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$SU(3)$ and isospin breaking effects on $B \to PPP$ amplitudes

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Several modes of B decays into three pseudoscalar octet mesons PPP are measured. These decays provide useful information for B decays in the standard model (SM). Some powerful tools in analyzing B decays are flavor $SU(3)$ and isospin symmetries. Such analyses are usually hampered by $SU(3)$ breaking effects due to a relatively large strange quark mass which breaks $SU(3)$ symmetry down to isospin symmetry. The isospin symmetry also breaks down when the up and down quark mass difference is nonzero. It is, therefore, interesting to find relations which are not sensitive to $SU(3)$ and isospin breaking effects. We find that the relations among several fully symmetric $B \to PPP$ decay amplitudes are not affected by first-order $SU(3)$ breaking effects due to a nonzero strange quark mass, and also some of them are not affected by first isospin breaking effects. These relations, therefore, hold to good precisions. The measurements for these relations can provide important information about B decays in the SM.

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I. INTRODUCTION

Several decay modes of B decays into three pseudoscalar octet mesons PPP have been measured [1,2]. $B \to PPP$ has been a subject of theoretical studies [3]. The new data have raised new interests in related theoretical studies [4–9]. With more data from LHCb, one can expect that the study of $B \to PPP$ will provide more important information for B decays in the standard model (SM).

A powerful tool to analyze B decays is flavor $SU(3)$ symmetry [10]. Some of the interesting features of using flavor $SU(3)$ are the predictions of relations among different decay modes which can be experimentally tested. The flavor $SU(3)$ symmetry is, however, expected to be only an approximate symmetry because u, d, and s quarks have different masses. Since the strange quark has a relatively larger mass compared with those of up and down quarks, it is the larger source of symmetry breaking. If up and down quark masses are neglected, a nonzero strange quark mass breaks flavor $SU(3)$ symmetry down to the isospin symmetry. When the up and down quark mass difference is kept, isospin symmetry is also broken. The $SU(3)$ breaking effect is at the level of 20% for the π and K decay constants f_π and f_K. For two-body pseudoscalar octet meson B decays, although there are some $SU(3)$ breakings [11], it works reasonably well, such as rate differences between some of the $\Delta S = 0$ and $\Delta S = 1$ two-body pseudoscalar meson B decays [12,13]. Recently, an analysis has also been carried out for $B \to PPP$ decays using flavor $SU(3)$. It has been shown that the decay and CP asymmetry patterns for the charged B^+ decays into $K^+K^-K^+$, $K^+K^-\pi^+$, $K^+\pi^-\pi^+$, and $\pi^+\pi^-\pi^+$ do not follow $SU(3)$ predictions. To explain the data, large $SU(3)$ breaking effects are needed [6,7]. Usually, isospin breaking effects are much smaller because the up and down quark masses are much smaller than the strange quark mass and the QCD scale.

Because of possible large flavor $SU(3)$ breaking effects for $B \to PPP$, the predicted relations among different decay modes can only provide limited information. One wonders whether there exist relations which are immune from $SU(3)$ or even isospin breaking effects due to u, d, and s quark mass differences. To this end, we carried out an analysis for $B \to PPP$ decays using flavor $SU(3)$ symmetry to identify possible relations and then included $SU(3)$ breaking effects due to a strange quark mass and also up and down quark masses to see whether some relations still remain to hold. We find that the relations between several fully symmetric $B \to PPP$ decay amplitudes studied in Ref. [9] are not affected by the flavor $SU(3)$ breaking effects due a nonzero strange quark mass, and some of them are not even affected by isospin breaking effects. These relations, when measured experimentally, can provide useful information about B decays in the SM. In the following, we provide some details.

II. $SU(3)$-CONSERVING AMPLITUDES

We start with the description of B decays into three pseudoscalar octet mesons from flavor $SU(3)$ symmetry.

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The leading quark-level effective Hamiltonian up to the one-loop level in electroweak interaction for hadronic charmless B decays in the SM can be written as

$$H_{\rm eff}^{q} = \frac{4G_F}{\sqrt{2}}\left[V_{ub}V_{uq}^{*}(c_1 O_1 + c_2 O_2) - \sum_{i=3}^{12}(V_{ub}V_{uq}^{*}c_i^{uc} + V_{tb}V_{tq}^{*}c_i^{tc})O_i\right], \quad (2.1)$$

where q can be d or s, and the coefficients $c_{1,2}$ and $c_i^{jk} = c_i^j - c_i^k$ (with j and k indicating the internal quark) are the Wilson coefficients (WCs). The tree WCs are of order one with $c_1 = -0.31$ and $c_2 = 1.15$. The penguin WCs are much smaller, with the largest one (c_6) equal to -0.05. These WCs have been evaluated by several groups [14]. V_{ij} are the KM matrix elements. In the above the factor, $V_{cb}V_{cq}^{*}$ has been eliminated using the unitarity property of the KM matrix.

The operators O_i are given by

$$\begin{aligned}
&O_1 = (\bar{q}_i u_j)_{V-A}(\bar{u}_i b_j)_{V-A}, \qquad O_2 = (\bar{q}u)_{V-A}(\bar{u}b)_{V-A},\\
&O_{3,5} = (\bar{q}b)_{V-A}\sum_{q'}(\bar{q}'q')_{V\mp A}, \qquad O_{4,6} = (\bar{q}_i b_j)_{V-A}\sum_{q'}(\bar{q}'_j q'_i)_{V\mp A},\\
&O_{7,9} = \frac{3}{2}(\bar{q}b)_{V-A}\sum_{q'} e_{q'}(\bar{q}'q')_{V\pm A}, \qquad O_{8,10} = \frac{3}{2}(\bar{q}_i b_j)_{V-A}\sum_{q'} e_{q'}(\bar{q}'_j q'_i)_{V\pm A},\\
&O_{11} = \frac{g_s}{16\pi^2}\bar{q}\sigma_{\mu\nu}G^{\mu\nu}(1+\gamma_5)b, \qquad O_{12} = \frac{Q_b e}{16\pi^2}\bar{q}\sigma_{\mu\nu}F^{\mu\nu}(1+\gamma_5)b,
\end{aligned} \quad (2.2)$$

where $(\bar{a}b)_{V-A} = \bar{a}\gamma_\mu(1-\gamma_5)b$, and $G^{\mu\nu}$ and $F^{\mu\nu}$ are the field strengths of the gluon and photon, respectively.

