(-2f + 3h + (f-1)n-1)\Omega^2r^2 + fg(n-2)(h + (f-1)n-1)))]r \\
& + g[g(n-3)(fg((f-1)^2n^2 + 2(f-1)(h-f)n + (h-1)(-4f + 3h + 1)) \\
& - h((n-4)(n-2)f^2 + 2(3h-n-1)(n-4)f + 15h^2 + n^2 - 6h(n+1) - 1)r^2\Omega^2)h^2 \\
& + r(f^2g^2(3h - f(n-2) + n+1)r(h')^2 \\
& + 2fgh(3h(-2f + 5h + (f-1)n-1)\Omega\Omega'r^3 + fg(n-1)(-2f + 3h + (f-1)n-1))h' \\
& + hr(r^2(3h(5h + f(n-2) - n-1)r^2\Omega^2 + fg(3h + f(n-2) - n-1))(\Omega')^2h^2 \\
& + 4fg(3h + f(n-2) - n-1)nr\Omega\Omega'h^2 \\
& + 2(3h + f(n-2) - n-1)(f(fh''g^2 + h^2r^2\Omega(2g\Omega'' - \Omega g'')) - gh^2r^2\Omega^2f'')]\} = 0
\end{aligned} \tag{A3}$$

$$\begin{aligned}
& r^2\{2fh(2\Pi^2 + h(2\Lambda r^2 + 2(n-3)\Pi^2 + h(n-5) + (3-n)(n+1)))g^2 \\
& + 2h(f^2gh(n-2)(n-1) - 2hr^2\Pi^2(\omega - \Omega)^2)g \\
& + fr[-fh^2r(g')^2 + fgh(2h(n-1) + rh')g' + ghf'(4gh(n-1) + 3hr'g' + 2grh') \\
& + g(-fgr(h')^2 + 2fghnh' + hr(h(4fg(\Pi')^2 - hr^2(\Omega')^2 + 2gf'' + 2fg'') + 2fgh''))] \\
& - 2f\alpha\{g^2((n-4)(n-2)(n-1)(f-1)^2 + 2(h-1)(n-5)(n-4)(f-1) + 3(h-1)^2(n-9))(n-3)h^2 \\
& + r[2g^2h(n-2)r(h(n-1) + rh')(f')^2 + g(4g(n-3)(h(n-5) + f(n-2)(n-1) + (3-n)(n+1))h^2 \\
& + r(-fg(n-2)r(h')^2 + 2gh(3h(n-3) + (-n-1)(n-3) + f(n-2)(3n-7))h' + hg' \\
& + 2(n-2)(h^3(\Omega')^2r^3 + fghh''r))f' \\
& - fhr(g')^2(h(h(n-5) + f(n-2)(n-1) + (3-n)(n+1)) + f(n-2)rh') \\
& + fg'(2g(h(n-5) + (-n-1)(n-3) + f(n-2)(n-1))(n-3)h^2 \\
& + r(-h^3(n-2)(\Omega')^2r^3 - fg(n-2)(h')^2r + 2fgh(n-2)h''r + gh(3h(n-3) + (3-n)(n+1) \\
& + f(n-2)(3n-5))h')) \\
& + g(fg(n-3)(3h - f(n-2) + n+1)r(h')^2 \\
& + fh(n-2)(3h(\Omega')^2r^4 + 2(gf'' + fg'')r^2 + 2g(3h + f(n-2) - n-1)(n-3))h' \\
& + hr(4fh^2(n-2)\Omega'\Omega''r^3 + h^2(-3h(n-5) + (n-3)(n+1) + f(n-2)(n+3))(\Omega')^2r^2 \\
& + 2h(h(n-5) + f(n-2)(n-1) + (3-n)(n+1))(gf'' + fg'') + fg(n-3)(6h + 2f(n-2) \\
& - 2(n+1)h''))\} = 0
\end{aligned} \tag{A4}$$

$$\Pi'' + \frac{\Pi'}{2fr}(2nfg'h + 2rghf' + rf(gh)') + \frac{\Pi(\omega - \Omega)^2}{f^2g} - \frac{(1 + (n-1)h)\Pi}{fhr^2} = 0 \tag{A5}$$

where a ' denotes differentiation with respect to r . In addition to these second-order ODEs, there is a first-order ODE in the form of a constraint equation, given by

$$\begin{aligned}
& r^2\left[4\left(\frac{1}{2}f^2gh(n-1)n + fg\left(h\left(\frac{1}{2}(n-1)(h-n+2\Pi^2-1) + \Lambda r^2\right) + \Pi^2\right) - h\Pi^2r^2(\omega - \Omega)^2\right) \right. \\
& \left. + fr\left((fg)'(rh' + 2hn) + 4fg\left(\frac{1}{2}(n-1)h' - hr(\Pi')^2\right)\right) + fh^2r^4(\Omega')^2\right] \\
& - 2\alpha f(n-1)[r(fg)'(rh'(3f(n-2) + 3h - n - 1) + 2h(n-2)(fn + h - n - 1)) \\
& + g(n-3)(2frh'(f(n-2) + 3h - n - 1) + h(2(f-1)(h-1)(n-2) + (f-1)^2(n-2)n + 3(h-1)^2)) \\
& + h^2r^4(\Omega')^2(3f(n-2) + 3h - n - 1)] = 0.
\end{aligned} \tag{A6}$$

APPENDIX B: PERTURBATIVE FIELDS

In this appendix we catalog all of the perturbative gravitational and scalar field solutions to the field equations in space-time dimension $D = n + 2$ for $n = 3, 5, 7, 9$. The fields are labeled as $F_{n,p}$, where p denotes the order in ϵ .

