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Newton-like equations for a radiating particle

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Second-order Newton equations of motion for a radiating particle are presented. It is argued that the trajectories obeying them also satisfy the Abraham-Lorentz-Dirac (ALD) equations for general 3D motions in the nonrelativistic and relativistic limits. The case of forces depending on only the proper time is here considered. For these properties to hold, it is sufficient that the external force be infinitely smooth and that a Landau-Lifshitz series formed with its time derivatives converges. This series defines in a special local way the effective forces entering the Newton equations. When the external force vanishes in an open vicinity of a given time, the effective one also becomes null. Thus, the proper solutions of the effective equations cannot show runaway or preacceleration effects. The Newton equations are numerically solved for a pulsed force given by an analytic function along the proper time axis. The simultaneous satisfaction of the ALD equations is numerically checked. Furthermore, a set of modified ALD equations for almost everywhere infinitely smooth forces, but including steplike discontinuities in some points, is also presented. The form of the equations supports the statement argued in a previous work, that the causal Lienard-Wiechert field solution surrounding a radiating particle implies that the effective force on the particle should instantaneously vanish when the external force is retired. The modified ALD equations proposed in the previous work are here derived in a generalized way including the same effect also when the force is instantly connected. The possibility of deriving a pointlike model showing a finite mass and an infinite electromagnetic energy from a reasonable Lagrangian theory is also started to be investigated here.

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I. INTRODUCTION

The search for a consistent formulation of the equations of motion for a radiating particle has been and still is a subject of intense research activity [1–26]. The complete understanding of the physics implied in the problem has presented hard theoretical difficulties. The existence of solutions with values growing without bound (runaway behavior) or preaccelerated motions in advance of the applied forces are two of the most debated issues associated with the Abraham-Lorentz-Dirac (ALD) equations. Important advances in the understanding of these properties have been by now obtained.

In relatively recent times, one line of thinking has been adopted by many authors which supports the original idea of the Lorentz model, that is, to accept the essential need of a finite spatial extension for the radiating particle, in order to solve for the mentioned difficulties of the ALD equations (see [1,2,8–10,25]). For example, in Refs. [9,10], the authors, after arguing about this requirement, derived a simple second-order equation for a radiating particle by assuming that it shows an internal structure. Their equation is claimed to be an exact one but for a structured particle. Another work following this viewpoint was Ref. [8], where it was argued that, for nonanalytically time-dependent forces, the ALD equations cannot be derived in an exact way from the equations of the coupled motion of the particle and its accompanying field. The author also introduced a correction to the ALD equations for suddenly changing forces. In it, the reaction force of the field on the particle is factored by a proper-time-dependent function $\eta(\tau)$ [8]. This change was claimed to get rid of both the runaway and the preaccelerated solutions. A large quantity of works by now have been devoted to investigating extended particle models. For example, the authors of Ref. [25] introduced a special extended structure for the particle which also eliminates the runaway and preaccelerated motions.

However, there also exists an alternative point of view in the literature. It is based on the notion that what is required to eliminate the unphysical properties of the ALD equations is to impose some "physical" constraints on the manifold of solutions for those equations. The first proponent of this approach was the same Dirac [5]. This viewpoint has also been argued in great detail in Ref. [11] and its contained references of the same author. In those works, criteria had been advanced for specifying exact solutions of the ALD equations not showing the undesirable properties.

More recently, another interesting approach which could be considered as compatible with the validity of the ALD

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equations was introduced in Ref. [26]. This work considers a particle of charge q and mass m having an extended structure but so that, in the point-particle limit, the charge and the mass both tend to vanish, by maintaining their ratio finite. In that special limit, the authors argued the validity of the ALD equation. It can be noted that this result is compatible with the more general possibility of the satisfaction of the ALD equations when the zero mass and charge limit is not assumed. Thus, it can yet be possible that the preaccelerated and runaway solutions could be eliminated by some means in the point-particle limit and the ALD equations could have a consistent formulation not requiring a reduction of the order approximation. Therefore, one relevant point (that was underlined in Ref. [26]) is required to be clarified in connection with the pointlike limit, for which Dirac showed the satisfaction of the ALD equations in the case of not taking the vanishing charge and mass. The point is whether it is possible to construct a reasonable physical model for the point particle showing the following Dirac conditions: a finite and positive total mass and an infinite electromagnetic energy in the particle pointlike limit. These are the essential conditions determined by Dirac in Ref. [5], allowing the satisfaction of the ALD equations. It can be concluded that the explicit construction of such a physical model for the charged point particle should justify the exact validity of the ALD equation for the finite charge and mass situation, without requiring one to do order-reduction approximations. This conclusion will extend the results of Ref. [26], which are limited to the case of infinitesimal values of the charges. In this sense, it can be underlined that the common vanishing limit for the charge and mass assumed in Ref. [26] helps show the validity of the ALD equations seemingly due the associated reduction of the electrostatic energy.

In the present work, at variance with the studies linked with the first view mentioned above, we will not consider that the radiating particles have a structure. Rather, we will present a curious Newton-type equation of motion for those particles that, when satisfied, directly obey the ALD equations. Therefore, the work can be considered as supporting, but without furnishing a definitive conclusion, the second viewpoint described before. The satisfaction of the ALD equations occurs when the force is an infinitely smooth function of time and a particular series (known as the Landau-Lifshitz series [27]), formed with the infinite sequence of its time derivatives at a given instant, converges. The effective Newton force at this instant is defined by the values of the series. Therefore, the effective force vanishes when the external forces is null within an open neighborhood of the given instant. Thus, the preacceleration or runaway solution when the external force vanishes is eliminated.

To motivate these equations, we first show their equivalence with the ALD ones in the nonrelativistic limit. Next, the equations are generalized to the relativistic case. Furthermore, we present the solution of the Newton equations for a pulselike unidirectional force, for which the above-mentioned particular series of their proper time derivatives converge in the whole proper time axis. The results do not show runaway or preaccelerated behaviors, since, as noted before, the effective force depends only on the local behavior around a proper time instant. The parameters of the squared pulse defining the force can be selected to arbitrarily approach in absolute values to the exact pulse force considered by Dirac in the classic paper [5]. The alternative limit solution solves the ALD equation, but its main difference with the one discussed by Dirac is that it defines a solution in the sense of the distributions. The case of forces being infinitely smooth, but eventually showing a numerable set of steplike discontinuities, is also discussed. A modified set of ALD equation is also derived, after some assumptions which generalize the one proposed in Ref. [28]. Then, the central idea advanced in that work gets support: The validity of the Lienard-Wiechert solutions for the electromagnetic field close around the particle, just after the external force is retired, implies that its acceleration should also suddenly disappear. This property is a natural consequence of the fact that the electromagnetic field in a sufficiently close small vicinity of the particle, with no forces exerted on it, is given by a Lorentz "boosted" Coulomb field, which does not produce any force on the central point particle.

The modified equations are here also solved for an external force in the form of a rigorously square pulse which exactly vanishes outside a given time interval. The solution predicts, in accordance with the discussion above, that after the forces discontinuously disappear at the end of the pulse the acceleration also instantly vanishes, in accordance with the absence of forces determined by the Lienard-Wiechert solution for the fields. This force pertains to the class of almost everywhere C^∞ functions showing steplike discontinuities and is exactly the one studied by Dirac in Ref. [5]. The solution is compared with the one associated with the previously studied analytical pulsed force. They show a very close appearance, indicating the presence of Dirac delta functions concentrated at the discontinuity points of the external force in the modified ALD equations. This indicates that the discontinuities of the acceleration do not contribute with finite terms to the momentum of the radiation, contrary to what could be supposed from the appearance of the Delta functions in the modified equations.

Finally, in this work, we also start the investigation of the possibility of the construction of a reasonable physical model for the charged point particle satisfying the above-mentioned Dirac conditions, that is, a model of the point particle showing a finite total mass by also having an infinite electrostatic energy in the pointlike limit. We first underline that no theory satisfying the weak energy condition can justify the above-cited Dirac conditions. However, it is also noted that Lagrangians showing a bounded from below energy density have the opportunity to validate such conditions. A particular Lagrangian is proposed as being constituted by two scalars fields, one of them interacting with an electromagnetic field. The energy density is not satisfying the weak energy condition but is bounded from below at all the spatial points. The model presents families of solutions showing constituting fields which shrink their spatial regions of definition to zero size. This follows when a parameter is tending to a limiting value. The solutions also show negative energy densities in the regions in which the fields take appreciable values. The dependence of the solutions on the parameters will be investigated elsewhere in a search for a concrete point-particle model satisfying the Dirac conditions. This result can justify the exact validity of the ALD equations for the finite values of the charge and mass of the radiating particle, generalizing in this way the results of Ref. [26]. It should be stressed that the discussion in this work does not represent an approximate reduction of the order procedure, as it was recently expressed in Ref. [12]. The Newton-like equations discussed here are argued in fact to be equivalent to the ALD equations.

The exposition proceeds as follows. In Sec. II, the Newton-like equations for the nonrelativistic motion are presented and shown to have common solutions with the ALD equations. Section III generalizes the discussion by constructing relativistic equations whose solutions satisfy the ALD ones. Section IV presents conditions such that, given an infinitely smooth external force, the effective Newton force associated with it becomes well defined. Furthermore, Sec. V exposes the numerical solution of the Newton second equations for a force being similar to a squared pulse but defined by an analytical function along all the proper time axis. Next, Sec. VI is devoted to deriving the modified ALD equations for forces defined almost everywhere by infinitely smooth functions but showing steplike discontinuities at a set of instants along the time axis. In Sec. VII, the modified ALD equations are solved for the rigorous squared pulse. Section VIII is devoted to presenting the elements of a Lagrangian theory from which it might be possible to construct a model for the Dirac pointlike particle satisfying the ALD equations. Finally, in the summary, the results are shortly reviewed and possible extensions of the work are commented.

II. NEWTON-LIKE EQUATION FOR THE NONRELATIVISTIC ALD EQUATION

Let us consider a general but nonrelativistic motion of a particle P along a space-time trajectory defined by a curve $x(\tau) = (x^1(t), x^2(t), x^3(t))$ and parameterized by the time t. For this motion, the nonrelativistic ALD equations take the standard form

$$ma^i(t) - f^i(t) = \kappa \frac{da^i(t)}{dt}, \tag{1}$$

$$v^i(t) = \dot{x}^i(t) = \frac{dx^i(t)}{dt}, \tag{2}$$

$$a^i(t) = \dot{v}^i(t) = \frac{dv^i(t)}{dt}, \tag{3}$$

where the index i has the three values $i = 1, 2, 3$ and $\frac{\kappa}{m}$ is the ALD time constant, equal to $c \times 10^{-22}$ cm in the natural units here employed. In what follows, without attempting to repeat the trial and error process which led to the mentioned solution, we will directly write its expression for afterwards arguing that it solves the above Eq. (1). The second-order Newton-like equations have the specific form

$$a^i(t) = \frac{1}{m}\sum_{n=0}^{\infty} \frac{d^n}{dt^n} f^i(t)\left(\frac{\kappa}{m}\right)^n. \tag{4}$$

Let us assume that the forces are infinitely smooth, that is, pertaining to C^∞ and that the series at the right-hand side converges in a certain interval of times. Then, for the time derivative of the acceleration we have

$$\begin{aligned}
\dot{a}^i(t) &= \frac{1}{m}\sum_{n=0} \frac{d^{n+1}}{dt^{n+1}} f^i(t)\left(\frac{\kappa}{m}\right)^n \\
&= \frac{1}{\kappa}\sum_{n=0} \frac{d^{n+1}}{dt^{n+1}} f^i(t)\left(\frac{\kappa}{m}\right)^{n+1} \\
&= \frac{1}{\kappa}\left(\sum_{n=0} \frac{d^n}{dt^n} f^i(t)\left(\frac{\kappa}{m}\right)^n - f^i(t)\right) \\
&= \frac{1}{\kappa}(m\, a^i(t) - f^i(t)),
\end{aligned} \tag{5}$$

where, in the following, a point over a quantity will mean a derivative over its defined temporal argument. Therefore, assuming that the series is well defined, the trajectory $x^i(t)$ solving Eq. (4) also satisfies the nonrelativistic ALD equations

$$m\, a^i(t) - f^i(t) = \kappa \dot{a}^i(t). \tag{6}$$

Now, a question appears about the existence of proper and helpful definitions of the effective force at the right-hand side of Eq. (4). Let us defer the discussion of this point to the next sections. There, we will derive a condition to be satisfied by the external force for the effective one to be well defined. In addition, we will construct an explicit example in which the force is infinitely smooth at all times, allowing one to calculate the effective one. In the coming section, we will generalize the discussion done here by determining a second-order covariant equation whose solution should also satisfy the relativistic ALD ones.