At the hadron level, the decay amplitude can be generically written as

$$A = \langle \text{final state}|H_{\rm eff}^{q}|\bar{B}\rangle = V_{ub}V_{uq}^{*}T(q) + V_{tb}V_{tq}^{*}P(q), \quad (2.3)$$

where $T(q)$ contains contributions from the tree as well as penguin due to the charm and up quark loop corrections to the matrix elements, while $P(q)$ contains contributions purely from one-loop penguin contributions. B indicates one of the B^+, B^0, and B_s^0. $B_i = (B^+, B^0, B_s^0)$ forms an $SU(3)$ triplet.

The flavor $SU(3)$ symmetry transformation properties for operators $O_{1,2}$, $O_{3\text{–}6,11,12}$, and $O_{7\text{–}10}$ are $\bar{3}_a + \bar{3}_b + 6 + \overline{15}$, $\bar{3}$, and $\bar{3}_a + \bar{3}_b + 6 + \overline{15}$, respectively. We indicate these representations by matrices in $SU(3)$ flavor space by $H(\bar{3})$, $H(6)$, and $H(\overline{15})$. For $q = d$, the nonzero entries of the matrices $H(i)$ are given by [12]

$$\begin{aligned}
&H(\bar{3})^2 = 1, \qquad H(6)_1^{12} = H(6)_3^{23} = 1, \qquad H(6)_1^{21} = H(6)_3^{32} = -1,\\
&H(\overline{15})_1^{12} = H(\overline{15})_1^{21} = 3, \qquad H(\overline{15})_2^{22} = -2, \qquad H(\overline{15})_3^{32} = H(\overline{15})_3^{23} = -1.
\end{aligned} \quad (2.4)$$

And for $q = s$, the nonzero entries are

$$\begin{aligned}
&H(\bar{3})^3 = 1, \qquad H(6)_1^{13} = H(6)_2^{32} = 1, \qquad H(6)_1^{31} = H(6)_2^{23} = -1,\\
&H(\overline{15})_1^{13} = H(\overline{15})_1^{31} = 3, \qquad H(\overline{15})_3^{33} = -2, \qquad H(\overline{15})_2^{32} = H(\overline{15})_2^{23} = -1.
\end{aligned} \quad (2.5)$$

These properties enable one to write the decay amplitudes for $B \to PPP$ decays in only a few $SU(3)$ invariant amplitudes [10]. Here, P is one of the mesons in the pseudoscalar octet meson $M = (M_{ij})$, which is given by

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \quad (2.6)$$

The construction of the $B \to PPP$ decay amplitude can be done order by order by using three M's, B, and the Hamiltonian H, and also derivatives on the mesons to form $SU(3)$. The $SU(3)$ conserving momentum-independent amplitudes can be constructed by the following.

For the $T(q)$ amplitude, we have [6]

$$\begin{aligned} T(q) = {}& a^T(\bar{3})B_i H^i(\bar{3})M^j_k M^k_l M^l_j + b^T(\bar{3})H^i(\bar{3})M^j_i B_j M^k_l M^l_k + c^T(\bar{3})H^i(\bar{3})M^l_i M^j_l M^k_j B_k \\ &+ a^T(6)B_i H^{ij}_k(6)M^k_j M^l_n M^n_l + b^T(6)B_i H^{ij}_k(6)M^k_l M^l_n M^n_j \\ &+ c^T(6)B_i H^{jk}_l(6)M^i_j M^n_k M^l_n + d^T(6)B_i H^{jk}_l(6)M^i_n M^l_j M^n_k \\ &+ a^T(\overline{15})B_i H^{ij}_k(\overline{15})M^k_j M^l_n M^n_l + b^T(\overline{15})B_i H^{ij}_k(\overline{15})M^k_l M^l_n M^n_j \\ &+ c^T(\overline{15})B_i H^{jk}_l(\overline{15})M^i_j M^n_k M^l_n + d^T(\overline{15})B_i H^{jk}_l(\overline{15})M^i_n M^l_j M^n_k. \end{aligned} \tag{2.7}$$

One can write a similar amplitude $P(q)$ for the penguin contributions.

The coefficients $a(i)$, $b(i)$, $c(i)$, and $d(i)$ are constants which contain the WCs and information about QCD dynamics. Expanding the above $T(q)$ amplitude, one can extract the decay amplitudes for specific decays in terms of these coefficients.

In the above, we have described how to obtain flavor $SU(3)$ amplitudes which are momentum independent. However, due to the three-body decay nature, in general, there is momentum dependence in the decay amplitudes. The momentum dependence can, in principle, be determined by analyzing Dalitz plots for the decays. The lowest-order terms with derivatives lead to two powers of momentum dependence. One can obtain relevant terms by taking two times of derivatives on each of the terms in Eq. (2.7) and then collecting them together. It has been shown [6] that there are six independent ways of taking derivatives for each of the terms listed in Eq. (2.7). For example, after taking derivatives for $B_i H^i(\bar{3})M^j_k M^k_l M^l_j$, we have the following independent terms:

$$\begin{aligned} &(\partial_\mu B_i)H^i(\bar{3})(\partial^\mu M^j_k)M^k_l M^l_j, \qquad (\partial_\mu B_i)H^i(\bar{3})M^j_k(\partial^\mu M^k_l)M^l_j, \qquad (\partial_\mu B_i)H^i(\bar{3})M^j_k M^k_l(\partial^\mu M^l_j), \\ &B_i H^i(\bar{3})(\partial_\mu M^j_k)(\partial^\mu M^k_l)M^l_j, \qquad B_i H^i(\bar{3})(\partial_\mu M^j_k)M^k_l(\partial^\mu M^l_j), \qquad B_i H^i(\bar{3})M^j_k(\partial_\mu M^k_l)(\partial^\mu M^l_j). \end{aligned} \tag{2.8}$$

The full list of the possible terms have been obtained in Ref. [6] Appendix B. We will not repeat them here.

Using the above $SU(3)$ decay amplitudes, one can find some interesting relations among different decays [6]. It has been recently pointed out that there are additional relations among the fully symmetric final states B-decay amplitudes $\mathcal{A}_{\rm FS}$ [9]. A study of these relations can provide further information about flavor $SU(3)$ symmetry in B decays.

The fully symmetric $B \to PPP$ amplitudes $\mathcal{A}_{\rm FS}$ is related to the usual decay amplitudes $A(P_1(p_1)P_2(p_2)P_3(p_3))$ for the final mesons $P_{1,2,3}$ carrying momenta $p_{1,2,3}$, for all three final mesons are distinctive, by

$$\begin{aligned} \mathcal{A}_{\rm FS}(P_1P_2P_3) = {}& \frac{1}{\sqrt{3}}(A(P_1(p_1)P_2(p_2)P_3(p_3)) \\ &+ A(P_1(p_2)P_2(p_3)P_3(p_1)) \\ &+ A(P_1(p_3)P_2(p_1)P_3(p_2))). \end{aligned} \tag{2.9}$$

For the cases that two of them or all three of them are identical particles, the identical particle factorial factors should be taken cared. Reference [9] discusses in detail how the fully symmetric amplitudes can be determined experimentally. We will not repeat the discussions here. We concentrate on how these amplitudes are derived in the framework of flavor $SU(3)$ symmetry and how they are affected by $SU(3)$ breaking effects due to finite quark masses for u, d, and s quarks.