1. $n = 3$ results

$$f_{3;2} = \frac{6\ell^4 + 20\ell^2 r^2 + 5r^4}{9}$$

$$\Omega_{3;2} = \frac{6\ell^4 + 4\ell^2 r^2 + r^4}{12}$$

$$\Pi_{3;3} = \frac{1540\ell^6 + 5548\ell^4 r^2 + 3935\ell^2 r^4 + 900r^6}{2016}$$

$$f_{3;4} = \left[\frac{\ell^2}{1270080} (1315860\ell^{14} + 11249595\ell^{12}r^2 + 36154839\ell^{10}r^4 + 51954798\ell^8r^6 + 40913902\ell^6r^8 + 18512283\ell^4r^{10} + 4631027\ell^2r^{12} + 514952r^{14}) - \frac{\alpha}{4762800} (38080980\ell^{14} + 165606315\ell^{12}r^2 + 564058887\ell^{10}r^4 + 809407494\ell^8r^6 + 616096806\ell^6r^8 + 272183829\ell^4r^{10} + 67620861\ell^2r^{12} + 7535096r^{14}) \right]$$

$$g_{3;4} = \left[\frac{\ell^2}{1270080} (872290\ell^{16} + 7971520\ell^{14}r^2 + 25939155\ell^{12}r^4 + 34403186\ell^{10}r^6 + 16099475\ell^8r^8 + 3052440\ell^6r^{10} + 149055\ell^4r^{12} + 35410\ell^2r^{14} + 3541r^{16}) - \frac{\alpha}{1587600} (5891650\ell^{16} + 47634040\ell^{14}r^2 + 173336925\ell^{12}r^4 + 217174806\ell^{10}r^6 + 98887425\ell^8r^8 + 21549240\ell^6r^{10} + 2947095\ell^4r^{12} + 650010\ell^2r^{14} + 65001r^{16}) \right]$$

$$h_{3;4} = \left[\frac{\ell^2 (22260\ell^{12} + 183645\ell^{10}r^2 + 311661\ell^8r^4 + 260694\ell^6r^6 + 123066\ell^4r^8 + 31869\ell^2r^{10} + 3541r^{12})}{1270080} - \frac{\alpha (1604820\ell^{12} + 2949765\ell^{10}r^2 + 2921877\ell^8r^4 + 1877358\ell^6r^6 + 787362\ell^4r^8 + 195003\ell^2r^{10} + 21667r^{12})}{529200} \right]$$

$$\Omega_{3;4} = \left[\frac{\ell^2}{2540160} (1598455\ell^{14} + 8795055\ell^{12}r^2 + 19004760\ell^{10}r^4 + 20982192\ell^8r^6 + 14139048\ell^6r^8 + 6020712\ell^4r^{10} + 1505178\ell^2r^{12} + 167242r^{14}) - \frac{\alpha}{635040} (2196535\ell^{14} + 11355375\ell^{12}r^2 + 22425240\ell^{10}r^4 + 22910832\ell^8r^6 + 14668248\ell^6r^8 + 6079512\ell^4r^{10} + 1505178\ell^2r^{12} + 167242r^{14}) \right]$$

$$\Pi_{3;5} = \left[\frac{\ell^2}{853493760} (1128452101\ell^{18} + 10678880150\ell^{16}r^2 + 41756607180\ell^{14}r^4 + 88872056182\ell^{12}r^6 + 115794392980\ell^{10}r^8 + 98320298706\ell^8r^{10} + 55586393870\ell^6r^{12} + 20395866890\ell^4r^{14} + 4416801537\ell^2r^{16} + 428716940r^{18}) - \frac{\alpha}{152562009600} (928722217423\ell^{18} + 8819401225090\ell^{16}r^2 + 33780454205940\ell^{14}r^4 + 71249955470946\ell^{12}r^6 + 92323311236940\ell^{10}r^8 + 78281035435638\ell^8r^{10} + 44381025035130\ell^6r^{12} + 16375305364670\ell^4r^{14} + 3570276679411\ell^2r^{16} + 349126822500r^{18}) \right]$$

2. $n = 5$ results

$$\begin{aligned}
f_{5;2} &= \frac{20\ell^6 + 105\ell^4 r^2 + 42\ell^2 r^4 + 7r^6}{50} \\
\Omega_{5;2} &= \frac{20\ell^6 + 15\ell^4 r^2 + 6\ell^2 r^4 + r^6}{60} \\
\Pi_{5;3} &= \frac{152581\ell^{10} + 767826\ell^8 r^2 + 1035891\ell^6 r^4 + 773598\ell^4 r^6 + 313026\ell^2 r^8 + 53970r^{10}}{257400} \\
f_{5;4} &= \left[\frac{\ell^2}{618377760000} (295557662400\ell^{20} + 3682611572640\ell^{18} r^2 + 17103932986140\ell^{16} r^4 \right. \\
&\quad + 36498131558460\ell^{14} r^6 + 47555951613885\ell^{12} r^8 + 41236810544215\ell^{10} r^{10} + 24529389790620\ell^8 r^{12} \\
&\quad + 10021950443882\ell^6 r^{14} + 2733953554501\ell^4 r^{16} + 455760809811\ell^2 r^{18} + 35065241462r^{20}) \\
&\quad - \frac{\alpha}{360720360000} (5730125681280\ell^{20} + 54710235259680\ell^{18} r^2 + 261420130519548\ell^{16} r^4 \\
&\quad + 542618701892124\ell^{14} r^6 + 683370729631161\ell^{12} r^8 + 581979609281499\ell^{10} r^{10} + 343653106261036\ell^8 r^{12} \\
&\quad \left. + 140092289643826\ell^6 r^{14} + 38200555296657\ell^4 r^{16} + 6369004169047\ell^2 r^{18} + 490071416494r^{20}) \right] \\
g_{5;4} &= \left[\frac{\ell^2}{618377760000} (198795488864\ell^{22} + 2586519623296\ell^{20} r^2 + 11564746522784\ell^{18} r^4 \right. \\
&\quad + 22932404913236\ell^{16} r^6 + 21539953150144\ell^{14} r^8 + 12564927910535\ell^{12} r^{10} + 4244040931130\ell^{10} r^{12} \\
&\quad + 677658846865\ell^8 r^{14} + 31598878220\ell^6 r^{16} + 7946956745\ell^4 r^{18} + 1222608730\ell^2 r^{20} + 87329195r^{22}) \\
&\quad - \frac{\alpha}{360720360000} (3517759244608\ell^{22} + 44211086834432\ell^{20} r^2 + 200305583802688\ell^{18} r^4 \\
&\quad + 363929580852292\ell^{16} r^6 + 319063756957088\ell^{14} r^8 + 180755354654875\ell^{12} r^{10} + 61031317677330\ell^{10} r^{12} \\
&\quad \left. + 10223405437245\ell^8 r^{14} + 700305076380\ell^6 r^{16} + 175106329125\ell^4 r^{18} + 26939435250\ell^2 r^{20} + 1924245375r^{22}) \right] \\
h_{5;4} &= \left[\frac{\ell^2}{123675552000} (769728960\ell^{18} + 9390693312\ell^{16} r^2 + 20983798836\ell^{14} r^4 + 25938182556\ell^{12} r^6 \right. \\
&\quad + 21295887327\ell^{10} r^8 + 12277654675\ell^8 r^{10} + 4978052794\ell^6 r^{12} + 1362335442\ell^4 r^{14} \\
&\quad + 227055907\ell^2 r^{16} + 17465839r^{18}) - \frac{\alpha}{369969600} (1593271680\ell^{18} + 3122255136\ell^{16} r^2 + 3848592748\ell^{14} r^4 \\
&\quad + 3585574564\ell^{12} r^6 + 2592129969\ell^{10} r^8 + 1419459613\ell^8 r^{10} + 565061926\ell^6 r^{12} + 153939630\ell^4 r^{14} \\
&\quad \left. + 25656605\ell^2 r^{16} + 1973585r^{18}) \right] \\
\Omega_{5;4} &= \left[\frac{\ell^2}{46378332000} (15058082990\ell^{20} + 106176159440\ell^{18} r^2 + 311179210446\ell^{16} r^4 \right. \\
&\quad + 505325794688\ell^{14} r^6 + 547149789758\ell^{12} r^8 + 420702504651\ell^{10} r^{10} + 234379619045\ell^8 r^{12} \\
&\quad + 93593083298\ell^6 r^{14} + 25525386354\ell^4 r^{16} + 4254231059\ell^2 r^{18} + 327248543r^{20}) \\
&\quad - \frac{\alpha}{1932430500} (19521646094\ell^{20} + 131737647392\ell^{18} r^2 + 353356087188\ell^{16} r^4 \\
&\quad + 537690140702\ell^{14} r^6 + 562006315442\ell^{12} r^8 + 424922932863\ell^{10} r^{10} + 235083023747\ell^8 r^{12} \\
&\quad \left. + 93647191352\ell^6 r^{14} + 25525386354\ell^4 r^{16} + 4254231059\ell^2 r^{18} + 327248543r^{20}) \right]
\end{aligned}$$

