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# Suggestion for measuring the weak dipole moment of $\tau$ lepton at $Z$ factory

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Three new observables for $Z$ factory to measure the weak dipole moment of $\tau$ lepton ($d_w^\tau$) are proposed. As those observables are employed at LEP-I, the new observables depend on the real and imaginary parts of $d_w^\tau$ linearly; thus, we calculate the slopes, i.e., the linear coefficients of the dependence, precisely and find that the obtained slopes are much lager than those employed at LEP-I. It means that the signal for the weak dipole moment $d_w^\tau$ may be enhanced quite a lot if one employs the new observables. Being superior to those at LEP-I, we recommend measuring the weak dipole moment $d_w^\tau$ in terms of the new observables if reanalyzing the data of LEP-I or doing the measurement at a possible $Z$ factory in the near future.

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## I. INTRODUCTION

The weak electric dipole moment of a charged lepton $l$ ($l = e, \mu, \tau$), $d_w^l$, which is called as the "weak dipole moment" of the lepton $l$ in this paper, merges in the $Zl^-l^+$ coupling; hence, the most sensitive place to observe it is at an $e^+e^-$ collider running at center-of-mass energy $\sqrt{S} = m_Z$, i.e., a $Z$ factory. At born level of the Standard Model (SM), the dipole moment is zero, although high-order correction may make it nonzero. In fact, the prediction of SM is that it is far below the present experiment sensitivity. Thus, a sizable nonzero signal of $d_w^l$ will be an unambiguous indication for new physics beyond SM. Moreover, the weak dipole moment $d_w^l$ is also used to parameterize a kind of the $CP$-violating effects relevant to $l$ lepton and $Z$ boson, which is known in many models beyond SM [1,2] to be proportional to the mass of the lepton $l$. Among the charged leptons, the lepton $\tau$ is the heaviest one and is "unstable" (in detector size); thus, the observation of $d_w^\tau$ is especially interesting. For this reason, LEP-I has made a lot of efforts on the observation of the weak dipole moment of $\tau$-lepton $d_w^\tau$ [3–9]. Therefore, it is sensible to improve the observation of the weak dipole moment $d_w^\tau$ further for reanalyzing the data of LEP-I and/or for observing it at a new $Z$ factory with a much higher luminosity. Note that besides ILC such a new $Z$ factory is indeed under consideration now [10]. Since we believe that the $\tau$ weak dipole moment $d_w^\tau$ may offer hints on the $CP$ violation beyond the Cabibbo-Kobayashi-Maskawa (CKM) phase in comparison with the other leptons and precisely measuring it may help to search for clues of new physics beyond SM, we would like to highlight the observation of the $\tau$ weak dipole moment $d_w^\tau$ in this paper.

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The observations of the weak dipole moment $d_w^\tau$ at LEP-I are based on $CP$-odd observables [11–13], which are constructed by the unit momenta of the decay products of $\tau^+$ and $\tau^-$, and it is found that the mean values of the observables are proportional to the real part of the weak dipole moment $d_w^\tau$. Namely, in most methods adopted by LEP-I, the flying directions of $\tau$ decay products are used for constructing the observables, but the information about the momenta and spins of the $\tau$ leptons is not used at all. Only two optimal observables, which depend on the directions and spins of the $\tau$ leptons, were employed in Ref. [6], where both the real and imaginary parts of $d_w^\tau$ were measured. The useful investigation for experimental measurements on the differential cross sections with different spins of $\tau$ leptons, as well as an asymmetry, which depend on the directions of $\tau$ leptons and their products, etc., is found in Refs. [14,15]. Based on these theoretical consideration, the weak dipole moment $d_w^\tau$ was measured by the ALEPH, OPAL, and L3 Collaborations at LEP-I [3–9], and as results and examples of the experimental observations, the best bound on $d_w^\tau$

$$\mathrm{Re}(d_w^\tau) < 0.5 \times 10^{-17}\ \mathrm{e\,cm},$$
$$\mathrm{Im}(d_w^\tau) < 1.1 \times 10^{-17}\ \mathrm{e\,cm} \tag{1}$$

was reached by the ALPEH Collaboration in 2003 [9].

In this paper, based on investigating the observables that describe the asymmetries relating to T-odd and T-even operators and to pursue improving the sensibility in measuring the weak dipole moment $d_w^\tau$, we show that at least three kinds of observables may raise the sensibility.

In order to suppress the background from SM, the relevant observables are constructed by applying the asymmetries related to particles and their antiparticles, etc. Furthermore, phase space of $\tau$-production processes is divided into two parts for the purpose of enhancing the signal of these observables. These newly constructed observables, to which the SM contribution is canceled, can be expressed as the linear functions of the real and imaginary parts of the weak dipole moment $d_w^\tau$. Their slope coefficients, i.e., the dependence on the couplings of $Z$ to leptons, the kinematics of the processes, as well as the decay modes of $\tau$ lepton, are calculated numerically. In comparison with the observables employed at LEP-I, it is shown that the new observables reflecting the asymmetries are much more sensitive to the real and imaginary parts of the weak dipole moment $d_w^\tau$ than those employed at LEP-I.