To understand why there are new relations between the fully symmetric amplitudes for different decay modes, let us consider $B^+ \to K^0\pi^+\pi^0$ and $B^0_d \to K^+\pi^0\pi^-$ decays as examples.

Expanding Eq. (2.7), one obtains

$$T(B^+ \to K^0\pi^+\pi^0) = \sqrt{2}(c(6) + d(6) + 2c(\overline{15}) + 2d(\overline{15})) \tag{2.10}$$

and

$$T(B^0 \to K^+\pi^-\pi^0) = T(B^+ \to K^0\pi^+\pi^0), \tag{2.11}$$

from which we get $T_{\rm FS}(B^+ \to K^0\pi^+\pi^0) = T_{\rm FS}(B^0 \to K^+\pi^-\pi^0)$.

As the decay amplitudes may have momentum dependence, we should also check if the equality of the above two amplitudes is equal when taking into account the momentum dependence in the amplitudes. Expanding the terms in Appendix B of Ref. [6], we find

$$T^p(B^+ \to K^0\pi^+\pi^0) = \alpha_1 p_B \cdot p_1 + \alpha_2 p_B \cdot p_2 + \alpha_3 p_B \cdot p_3 + \alpha_4 p_1 \cdot p_2 + \alpha_5 p_1 \cdot p_3 + \alpha_6 p_2 \cdot p_3,$$
$$T^p(B_d^0 \to K^+\pi^0\pi^-) = \beta_1 p_B \cdot p_1 + \beta_2 p_B \cdot p_2 + \beta_3 p_B \cdot p_3 + \beta_4 p_1 \cdot p_2 + \beta_5 p_1 \cdot p_3 + \beta_6 p_2 \cdot p_3. \quad (2.12)$$

The coefficients α_i and β_i are given by

$$\alpha_1 = \sqrt{2}(c'(6)_2 + 2c'(\overline{15})_2 + d'(6)_3 + 2d'(\overline{15})_3),$$
$$\alpha_2 = \frac{1}{\sqrt{2}}(-b'(6)_1 + b'(6)_2 - 3b'(\overline{15})_1 + 3b'(\overline{15})_2 + c'(\bar{3})_2 - c'(\bar{3})_3 + c'(6)_1$$
$$+ c'(6)_3 + c'(\overline{15})_1 + 3c'(\overline{15})_3 + d'(6)_1 + d'(6)_3 + 5d'(\overline{15})_1 - d'(\overline{15})_3),$$
$$\alpha_3 = \frac{1}{\sqrt{2}}(b'(6)_1 - b'(6)_2 + 3b'(\overline{15})_1 - 3b'(\overline{15})_2 - c'(\bar{3})_2 + c'(\bar{3})_3 + c'(6)_1 + c'(6)_3$$
$$+ 3c'(\overline{15})_1 + c'(\overline{15})_3 + d'(6)_1 + 2d'(6)_2 - d'(6)_3 - d'(\overline{15})_1 + 4d'(\overline{15})_2 + d'(\overline{15})_3),$$
$$\alpha_4 = \frac{1}{\sqrt{2}}(-b''(6)_2 + b''(6)_3 - 3b''(\overline{15})_2 + 3b''(\overline{15})_3 + c''(\bar{3})_1 - c''(\bar{3})_2 + c''(6)_1 + c''(6)_3$$
$$+ c''(\overline{15})_1 + 3c''(\overline{15})_2 - d''(6)_1 + 2d''(6)_2 + d''(6)_3 + d''(\overline{15})_1 - d''(\overline{15})_2 + 4d''(\overline{15})_3),$$
$$\alpha_5 = \frac{1}{\sqrt{2}}(b''(6)_2 - b''(6)_3 + 3b''(\overline{15})_2 - 3b''(\overline{15})_3 - c''(\bar{3})_1 + c''(\bar{3})_2 + c''(6)_1$$
$$+ c''(6)_3 + 3c''(\overline{15})_1 + c''(\overline{15})_2 + d''(6)_1 + d''(6)_3 - d''(\overline{15})_1 + d''(\overline{15})_2),$$
$$\alpha_6 = \sqrt{2}(c''(6)_2 + 2c''(\overline{15})_3 + d''(6)_1 + 2d''(\overline{15})_1) \quad (2.13)$$

and

$$\beta_1 = \sqrt{2}(c'(6)_2 + 2c'(\overline{15})_2 + d'(6)_3 + 2d'(\overline{15})_3),$$
$$\beta_2 = \frac{1}{\sqrt{2}}(b'(6)_1 - b'(6)_2 + b'(\overline{15})_1 - b'(\overline{15})_2 + c'(\bar{3})_2 - c'(\bar{3})_3 + c'(6)_1 + c'(6)_3$$
$$+ c'(\overline{15})_1 + 3c'(\overline{15})_3 + d'(6)_1 + 2d'(6)_2 - d'(6)_3 - 3d'(\overline{15})_1 + 4d'(\overline{15})_2 + 3d'(\overline{15})_3),$$
$$\beta_3 = \frac{1}{\sqrt{2}}(-b'(6)_1 + b'(6)_2 - b'(\overline{15})_1 + b'(\overline{15})_2 - c'(\bar{3})_2 + c'(\bar{3})_3 + c'(6)_1$$
$$+ c'(6)_3 + 3c'(\overline{15})_1 + c'(\overline{15})_3 + d'(6)_1 + d'(6)_3 + 7d'(\overline{15})_1 - 3d'(\overline{15})_3),$$
$$\beta_4 = \frac{1}{\sqrt{2}}(b''(6)_2 - b''(6)_3 + b''(\overline{15})_2 - b''(\overline{15})_3 + c''(\bar{3})_1 - c''(\bar{3})_2 + c''(6)_1$$
$$+ c''(6)_3 + c''(\overline{15})_1 + 3c''(\overline{15})_2 + d''(6)_1 + d''(6)_3 - 3d''(\overline{15})_1 + 7d''(\overline{15})_2),$$
$$\beta_5 = \frac{1}{\sqrt{2}}(-b''(6)_2 + b''(6)_3 - b''(\overline{15})_2 + b''(\overline{15})_3 - c''(\bar{3})_1 + c''(\bar{3})_2 + c''(6)_1 + c''(6)_3$$
$$+ 3c''(\overline{15})_1 + c''(\overline{15})_2 - d''(6)_1 + 2d''(6)_2 + d''(6)_3 + 3d''(\overline{15})_1 - 3d''(\overline{15})_2 + 4d''(\overline{15})_3),$$
$$\beta_6 = \sqrt{2}(c''(6)_2 + 2c''(\overline{15})_3 + d''(6)_1 + 2d''(\overline{15})_1). \quad (2.14)$$