$$\begin{aligned} \Pi_{5;5} = & \left[\frac{\ell^2}{453635740958400000} (337633104499816268\ell^{26} + 4271077136958547132\ell^{24}r^2 \right. \\ & + 23112600885726752792\ell^{22}r^4 + 72350175319616674844\ell^{20}r^6 + 151159034180233581556\ell^{18}r^8 \\ & + 228077768805842399885\ell^{16}r^{10} + 258620332265962810395\ell^{14}r^{12} + 223871564194330948248\ell^{12}r^{14} \\ & + 147920037831416512914\ell^{10}r^{16} + 73557029936994344181\ell^8r^{18} + 26698941169899723207\ell^6r^{20} \\ & + 6684077146010747418\ell^4r^{22} + 1032668691070620996\ell^2r^{24} + 74253000574956420r^{26}) \\ & - \frac{\alpha}{4498554431170800000} (88843628024978738912\ell^{26} + 1122507865343630559388\ell^{24}r^2 \\ & + 6023372115583107270188\ell^{22}r^4 + 18720948444225766469276\ell^{20}r^6 \\ & + 38914019934303990489424\ell^{18}r^8 + 58601438194884845712305\ell^{16}r^{10} \\ & + 66469446985987754507715\ell^{14}r^{12} + 57615205604166626638152\ell^{12}r^{14} \\ & + 38129098662142936918386\ell^{10}r^{16} + 18989139612729591541089\ell^8r^{18} \\ & + 6901386814262466690003\ell^6r^{20} + 1729600335134757290322\ell^4r^{22} \\ & \left. + 267442276275622329084\ell^2r^{24} + 19242295861536333180r^{26}) \right] \end{aligned}$$

3. $n = 7$ results

$$f_{7;2} = \frac{70\ell^8 + 504\ell^6r^2 + 252\ell^4r^4 + 72\ell^2r^6 + 9r^8}{245}$$

$$\Omega_{7;2} = \frac{70\ell^8 + 56\ell^6r^2 + 28\ell^4r^4 + 8\ell^2r^6 + r^8}{280}$$

$$\begin{aligned} \Pi_{7;3} = & \frac{1}{62375040} (28301260\ell^{14} + 181648328\ell^{12}r^2 + 342224260\ell^{10}r^4 + 415360056\ell^8r^6 + 331552839\ell^6r^8 \\ & + 168013320\ell^4r^{10} + 48974940\ell^2r^{12} + 6252120r^{14}) \end{aligned}$$

$$\begin{aligned} f_{7;4} = & \left[\frac{\ell^2}{371502619488000} (96780316472840\ell^{26} + 1589105610832740\ell^{24}r^2 + 9521410725967140\ell^{22}r^4 \right. \\ & + 25965374820716045\ell^{20}r^6 + 45556296988624565\ell^{18}r^8 + 56539135502871164\ell^{16}r^{10} \\ & + 51618427799679364\ell^{14}r^{12} + 35302266172890626\ell^{12}r^{14} + 18211924445078266\ell^{10}r^{16} \\ & + 7062641146585640\ell^8r^{18} + 2018090265782824\ell^6r^{20} + 403651927726529\ell^4r^{22} \\ & + 50460227131621\ell^2r^{24} + 2968444009876r^{26}) \\ & - \frac{\alpha}{835880893848000} (16866537231824280\ell^{26} + 232049118692571180\ell^{24}r^2 + 1414110709187636460\ell^{22}r^4 \\ & + 3716883650942351175\ell^{20}r^6 + 6304111318629442175\ell^{18}r^8 + 7700650359344929844\ell^{16}r^{10} \\ & + 6986591021140767244\ell^{14}r^{12} + 4767825442882086806\ell^{12}r^{14} + 2458034340313267006\ell^{10}r^{16} \\ & + 953081702075970200\ell^8r^{18} + 272332684743145624\ell^6r^{20} + 54472333166578679\ell^4r^{22} \\ & \left. + 6809680934566771\ell^2r^{24} + 400602893601676r^{26}) \right] \end{aligned}$$