III. THE RELATIVISTIC GENERALIZATION

Let us consider a force in the instant rest frame of the particle, written in the way

$$f_e{}^\mu(\tau) = (0, f_e^i(\tau)), \tag{7}$$

where the spatial components in the rest frame $f_e^i(\tau)$ are functions of the proper time given in the form suggested by the discussion in the previous section:

$$f_e{}^i(\tau) = \frac{1}{m}\sum_{m=0}^{n}\frac{d^m}{d\tau^m}f^i(\tau)\left(\frac{\kappa}{m}\right)^m, \tag{8}$$

and $f^i(\tau)$ are the components of the external forces exerted on the particle in the rest frame. Note that in the rest frame the zeroth component of the external force is always equal to zero. Then, the time derivatives of arbitrary order of these components will automatically vanish also.

We will assume an inertial observer's inertial frame O and determine a particular Lorentz transformation that, at any value of the proper time of the particle τ, links the coordinates of the observer's and the proper one. For this purpose, imagine the trajectory of the particle $y^\mu(\tau)$ as a function of τ, as seen by the observer. Then, assuming that the particle starts moving at the origin of the observer's frame at $\tau = 0$, we can divide the whole proper time interval of its movement in N equal intervals of size $\epsilon = \frac{\tau}{N}$. Let us now construct a Poincaré inhomogeneous transformation which relates the instantaneous rest frame and the observer's. To start the discussion, we can first recall the expression for a general Lorentz boost (a Lorentz transformation without rotation):

$$B^\mu{}_\nu \equiv \begin{pmatrix} \gamma & \gamma v^j \\ \gamma v^i & \delta^i_j + (\gamma - 1)\frac{v^i v^j}{v^2} \end{pmatrix}, \tag{9}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad v^2 = v^j v^j. \tag{10}$$

Considering that the transformation is associated with an infinitesimal increment in velocity dv^i, the expression reduces to

$$B^\mu{}_\nu \equiv \begin{pmatrix} 1 & dv^j \\ dv^i & \delta^i_j \end{pmatrix}. \tag{11}$$

But by defining the 4-velocity of the rest frame at the proper time τ, and its corresponding increment in a proper time $d\tau$, as given by

$$u^\mu = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}, \qquad du^\mu = \begin{pmatrix} 0 \\ d\vec{v} \end{pmatrix},$$

the infinitesimal boost in the rest frame can be written in the form

$$B^\mu{}_\nu(u, du) \equiv \delta^\mu{}_\nu - u^\mu du_\nu + du^\mu u_\nu. \tag{12}$$

Then, after performing a Lorentz or Poincaré transformation to an arbitrary reference frame, the infinitesimal transformations between two successive rest frames separated by a small proper time interval $d\tau$ are defined by the same covariant formula but in terms of the 4-velocity and its increment in the form

$$u^\mu \equiv \frac{1}{\sqrt{1-v^2}}(1, \vec{v}(\tau)) = (\gamma, \gamma\vec{v}(\tau)), \tag{13}$$

$$du^\mu \equiv \left(\frac{d}{d\tau}\gamma, \frac{d}{d\tau}(\gamma\vec{v}(\tau))\right)d\tau, \tag{14}$$

$$\begin{aligned}\vec{v}(\tau) &\equiv (v^1(\tau), v^2(\tau), v^3(\tau)) \\ &= -(v_1(\tau), v_2(\tau), v_3(\tau)),\end{aligned} \tag{15}$$

where the metric tensor will be assumed in the convention

$$g^\mu{}_\nu \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{16}$$

Having the expression for the Lorentz boost which transforms (in the observer's frame) between two contiguous proper references frames (being associated with a small difference of proper times $d\tau$), we can combine a large set of infinitesimal successive Poincaré transformations in the form

$$U(\Lambda, y(\tau)) = \lim_{N\to\infty}\left(\prod_{n=1}^{N} U(B(u(n\epsilon), du(n\epsilon)), dy(n\epsilon))\right), \tag{17}$$

to construct a finite Poincaré transformation in the explicit form

$$\begin{aligned} x'^\mu &= \lim_{N\to\infty}\left(\prod_{n=1}^{N} B\left(u\left(\frac{n}{N}\tau\right), du\left(\frac{n}{N}\tau\right)\right)\right)^\mu_{\ \nu} x^\nu + y^\mu(\tau) \\ &= \Lambda^\mu_\nu(\tau)x^\nu + y^\mu(\tau), \\ y^\mu(\tau) &= dy^\mu(\epsilon) + \lim_{N\to\infty}\sum_{m=2}^{N}\left(\prod_{n=1}^{m-1} B\left(u\left(\frac{n}{N}\tau\right), du\left(\frac{n}{N}\tau\right)\right)\right)^\mu_{\ \nu} dy^\nu(m\epsilon), \end{aligned} \tag{18}$$

which defines a global transformation from the rest system to the observer's reference frame. Note that the $u(\frac{n}{N}\tau)$, $du(\frac{n}{N}\tau)$, and $dy^\nu(n\epsilon)$ are 4-velocities, their differential and the change in the four coordinates of the particle, at the proper time value $\frac{n}{N}\tau$, that is, at an intermediate point of the trajectory. Thus, their values define contributions to the total coordinate $y^\mu(\tau)$ of the particle at the proper time τ, which "should" be transformed by all the infinitesimal transformations ahead of the time $n\epsilon$, to define their contributions to the total coordinate of the particle $y^\mu(\tau)$. The product of the boosts is assumed to be ordered, with the index n growing from left to right and

$$u^\mu\left(\frac{N}{N}\tau\right) = u^\mu(\tau) = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}.$$

Now, the force in the observer's frame will be defined by the total Lorentz transformation [determining the Poincaré one in (18)] of the force written in the rest frame, that is, by the formula

$$\mathcal{F}^\mu(\tau) = \Lambda^\mu{}_\nu(\tau) f_e{}^\nu(\tau), \tag{19}$$

in which $\Lambda^\mu_\nu(\tau)$ is given by

$$\Lambda^\mu{}_\nu(\tau) = \lim_{N\to\infty}\left(\prod_{n=1}^{N} B\left(u\left(\frac{n}{N}\tau\right), du\left(\frac{n}{N}\tau\right)\right)\right)^\mu_\nu. \tag{20}$$

This formula for the forces gives them a covariant definition. It can be argued as follows. Let us consider the same construction of the transformation $\Lambda^\mu{}_\nu(\tau)$ but defined in another arbitrary observer's frame and denote it as $\tilde{\Lambda}^\alpha{}_\beta(\tau)$. Then, the expressions of the forces in the two considered frames are given as

$$\mathcal{F}^\mu(\tau) = \Lambda^\mu{}_\nu(\tau) f_e{}^\nu(\tau), \tag{21}$$

$$\tilde{\mathcal{F}}^\mu(\tau) = \tilde{\Lambda}^\mu{}_\nu(\tau) f_e{}^\nu(\tau), \tag{22}$$

and after multiplying by the inverses of $\Lambda^\alpha{}_\beta(\tau)$ and $\tilde{\Lambda}^\alpha{}_\beta(\tau)$, it follows that

$$\begin{aligned}
\Lambda_\nu{}^\mu(\tau)\mathcal{F}^\nu(\tau) &= f_e{}^\mu(\tau), \\
\tilde{\Lambda}_\nu{}^\mu(\tau)\tilde{\mathcal{F}}^\nu(\tau) &= f_e{}^\mu(\tau), \\
\tilde{\Lambda}_\nu{}^\mu(\tau)\tilde{\mathcal{F}}^\nu(\tau) &= \Lambda_\nu{}^\mu(\tau)\mathcal{F}^\nu(\tau), \\
\tilde{\mathcal{F}}^\mu(\tau) &= \tilde{\Lambda}_\alpha{}^\mu(\tau)\Lambda_\nu{}^\alpha(\tau)\mathcal{F}^\nu(\tau) \\
&= \hat{\Lambda}_\nu{}^\mu(\tau)\mathcal{F}^\nu(\tau),
\end{aligned} \tag{23}$$

which indicates that the defined forces in two arbitrary observer's frames are related by the Lorentz transformation $\hat{\Lambda}_\nu{}^\mu(\tau)$ linking both reference systems. Therefore, the force is defined as a Lorentz vector, and the following covariant Newton-like equation will be considered:

$$a^\mu(\tau) = \frac{1}{m}\mathcal{F}^\mu(\tau). \tag{24}$$

The connection of this equation with the ALD one will be discussed in the following subsection. It should be noted that, when the motion is defined by a force that does not maintain the velocities of the particles along a definite direction, the analytic form of the force becomes complicated to determine. This is due to the fact that the Lorentz boosts associated to velocities oriented in different directions do not commute. This makes the analytic determination more difficult.

For the explicit solution of the examples to be further considered here, in which the motion is collinear, the forces can be explicitly written, since the set of Lorentz boosts along a fixed direction is a group, whose elements are given in (9). Then, a Lorentz transformation Λ^μ_ν expressing the coordinates of the observer's frame in terms of the rest one in this simpler case can be chosen in the form

$$\Lambda^\mu{}_\nu(\tau) \equiv \begin{pmatrix} \gamma & \gamma v^j \\ \gamma v^i & \delta^{ij} + (\gamma-1)\frac{v^i v^j}{v^2} \end{pmatrix}, \tag{25}$$

and correspondingly the formula for the force becomes

$$\begin{aligned}
\mathcal{F}^\mu(\tau) &= \Lambda^\mu{}_\nu(\tau) f_e{}^\nu(\tau) \\
&= \left(\gamma(v)\vec{v}(\tau)\vec{f}_e(\tau), \vec{f}_e(\tau) + (\gamma-1)\frac{\vec{v}\,\vec{f}_e(\tau)}{v^2}\vec{v}\right).
\end{aligned} \tag{26}$$

Below, in this subsection, we enumerate some properties and conventions that can be helpful to specify for what follows. The previous discussion determines that, in the rest frame of the particle, these relations are valid:

$$a^\mu(\tau)u_\mu(\tau) = 0, \qquad a^0(\tau) = 0. \tag{27}$$

In this same rest system, the explicit form of the projection operator over the three-space being orthogonal to the 4-velocity is

$$P^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix}. \tag{28}$$

Finally, for the sake of definiteness, let us explicitly collect here some properties of the Lorentz transformation defining the four vectors $a^\nu(\tau)$ and $\mathcal{F}^\nu(\tau)$ in the observer's frame:

$$a'^\mu(\tau) = \Lambda^\mu{}_\nu(\tau) a^\nu(\tau), \tag{29}$$

$$\mathcal{F}'^{\mu}(\tau) = \Lambda^{\mu}{}_{\nu}(\tau)\mathcal{F}^{\nu}(\tau),$$
$$u'^{\mu}(\tau) = \Lambda^{\mu}{}_{\nu}(\tau)u^{\nu}(\tau) \equiv \frac{1}{\sqrt{1-v'^2}}(1, \vec{v}'(\tau)), \quad (30)$$

$$P'^{\mu\nu} = g^{\mu\nu} - u'^{\mu}u'^{\nu}, \quad (31)$$

$$\vec{v}'(\tau) \equiv \frac{d}{dt'}x'^{i}(\tau), \quad i = 1, 2, 3, \quad (32)$$

$$x'^0 = t', \quad (33)$$

$$d\tau = \sqrt{1-v'^2}dt'. \quad (34)$$

Let us delete in what follows the "tilde" over the quantities in a general system of coordinates, in order to simplify the notation in the next discussion. When the special rest system will be considered, it will be explicitly noticed.