One can see from the above that $T^p(B^+ \to K^0\pi^+\pi^0)$ is no longer equal to $T^p(B_d^0 \to K^+\pi^0\pi^-)$. However, one can readily see from the above equations that

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3, \qquad \alpha_4 + \alpha_5 + \alpha_6 = \beta_4 + \beta_5 + \beta_6. \quad (2.15)$$

This fact makes the fully symmetric amplitudes satisfy

$$T^p(B^+ \to K^0\pi^+\pi^0)_{\mathrm{FS}} = T^p(B_d^0 \to K^+\pi^0\pi^-)_{\mathrm{FS}}. \quad (2.16)$$

Similarly, the penguin amplitudes P and P^p have the same properties discussed above for the tree amplitudes T and T^p.

The total fully symmetric amplitudes $\mathcal{A}_{\rm FS} = V_{ub}V_{uq}^* \times (T_{\rm FS} + T_{\rm FS}^p) + V_{tb}V_{tq}^*(P_{\rm FS} + P_{\rm FS}^p)$ then have the relation

$$\mathcal{A}(B^+ \to K^0\pi^+\pi^0)_{\rm FS} = \mathcal{A}(B_d^0 \to K^+\pi^0\pi^-)_{\rm FS}. \qquad (2.17)$$

Enlarging the amplitudes to fully symmetric ones, indeed, produces more relations.

Expanding Eq. (2.7) and the equations in Appendix B of Ref. [6], we obtain the following relations confirming those obtained in Ref. [9]. For $\bar{b} \to \bar{s}$ induced $B \to PPP$ amplitudes, we have

(1) $B \to K\pi\pi$

$$S1.1 = \mathcal{A}(B^+ \to K^0\pi^+\pi^0)_{\rm FS} - \mathcal{A}(B^0 \to K^+\pi^0\pi^-)_{\rm FS} = 0,$$
$$S1.2 = \sqrt{2}\mathcal{A}(B^+ \to K^0\pi^+\pi^0)_{\rm FS} - \mathcal{A}(B^0 \to K^0\pi^+\pi^-)_{\rm FS} + 2\mathcal{A}(B^0 \to K^0\pi^0\pi^0)_{\rm FS} = 0,$$
$$S1.3 = \sqrt{2}\mathcal{A}(B^0 \to K^+\pi^0\pi^-)_{\rm FS} + \mathcal{A}(B^+ \to K^+\pi^+\pi^-)_{\rm FS} - 2\mathcal{A}(B^+ \to K^+\pi^0\pi^0)_{\rm FS} = 0.$$

(2) $B \to KK\bar{K}$

$$S2.1 = -\mathcal{A}(B^+ \to K^+K^+K^-)_{\rm FS} + \mathcal{A}(B^+ \to K^+K^0\bar{K}^0)_{\rm FS} + \mathcal{A}(B^0 \to K^0K^+K^-)_{\rm FS} - \mathcal{A}(B^0 \to K^0K^0\bar{K}^0)_{\rm FS} = 0.$$

(3) $B_s^0 \to \pi K\bar{K}$

$$S3.1 = \sqrt{2}\mathcal{A}(B_s^0 \to \pi^0K^+K^-)_{\rm FS} - \sqrt{2}\mathcal{A}(B_s^0 \to \pi^0K^0\bar{K}^0)_{\rm FS} - \mathcal{A}(B_s^0 \to \pi^-K^+\bar{K}^0)_{\rm FS} - \mathcal{A}(B_s^0 \to \pi^+K^-K^0)_{\rm FS} = 0.$$

(4) $B_s^0 \to \pi\pi\pi$

$$S4.1 = 2\mathcal{A}(B_s^0 \to \pi^0\pi^0\pi^0)_{\rm FS} - \mathcal{A}(B_s^0 \to \pi^0\pi^+\pi^-)_{\rm FS} = 0.$$

For $\bar{b} \to \bar{d}$ induced $B \to PPP$ amplitudes, we have

(1) $B \to \pi K\bar{K}$

$$D1.1 = -\sqrt{2}\mathcal{A}(B^0 \to \pi^0K^+K^-)_{\rm FS} + \mathcal{A}(B^0 \to \pi^+K^0K^-)_{\rm FS} - \mathcal{A}(B^+ \to \pi^+K^+K^-)_{\rm FS} + \sqrt{2}\mathcal{A}(B^0 \to \pi^0K^0\bar{K}^0)_{\rm FS} + \mathcal{A}(B^0 \to \pi^-K^+\bar{K}^0)_{\rm FS} + \mathcal{A}(B^+ \to \pi^+K^0\bar{K}^0)_{\rm FS} - \sqrt{2}\mathcal{A}(B^+ \to \pi^0K^+\bar{K}^0)_{\rm FS} = 0.$$

(2) $B \to \pi\pi\pi$

$$D2.1 = 2\mathcal{A}(B^0 \to \pi^0\pi^0\pi^0)_{\rm FS} - \mathcal{A}(B^0 \to \pi^+\pi^0\pi^-)_{\rm FS} = 0,$$
$$D2.2 = 2\mathcal{A}(B^+ \to \pi^+\pi^0\pi^0)_{\rm FS} - \mathcal{A}(B^+ \to \pi^-\pi^+\pi^+)_{\rm FS} = 0.$$

(3) $B_s^0 \to K\pi\pi$

$$D3.1 = -2\mathcal{A}(B_s^0 \to \bar{K}^0\pi^0\pi^0)_{\rm FS} + \mathcal{A}(B_s^0 \to \bar{K}^0\pi^+\pi^-)_{\rm FS} - \sqrt{2}\mathcal{A}(B_s^0 \to K^-\pi^+\pi^0)_{\rm FS} = 0.$$

In the above, we have considered some relations among decay processes with the same ΔS. There are also some other relations among tree and penguin amplitudes but with different ΔS. Some of them will be discussed later in the Conclusions.