$$\begin{aligned}
 g_{7;4} = & \left[\frac{\ell^2}{371502619488000} (63735166423850\ell^{28} + 1080779423604460\ell^{26}r^2 + 6104455301649630\ell^{24}r^4 \right. \\
 & + 15807036948862760\ell^{22}r^6 + 21693866896535335\ell^{20}r^8 + 20860826548147390\ell^{18}r^{10} \\
 & + 13769626720091151\ell^{16}r^{12} + 5953776409345128\ell^{14}r^{14} + 1535551141626138\ell^{12}r^{16} \\
 & + 193867029381396\ell^{10}r^{18} + 10157293967310\ell^8r^{20} + 2709965597136\ell^6r^{22} + 508118549463\ell^4r^{24} \\
 & \left. + 59778652878\ell^2r^{26} + 3321036271r^{28} \right) \\
 & - \frac{\alpha}{835880893848000} (10750577761041750\ell^{28} + 178690917717167220\ell^{26}r^2 + 1006526355431407410\ell^{24}r^4 \\
 & + 2361158122174494840\ell^{22}r^6 + 3036925736949961845\ell^{20}r^8 + 2850729294425536090\ell^{18}r^{10} \\
 & + 1868492713518324621\ell^{16}r^{12} + 807995702469461688\ell^{14}r^{14} + 209605974589864398\ell^{12}r^{16} \\
 & + 27209163001515516\ell^{10}r^{18} + 1738728305273610\ell^8r^{20} + 463697435964336\ell^6r^{22} + 86943269243313\ell^4r^{24} \\
 & \left. + 10228619910978\ell^2r^{26} + 568256661721r^{28} \right) \\
 h_{7;4} = & \left[\frac{\ell^2}{53071802784000} (153269396280\ell^{24} + 2477855239860\ell^{22}r^2 + 6500468085540\ell^{20}r^4 + 9976953706005\ell^{18}r^6 \right. \\
 & + 10884488013685\ell^{16}r^8 + 9029812070708\ell^{14}r^{10} + 5829937982596\ell^{12}r^{12} + 2930312879858\ell^{10}r^{14} \\
 & + 1128813864010\ell^8r^{16} + 322614952040\ell^6r^{18} + 64522990408\ell^4r^{20} + 8065373801\ell^2r^{22} + 474433753r^{24}) \\
 & - \frac{\alpha}{17058793752000} (86486350630680\ell^{24} + 182245183220100\ell^{22}r^2 + 260482463266740\ell^{20}r^4 \\
 & + 301178906390505\ell^{18}r^6 + 289138698210505\ell^{16}r^8 + 227450874603524\ell^{14}r^{10} + 143878222471828\ell^{12}r^{12} \\
 & + 71803208222954\ell^{10}r^{14} + 27603213721570\ell^8r^{16} + 7886010815720\ell^6r^{18} + 1577202163144\ell^4r^{20} \\
 & \left. + 197150270393\ell^2r^{22} + 11597074729r^{24} \right) \\
 \Omega_{7;4} = & \left[\frac{\ell^2}{25474465336320} (4684576834175\ell^{26} + 39868495607983\ell^{24}r^2 + 144325351642808\ell^{22}r^4 \right. \\
 & + 301265955958360\ell^{20}r^6 + 438282129396128\ell^{18}r^8 + 473620370087280\ell^{16}r^{10} + 391916979706512\ell^{14}r^{12} \\
 & + 252029223792528\ell^{12}r^{14} + 126161196878808\ell^{10}r^{16} + 48499288297880\ell^8r^{18} + 13856939513680\ell^6r^{20} \\
 & + 2771387902736\ell^4r^{22} + 346423487842\ell^2r^{24} + 20377852226r^{26}) \\
 & - \frac{\alpha}{2122872111360} (30060965769595\ell^{26} + 246684097210475\ell^{24}r^2 + 809950806744472\ell^{22}r^4 \\
 & + 1586524546896824\ell^{20}r^6 + 2238845007906592\ell^{18}r^8 + 2387640938645040\ell^{16}r^{10} \\
 & + 1965266521267920\ell^{14}r^{12} + 1261282443509712\ell^{12}r^{14} + 630948024962424\ell^{10}r^{16} \\
 & + 242504796816952\ell^8r^{18} + 69284697568400\ell^6r^{20} + 13856939513680\ell^4r^{22} + 1732117439210\ell^2r^{24} \\
 & \left. + 101889261130r^{26} \right) \\
 \Pi_{7;5} = & \left[\frac{\ell^2}{21848348422059134976000} (8982162145045598581865\ell^{34} + 141066092907373532097690\ell^{32}r^2 \right. \\
 & + 957479816306705256586845\ell^{30}r^4 + 3856281625419549220592040\ell^{28}r^6 \\
 & + 10789673943068638679272455\ell^{26}r^8 + 22792336247876448363452550\ell^{24}r^{10} \\
 & \left. + 37918308373546561272000663\ell^{22}r^{12} + 50757619733081920144022472\ell^{20}r^{14} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ 55227488952051138771625005\ell^{18}r^{16} + 48974717480356234743070466\ell^{16}r^{18} \\
 &+ 35284523399336392108933005\ell^{14}r^{20} + 20477493617041202568028488\ell^{12}r^{22} \\
 &+ 9433307528652764598206937\ell^{10}r^{24} + 3370660892553798052350570\ell^8r^{26} \\
 &+ 900812314937847677993001\ell^6r^{28} + 169475537803320197519400\ell^4r^{30} \\
 &+ 20022251781128501589300\ell^2r^{32} + 1117614384817280519400r^{34}) \\
 &- \frac{\alpha}{795644021703320165376000} (21464560511021140995490385\ell^{34} \\
 &+ 336419204700137912471998890\ell^{32}r^2 + 2265714121070327795119738725\ell^{30}r^4 \\
 &+ 9045794851073754077922660840\ell^{28}r^6 + 25157542057967397600077180895\ell^{26}r^8 \\
 &+ 53007966747953063314567065590\ell^{24}r^{10} + 88148563702087347039590470383\ell^{22}r^{12} \\
 &+ 118053515649034646376315606792\ell^{20}r^{14} + 128548831954201729835945339445\ell^{18}r^{16} \\
 &+ 114084293382704825962729579346\ell^{16}r^{18} + 82251607219034803579127543445\ell^{14}r^{20} \\
 &+ 47764004511303199947728571528\ell^{12}r^{22} + 22014741352249962854054900097\ell^{10}r^{24} \\
 &+ 7869719713738135878103963770\ell^8r^{26} + 2104015408041561787435884881\ell^6r^{28} \\
 &+ 395980508360006822009399400\ell^4r^{30} + 46797072097258916197449300\ell^2r^{32} \\
 &+ 2612928661974905984879400r^{34}) \Big]
 \end{aligned}$$