A. The satisfaction of the relativistic ALD equations

Now, consider that the above-defined Newton equations have a well-defined trajectory solving them, and then study the question about whether or not this solution could also satisfy the ALD equations. For this purpose, let us evaluate the time derivative of the acceleration in the proper frame:

$$\frac{d}{d\tau}a^{\mu}(\tau) = \frac{1}{m}\frac{d}{d\tau}\mathcal{F}^{\mu}(\tau) = \frac{1}{m}\frac{d}{d\tau}(\Lambda^{\mu}{}_{\nu}(\tau)f_e{}^{\nu}(\tau)), \quad (35)$$

by considering the definition of $\Lambda^{\mu}{}_{\nu}(\tau)$ as follows:

$$\begin{aligned}\frac{d}{d\tau}(\Lambda^{\mu}{}_{\nu}(\tau)f_e{}^{\nu}(\tau)) &= \frac{d}{d\tau}\left(\lim_{\epsilon\to\infty}\left(\prod_{n=1}^{\frac{\tau}{\epsilon}}B(u(n\epsilon), du(n\epsilon))\right)^{\mu}{}_{\nu}f_e{}^{\nu}(\tau)\right)\\
&= \Lambda^{\mu}{}_{\nu}(\tau)\lim_{\delta\to0}\left(\frac{B(u(\tau+\delta), du(\tau+\delta))^{\nu}{}_{\alpha}f_e{}^{\alpha}(\tau+\delta) - f_e^{\nu}(\tau)}{\delta}\right)\\
&= \Lambda^{\mu}{}_{\nu}(\tau)\lim_{\delta\to0}\left(\frac{B(u(\tau+\delta), du(\tau+\delta))^{\nu}{}_{\alpha}f_e^{\alpha}(\tau+\delta) - f_e^{\nu}(\tau)}{\delta}\right)\\
&= \Lambda^{\mu}{}_{\alpha}(\tau)\left(\left(-u^{\alpha}(\tau)\frac{d}{d\tau}u(\tau)_{\nu} + \frac{d}{d\tau}u^{\alpha}(\tau)u(\tau)_{\nu}\right)f_e^{\nu}(\tau) + \frac{d}{d\tau}f_e{}^{\alpha}(\tau)\right),\end{aligned}$$

where all the quantities at the right of $\Lambda^{\mu}{}_{\alpha}(\tau)$ in the last line are defined in the rest frame. But, in this system of coordinates

$$u^{\mu}(\tau) = \begin{pmatrix}1\\ \vec{0}\end{pmatrix}.$$

After recalling that also in this frame

$$\frac{d}{d\tau}(\gamma(v))|_{v=0} = -\frac{1}{2}\gamma^{\frac{3}{2}}(0)\left(2v\frac{d}{d\tau}v\right)|_{v=0} = 0, \quad (36)$$

it follows for the derivative of the acceleration that

$$\frac{d}{d\tau}u^{\mu}(\tau) = \begin{pmatrix}0\\ \frac{d}{d\tau}\vec{v}\end{pmatrix}.$$

By using these relations, it is possible to write

$$\begin{aligned}&\frac{1}{m}\frac{d}{d\tau}(\Lambda^{\mu}{}_{\nu}(\tau)f_e{}^{\nu}(\tau))\\
&= \frac{\Lambda^{\mu}{}_{\alpha}(\tau)}{m}\left(-u^{\alpha}(\tau)\frac{d}{d\tau}u(\tau)_{\nu}f_e{}^{\nu}(\tau) + \frac{d}{d\tau}f_e{}^{\alpha}(\tau)\right)\\
&= \frac{\Lambda^{\mu}{}_{\alpha}(\tau)}{m}\left(u^{\alpha}(\tau)a^{i}(\tau)a^{i}(\tau) + \delta^{\alpha}_{i}\frac{m}{\kappa}(ma^{i}(\tau) - f^{i}(\tau))\right),\end{aligned}$$

in which we have employed the property of the considered force given in (5). After taking into account that $a^{i}(\tau)a^{i}(\tau) = -a^{\mu}(\tau)a_{\mu}(\tau)$, where $a^{i}(\tau)$ are the spatial components of the acceleration in the rest frame and the temporal one vanishes, it follows that

$$\begin{aligned}\frac{1}{m}\frac{d}{d\tau}(\Lambda^{\mu}{}_{\nu}(\tau)f_e{}^{\nu}(\tau)) &= \Lambda^{\mu}{}_{\alpha}(\tau)(-u^{\alpha}(\tau)a^{\mu}(\tau)a_{\mu}(\tau)\\
&\quad + \delta^{\alpha}{}_{i}\frac{1}{\kappa}(ma^{i}(\tau) - f^{i}(\tau))).\end{aligned}$$

But, since $u^{\alpha}(\tau)$ is the 4-velocity of the particle in the rest frame, the vector $\Lambda^{\mu}_{\alpha}(\tau)u^{\alpha}(\tau)$ is the 4-velocity in the observer's frame, and the previous expression can be expressed in the form

$$\kappa\left(\frac{d}{d\tau}a^{\mu}(\tau) + u^{\mu}(\tau)a^{\nu}(\tau)a_{\nu}(\tau)\right) = ma^{\mu}(\tau) - f^{\mu}(\tau).$$

Therefore, it follows that the satisfaction of the proposed Newton-like equations implies the corresponding satisfaction of the Abraham-Lorentz-Dirac ones, also in the relativistic case. We have directly checked the satisfaction of the ALD equation for the case of the collinear motion in which the force is explicitly defined by (26). The explicit solutions in the coming sections all refer to the collinear motions.

IV. THE CLASS OF FORCES

Let us consider the series defining the components of the effective forces in the rest frame system in the form

$$S^i_\infty(\tau) = \sum_{m=0}^{\infty} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m \tag{37}$$

and assume the three functions $f^i(t)$ as pertaining to the space of infinitely smooth functions C^∞ with the additional condition that the series converges in an open region of proper time values. We will argue below that the set of all such series is not vanishing and, moreover, that it is a large class of functions. For this purpose, consider a decomposition of the series in a sum over a finite number of terms up to a largest index $m = m_f$, plus the rest of the series as

$$S^i_\infty(\tau) = \sum_{m=0}^{m_f} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m + \sum_{m=m_f}^{\infty} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m. \tag{38}$$

Then, assume that the only restriction on the functions $f^i(\tau)$ is that their times derivatives of arbitrary order, and at any time within the mentioned open region, are bounded by a constant M, for all the orders higher than a given number $m(M)$. Then, select $m_f = m(M)$ which allows one to write the inequalities

$$\begin{aligned} |S^i_\infty(\tau)| &\le \left|\sum_{m=0}^{m_f} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m\right| + \left|\sum_{m=m_f}^{\infty} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m\right| \\ &\le \left|\sum_{m=0}^{m_f} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m\right| + M\left|\sum_{m=m_f}^{\infty} \left(\frac{\kappa}{m}\right)^m\right| \\ &= \left|\sum_{m=0}^{m_f} \frac{d^m}{d\tau^m} f^i(\tau) \left(\frac{\kappa}{m}\right)^m\right| + M\left(\frac{\kappa}{m}\right)^{m_f} \left|\frac{1}{1-\frac{\kappa}{m}}\right| < \infty. \end{aligned} \tag{39}$$

Thus, the series defining the effective forces are convergent at all time values, with the unique condition that the time derivatives of arbitrary order of the external forces are uniformly bounded for all orders and the constant $\frac{\varkappa}{m} < 1$. These constraints seem not to be strong ones. By example, it is known that, when all the time derivatives of a given function at a point are bounded, the function admits a Taylor expansion that converges to the value of the function in a neighborhood of the considered point. That is, the class of external forces for which the effective forces are well defined includes a large set of smooth functions.

V. A SMOOTH REGULARIZATION OF THE DIRAC CONSTANT FORCE PULSE

In this section, we will solve the effective Newton equations for an external force which constitutes a regularization of a rigorously constant force acting only during a specified time interval of duration T and exactly vanishing outside this time lapse. The regularization will be defined by a time interval t_o assumed to be very much shorter than T. It will be found that the parameter t_o can be as short as 10 times the extremely short characteristic time $\frac{\varkappa}{m}$ being associated to the radiation reaction forces in the ALD equations. However, in order to numerically evidence the exact satisfaction of the ALD equations by the solutions found for the effective Newton equations, larger values of $\frac{\varkappa}{m}$ will be assumed. This will avoid the extremely small values of the terms entering the series defining the effective forces, when the "electromagnetic" values of $\frac{\varkappa}{m}$ are assumed. The considered form of the force represents a regularization of the one employed by Dirac in his classical work [5], in order to illustrate the appearance of runaway solutions in the ALD equations. For the values of the parameters giving a force in the form of a square pulse, it will be shown that the solution predicted by the effective Newton equations does not show the runaway, nor the preaccelerated behavior, exhibited by the solutions derived by Dirac. It will also be numerically checked that these solutions also satisfy the ALD equations.