Note that we have different normalizations than those used in Ref. [9] for some of the final meson states and also identical particle combinatorial factors. One can easily obtain relations that are in the form used in Ref. [9] by multiplying a "-1" by the amplitudes when π^-, K^-, π^0 appear each time as one of the final states, and a factor $1/\sqrt{2}$ and $1/\sqrt{6}$ in our formulation for the corresponding amplitudes, respectively, when the decays involve two and three identical particles.

III. $SU(3)$ AND ISOSPIN BREAKING DUE TO QUARK MASS DIFFERENCES

The main source for flavor $SU(3)$ symmetry breaking effects comes from the difference in the masses of u, d, and s quarks. Under $SU(3)$, the mass matrix can be viewed as combinations of representations from $3 \times \bar{3}$, to matching the $(u,\ d,\ s)$ transformation property as a fundamental representation, which contains one and eight irreducible representations. The diagonalized mass matrix can be expressed as a linear combination of the identity matrix I and the Gell-Mann matrices λ_3 and λ_8. We have

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = \frac{1}{3}(m_u + m_d + m_s)I + \frac{1}{2}(m_u - m_d)X + \frac{1}{6}(m_u + m_d - 2m_s)W, \qquad (3.1)$$

with X and W given by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \qquad (3.2)$$

Compared with s-quark mass m_s, the u and d quark masses $m_{u,d}$ are much smaller, and $SU(3)$ breaking effects due to a nonzero m_s dominate the $SU(3)$ breaking effects. When the up and down quark mass difference is neglected, the residual symmetry of $SU(3)$ becomes the isospin symmetry. In that case, when studying $SU(3)$ breaking effects, the term proportional to X can be dropped. The identity I part contributes to the B-decay amplitudes in a similar way as that given in Eq. (2.7), which can be absorbed into the coefficients $a(i)$ to $d(i)$. Only the W piece will contribute to the $SU(3)$ breaking effects. We will first discuss this case to first order in W and then also study the isospin breaking effects by including the first-order term proportional to X.

A. SU(3) breaking due a nonzero m_s

To construct the relevant decay amplitudes for $B \to PPP$ decays, one first breaks the contraction of indices at any joint in Eq. (2.7), inserts a W in between, and then contracts all indices appropriately. For example, corresponding to the first term in Eq. (2.7), there are two ways to insert W:

$$B_i H^a(\bar{3}) W_a^i M_k^j M_l^k M_j^l, \qquad B_i H^i(\bar{3}) M_k^j M_l^k M_j^a W_a^l. \tag{3.3}$$

The full list of possible independent terms is given in Appendix A of Ref. [6].

Extracting the $SU(3)$ breaking terms for $B^+ \to K^0\pi^+\pi^0$ and $B^0 \to K^+\pi^0\pi^-$ decays, we have the corrections for the decay amplitudes ΔT as

$$\begin{aligned}\triangle T(B^+ \to K^0\pi^+\pi^0) = \sqrt{2}(&c_1^T(6) + \sqrt{2}c_2^T(6) - 2c_3^T(6) + \sqrt{2}c_4^T(6) + c_5^T(6) + c_1^T(\overline{15})\\ &+ c_2^T(\overline{15}) - 2c_3^T(\overline{15}) + c_4^T(\overline{15}) + c_5^T(\overline{15}) + d_1^T(6) + d_2^T(6) - 2d_3^T(6) + d_4^T(6)\\ &+ d_5^T(6) + d_1^T(\overline{15}) + d_2^T(\overline{15}) - 2d_3^T(\overline{15}) + d_4^T(\overline{15}) + d_5^T(\overline{15}))\end{aligned} \tag{3.4}$$

and

$$\triangle T(B^0 \to K^+\pi^0\pi^-) = \triangle T(B^+ \to K^0\pi^+\pi^0), \tag{3.5}$$

which leads to the equality of the fully symmetric amplitudes for these two decays. Therefore,

$$S1.1 = \mathcal{A}(B^0 \to K^+\pi^0\pi^-)_{\rm FS} - \mathcal{A}(B^+ \to K^0\pi^+\pi^0)_{\rm FS} = 0 \tag{3.6}$$

still holds.

Note that even $SU(3)$ breaking effects affect each of the decay amplitudes; the relation of the fully symmetric amplitudes of these two decays is not affected by the first-order $SU(3)$ breaking effects. Expanding the terms in Appendix A of Ref. [6], one can study the relations discussed above. We find that all the relations among the fully symmetric amplitudes still hold; that is,

$$\begin{aligned}&S1.1 = 0, \quad S2.1 = 0, \quad S1.3 = 0\\ &S2.1 = 0, \quad S3.1 = 0, \quad S4.1 = 0,\\ &D1.1 = 0, \quad D2.1 = 0, \quad D2.2 = 0, \quad D3.1 = 0\end{aligned} \tag{3.7}$$

are still true even if one includes $SU(3)$ breaking effects due a nonzero strange quark mass. This actually is not a surprise because the relations discussed can be obtained by isospin symmetry considerations.

Experimental verification of these relations may provide important tests for the validity of flavor $SU(3)$ for B decays.

B. Isospin breaking due to the up and down quark mass difference

It would be interesting to investigate what happens when the mass difference between the up and down quarks, which breaks isospin symmetry, is also included. We now discuss these isospin breaking effects for the relations discussed before.

One can obtain the corrections by replacing W by X in Appendix A of Ref. [6]. We indicate the coefficients in a similar way as that of $SU(3)$ breaking effects due to a nonzero m_s but with a superscript I to indicate the effects of isospin breaking, for example, for tree operator corrections by $a_i^{T^I}$, $b_i^{T^I}$, $c_i^{T^I}$, and $d_i^{T^I}$. The correction to the decay amplitude will also be indicated by a superscript I, ΔT^I.

Expanding all terms, we obtain the corrections due to isospin breaking effects for all the decay amplitudes discussed previously. We find that, with the exception that the relation $S4.1$ still holds, all other relations for the $B \to PPP$ decay amplitudes induced by $\bar{b} \to \bar{s}$ and $\bar{b} \to \bar{d}$ interactions discussed earlier are broken.