4. $n = 9$ results

$$f_{9;2} = \frac{252\ell^{10} + 2310\ell^8r^2 + 1320\ell^6r^4 + 495\ell^4r^6 + 110\ell^2r^8 + 11r^{10}}{1134}$$

$$\Omega_{9;2} = \frac{252\ell^{10} + 210\ell^8r^2 + 120\ell^6r^4 + 45\ell^4r^6 + 10\ell^2r^8 + r^{10}}{1260}$$

$$\begin{aligned}
 \Pi_{9;3} = &\frac{1}{1025589600} (356786123\ell^{18} + 2750576510\ell^{16}r^2 + 6313029495\ell^{14}r^4 + 10221019320\ell^{12}r^6 \\
 &+ 11878432160\ell^{10}r^8 + 9822117536\ell^8r^{10} + 5636851220\ell^6r^{12} + 2135373240\ell^4r^{14} + 480720240\ell^2r^{16} \\
 &+ 48768720r^{18})
 \end{aligned}$$

$$\begin{aligned}
 f_{9;4} = &\left[\frac{\ell^2}{859498662576153600} (133340623834666248\ell^{32} + 2720901281792621928\ell^{30}r^2 \right. \\
 &+ 19770363913882326030\ell^{28}r^4 + 63513374918908684770\ell^{26}r^6 + 135934002405968654888\ell^{24}r^8 \\
 &+ 213952704807502670100\ell^{22}r^{10} + 257756682509308369842\ell^{20}r^{12} + 242609187466772524790\ell^{18}r^{14} \\
 &+ 180579919814518993740\ell^{16}r^{16} + 107062480455455025288\ell^{14}r^{18} + 50684524900929500343\ell^{12}r^{20} \\
 &+ 19068904767424154265\ell^{10}r^{22} + 5608737609387866100\ell^8r^{24} + 1246433118183419250\ell^6r^{26} \\
 &+ 196811905684882635\ell^4r^{28} + 19681794632595465\ell^2r^{30} + 937254465799230r^{32}) \\
 &- \frac{\alpha}{2363621322084422400} (51858708461340364584\ell^{32} + 919343053698617972424\ell^{30}r^2 \\
 &+ 6755425175241524371590\ell^{28}r^4 + 20763976427014460249610\ell^{26}r^6 \\
 &+ 42888828387704309354344\ell^{24}r^8 + 66469172496452595919140\ell^{22}r^{10} \\
 &+ 79612560563631174610986\ell^{20}r^{12} + 74782307435370609291070\ell^{18}r^{14}
 \end{aligned}$$