The explicit form of the "regularized" pulse defining the force in the rest frame of the particle will be

$$\begin{aligned} f(\tau, t_o, T) &= f_o \int_0^T ds \exp\left(-\frac{(\tau - s)^2}{f_1^2 t_o^2}\right) \\ &= 5\sqrt{\pi} t_o f_o \left(\mathrm{Erf}\left(\frac{(T-\tau)}{f_1 t_o}\right) + \mathrm{Erf}\left(\frac{\tau}{f_1 t_o}\right)\right), \end{aligned} \tag{40}$$

$$f_o = \frac{1}{10000}, \qquad f_1 = 10, \tag{41}$$

where Erf is the error function. The defined force is depicted in Fig. 1 for the chosen values of $t_o = 1$ cm and $T = 1000$ cm. Note that we are expressing the time in normal units. Let us now consider the series defining the effective force $f_e(\tau)$, which is associated with the external force $f(\tau)$:

$$f_e(\tau) = S_\infty(\tau) = \sum_{m=0}^{\infty} \frac{d^m}{d\tau^m} f(\tau) \left(\frac{\kappa}{m}\right)^m. \tag{42}$$

But, given $\frac{\kappa}{m} < 1$ (which is extremely well satisfied by the case of the electromagnetic ALD equation), the series defining the effective force will converge just by only requiring that the time derivatives of arbitrary order are

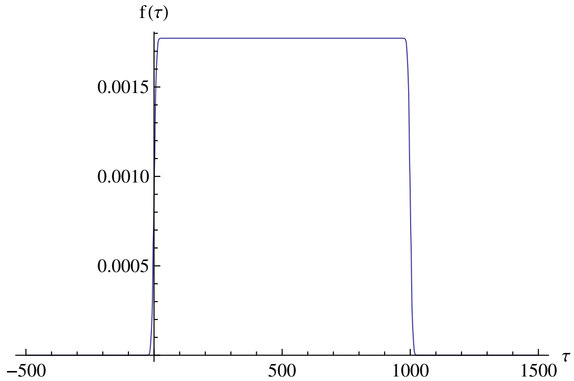

FIG. 1 (color online). The plot illustrates the form of the external force defined by a smooth function of the proper time along the whole real axis. Note that the force can be seen as an infinitely differentiable regularization of an exact square pulse showing the discontinuities at the times $\tau = 0$ s and $\tau = 1000$ s.

uniformly bounded for all time values. The satisfaction of this condition for the specific form to be considered for the force, after fixing $t_o = 1$ cm and $T = 1000$ cm, is evidenced in Fig. 2. It shows the plots of the time derivatives of orders $n = 1, 2, 3, \ldots, 9, 10$. It can be observed that all the depicted time derivatives are bounded, and moreover the bound decreases when the order of the derivatives increases. This behavior is maintained for higher orders, up to values in which the numerical precision becomes degraded in our evaluation.

However, before assuming the form of the force in (41) by fixing the values of t_o and T, it can be argued that the

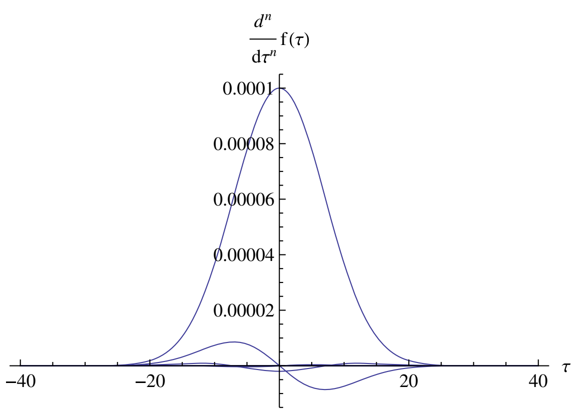

FIG. 2 (color online). The figure shows the plots of the proper time derivatives up to order 10, of the force function associated to the defined analytic regularization of the square pulse. The plot was done in a range of the origin of coordinates in which the derivatives show the higher values, since the pulse is transiting to attain its constant nonvanishing value. As can be observed, all these derivatives are bounded functions of the proper time. Even more, the bounds for the derivatives decrease with their order. This behavior is maintained up to high values of the derivatives of order 50, where the numerical errors started distorting the numerical results.

effective pulselike forces are also well defined for extremely short "rising" times t_o and arbitrarily large time lapses T of the pulses. This property can be argued after performing the changes of variables

$$\tau = t_o x, \qquad s = t_o y, \tag{43}$$

which allows one to express the force f in the form

$$\begin{aligned} f(\tau, t_o, T) &= f^*\left(x, t_o, \frac{T}{t_o}\right) \\ &= f_o t_o \int_0^{\frac{T}{t_o}} dy \exp\left(-\left(\frac{x-y}{f_1}\right)^2\right) \\ &= 5\sqrt{\pi} t_o f_o \left(\mathrm{Erf}\left(\frac{(\frac{T}{t_o} - x)}{f_1}\right) + \mathrm{Erf}\left(\frac{x}{f_1}\right)\right). \end{aligned} \tag{44}$$

But the implemented change of variables allows one to write for the effective force series

$$\begin{aligned} f_e(\tau) &= \sum_{m=0}^{\infty} \frac{d^m}{dt^m} f(t, t_o, T) \left(\frac{\kappa}{m}\right)^m \\ &= \sum_{m=0}^{\infty} \frac{d^m}{dx^m} f^*\left(x, t_o, \frac{T}{t_o}\right) \left(\frac{\kappa}{m} \frac{1}{t_o}\right)^m. \end{aligned} \tag{45}$$

The last line of this relation again indicates that with the rising time being so short as to merely satisfy

$$t_o > \frac{\kappa}{m} \tag{46}$$

the effective force as a function of the time (or, equivalently, the variable x) becomes well defined if the arbitrary derivatives over x of the function $f^*(x, t_o, \frac{T}{t_o})$ are uniformly bounded for all the considered values of the variable x. But, Fig. 3 illustrates that the values of these derivatives as functions of x up to order 10 are bounded at all the values of the x variable. This behavior is valid up to high orders for which the numerical precision of the evaluations starts to become degraded. Therefore, the results indicate that the regularized constant force pulses, constructed in the described analytic way, well define effective forces for very fast pulses. These pulses can rise so rapidly as a few times the ALD time constant of nearly 10^{-24} s in the electromagnetic case.

A. Solutions of the Newton equations for the regularized pulsed force

Now, the equations for the coordinates as functions of the proper time for the pulsed force with the time parameters $t_o = 1$ cm and $T = 1000$ cm will be numerically solved. With the use of their definition (26), these equations can be written in the form

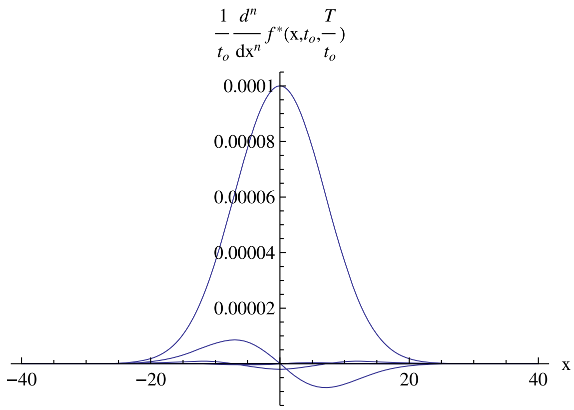

FIG. 3 (color online). The behavior of the derivatives with respect to the variable x of the auxiliary functions f^* (for $\frac{T}{t_0} \gg 1$). The plots show derivatives up to order 10 of the function f^* being associated with the defined analytic regularization of the square pulse. The graphic corresponds to a range of the origin of coordinates in which the derivatives show the higher values. Again, all these derivatives are bounded functions of the coordinate x independently of the values of T and t_0. The restriction $\frac{T}{t_0} \gg 1$ used for the plot was chosen only to separate the rising and lowering time intervals of the pulsed force.

$$a^\mu(\tau) = \frac{1}{m}\mathcal{F}^\mu(\tau), \tag{47}$$

$$\mathcal{F}^\mu(\tau) = \left(\gamma \vec{v}.\vec{f}_e, \vec{f}_e + (\gamma - 1)\frac{\vec{v}.\vec{f}_e}{v^2}\vec{v}\right), \tag{48}$$

$$\vec{f}_e = \sum_{m=0}^{\infty} \frac{d^m}{dt^m} f(\tau, t_o, T)\left(\frac{\kappa}{m}\right)^m. \tag{49}$$

In this collinear motion case we have the definitions

$$u^\mu(\tau) \equiv (\gamma, \gamma\vec{v}), \tag{50}$$

$$a^\mu(\tau) \equiv \left(\frac{d}{d\tau}\gamma, \frac{d}{d\tau}(\gamma\vec{v})\right). \tag{51}$$

The following two Newton equations can be explicitly written in the form

$$\frac{d}{d\tau}u(\tau) = \frac{1}{m}\gamma f_e(\tau), \tag{52}$$

$$\frac{d}{d\tau}u^0(\tau) = \frac{1}{m}\gamma f_e(\tau) v(\tau), \tag{53}$$

$$\gamma(\vec{v}) = \sqrt{1 - v^2}, \qquad v^2 = \vec{v}\,\vec{v}\,. \tag{54}$$

But, employing the definitions of the spatial velocity $\vec{v}$, 4-velocity u^μ, and the effective forces f_e

$$\vec{v}(\tau) = \frac{d}{dt}\vec{x} = \frac{d}{\gamma d\tau}\vec{x}(\tau), \tag{55}$$

$$u^\mu = \frac{d}{d\tau}x^\mu(\tau), \tag{56}$$

$$\vec{f}_e = \sum_{m=0}^{\infty} \frac{d^m}{d\tau^m} f(\tau, t_o, T)\left(\frac{\kappa}{m}\right)^m, \tag{57}$$

the equations for the position and the time describing the trajectories $(t(\tau), x(\tau))$ which solve the Newton equations become

$$\begin{aligned} \frac{d^2}{d\tau^2}x(\tau) &= \frac{1}{m}\frac{d}{d\tau}t(\tau)\sum_{m=0}^{\infty}\frac{d^m}{d\tau^m}f(\tau, t_o, T)\left(\frac{\kappa}{m}\right)^m, \\ \frac{d^2}{d\tau^2}t(\tau) &= \frac{1}{m}\frac{d}{d\tau}x(\tau)\sum_{m=0}^{\infty}\frac{d^m}{d\tau^m}f(\tau, t_o, T)\left(\frac{\kappa}{m}\right)^m. \end{aligned} \tag{58}$$

These equations are now solved for the particular values of the parameters

$$t_o = 1 \text{ cm}, \tag{59}$$

$$T = 1000 \text{ cm}. \tag{60}$$

The constant $\frac{\kappa}{m}$ will be set to a value being in fact very much higher than the one associated with the electron motion, which is nearly $10^{-24}c$ cm. The chosen specific value $\frac{\kappa}{m} = 0.8$ cm will help to avoid extremely small higher-order contributions in powers of $\frac{\kappa}{m}$ in the numerical solution of these equations. It can be noticed that very much larger values of $\frac{\kappa}{m}$ with respect to the one associated with the electron are also of physical interest, for example, when considering the radiation of small moving objects in the air. The solutions of the equations were considered for the following initial conditions:

$$x(-500) = 0, \tag{61}$$

$$\frac{d}{d\tau}x(-500) = 0.2. \tag{62}$$

That is, at a proper time value of −500 cm, the particle is situated at the origin of coordinates with a velocity given the proper time derivative of its coordinates (the spatial component of the 4-velocity) equal to 0.2. It can be noted that this problem is similar to the one considered by Dirac to illustrate the appearance of preacceleration in the solutions of the ALD equations [5]. The main difference between the two situations is that here the pulse is not rigorously squared with discontinuous steplike transitions but defined by an analytic function along the whole time axis. The form of this pulse was shown in Fig. 1. The high value of the ratio $\frac{T}{t_o}$ gives to this function the approximate