In fact, each of the decay modes relevant in $S4.1$ is affected by isospin breaking effects,

$$\begin{aligned}\Delta T^I(B_s^0 \to \pi^0\pi^0\pi^0) &= \frac{\sqrt{2}}{2}(a_2^{T^I}(\bar{3}) + 2a_2^{T^I}(\overline{15}) + 2a_3^{T^I}(\overline{15}) + b_2^{T^I}(\overline{15}) + b_3^{T^I}(\overline{15}) + b_4^{T^I}(\overline{15}) + b_5^{T^I}(\overline{15})),\\ \Delta T^I(B_s^0 \to \pi^+\pi^-\pi^0) &= \sqrt{2}(a_2^{T^I}(\bar{3}) + 2a_2^{T^I}(\overline{15}) + 2a_3^{T^I}(\overline{15}) + b_2^{T^I}(\overline{15}) + b_3^{T^I}(\overline{15}) + b_4^{T^I}(\overline{15}) + b_5^{T^I}(\overline{15})),\end{aligned} \tag{3.8}$$

but they are affected in such a way that the equality of the amplitudes is not affected. That is, we still have

$$S4.1 = 2\mathcal{A}(B_s^0 \to \pi^0\pi^0\pi^0)_{\rm FS} - \mathcal{A}(B_s^0 \to \pi^0\pi^+\pi^-)_{\rm FS} = 0. \tag{3.9}$$

This makes this relation special because this relation is not affected by first-order $SU(3)$ breaking effects due to a nonzero strange quark and isospin breaking due to the up and down quark mass difference. It should hold to a high precision. An experimental test of this relation can provide important information about $B \to PPP$.

We also found some other interesting relations where even isospin violating effects are included; namely, the corrections for some of the relations discussed above are related to others. For the $b \to s$ interaction induced decay modes, we have an additional relation which relates $S1.2$ and $S1.3$ because the isospin breaking effects satisfy

$$\begin{aligned}&[\sqrt{2}\Delta T^I(B^+ \to K^0\pi^+\pi^0) - \Delta T^I(B^0 \to K^0\pi^+\pi^-) + 2\Delta T^I(B^0 \to K^0\pi^0\pi^0)]\\&\quad = -[\sqrt{2}\Delta T^I(B^0 \to K^+\pi^-\pi^0) + \Delta T^I(B^+ \to K^+\pi^+\pi^-) - 2\Delta T^I(B^+ \to K^+\pi^0\pi^0)].\end{aligned} \tag{3.10}$$

Although the right-hand sides of $S1.2$ and $S1.3$ are not zero anymore, the above relation leads to

$$S1.2 = -S1.3 \neq 0. \tag{3.11}$$

For the $b \to d$ interactions induced decay modes, we have

$$\begin{aligned}&[2\Delta T^I(B^+ \to \pi^0\pi^0\pi^+) - \Delta T^I(B^+ \to \pi^-\pi^+\pi^+)] = -[-2\Delta T^I(B_s^0 \to \bar{K}^0\pi^0\pi^0) + \Delta T^I(B_s^0 \to \bar{K}^0\pi^+\pi^-)\\&\qquad - \sqrt{2}\Delta T^I(B_s^0 \to K^-\pi^+\pi^0)],\\&\sqrt{2}[2\Delta T^I(B^0 \to \pi^0\pi^0\pi^0) - \Delta T^I(B^0 \to \pi^+\pi^-\pi^0)] = -[2\Delta T^I(B^+ \to \pi^0\pi^0\pi^+) - \Delta T^I(B^+ \to \pi^-\pi^+\pi^+)].\end{aligned} \tag{3.12}$$

Because of isospin breaking effects, the right-hand sides of $D2.1$, $D2.2$, and $D3.1$ are nonzero. However, the above relations imply

$$\sqrt{2}D2.1 = -D2.2 \neq 0, \qquad D2.2 = -D3.1 \neq 0. \tag{3.13}$$

We would like to emphasize that since the above relations hold even when first-order isospin effects have been taken into account, they can provide useful information about B decays in the SM in a way independent of flavor $SU(3)$ and isospin breaking effects.

C. Momentum-dependent $SU(3)$ and isospin breaking amplitudes

There are also momentum-dependent amplitudes at the same order to the $SU(3)$ and isospin breaking effects discussed in the previous subsections. We find that all the relations Eq. (3.7) and Eqs. (3.9), (3.11), and (3.13) still hold when $SU(3)$ breaking effects are due, respectively, to a nonzero m_s and when isospin breaking effects are due to the m_u and m_d mass difference discussed earlier. The analysis is similar to the case for $SU(3)$ conserving momentum amplitudes. We will not give details here but just outline how the analysis can be carried out. The leading ones are constructed by taking two powers of derivatives on each term of the $SU(3)$ breaking amplitudes which have been shown in Appendix A of Ref. [6]. For example, for the term, $a_1^T(\bar{3})B_i H^a(\bar{3}) W_a^i M_k^j M_l^k M_j^l$, we obtain the following six independent terms with two derivatives:

$$\begin{aligned}&a_1'(\bar{3})_1(\partial_\mu B_i)H^a(\bar{3})W_a^i(\partial_\mu M_k^j)M_l^k M_j^l, \qquad a_1'(\bar{3})_2(\partial_\mu B_i)H^a(\bar{3})W_a^i M_k^j(\partial_\mu M_l^k)M_j^l,\\&a_1'(\bar{3})_3(\partial_\mu B_i)H^a(\bar{3})W_a^i M_k^j M_l^k(\partial_\mu M_j^l), \qquad a_1''(\bar{3})_1 B_i H^a(\bar{3})W_a^i(\partial_\mu M_k^j)(\partial_\mu M_l^k)M_j^l,\\&a_1''(\bar{3})_2 B_i H^a(\bar{3})W_a^i(\partial_\mu M_k^j)M_l^k(\partial_\mu M_j^l), \qquad a_1''(\bar{3})_3 B_i H^a(\bar{3})W_a^i M_k^j(\partial_\mu M_l^k)(\partial_\mu M_j^l).\end{aligned} \tag{3.14}$$

Here, $a_1'(\bar{3})_1, a_1'(\bar{3})_2, a_1'(\bar{3})_3, a_1''(\bar{3})_1, a_1''(\bar{3})_2, a_1''(\bar{3})_3$ are constants. We then extend a similar definition of constants for other $SU(3)$ breaking terms in Appendixes A and B of Ref. [6]. There are both tree and penguin amplitudes which can be further labeled by superscripts T and P. We will omit writing them out with the understanding that what is described below will work for both tree and penguin amplitudes. One obtains the relevant terms for isospin breaking effects by replacing W by X.