$$\begin{aligned}
 & + 55625640941623976375420\ell^{16}r^{16} + 32972881325499331952424\ell^{14}r^{18} \\
 & + 15608925102093402787239\ell^{12}r^{20} + 5872467180388490650745\ell^{10}r^{22} \\
 & + 1727276938801702287300\ell^8r^{24} + 383856087397493107250\ell^6r^{26} \\
 & + 60611238678141544155\ell^4r^{28} + 6061339452062773545\ell^2r^{30} + 288644544654444590r^{32} \Big] \\
 g_{9;4} = & \left[\frac{\ell^2}{859498662576153600} (84495843563523392\ell^{34} + 1768012312961670776\ell^{32}r^2 \right. \\
 & + 11936492827687331904\ell^{30}r^4 + 37242538451355126890\ell^{28}r^6 + 65138644777217991980\ell^{26}r^8 \\
 & + 85422451772928300030\ell^{24}r^{10} + 82904154261107854584\ell^{22}r^{12} + 58562593998521246634\ell^{20}r^{14} \\
 & + 29286912971461691700\ell^{18}r^{16} + 9866747321987487030\ell^{16}r^{18} + 2041707149885182032\ell^{14}r^{20} \\
 & + 219306855955192929\ell^{12}r^{22} + 14460723267227622\ell^{10}r^{24} + 4017026312889975\ell^8r^{26} \\
 & + 845689750082100\ell^6r^{28} + 126853462512315\ell^4r^{30} + 12081282144030\ell^2r^{32} + 549149188365r^{34}) \\
 & - \frac{\alpha}{787873774028140800} (10942174841871425152\ell^{34} + 225620421791864115016\ell^{32}r^2 \\
 & + 1510935255902133245184\ell^{30}r^4 + 4208312886338787616390\ell^{28}r^6 + 6894619427482858451380\ell^{26}r^8 \\
 & + 8848332307639603148370\ell^{24}r^{10} + 8534331180859836820104\ell^{22}r^{12} + 6020530420354129493814\ell^{20}r^{14} \\
 & + 3011536544232528981900\ell^{18}r^{16} + 1015910734943704195530\ell^{16}r^{18} + 211151074883282761392\ell^{14}r^{20} \\
 & + 23184156465480324699\ell^{12}r^{22} + 1720346092586126082\ell^{10}r^{24} + 477878417772403725\ell^8r^{26} \\
 & + 100605982688927100\ell^6r^{28} + 15090897403339065\ell^4r^{30} + 1437228324127530\ell^2r^{32} + 65328560187615r^{34}) \Big] \\
 h_{9;4} = & \left[\frac{\ell^2}{95499851397350400} (149725307211480\ell^{30} + 3015895473831240\ell^{28}r^2 + 8797952237444370\ell^{26}r^4 \right. \\
 & + 15541844723695290\ell^{24}r^6 + 20300574058072308\ell^{22}r^8 + 21034536810890148\ell^{20}r^{10} \\
 & + 17783794333160310\ell^{18}r^{12} + 12381092092392030\ell^{16}r^{14} + 7089196246488360\ell^{14}r^{16} \\
 & + 3310396957607208\ell^{12}r^{18} + 1241596238094513\ell^{10}r^{20} + 365184210262725\ell^8r^{22} + 81152046725050\ell^6r^{24} \\
 & + 12813481061850\ell^4r^{26} + 1281348106185\ell^2r^{28} + 61016576485r^{30}) \\
 & - \frac{\alpha}{29180510149190400} (162655860396589560\ell^{30} + 362008206104224680\ell^{28}r^2 \\
 & + 572645003750910090\ell^{26}r^4 + 763224818178352530\ell^{24}r^6 + 877801432409857956\ell^{22}r^8 \\
 & + 862035819332396436\ell^{20}r^{10} + 713514589774183470\ell^{18}r^{12} + 492832261815045110\ell^{16}r^{14} \\
 & + 281416001757949320\ell^{14}r^{16} + 131301969045434376\ell^{12}r^{18} + 49236215518455161\ell^{10}r^{20} \\
 & + 14481164174921325\ell^8r^{22} + 3218036483315850\ell^6r^{24} + 508111023681450\ell^4r^{26} \\
 & + 50811102368145\ell^2r^{28} + 2419576303245r^{30}) \Big] \\
 \Omega_{9;4} = & \left[\frac{\ell^2}{59687407123344000} (6598585567818972\ell^{32} + 64940197030107672\ell^{30}r^2 + 273777806813998500\ell^{28}r^4 \right. \\
 & + 681865813132481250\ell^{26}r^6 + 1221884704722504550\ell^{24}r^8 + 1674173405090497584\ell^{22}r^{10} \\
 & + 1807245829726785084\ell^{20}r^{12} + 1565834578187766250\ell^{18}r^{14} + 1101647489722639650\ell^{16}r^{16} \\
 & + 632417867298394200\ell^{14}r^{18} + 295268817053306040\ell^{12}r^{20} + 110708213943786165\ell^{10}r^{22} \\
 & \left. + 110708213943786165\ell^8r^{24} + 110708213943786165\ell^6r^{26} + 110708213943786165\ell^4r^{28} + 110708213943786165r^{30} \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ 32561239395231225\ell^8 r^{24} + 7235830976718050\ell^6 r^{26} + 1142499627902850\ell^4 r^{28} \\
 &+ 114249962790285\ell^2 r^{30} + 5440474418585r^{32}) \\
 &- \frac{\alpha}{532923277887000} (8546242063804172\ell^{32} + 81523891654804872\ell^{30} r^2 + 307318194150454500\ell^{28} r^4 \\
 &+ 715769933670769250\ell^{26} r^6 + 1245320391503010550\ell^{24} r^8 + 1686136326474110784\ell^{22} r^{10} \\
 &+ 1811885990514247284\ell^{20} r^{12} + 1567209651830709250\ell^{18} r^{14} + 1101953061643293650\ell^{16} r^{16} \\
 &+ 632466115496392200\ell^{14} r^{18} + 295273641873105840\ell^{12} r^{20} + 110708443697109965\ell^{10} r^{22} \\
 &+ 32561239395231225\ell^8 r^{24} + 7235830976718050\ell^6 r^{26} + 1142499627902850\ell^4 r^{28} \\
 &+ 114249962790285\ell^2 r^{30} + 5440474418585r^{32}) \Big] \\
 \Pi_{9,5} = &\left[\frac{\ell^2}{12928610477426464084962297600000} (2963660262582405195600812804780\ell^{42} \right. \\
 &+ 54960718928326817581432660068800\ell^{40} r^2 + 438680016278348277072828543787740\ell^{38} r^4 \\
 &+ 2095459575089784573411052833767630\ell^{36} r^6 + 7126896792786249923435515161591030\ell^{34} r^8 \\
 &+ 18828704309439914264745324330571272\ell^{32} r^{10} + 40318687255770901091487447290055224\ell^{30} r^{12} \\
 &+ 71564603098088232189439094493589402\ell^{28} r^{14} + 106606113471306396324075091246911420\ell^{26} r^{16} \\
 &+ 134129087251912145628690383503063720\ell^{24} r^{18} + 142873004760249240821838450887004074\ell^{22} r^{20} \\
 &+ 128775176906725704442407617007965163\ell^{20} r^{22} + 97921202768932011707091446069275799\ell^{18} r^{24} \\
 &+ 62477592200815789501550742105430020\ell^{16} r^{26} + 33171546764185616749586054262623220\ell^{14} r^{28} \\
 &+ 14480110437447756571401543030317327\ell^{12} r^{30} + 5107957999464578930096857586903749\ell^{10} r^{32} \\
 &+ 1419986713892478010760754706555122\ell^8 r^{34} + 299476749663332693852942194632980\ell^6 r^{36} \\
 &+ 45035990011802132234460421933080\ell^4 r^{38} + 4302621620873952400266415533360\ell^2 r^{40} \\
 &+ 196278755076119819922240261840r^{42}) \\
 &- \frac{\alpha}{6691827369268356151636800000} (188034260736405759920667016480\ell^{42} \\
 &+ 3478119834944534785677233867800\ell^{40} r^2 + 27521476972944344576431164066840\ell^{38} r^4 \\
 &+ 129974087795678824227161887469830\ell^{36} r^6 + 438663601013588577076579436614230\ell^{34} r^8 \\
 &+ 1154908376153092917794909859416952\ell^{32} r^{10} + 2470488821740365277752497363025784\ell^{30} r^{12} \\
 &+ 4385163888580609561595948244197282\ell^{28} r^{14} + 6535069763202647770491146331262470\ell^{26} r^{16} \\
 &+ 8226462646319309998709010636408020\ell^{24} r^{18} + 8767061720853343920255745069666384\ell^{22} r^{20} \\
 &+ 7905447916801781766415703529158883\ell^{20} r^{22} + 6013636958064400713013562652664609\ell^{18} r^{24} \\
 &+ 3838221370012324936903467456571320\ell^{16} r^{26} + 2038449297489327713347987874728770\ell^{14} r^{28} \\
 &+ 890063930280989733044566047178207\ell^{12} r^{30} + 314051732713817341652961541727309\ell^{10} r^{32} \\
 &+ 87324111529482417321012534860802\ell^8 r^{34} + 18420525186923716655392197784180\ell^6 r^{36} \\
 &+ 2770650184390283183984243528280\ell^4 r^{38} + 264748247119254298796127491760\ell^2 r^{40} \\
 &+ 12079435074425924874820591440r^{42}) \Big]
 \end{aligned}$$