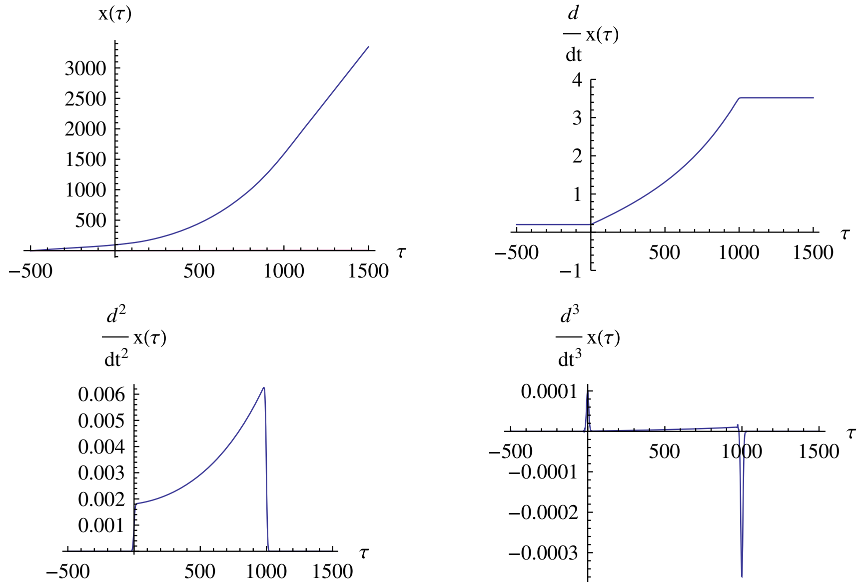

FIG. 4 (color online). The top left figure depicts the proper time evolution of the coordinates of the particle upon which the force is acting. The top right one then shows the velocity of the particle in the same proper time interval. It is clear that the motion tends to be uniform, outside of the interval (0,1000). The bottom left figure shows the behavior of the acceleration which tends to vanish outside the times in which the pulse gets appreciable values. The last graph presents the dependence of the time derivative of the acceleration. This quantity tends to be peaked around the instants at which the pulse drastically changes its value.

pulselike appearance. The set of plots in Fig. 4 shows, in first place, the proper time evolution of the coordinate of the particle, indicating that the motion is nearly free before the time interval of the pulse, for becoming accelerated for the times in which the force is nearly constant. When the time is large and outside the region in which the force is constant, the solution becomes again a uniform motion as illustrated by the vanishing of the acceleration in this zone. Note that the solution of the Newton equations does not exhibit the preacceleration effect, nor the runaway motions after the pulse is passed, as was the case in the Dirac solution of the ALD equations [5]. This is not a strange result given that the effective force tends to vanish outside the pulse interval. However, this example allows one to numerically check that the obtained solution also satisfies the ALD equations. This is clearly illustrated in Fig. 5. It shows the plot of the spatial component of the ALD equations

$$E_{\mathrm{ALD}}(\tau) = ma^1(\tau) - f^1(\tau) - \varkappa\left(\frac{d}{d\tau}a^1(\tau) + a^\nu(\tau)a_\nu(\tau)u^1(\tau)\right), \qquad (63)$$

in common with the plot of the time derivative of the acceleration term of the ALD equations, which is the term of the equations showing the smaller values along the times axis. As can be noticed, the values of the function $E_{\mathrm{ALD}}(\tau)$ cannot be noticed in comparison with the values of the third derivative term. This indicates that the ALD equations are satisfied within the precision of the numerical approximation of the solution, confirming the general derivation presented before. Therefore, an interesting conclusion arises: The obtained solution of the effective Newton equation also satisfies the ALD equation, avoiding the appearance of the preacceleration or runaway effects.

This example of an analytically regularized pulse leads to an idea about how to justify a modification of the ALD equations for the case of nonanalytically defined forces,

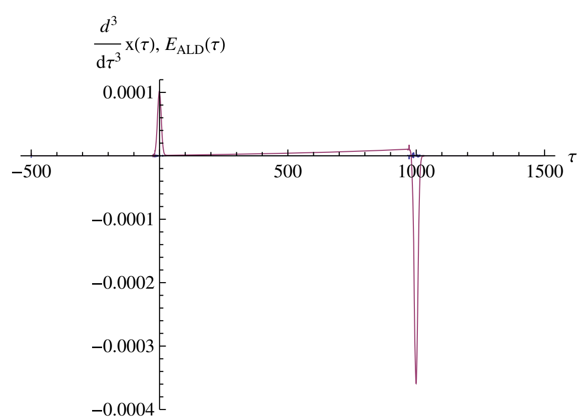

FIG. 5 (color online). The figure shows two plots in common: the value of the derivative of the acceleration, which is the smaller term among the various contributions to the ALD equations, and the function E_{ALD} which vanishing implies the satisfaction of the spatial ALD equation. The fact that the values of the function E_{ALD} cannot be noticed in the plot checks the very approximate satisfaction of the spatial ALD equation.

presented in Ref. [28]. This point will discussed in the next section.

VI. MODIFIED ALD EQUATIONS FOR FORCES WITH SUDDEN CHANGES

We will now assume the existence of a sequence of forces $f_k^\mu(\tau)$, $k = 1, 2, \ldots, \infty$, defined in all the proper time axis. They will also show, for each k, convergent values of the series defining their effective forces $f_{e,k}^\mu(\tau)$, $k = 1, 2, \ldots, \infty$. The first purpose of this section will be to argue that, assuming that the sequence $f_k^\mu(\tau)$ converges to a piecewise continuous limiting force $f^\mu(\tau)$, the sequence of solutions of the corresponding effective Newton equations will tend to satisfy a set of modified ALD equations. It will follow that these equations just generalize the ones which were proposed in Ref. [28]. In that work, those equations were simply advanced under the basis of an idea: When a force acting over a radiating classical electron is instantly removed, the Lienard-Wiechert solution for the electromagnetic field surrounding the electron within a sufficiently close neighborhood of its position should be instantly and exactly defined by a Lorentz boost transformed Coulomb field. But such fields are known to exert a vanishing 4-force over the central core of the electron. This simple observation strongly suggests that the mechanical equations driving the electron motion should be able to reproduce this effect, that is, to instantaneously lead the acceleration of the particle to vanish when a force is removed in an extremely rapid way. The equations of motion found here implement this property.

Let us first consider that the proper time axis is subdivided in a denumerable set of contiguous intervals by the specific sequence of increasing values of times

$$(\tau_o, \tau_o^{(k)*}, \tau_1, \tau_1^{(k)*}, \tau_2, \tau_2^{(k)*}, \ldots, \tau_n, \tau_n^{(k)*}, \ldots). \quad (64)$$

The general intervals $(\tau_n, \tau_n^{(k)*})$ for arbitrary n values will be called the "transition" intervals, in which the arbitrary time derivatives for the forces of the sequence of $f_k^\mu(\tau)$ for all k values will be assumed as well defined, by also determining bounded values for the effective forces $f_{e,k}^\mu(\tau)$ for all values of k. In the limit $k \to \infty$ we will consider that all the times τ_n not showing the superindex k will remain constant, and the other instants will have limits $\tau_n^{(k)*} \to \tau_n$. In the limit $k \to \infty$, the sequence of forces will be assumed to approach a piecewise discontinuous limiting force at the points τ_n for all n values.

On the opposite way, within all the intervals $(\tau_n^{(k)*}, \tau_{n+1})$ the sequence of force functions is assumed to tend to an infinitely smooth function of the proper time, by also leading to well-defined effective forces. In particular, if the force functions are given by polynomial functions of the proper time with maximal order $N_{\max}$, the series defining the effective forces, having a finite number of terms, will always correctly define the effective forces.

Because of the assumption about the existence of the sequence of effective forces along the whole time axis for all finite k values, the effective Newton equations will be properly defined for all k and can be solved by simply integrating over the proper time. We assumed that the sequence of the effective forces $f_{e,k}^\mu(\tau)$ also tends to be piecewise continuous and bounded at all the time axis. Then, this condition will imply that the time integrals of the forces within all the transition intervals $(\tau_n, \tau_n^{(k)*})$ should vanish when these intervals shrink in the limit $k \to \infty$. Therefore, since integrals of this sort will define the discontinuity in the 4-velocities in the limit $k \to \infty$, it follows that the 4-velocities should be continuous in that limit. This defines boundary conditions at all the t_n points, to be considered in addition to the continuity of the coordinates at all these points.

A. The modified ALD equations

Consider now the sequence of the solutions of the effective Newton equations $x^{(k)\mu}(\tau)$ for all values of k. From the previous discussion, it is clear that, for each k value, the 4-velocities as functions of the time will not tend to develop discontinuities across the transition intervals when they reduce their sizes in the limit $k \to \infty$. This follows because the effective forces are assumed to exist and to be also bounded. Thus, the impulses of these forces during the shrinking transition intervals should tend to zero sizes. Therefore, we will have that the sequence of solutions tends to be piecewise smooth trajectory $x^{(k)\mu}(\tau)$, showing also a continuous velocity at the transition points in the $k \to \infty$ limit.

Strictly inside all the intervals $(\tau_n^{(k)*}, \tau_{n+1})$, the solutions $x^{(k)\mu}(\tau)$ can be constructed as satisfying the $k \to \infty$ limit of the Newton equation, since the limiting $k \to \infty$ values of the forces and the effective forces within these intervals are assumed to be smooth and also bounded. Then, in these zones they will also satisfy the ALD equation. Therefore, within each of these intervals the limiting $k \to \infty$ trajectory $x^\mu(\tau)$ should satisfy the effective Newton equations

$$\begin{aligned} m a^\mu(\tau) &= \mathcal{F}^{(n)\mu}(\tau) \\ &= \Lambda_\nu^\mu(\tau) f_e^{(n)\nu}(\tau), \end{aligned} \quad (65)$$

in which the effective forces $f_e^{(n)\mu}$ are defined by

$$f_e^{(n)\mu} \equiv \begin{pmatrix} 0 \\ \vec{f}_e^{\,(n)}(\tau) \end{pmatrix}, \quad (66)$$

$$\vec{f}_e^{\,(n)}(\tau) = \sum_{m=0}^{\infty} \frac{d^m}{dt^m} \vec{f}^{(n)}(\tau) \left(\frac{\kappa}{m}\right)^m, \quad (67)$$

with $\vec{f}^{(n)}(\tau)$ being the limit $k \to \infty$ of the sequence of forces $\vec{f}_k^{(n)}(\tau)$, for the time taken within the interval (τ_n, τ_{n+1}). These equations, in conjunction with the boundary condition of equal 4-velocities at both sides of each of the boundary points of all the intervals, define the new set of equations for forces showing finite discontinuities. This conclusion follows because the specification of the coordinates and velocities at any spatial point and time (outside the boundaries) fully determines the solutions of the Newton equations within all the intervals. Then, the continuity of the position and velocities at both sides of the times of discontinuities assures the uniqueness of the solution for all times.

The described equations show steplike discontinuities of the force values after the limit $k \to \infty$, at the transition points τ_n. These discontinuities define corresponding steplike changes in the accelerations at the left τ_n^- and right τ_n^+ of the transition instants, with values

$$ma^\mu(t_n^-) \equiv \Lambda_\nu^\mu(\tau) f_e^{(n-1)\nu}(\tau)|_{\tau=\tau_n^-}, \tag{68}$$

$$ma^\mu(t_n^+) \equiv \Lambda_\nu^\mu(\tau) f_e^{(n)\nu}(\tau)|_{\tau=\tau_n^+}, \tag{69}$$

where the coordinates and velocities are continuous at both sides of the transition point τ_n. This property defines a clear deviation from the case of the solutions showing preacceleration or runaway behavior, which are assumed to have continuous accelerations [5].