Expanding all possible terms, one obtains the amplitudes. Taking the amplitudes in $S1.1$ for illustration, one finds that the momentum-dependent amplitudes can be written as

$$\Delta T^p(B^+ \to K^0\pi^+\pi^0) = \Delta\alpha_1 p_B \cdot p_1 + \Delta\alpha_2 p_B \cdot p_2 + \Delta\alpha_3 p_B \cdot p_3 + \Delta\alpha_4 p_1 \cdot p_2 + \Delta\alpha_5 p_1 \cdot p_3 + \Delta\alpha_6 p_2 \cdot p_3,$$
$$\Delta T^p(B_d^0 \to K^+\pi^0\pi^-) = \Delta\beta_1 p_B \cdot p_1 + \Delta\beta_2 p_B \cdot p_2 + \Delta\beta_3 p_B \cdot p_3 + \Delta\beta_4 p_1 \cdot p_2 + \Delta\beta_5 p_1 \cdot p_3 + \Delta\beta_6 p_2 \cdot p_3. \quad (3.15)$$

The coefficients $\Delta\alpha_i, \Delta\beta_i$ are collections of coefficients from all possible terms. Our detailed calculations show that

$$\begin{aligned}
\Delta\alpha_1 + \Delta\alpha_2 + \Delta\alpha_3 &= \Delta\beta_1 + \Delta\beta_2 + \Delta\beta_3 \\
&= \sqrt{2}[2c_1'(\overline{15})_1 + 2c_1'(\overline{15})_2 + 2c_1'(\overline{15})_3 + 2c_2'(\overline{15})_1 + 2c_2'(\overline{15})_2 + 2c_2'(\overline{15})_3 - 4c_3'(\overline{15})_1 \\
&\quad - 4c_3'(\overline{15})_2 - 4c_3'(\overline{15})_3 + 2c_4'(\overline{15})_1 + 2c_4'(\overline{15})_2 + 2c_4'(\overline{15})_3 + 2c_5'(\overline{15})_1 + 2c_5'(\overline{15})_2 + 2c_5'(\overline{15})_3 \\
&\quad + c_1'(6)_1 + c_1'(6)_2 + c_1'(6)_3 + c_2'(6)_1 + c_2'(6)_2 + c_2'(6)_3 - 2c_3'(6)_1 - 2c_3'(6)_2 - 2c_3'(6)_3 + c_4'(6)_1 \\
&\quad + c_4'(6)_2 + c_4'(6)_3 + c_5'(6)_1 + c_5'(6)_2 + c_5'(6)_3 + 2d_1'(\overline{15})_1 + 2d_1'(\overline{15})_2 + 2d_1'(\overline{15})_3 + 2d_2'(\overline{15})_1 \\
&\quad + 2d_2'(\overline{15})_2 + 2d_2'(\overline{15})_3 - 4d_3'(\overline{15})_1 - 4d_3'(\overline{15})_2 - 4d_3'(\overline{15})_3 + 2d_4'(\overline{15})_1 + 2d_4'(\overline{15})_2 + 2d_4'(\overline{15})_3 \\
&\quad + 2d_5'(\overline{15})_1 + 2d_5'(\overline{15})_2 + 2d_5'(\overline{15})_3 + d_1'(6)_1 + d_1'(6)_2 + d_1'(6)_3 + d_2'(6)_1 + d_2'(6)_2 + d_2'(6)_3 \\
&\quad - 2d_3'(6)_1 - 2d_3'(6)_2 - 2d_3'(6)_3 + d_4'(6)_1 + d_4'(6)_2 + d_4'(6)_3 + d_5'(6)_1 + d_5'(6)_2 + d_5'(6)_3] \\
\Delta\alpha_4 + \Delta\alpha_5 + \Delta\alpha_6 &= \Delta\beta_4 + \Delta\beta_5 + \Delta\beta_6 \\
&= \sqrt{2}[2c_1''(\overline{15})_1 + 2c_1''(\overline{15})_2 + 2c_1''(\overline{15})_3 + 2c_2''(\overline{15})_1 + 2c_2''(\overline{15})_2 + 2c_2''(\overline{15})_3 - 4c_3''(\overline{15})_1 \\
&\quad - 4c_3''(\overline{15})_2 - 4c_3''(\overline{15})_3 + 2c_4''(\overline{15})_1 + 2c_4''(\overline{15})_2 + 2c_4''(\overline{15})_3 + 2c_5''(\overline{15})_1 + 2c_5''(\overline{15})_2 + 2c_5''(\overline{15})_3 \\
&\quad + c_1''(6)_1 + c_1''(6)_2 + c_1''(6)_3 + c_2''(6)_1 + c_2''(6)_2 + c_2''(6)_3 - 2c_3''(6)_1 - 2c_3''(6)_2 - 2c_3''(6)_3 + c_4''(6)_1 \\
&\quad + c_4''(6)_2 + c_4''(6)_3 + c_5''(6)_1 + c_5''(6)_2 + c_5''(6)_3 + 2d_1''(\overline{15})_1 + 2d_1''(\overline{15})_2 + 2d_1''(\overline{15})_3 + 2d_2''(\overline{15})_1 \\
&\quad + 2d_2''(\overline{15})_2 + 2d_2''(\overline{15})_3 - 4d_3''(\overline{15})_1 - 4d_3''(\overline{15})_2 - 4d_3''(\overline{15})_3 + 2d_4''(\overline{15})_1 + 2d_4''(\overline{15})_2 + 2d_4''(\overline{15})_3 \\
&\quad + 2d_5''(\overline{15})_1 + 2d_5''(\overline{15})_2 + 2d_5''(\overline{15})_3 + d_1''(6)_1 + d_1''(6)_2 + d_1''(6)_3 + d_2''(6)_1 + d_2''(6)_2 + d_2''(6)_3 \\
&\quad - 2d_3''(6)_1 - 2d_3''(6)_2 - 2d_3''(6)_3 + d_4''(6)_1 + d_4''(6)_2 + d_4''(6)_3 + d_5''(6)_1 + d_5''(6)_2 + d_5''(6)_3].
\end{aligned}$$

With these facts, after symmetrizing the amplitude to the fully symmetric one, we find $S1.1 = 0$ still holds. We find that the other relations of Eq. (3.7) also hold.

In a very similar way, one can obtain the momentum-dependent corrections to the isospin breaking effects by replacing W by X as has been done for the $SU(3)$ case. We find that all the relations of Eqs. (3.9), (3.11), and (3.13) still hold.

Before we close this section, we would like to make a comment about the finite mass effects of m_π^2 and m_K^2. In the practical extraction of the amplitudes, one should also consider $SU(3)$ corrections in phase space due to final state meson mass differences, which come in order $m_{\pi,K}^2/m_{B,B_s}^2$ since $m_\pi^2 \sim m_u, m_d$ and $m_K^2 \sim m_s$ which are the same order of the $SU(3)$ and isospin breaking effects considered earlier. This can be done systematically when extracting the fully symmetric amplitudes by Dalitz plot analysis. In the momentum-dependent amplitudes discussed in Sec. II, when expressing the amplitudes, for example, those in Eq. (2.12), in terms of the s, t, and u variables, the terms proportional to m_π^2 and m_K^2 will be generated. However, these will not generate new terms compared with those already included in the $SU(3)$ and isospin breaking effects considered earlier in this section. The conclusions drawn above will not be changed.