APPENDIX C: CONSERVED CHARGES AND THERMODYNAMIC QUANTITIES

Here we catalog the thermodynamic charges and potentials entering the first law of thermodynamics in spacetime dimension $D = n + 2$, for $n = 3, 5, 7, 9$. As the boson star temperature is zero, the first law reads $dM = \omega dJ$.

1. $n = 3$

$$M_3 = \frac{5\pi\ell^2\epsilon^2}{24} + \frac{77951\pi\ell^4\epsilon^4}{508032(\ell^2 - 4\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_3 = \frac{\pi\ell^3\epsilon^2}{24} + \frac{83621\pi\ell^5\epsilon^4}{2540160(\ell^2 - 4\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_3 = 5 - \frac{15\ell^2\epsilon^2}{28(\ell^2 - 4\alpha)} - \frac{\ell^4(3211271921\ell^2 - 14548856100\alpha)\epsilon^4}{5085400320(\ell^2 - 4\alpha)^3} + \mathcal{O}(\epsilon^6)$$

2. $n = 5$

$$M_5 = \frac{7\pi^2\ell^4\epsilon^2}{160} + \frac{314018183\pi^2\ell^6\epsilon^4}{17667936000(\ell^2 - 24\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_5 = \frac{\pi^2\ell^5\epsilon^2}{160} + \frac{327248543\pi^2\ell^7\epsilon^4}{123675552000(\ell^2 - 24\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_5 = 7 - \frac{514\ell^2\epsilon^2}{2145(\ell^2 - 24\alpha)} - \frac{\ell^4(1023791506512739\ell^2 - 26707161768137988\alpha)\epsilon^4}{5355421941870000(\ell^2 - 24\alpha)^3} + \mathcal{O}(\epsilon^6)$$

3. $n = 7$

$$M_7 = \frac{3\pi^3\ell^6\epsilon^2}{560} + \frac{1100829437\pi^3\ell^8\epsilon^4}{943498716160(\ell^2 - 60\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_7 = \frac{\pi^3\ell^7\epsilon^2}{1680} + \frac{10188926113\pi^3\ell^9\epsilon^4}{76423396008960(\ell^2 - 60\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_7 = 9 - \frac{4135\ell^2\epsilon^2}{37128(\ell^2 - 60\alpha)} - \frac{\ell^4(1188837899283033191\ell^2 - 76263642821315268516\alpha)\epsilon^4}{20666078485800523776(\ell^2 - 60\alpha)^3} + \mathcal{O}(\epsilon^6) \quad (C1)$$

4. $n = 9$

$$M_9 = \frac{11\pi^4\ell^8\epsilon^2}{24192} + \frac{1065755263141\pi^4\ell^{10}\epsilon^4}{20836331213967360(\ell^2 - 112\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_9 = \frac{\pi^4\ell^9\epsilon^2}{24192} + \frac{1088094883717\pi^4\ell^{11}\epsilon^4}{229199643353640960(\ell^2 - 112\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_9 = 11 - \frac{754\ell^2\epsilon^2}{14535(\ell^2 - 112\alpha)} - \frac{\ell^4(971718029741243000453\ell^2 - 115487747450683234769436\alpha)\epsilon^4}{58241769402253879260000(\ell^2 - 112\alpha)^3} + \mathcal{O}(\epsilon^6)$$

APPENDIX D: KRETSCHMANN SCALAR**1. $n = 3$**

$$\begin{aligned}
K_3 = & \frac{1}{\alpha^2 \ell^4 (144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2)} [\ell^2 (-(144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2)^{3/2}) \\
& - 2\ell^2 (48\alpha^2 + 3\ell^4 - 24\alpha\ell^2 + 8\alpha\ell^2 q_0^2) \sqrt{144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2} \\
& + (24\alpha^2 + 4\ell^4 - 12\alpha\ell^2 + 8\alpha\ell^2 q_0^2) (144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2) \\
& + (48\alpha^2 + 3\ell^4 - 24\alpha\ell^2 + 8\alpha\ell^2 q_0^2)^2] \tag{D1}
\end{aligned}$$

2. $n = 5$

$$\begin{aligned}
K_5 = & \frac{1}{48\alpha^2 \ell^4 (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)} \\
& \times [-2\ell^2 (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)^{3/2} \\
& - 4\ell^2 (2880\alpha^2 + 5\ell^4 - 240\alpha\ell^2 + 48\alpha\ell^2 q_0^2) \sqrt{14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2} \\
& - (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)^2 + (17280\alpha^2 + 37\ell^4 - 1440\alpha\ell^2 \\
& + 576\alpha\ell^2 q_0^2) (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2) \\
& + 2(2880\alpha^2 + 5\ell^4 - 240\alpha\ell^2 + 48\alpha\ell^2 q_0^2)^2] \tag{D2}
\end{aligned}$$

3. $n = 7$

$$\begin{aligned}
K_7 = & \frac{1}{225\alpha^2 \ell^4 (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)} \\
& \times [-2\ell^2 (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)^{3/2} \\
& - 4\ell^2 (25200\alpha^2 + 7\ell^4 - 840\alpha\ell^2 + 120\alpha\ell^2 q_0^2) \sqrt{176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2} \\
& - 2(176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)^2 \\
& + (378000\alpha^2 + 114\ell^4 - 12600\alpha\ell^2 + 3600\alpha\ell^2 q_0^2) (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2) \\
& + 2(25200\alpha^2 + 7\ell^4 - 840\alpha\ell^2 + 120\alpha\ell^2 q_0^2)^2] \tag{D3}
\end{aligned}$$