Now, let us also argue that the effective equations (65), complemented with the continuity of the coordinates and velocities, can be also expressed in the alternative way

$$\begin{aligned} ma^\mu(\tau) - f^\mu(\tau) = \varkappa\left(\frac{d}{d\tau}a^\mu(\tau) + a^\nu(\tau)a_\nu(\tau)u^\mu(\tau)\right) \\ + \frac{\varkappa}{m}\sum_{n=0}^{\infty}(\mathcal{F}^\mu(t_n^-)\delta^{(D,-)}(\tau-\tau_n) \\ - \mathcal{F}^\mu(\tau_n^+)\delta^{(D,+)}(\tau-\tau_n)), \end{aligned} \tag{70}$$

where the Dirac Delta function, say, $\delta^{(D,-)}(\tau)$, is defined as a linear functional determined by a sequence of functions of time $\delta_k^{(D,-)}(\tau), k = 1, 2, 3, \ldots,$ all having time integrals equal to the unit for any k value. Their support for each k is defined by intervals $(-\epsilon_k, 0)$, with positive ϵ_k, which tend to vanish in the limit $k \to \infty$. The linear functional acting over a possibly piecewise continuous function $g(\tau)$ is then defined as the integral

$$\lim_{\tau\to 0^-}(g(\tau)) = \lim_{k\to\infty}\int d\tau\, \delta_k^{(D,-)}(\tau)g(\tau). \tag{71}$$

Therefore, if the function g has a discontinuity of the step function type, the integral of the function $\delta^{(D,-)}(\tau-\tau_n)$ will give the value of the limit of g at the left of the point τ_n. The right Dirac Delta $\delta^{(D,+)}(\tau)$ is defined in a similar way but with support of the form $(0, \epsilon_k)$ with all ϵ_k again positive.

Now let us consider the acceleration $a^\mu(\tau)$ defined by the effective force in a sufficiently small open neighborhood B_n of each instant t_n. The expression for it can be written in the form

$$a^\mu(\tau) = \frac{\mathcal{F}^{(n-1)\mu}(\tau)}{m}\Theta^{(-)}(\tau_{n.} - \tau) + \frac{\mathcal{F}^{(n)\mu}(\tau)}{m}\Theta^{(+)}(\tau - \tau_{n.}), \tag{72}$$

where the special Heaviside-like functions are defined as $\Theta^{(\pm)}(\tau) = \int_{-\infty}^{\tau} ds \delta^{(D,\pm)}(s)$. Therefore, let us search for the equation satisfied by this expression for the acceleration. The derivative of the acceleration within the neighborhood B_n takes the form

$$\begin{aligned} \frac{d}{d\tau}a^\mu(\tau) &= \left(\frac{d}{d\tau}\left(\frac{\mathcal{F}^{(n-1)\mu}(\tau)}{m}\right)\Theta^{(-)}(\tau_{n.}-\tau) + \frac{d}{d\tau}\left(\frac{\mathcal{F}^{(n)\mu}(\tau)}{m}\right)\Theta^{(+)}(\tau-\tau_{n.}) \right. \\ &\quad \left. + \frac{\mathcal{F}^{(n-1)\mu}(\tau)}{m}\frac{d}{d\tau}\Theta^{(-)}(\tau_{n.}-\tau) + \frac{\mathcal{F}^{(n)\mu}(\tau)}{m}\frac{d}{d\tau}\Theta^{(+)}(\tau-\tau_{n.})\right) \\ &= \left(\frac{d}{d\tau}\left(\frac{\mathcal{F}^{(n-1)\mu}(\tau)}{m}\right)\Theta^{(-)}(\tau_{n.}-\tau) + \frac{d}{d\tau}\frac{\mathcal{F}^{(n)\mu}(\tau)}{m}\Theta^{(+)}(\tau-\tau_{n.}) \right. \\ &\quad \left. - \frac{\mathcal{F}^{(n-1)\mu}(\tau)}{m}\delta^{(D,-)}(\tau-\tau_n) + \frac{\mathcal{F}^{(n)\mu}(\tau)}{m}\delta^{(D,+)}(\tau-\tau_n)\right). \end{aligned} \tag{73}$$

Henceforth, after substituting this expression in (70), and considering that the ALD equations are satisfied at the interior points of all the intervals (τ_n, τ_{n+1}), it follows that

$$\begin{aligned}&\Big(\Big(ma^{\mu}(\tau)-f^{\mu(n-1)}(\tau)-\varkappa\Big(\frac{d}{d\tau}a^{\mu}(\tau)+a^{\nu}(\tau)a_{\nu}(\tau)u^{\mu}(\tau)\Big)\Big)\Theta^{(-)}(\tau_{n.}-\tau)\\&+\Big(ma^{\mu}(\tau)-f^{\mu(n)}(\tau)-\varkappa\Big(\frac{d}{d\tau}a^{\mu}(\tau)+a^{\nu}(\tau)a_{\nu}(\tau)u^{\mu}(\tau)\Big)\Big)\Theta^{(+)}(\tau-\tau_{n.})\\&-\frac{\varkappa\mathcal{F}^{(n-1)\mu}(\tau)}{m}\delta^{(D,-)}(\tau-\tau_n)+\frac{\varkappa\mathcal{F}^{(n)\mu}(\tau)}{m}\delta^{(D,+)}(\tau-\tau_n)\\&+\frac{\varkappa}{m}(\mathcal{F}^{\mu}(\tau_n^-)\delta^{(D,-)}(\tau-\tau_n)-\mathcal{F}^{\mu}(\tau_n^+)\delta^{(D,+)}(\tau-\tau_n)\Big)=0. \end{aligned}\tag{74}$$

Thus, all the terms in the above equations add to zero around each transition time t_n. The first two lines vanish because the trajectories solving the Newton equations within each of the intervals (τ_n, τ_{n+1}) also satisfy the ALD equations within each neighborhood B_n except at the point t_n. The last two terms also cancel between themselves, since the Delta functions allow one to evaluate the argument of the functions multiplying them at their support point. Therefore, after considering that the modifying terms of the ALD equations vanish outside all the vicinities B_n, it follows that the limiting solution along the whole axis satisfies the modified ALD equations

$$\begin{aligned}ma^{\mu}(\tau)-f^{\mu}(\tau)=&\varkappa\Big(\frac{d}{d\tau}a^{\mu}(\tau)+a^{\nu}(\tau)a_{\nu}(\tau)u^{\mu}(\tau)\Big)\\&+\varkappa\sum_{n=0}^{\infty}(a^{\mu}(\tau_n^-)\delta^{(D,-)}(\tau-\tau_n)\\&-a^{\mu}(\tau_n^+)\delta^{(D,+)}(\tau-\tau_n)).\end{aligned}\tag{75}$$

B. The solution for the constant force pulse

Consider now the solution of the modified ALD equations for the case of an exact square pulse of the similar form as the before-considered analytic one and coinciding in form with the one employed by Dirac in Ref. [5]. The same initial conditions for the position and velocities are fixed:

$$x(-500)=0,\tag{76}$$

$$\frac{d}{d\tau}x(-500)=0.2.\tag{77}$$

The Newton equations at the interior points of any of the three intervals in which the time axis is decomposed by the two instants $\tau_o=0$ and $\tau_1=T$ have basically the same form as Eq. (58):

$$\begin{aligned}\frac{ds^2}{d\tau^2}x(\tau)&=\frac{1}{m}\frac{d}{d\tau}t(\tau)f_P(\tau,T),\\\frac{d^2}{d\tau^2}t(\tau)&=\frac{1}{m}\frac{d}{d\tau}x(\tau)f_P(\tau,T),\end{aligned}\tag{78}$$

where the force now defining the exact pulse of constant amplitude is given by the formula

$$f_P(\tau,T)=f^p\Theta(\tau)\Theta(T-\tau),\tag{79}$$

in which $\Theta(\tau)$ is the Heaviside function. The width of the pulse was chosen as given by the same parameter T defining the width of the analytically regularized pulse in past sections. The constant amplitude of the pulse f^p will be approximately coinciding with the height of the regularized pulse, by selecting its magnitude as

$$f^p=f\Big(\frac{T}{2},t_o,T\Big),\tag{80}$$

that is, by the height of the pulse at a time equal to half of its approximate width T. The parameters for the numerical evaluation will also coincide with values selected before $t_o=1$ and $T=1000$. The solution of Eq. (78) for the time dependence of the coordinate, velocity, acceleration, and its time derivative are jointly plotted in Fig. 6, for both pulses: the one of exactly constant amplitude and the analytically regularized one. The plots evidence that these quantities are closely similar for both forces. A small difference starts to be noticed in the curve for the time derivative of the acceleration. It shows that only in the close neighborhood of the transition points, in which the forces suddenly change, is the time derivative of the acceleration presenting a difference. For making the comparison clearer, the plot associated with the derivative of the acceleration for the pulse of exact constant amplitude is slightly shifted along the vertical axis away from the zero values. This permits one to note that the magnitude of the derivative of the acceleration at the interior points of the interval $(0,T)$ closely coincide for both solutions. However, near the transition points, the smooth pulse solution presents a peaked behavior. This is a numerical confirmation for the validity of the derived modified ALD equations, since these peaked dependences are necessary for reproducing the discontinuities in the acceleration associated with the modified equations. That is, when the regularized pulse is gradually made to be even more similar to the constant pulse, it should be expected that the "spikes" appearing at the transition points for the

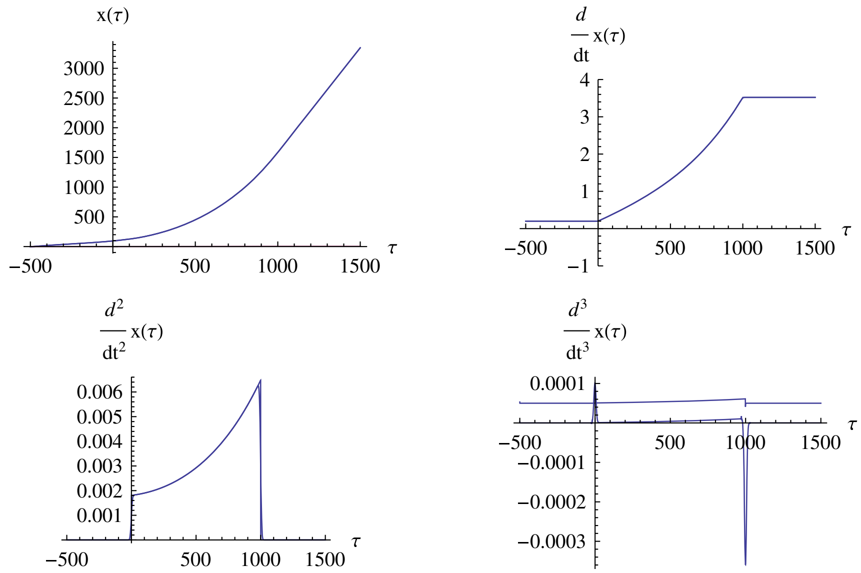

FIG. 6 (color online). The array of figures shows in each graphic two plots: one associated with the analytic pulse of force and the other related to the exact square pulse solving the modified ALD equations. The top left figure shows the coordinates of both solutions. Note the close coincidence of the coordinates. The top right curve for the velocities also illustrates the similarity between the solutions for this quantity. The bottom left plot presents the results for the accelerations for the two solutions. The nonconstant behavior of the acceleration shows that the motion is relativistic. Finally, the bottom right plot compares the time derivatives of the acceleration for both solutions. For better evidencing the difference, the quantity associated with the exactly square pulse is shifted in a positive constant along the vertical axis. It can be noted that in the internal points of the interval (0, 1000) the two evaluations tend to coincide. However, in the points close to the boundaries of this interval, the derivative of the acceleration associated with the analytic pulse develops peaked values. These values give account of the developing of the Dirac functions entering the modified ALD equations. The analytic solution, obeying the exact ALD equations along the whole axis, should generate these Dirac functions, in order to imply the modified equations in the limit.