IV. CONCLUSIONS AND DISCUSSION

Flavor $SU(3)$ and isospin symmetries have been considered to be powerful tools in analyzing B decays. Such analyses are usually hampered by a relatively large strange quark mass which breaks $SU(3)$ symmetry down to isospin symmetry. The isospin symmetry also breaks down when the up and down quark mass difference is kept. It is, therefore, interesting to find relations which are not sensitive to $SU(3)$ and isospin breaking effects. We have carried out detailed analyses including $SU(3)$ and isospin breaking effects due to u, d, and s quark mass differences for $B \to PPP$ decays. We have found that a class of relations in fully symmetric amplitudes is not broken by $SU(3)$ breaking effects due to a nonzero strange quark mass, and the relations

$$S4.1 = 0, \quad S1.2 + S1.3 = 0,$$
$$\sqrt{2}D2.1 + D2.2 = 0, \quad D2.2 + D3.1 = 0 \tag{4.1}$$

hold even when the isospin breaking effects due to the up and down quark mass difference are included. The measurements for these relations will provide important information about B decays in the SM.

We would like to end the paper by commenting on $SU(3)$ breaking effects on the U-spin symmetry relations in the following:

$$T_{\triangle s=-1}(B^+ \to K^+K^+K^-) = T_{\triangle s=0}(B^+ \to \pi^+\pi^+\pi^-),$$
$$T_{\triangle s=-1}(B^+ \to K^+\pi^+\pi^-) = T_{\triangle s=0}(B^+ \to \pi^+K^+K^-). \tag{4.2}$$

The momentum-dependent terms also respect the above relations. The above equalities also hold for the fully symmetric amplitudes for corresponding pairs of decay modes in the $SU(3)$ limit. These relations imply in the SM that the CP-violating rate asymmetries defined by $A_{\mathrm{asy}} = \Gamma(B \to PPP) - \Gamma(\bar{B} \to \bar{P}\bar{P}\bar{P})$ are equal but opposite in sign for each pair of decay modes above.

For the fully symmetric amplitudes of these decays modes, we also have

$$T_{\triangle s=-1}(B^+ \to K^+K^+K^-)_{\mathrm{FS}} = T_{\triangle s=-1}(B^+ \to K^+\pi^+\pi^-)_{\mathrm{FS}},$$
$$T_{\triangle s=0}(B^+ \to \pi^+\pi^+\pi^-)_{\mathrm{FS}} = T_{\triangle s=0}(B^+ \to \pi^+K^+K^-)_{\mathrm{FS}}. \tag{4.3}$$

Unlike the other fully symmetric amplitudes studied in the previous sections, the relations in Eqs. (4.2) and (4.3) are broken when $SU(3)$ breaking effects due to a nonzero strange quark mass are included. Therefore, there may be sizeable deviation for these relations. The relations in Eq. (4.2) have been discussed recently. It was found that, indeed, there are large $SU(3)$ breaking effects [5–7]. The relations in Eqs. (4.2) and (4.3) will not provide as much insight as those from the fully symmetric amplitudes which still hold when isospin breaking effects are included, as discussed earlier.

However, we have found that the $SU(3)$ breaking effects due to a nonzero strange quark mass and the isospin breaking effects due to the difference of the up and down quark masses are equal for some of the above relations with

$$\begin{aligned}\Delta T(B^+ \to K^+\pi^+\pi^-) - \Delta T(B^+ \to K^+K^+K^-) &= -[\Delta T(B^+ \to \pi^+K^+K^-) - \Delta T(B^+ \to \pi^+\pi^+\pi^-)]\\ &= 3[a_4^T(6) + 3a_4^T(\overline{15}) + b_3^T(\bar{3}) + b_4^T(6) + 3b_4^T(\overline{15}) + c_2^T(\bar{3}) - c_2^T(6) + c_3^T(6)\\ &\quad - c_5^T(6) - c_2^T(\overline{15}) - c_3^T(\overline{15}) - 2c_4^T(\overline{15}) + 3c_5^T(\overline{15}) - d_5^T(6) + 3d_5^T(\overline{15})],\end{aligned} \tag{4.4}$$

and the isospin breaking effects satisfy

$$\begin{aligned}\Delta T^I(B^+ \to K^+\pi^+\pi^-) - \Delta T^I(B^+ \to K^+K^+K^-) &= -[\Delta T^I(B^+ \to \pi^+K^+K^-) - \Delta T^I(B^+ \to \pi^+\pi^+\pi^-)]\\ &= -[a_4^{T^I}(6) + 3a_4^{T^I}(\overline{15}) + b_3^{T^I}(\bar{3}) + b_4^{T^I}(6) + 3b_4^{T^I}(\overline{15}) + c_2^{T^I}(\bar{3})\\ &\quad - c_2^{T^I}(6) + c_3^{T^I}(6) - c_5^{T^I}(6) - c_2^{T^I}(\overline{15}) - c_3^{T^I}(\overline{15}) - 2c_4^{T^I}(\overline{15})\\ &\quad + 3c_5^{T^I}(\overline{15}) - d_5^{T^I}(6) + 3d_5^{T^I}(\overline{15})].\end{aligned} \tag{4.5}$$

With the momentum-dependent corrections to the $SU(3)$ and isospin breaking effects and detailed analyses similar to those carried out in Sec. III, we have found that the following relation is still true to the order we are considering:

$$\Delta T^p(B^+ \to K^+\pi^+\pi^-) - \Delta T^p(B^+ \to K^+K^+K^-)$$
$$= -[\Delta T^p(B^+ \to \pi^+K^+K^-) - \Delta T^p(B^+ \to \pi^+\pi^+\pi^-)]. \tag{4.6}$$

Here, ΔT^p includes both the momentum-dependent corrections to the $SU(3)$ and the isospin breaking effects.

The above leads to the following relation, which is not affected by first-order $SU(3)$ breaking effects due to strange up and down quark mass differences,

$$T(B^+ \to K^+\pi^+\pi^-)_{\mathrm{FS}} - T(B^+ \to K^+K^+K^-)_{\mathrm{FS}}$$
$$= T(B^+ \to \pi^+\pi^+\pi^-)_{\mathrm{FS}} - T(B^+ \to \pi^+K^+K^-)_{\mathrm{FS}} \neq 0, \tag{4.7}$$

and similarly for penguin amplitudes P_{FS}.

When the relevant decay amplitudes are measured precisely, one can also obtain useful information for B decays in the SM.

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