4. $n = 9$

$$\begin{aligned}
K_9 = & \frac{1}{28224\alpha^2 \ell^4 (112896\alpha^2 - 14336\alpha^2 q'_h(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)} [-270\ell^2 (112896\alpha^2 - 14336\alpha^2 q'_h(0)^2) \\
& + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)^{3/2} - 60\ell^2 (112896\alpha^2 + 9\ell^4 - 2016\alpha\ell^2 + 224\alpha\ell^2 q_0^2) \\
& \times \sqrt{112896\alpha^2 - 14336\alpha^2 q'_h(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2} - 1215(112896\alpha^2 - 14336\alpha^2 q'_h(0)^2 + 9\ell^4 \\
& - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)^2 + (142248960\alpha^2 + 11835\ell^4 - 2540160\alpha\ell^2 + 564480\alpha\ell^2 q_0^2) \\
& \times (112896\alpha^2 - 14336\alpha^2 q'_h(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2) + 10(112896\alpha^2 + 9\ell^4 - 2016\alpha\ell^2 + 224\alpha\ell^2 q_0^2)^2] \tag{D4}
\end{aligned}$$

- [1] D. J. Kaup, *Phys. Rev. D* **172**, 1331 (1968).
- [2] R. Ruffini and S. Bonazzola, *Phys. Rev.* **187**, 1767 (1969).
- [3] O. Kichakova, J. Kunz, and E. Radu, *Phys. Lett. B* **728**, 328 (2014).
- [4] P. Jetzer, *Phys. Rep.* **220**, 163 (1992).
- [5] A. Bernal, J. Barranco, D. Alic, and C. Palenzuela, *Phys. Rev. D* **81**, 044031 (2010).
- [6] E. Berti and V. Cardoso, *Int. J. Mod. Phys. D***15**, 2209 (2006).
- [7] M. Kesden, J. Gair, and M. Kamionkowski, *Phys. Rev. D* **71**, 044015 (2005).
- [8] S. L. Liebling and C. Palenzuela, *Living Rev. Relativity* **15**, 6 (2012).
- [9] O. J. C. Dias, G. T. Horowitz, and J. E. Santos, *Classical Quantum Gravity* **29**, 194002 (2012).
- [10] P. Bizon and A. Rostworowski, *Phys. Rev. Lett.* **107**, 031102 (2011).
- [11] J. Jalmuzna, A. Rostworowski, and P. Bizon, *Phys. Rev. D* **84**, 085021 (2011).
- [12] M. Maliborski and A. Rostworowski, *Phys. Rev. Lett.* **111**, 051102 (2013).
- [13] A. Buchel, S. L. Liebling, and L. Lehner, *Phys. Rev. D* **87**, 123006 (2013).
- [14] O. J. C. Dias, G. T. Horowitz, D. Marolf, and J. E. Santos, *Classical Quantum Gravity* **29**, 235019 (2012).
- [15] D. Astefanesei and E. Radu, *Nucl. Phys.* **B665**, 594 (2003).
- [16] Y. Brihaye, B. Hartmann, and S. Tojiev, *Classical Quantum Gravity* **30**, 115009 (2013).
- [17] S. Stotyn, M. Park, P. McGrath, and R. B. Mann, *Phys. Rev. D* **85**, 044036 (2012).
- [18] O. J. C. Dias, G. T. Horowitz, and J. E. Santos, *J. High Energy Phys.* **07** (2011) 115.
- [19] M. Colpi, S. L. Shapiro, and I. Wasserman, *Phys. Rev. Lett.* **57**, 2485 (1986).
- [20] J.-W. Ho, F. C. Khanna, and C. H. Lee, [arXiv:gr-qc/0207073](https://arxiv.org/abs/gr-qc/0207073).
- [21] Y. Brihaye, B. Hartmann, and E. Radu, *Phys. Lett. B* **607**, 17 (2005).
- [22] O. J. C. Dias, P. Figueras, S. Minwalla, P. Mitra, R. Monteiro, and J. E. Santos, *J. High Energy Phys.* **08** (2012) 117.
- [23] G. Fodor, P. Forgacs, and M. Mezei, *Phys. Rev. D* **82**, 044043 (2010).
- [24] R.-G. Cai and J.-Y. Ji, *Phys. Rev. D* **58**, 024002 (1998).
- [25] B. Hartmann and J. Riedel, *Phys. Rev. D* **87**, 044003 (2013).
- [26] J. Bjoraker and Y. Hosotani, *Phys. Rev. Lett.* **84**, 1853 (2000).
- [27] S. Stotyn and R. B. Mann, *J. Phys. A* **45**, 374025 (2012).
- [28] S. Stotyn, M. Chanona, and R. B. Mann, *Phys. Rev. D* **89**, 044018 (2014).
- [29] S. Stotyn, C. Danielle Leonard, M. Oltean, L. J. Henderson, and R. B. Mann, *Phys. Rev. D* **89**, 044017 (2014).
- [30] D. Lovelock, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).
- [31] S. Deser and Z. Yang, *Classical Quantum Gravity* **6**, L83 (1989).
- [32] J. Crisostomo, R. Troncoso, and J. Zanelli, *Phys. Rev. D* **62**, 084013 (2000).
- [33] B. Hartmann, B. Kleihaus, J. Kunz, and M. List, *Phys. Rev. D* **82**, 084022 (2010).
- [34] Y. Brihaye, [arXiv:1108.2779](https://arxiv.org/abs/1108.2779).
- [35] Y. Brihaye and E. Radu, *J. High Energy Phys.* **09** (2008) 006.
- [36] A. Ashtekar and S. Das, *Class. Quantum Gra.* **17**, L17 (2000).
- [37] S. Das and R. B. Mann, *J. High Energy Phys.* **08** (2000) 033.
- [38] H. P. Pfeiffer, [arXiv:gr-qc/0510016](https://arxiv.org/abs/gr-qc/0510016).
- [39] R. C. Myers and B. Robinson, *J. High Energy Phys.* **08** (2010) 067.
- [40] Y. Brihaye and J. Riedel, *Phys. Rev. D* **89**, 104060 (2014).