derivatives of the acceleration are associated with a regularization of the Dirac delta functions entering in the distributional form of the modified ALD equations. The enhancing of the peaks when the regularized pulse tends to be closer to the exact constant ones is illustrated in Fig. 7. It shows the derivative of the acceleration for two solutions with almost all the parameters identical to the ones considered before but differing only in the value determining the pulse rise time t_o. The values selected for this parameter were $t_o = 1$ cm and $t_o = 2$ cm. That is, the pulse with $t_o = 1$ will have a rising time 2 times smaller than the one with $t_o = 2$ m. In the figure, there is a curve which shows the smaller peaks at the right and left of the figure, and the other one presents the higher peaks at both sides. The curve with the larger values is associated with the pulse $t_o = 1$, and the curve with smaller values is related to the slower rising pulse. Thus, it is clear that, when the analytic pulse tends to approach the exactly constant one, the solution tends to show time derivatives with the appearance of regularizations of the Dirac Delta functions. This should be the case if the modified ALD equations are implied by the limit of the exact ALD equations. The absence of the peaks in the solution linked with the square pulse is associated with the fact that the modified equations were solved in their nondistributional form. That is, they were solved at the interior

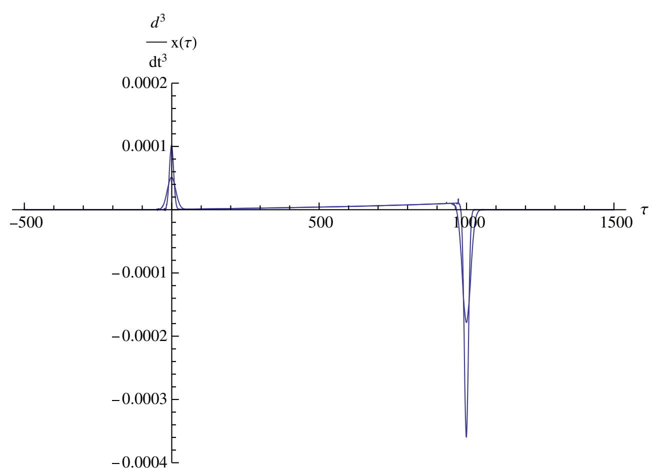

FIG. 7 (color online). The superposed plots correspond to two analytic pulsed forces of nearly equal height and width, but one having a rising time double that of the other. One curve is related with the larger in absolute value left and right peaks, and the other is associated with lower ones at both sides. The curve showing the higher peaks corresponds to the pulse with a shorter rising time, which illustrates how the Dirac Delta functions appearing in the modified ALD equations are gradually generated as the pulse approaches the exactly squared form.

points, and the boundary conditions for the coordinates and velocities were imposed across the two transition points. These boundary conditions were argued above to be equivalent to the presence of Dirac Delta functions with supports in the transition points in the second form of the modified equations.

VII. ON THE DIRAC POINT-PARTICLE MODEL

In this ending section, let us make a proposal of a Lagrangian looking to be able to justify the Dirac point particle satisfying the ALD equations, that is, a pointlike particle which, in spite of showing an infinite electrostatic energy outside of its singular point, is capable to exhibit a finite total energy (mass). As discussed in the introduction, the possibility of constructing a reasonable physical model for the point particle will support the exact validity of the ALD equations without assuming an approximate reduction of the order. If such is the case, the validity of the ALD equations for a particle having a finite mass-to-charge ratio in the limit of zero mass and charge argued in Ref. [26] will be also extended to particles having finite charge and mass.

It is clear that the searched model (before passing to the pointlike limit) should not be able to satisfy the weak energy condition. This needs to be the case, because this rule requires the energy density to be positive in a local way in any Lorentz frame. That means $T_{\mu\nu}u^\mu u^\nu \geq 0$ for all possible 4-velocities u^μ of the considered frame. Then, the integral of the energy over the extension of the particle (as defined before taking the pointlike limit) should be always positive and, thus, unable to compensate for the positive energy electrostatic contribution (tending to infinity in the pointlike limit) in defining a finite rest mass.

However, the required properties of the pointlike Dirac particle, a finite total mass and an infinite electrostatic energy outside it (in the pointlike limit), perhaps might be properly modeled by a system showing a bounded from below energy density but being able to exhibit negative values for the energy density in some points of the space. In this short section, we present an example of such a model and discuss some of its properties and solutions.

The model to be employed is based in two scalar massless fields ψ and ϕ. The first one interacts with an electromagnetic field, and the second is not charged. Both fields are assumed to interact. They are described by the Lagrangian

$$\begin{aligned}\mathcal{L} = &-(\partial_\mu + ieA_\mu)\psi^*(\partial^\mu - ieA^\mu)\psi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ &+ \frac{g}{2}\psi^*\psi\phi^2 - \lambda_1(\psi^*\psi)^2 - \lambda_2\phi^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \qquad (81)\end{aligned}$$

in which the metric is considered as a general curvilinear one but which can be continuously deformed to the Minkowski expression

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (82)$$

and natural units are also employed, with $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$. The action of the problem has the expression

$$I = \int dx\sqrt{-g}\mathcal{L} \qquad (83)$$

in terms of the metric tensor and the defined fields and g is the determinant of the metric tensor $g_{\mu\nu}$. The following general relations for $g_{\mu\nu}$:

$$\begin{aligned} g &= \mathrm{Det}\{g_{\mu\nu}\}, \\ g_{\mu\alpha}g^{\alpha\nu} &= \delta_\mu^\nu, \\ dg &= gg^{\mu\nu}dg_{\mu\nu} = -gg_{\mu\nu}dg^{\mu\nu}, \qquad (84)\end{aligned}$$

help to evaluate the derivative of the action over $g^{\mu\nu}$ in the form

$$\begin{aligned}\frac{\delta I}{\delta g^{\mu\nu}(y)} &= \int dx\sqrt{-g(x)}\left(\frac{\delta\mathcal{L}(x)}{\delta g^{\mu\nu}(y)} - \frac{\mathcal{L}(x)}{2}g_{\mu\nu}\delta^4(x-y)\right) \\ &= -\frac{1}{2}\sqrt{-g(y)}T_{\mu\nu}(y), \qquad (85)\end{aligned}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. The Lagrangian $\mathcal{L}(x)$ can be expressed as a function of $g^{\mu\nu}$ in a symmetrized form in the indices μ ν, as follows:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}(\partial_\mu + ieA_\mu)\psi^*(g^{\mu\nu} + g^{\nu\mu})(\partial_\nu - ieA_\nu)\psi \\ &-\frac{1}{2}\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi + \frac{g}{2}\psi^*\psi\phi^2 - \lambda_1(\psi^*\psi)^2 - \lambda_2\phi^4 \\ &-\frac{1}{4}F_{\mu\nu}\frac{(g^{\mu\alpha} + g^{\alpha\mu})}{2}\frac{(g^{\nu\beta} + g^{\beta\nu})}{2}F_{\alpha\beta} \\ &+\frac{1}{2\alpha}\left(\partial_\beta\frac{(g^{\beta\alpha} + g^{\alpha\beta})}{2}A_\alpha\right)^2. \qquad (86)\end{aligned}$$

Therefore, taking the variation of this action over the metric will give a symmetric result in the Lorentz indices of the metric increments. Thus, the vanishing of the changes in the action will imply the corresponding vanishing of the coefficients of the differentials of the metric after restricted to be symmetric changes.

For the equations of motions of the fields it follows that

$$\frac{\delta \mathcal{L}}{\delta \psi^*} = 0 = (\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\psi + g\frac{\phi^2}{2}\psi - 2\lambda_1(\psi^*\psi)\psi,$$
$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 = \partial_\mu\partial^\mu\phi + g\frac{\psi^*\psi}{2}\phi - 4\lambda_2\phi^3,$$
$$\frac{\delta \mathcal{L}}{\delta A^\mu} = 0 = \partial^\nu F_{\nu\mu} - \frac{1}{\alpha}\partial_\mu\partial_\nu A^\nu - ie\psi^*(\vec{\partial}_\mu - \overleftarrow{\partial}_\mu)\psi - 2e^2A_\mu\psi^*\psi. \quad (87)$$

Let us consider in what follows the finding of static solutions which in addition do not show electric currents. For this purpose we will assume the following properties for the fields:

$$\psi = \psi^*,$$
$$A^\mu = (V(x), 0, 0, 0),$$
$$\partial_0 V(x) = \partial_0\psi = \partial_0\psi^* = \partial_0\phi = 0. \quad (88)$$

In this particular case, the equations of motions and the action reduce to

$$0 = \left(\nabla^2 + e^2V^2 + g\frac{\phi^2}{2} - 2\lambda_1(\psi^*\psi)\right)\psi,$$
$$0 = \nabla^2\phi + g\psi^*\psi\phi - 4\lambda_2\phi^3,$$
$$0 = \nabla^2 V - 2e^2V\psi^*\psi,$$
$$\mathcal{L} = -\vec{\nabla}\psi.\vec{\nabla}\psi - e^2V^2\psi^2 - \frac{1}{2}\vec{\nabla}\phi.\vec{\nabla}\phi + \frac{g}{2}\psi^2\phi^2 - \lambda_1\psi^4 - \lambda_2\phi^4 + \frac{1}{2}\vec{\nabla}V.\vec{\nabla}V, \quad (89)$$

where the 3-vector form of the spatial derivatives has been used $\vec{\nabla} = (\partial_1, \partial_2, \partial_2)$ and the point designation for the scalar products of 3-vectors was employed. It was also assumed that the field configuration shows rotation invariance. In this case, the appearing Laplacian operator reduces to

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r,$$
$$\partial_r = \frac{\partial}{\partial r}, \quad (90)$$

where (r, θ, φ) are the usual spherical coordinates. After evaluating the derivatives over the metric of the Lagrangian (86) and calculating the integrals in a direct way, thanks to the appearance of Dirac delta functions, the derivative of the action and energy-momentum tensor can be written in the form

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi + e^2V^2\psi^2\delta_\mu{}^0\delta_\nu{}^0\frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}(\partial_\mu V.\partial_\nu V - \delta_\mu{}^0\delta_\nu{}^0\vec{\nabla}\psi.\vec{\nabla}\psi) + g_{\mu\nu}\mathcal{L}, \quad (91)$$

where it should be remembered that all the time derivatives ∂_0 vanish. Also note that the metric tensor has been evaluated to its simple form in Minkowski space (82). Consider now a 4-velocity $u^\mu = \frac{1}{\sqrt{1-v^2}}(1, \vec{v})$ which can be associated with the arbitrary Lorentz reference frame which moves with respect to the observer's frame. Then, the weak energy condition for the system under consideration requires that the energy density of the system in any of such Lorentz frames should be positive. That is, the energy-momentum tensor should satisfy $u^\mu T_{\mu\nu}u^\nu \geq 0$. In the considered example, this relation can be written as follows:

$$u^\mu T_{\mu\nu}u^\nu = u^\mu\partial_\mu\psi u^\nu\partial_\nu\psi + e^2V^2\psi^2u^0u^0\frac{1}{2}u^i\partial_i\phi u^j\partial_j\phi + \frac{1}{2(1-v^2)}(\delta^{ij} - v^iv^j)\partial_iV\partial_jV + \frac{1}{2}\left(\frac{1}{(1-v^2)} - 1\right)\vec{\nabla}V.\vec{\nabla}V + \vec{\nabla}\psi.\vec{\nabla}\psi + e^2V^2\psi^2 + \frac{1}{2}\vec{\nabla}\phi.\vec{\nabla}\phi + \lambda_1\psi^4 + \lambda_2\phi^4 - \frac{g}{2}\psi^2\phi^2 \geq 0. \quad (92)$$

It can be noted that the derived expression is fully satisfied when the interaction between the two scalar fields vanishes for $g = 0$. However, when these fields interact, the weak energy condition is also satisfied only when the coupling constant obeys

$$\frac{g}{4\sqrt{\lambda_1\lambda_2}} \leq 1. \quad (93)$$

But, when the coupling is sufficiently large to satisfy $\frac{g}{4\sqrt{\lambda_1\lambda_2}} \geq 1$, the energy density can be negative in the regions where the two scalar fields are not vanishing. Therefore, for the region of couplings $g \geq 4\sqrt{\lambda_1\lambda_2}$, this theory might allow one to construct a pointlike model for a particle. For this to happen, it is needed to find a family of localized solutions of the static Lagrange equations (89), whose region of nonvanishing scalar fields tends to shrink to vanishing sizes as some suitable regularization parameter is varied. It is also possible here to illustrate the existence of localized solutions of Eq. (89). For example, let us fix by the moment the parameters as $\lambda_1 = \lambda_2 = 1$ and define the

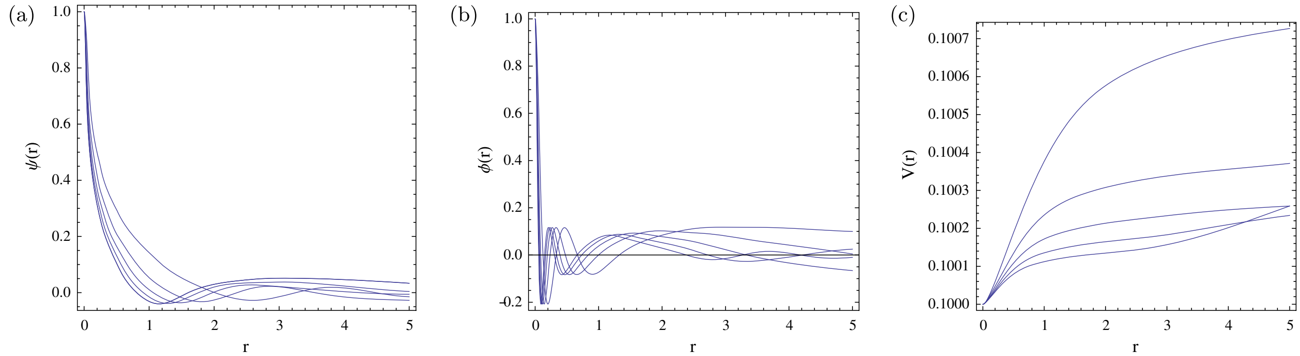

FIG. 8 (color online). All the plots show the values of the scalar and electromagnetic fields for static rotationally symmetric solution of Eq. (89). The parameter values common to all the solutions are $\lambda_1 = \lambda_2 = 1$, and the boundary conditions are defined in (94). (a) shows five curves of the scalar field ψ for increasing values of the interaction parameter $g = 10^3, 2\times 10^3, 3\times 10^3, 4\times 10^3, 5\times 10^3$. For larger values of g the rising parts of the curves tends to be closer to the vertical axis. (b) shows the values of the scalar field ϕ, which exhibit a similar behavior than the ones for ψ. Finally, (c) illustrates the variations the electrostatic potential for increasing g values. The potential plots shown, in general, decrease for equal radial position with the increase of g.

boundary condition at the origin of coordinates for the fields in the form

$$\begin{aligned} &\psi(0) = 1, \qquad \partial_r\psi(0) = 0, \\ &\phi(0) = 1, \qquad \partial_r\phi(0) = 0, \\ &V(0) = 0.1, \qquad \partial_r V(0) = 0. \end{aligned} \tag{94}$$

The numerical solutions of Eq. (89) for various values of the interaction coupling g are plotted in Fig. 8. The curves show that localized solutions exist for a wide range of values of the interaction coupling. The scalar field intensities tend to be higher at points close to the origin as the coupling g increases. Figure 9 also shows the energy density as a function of the radial distance for the considered solutions. It can be seen how this quantity takes negative values near the origin. This behavior is compatible with the possibility of constructing a pointlike

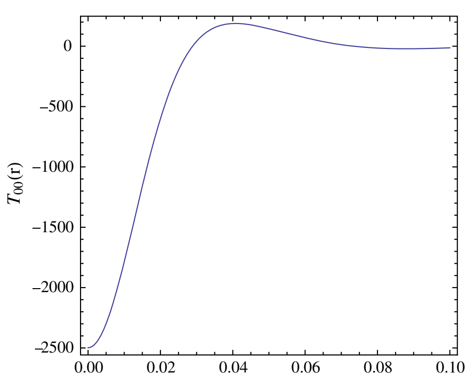

FIG. 9 (color online). The figure shows the radial dependence of the energy density of a solution with the same common parameters as the ones considered in Fig. 8 and a scalar field coupling of value $g = 5\times 10^3$. Note that the energy density becomes negative in regions close to the origin.

model within the considered Lagrangian system. At this point, it can be noted that the model is invariant under the scale transformation

$$\begin{aligned} \psi_\lambda(x_\lambda) &= \lambda\psi(\lambda x_\lambda), \\ \phi_\lambda(x_\lambda) &= \lambda\phi(\lambda x_\lambda), \\ A^\mu_\lambda(x_\lambda) &= \lambda A^\mu(\lambda x_\lambda), \\ x^\mu_\lambda &= \frac{1}{\lambda}x^\mu, \end{aligned} \tag{95}$$

for arbitrary values of the parameter λ. Therefore, applying the transformation to any solution having its scalar fields concentrated in a given region of volume V implies that the new fields take appreciable values within a smaller zone of volume $\frac{V}{\lambda^3}$ when λ is large. Thus, solutions having their scalar fields being concentrated at arbitrary small radial distances exist in the Lagrangian scheme. This is a basic requirement upon a theory being able to realize the point-particle model which satisfies the conditions for validating the ALD equations according to Ref. [5]. However, if the parameters of the model are fixed, the highly concentrated solutions show infinite values of the energy when $\lambda \to \infty$, if the energy of the original scale nontransformed solution is finite. This property can be directly seen from the formula of the total energy in the rest system of the solution

$$\begin{aligned} E = \int d\vec{x}T_{00} = \int d\vec{x}\bigg(&\vec{\nabla}\psi.\vec{\nabla}\psi + e^2V^2\psi^2 + \frac{1}{2}\vec{\nabla}\phi.\vec{\nabla}\phi \\ &+ e^2V^2\psi^2 + \frac{1}{2}\partial_i V\partial_i V + \lambda_1\psi^4 + \lambda_2\phi^4 - \frac{g}{2}\psi^2\phi^2\bigg). \end{aligned} \tag{96}$$

After performing the scale transformation in the above formula for the total energy, it follows that

$$\begin{aligned} E &= \int d\vec{x}T_{00} \\ &= \frac{1}{\lambda}\int d\vec{x}_\lambda\bigg(\vec{\nabla}_\lambda\psi_\lambda.\vec{\nabla}_\lambda\psi_\lambda + e^2V_\lambda^2\psi_\lambda^2 + \frac{1}{2}\vec{\nabla}\phi_\lambda.\vec{\nabla}\phi_\lambda \\ &\quad + e^2V_\lambda^2\psi_\lambda^2 + \frac{1}{2}\vec{\nabla}_\lambda V_\lambda.\vec{\nabla}_\lambda V_\lambda + \lambda_1\psi_\lambda^4 + \lambda_2\phi_\lambda^4 - \frac{g}{2}\psi_\lambda^2\phi_\lambda^2\bigg) \\ &= \frac{1}{\lambda}E_\lambda. \end{aligned} \tag{97}$$

Therefore, since E is the energy of the initial state, it follows that, assuming that E is finite, the energy of the transformed field configuration E_λ should tend to infinity when λ increases without limit. However, the energy density of the theory can take negative values. Then, if the parameters can be properly expressed also as functions of λ, it is not ruled out that the energy E (now being a function of λ) could be chosen as tending to zero as $\frac{1}{\lambda}$, in order to define a finite energy solution having the scalar fields concentrated at the origin. However, it should be also required to satisfy the condition that the electrostatic energy residing outside the singular point should tend to infinity in the desired limit, as describing a point charge. In connection with this point, it is possible to call attention to an interesting property of the family of solutions which are generated by making arbitrary scale transformations. Assume that the already-scaled solution shows a net charge q when it is observed from a large distance from the origin of coordinates. In this situation, the spatial dependence of the electric potential at those large distances from the central field distributions will have the Coulomb component

$$V_\lambda(x_\lambda) = \frac{q^2}{|\vec{x}_\lambda|}, \tag{98}$$

apart from a possible additive constant. Now, after considering the scale transformations we may write

$$\vec{x} = \lambda\vec{x}_\lambda, \tag{99}$$

$$V_\lambda(x_\lambda) = \lambda V(\lambda x_\lambda). \tag{100}$$

But, these relations allow one to write the Coulomb part of the potential $V(x)$ seen in the nonscaled coordinates x as follows:

$$V_\lambda(x_\lambda) = \frac{\lambda q^2}{|\vec{x}|} = \lambda V(\lambda x_\lambda), \tag{101}$$

$$V(x) = \frac{q^2}{|\vec{x}|}. \tag{102}$$

Thus, the Coulomb component of the electrostatic field of a given solution is invariant under the scale changes. This property can be of help in finding a family of parameter sets justifying the existence of a model with a pointlike limit exhibiting the mentioned Dirac conditions.

However, in finding the required set of field configurations showing an infinite electrostatic energy and a finite total mass in the limit, a careful discussion of the parameter dependence of the solutions of the model's equations becomes necessary. This task is out of the general objective of the present work, and it will be considered elsewhere.

VIII. SUMMARY

The work presents second-order Newton-like equations of motion for a radiating particle. It was argued that the trajectories obeying them exactly also satisfy the ALD equations. Forces which depend only on the proper time were considered by now. A condition for these equations to be defined in a given time interval is derived: It is sufficient that the external force becomes infinitely smooth and also that a particular series defined by the infinite sequence of its time derivatives converges to a bounded function. This series defines in a local way the effective force determining the Newton effective equation, in a way that the existing solutions of such effective equations do not show runaway or preacceleration effects. The Newton equations were numerically solved for a pulselike force given by an analytic function on the whole proper time axis. The satisfaction of the ALD equations by the obtained solution is numerically checked. In addition, a set of modified ALD equations was derived for almost infinitely smooth forces, which, however, show steplike discontinuities. The form of these equations supported the statement argued in a former work, that the Lienard-Wiechert field surrounding a radiating particle should determine that the effective force on the particle instantaneously vanishes, when the external force is suddenly removed. The modified ALD equations argued in the former study are here derived in a more general form, in which a suddenly applied external force is also instantly creating an effective nonvanishing acceleration. The work is expected to be extended in some directions. For example, one issue which seems of interest to define is whether or not the class of external forces which also show well-defined effective forces constitutes a dense subset (within an appropriate norm) within the set of forces defined by continuous proper time functions. This property could help to understand if the ALD equations for any continuous time-dependent force can always exhibit a solution approximately solving second-order Newton equations and then not show preaccelerated or runaway behavior. If this question can be answered in a positive sense, it will support the possibility of making full sense of the ALD equations for a pointlike particle. Another important extension is to continue the study initiated in Sec. VII, directed to find a plausible realization of the pointlike charged particle model in a framework of a Lagrangian theory of fields. The finding of such a model can furnish support to the exact validity of the ALD equations without needing the approximate recourse of reducing the order of the equations. It can be again underlined that the approach discussed here does not correspond to an approximate reduction of the order procedure, as was recently estimated in Ref. [12]. The Newton-like equations derived here are, in fact, equivalent to the ALD equations.

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