The paper is organized as follows. In Sec. II, three new kinds of observables are constructed, and important properties of them are pointed out. Observables employed at LEP-I are reviewed in Sec. III. In Sec. IV, the precise numerical values for the slope coefficients of the new observables, as well as those employed at LEP-I, are presented, and comparisons with that of previous observables are made. Brief discussions and a conclusion are at the end of this section.

## II. NEW OBSERVABLES SENSITIVE TO THE WEAK DIPOLE MOMENT $d_w^\tau$

A kind of $CP$-violating effects in the process $e^+e^- \to \tau^+\tau^-$ at Z pole is parameterized by the $\tau$ lepton weak dipole moment $d_w^\tau$ via the effective vertex for $Z\tau^+\tau-$ coupling as follows:

$$\mathcal{L}_{\rm eff} = ie\bar{\tau}\left[\gamma^\mu V_Z + \gamma^\mu\gamma_5 A_Z + \frac{d_w^\tau}{e}k_\nu\sigma^{\mu\nu}\gamma_5\right]\tau Z_\mu, \quad (2)$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu],$$

$k$ is the momentum of the gauge boson $Z$;

$$V_Z = \frac{-1+4s_w^2}{4s_wc_w}, \qquad A_Z = -\frac{1}{4s_wc_w} \quad (3)$$

are vector and axial vector couplings of gauge boson $Z$ to the charged leptons in SM, respectively; $s_w = \sin\theta_w$, $c_w = \cos\theta_w$, where $\theta_w$ is the weak mixing angle. Note that here we highlight that of $\tau$ lepton, so the coupling of $Z$ boson to electron is assumed as that predicted by SM at tree level.

Now let us consider the process

$$\begin{aligned} e^-(p_1) + e^+(p_2) &\to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \\ &\to a(q^-) + \bar{b}(q^+) + \nu_\tau(k_1) + \bar{\nu}_\tau(k_2) \end{aligned} \quad (4)$$

at Z pole, where $a$ and $\bar{b}$ are mesons originating from $\tau^-$ and $\tau^+$ decays, respectively. The relevant Feynman diagram for this process is shown in Fig. 1. In Eq. (4), $p_1$ and $p_2$ are 4-momenta of incoming $e^-$ and $e^+$; $p_{\tau^-}$ and $p_{\tau^+}$ are 4-momenta of outgoing $\tau^-$ and $\tau^+$ leptons; $q^-$, $q^+$, $k_1$, and $k_2$ are 4-momenta of particles $a$, $\bar{b}$ and neutrinos $\nu_\tau$, $\bar{\nu}_\tau$ in final state, respectively. In this paper, we also denote $\vec{p}$ as the 3-momentum of incoming $e^-$, $\vec{p}_{\tau^-}$ as the 3-momentum of outgoing $\tau^-$, $\vec{q}^-$ and $\vec{q}^+$ as the 3-momenta of final particles $a$ and $\bar{b}$, respectively. Moreover, the reference frame here is chosen such that the incoming $e^-$ is along with the $z$ axis, and the momentum of the outgoing $\tau^-$ is in the $x-z$ plane; the angel $\theta_{\tau^-}$ is determined by the two momenta $\vec{p}_{\tau^-}$ and $\vec{p}$ as $\cos\theta_{\tau^-} = \frac{\vec{p}_{\tau^-}}{|\vec{p}_{\tau^-}|}\cdot\frac{\vec{p}}{|\vec{p}|}$ (see Fig. 3 in the Appendix). The amplitude for the process is expressed as

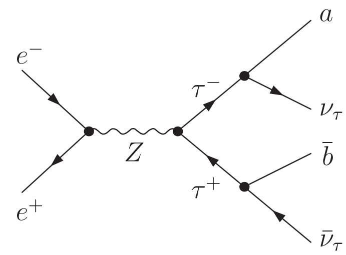


FIG. 1. Feynman diagram for process $e^-e^+ \to \tau^-\tau^+ \to a\bar{b}\nu_\tau\bar{\nu}_\tau$.

$$\begin{aligned} M_1^{a\bar{b}} &= ie\bar{v}(p_2)\gamma^\mu(V_Z+\gamma^5A_Z)u(p_1)\chi^Z_{\mu\nu} \\ &\times \frac{G}{\sqrt{2}}\bar{u}(k_1)\gamma^\alpha(1-\gamma^5)J^a_\alpha\chi^{\tau^-} \\ &\times ie\left(\gamma^\nu V_Z + \gamma^\nu\gamma_5A_Z + \frac{d_w^\tau}{e}k_\delta\sigma^{\delta\nu}\gamma_5\right) \\ &\times \chi^{\tau^+}\frac{G}{\sqrt{2}}\gamma^\beta(1-\gamma^5)v(k_2)J^{\bar{b}}_\beta. \end{aligned} \quad (5)$$

Here $G$ is the Fermi coupling constant, and $\chi^Z_{\mu\nu}$ is the propagator of gauge boson $Z$:

$$\chi^Z_{\mu\nu} = \frac{i(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_Z^2})}{k^2 - m_Z^2 + {\rm i}m_Z\Gamma_Z}. \quad (6)$$

$\Gamma_Z$ is the total decay width of gauge boson $Z$, and $k = p_1 + p_2$ is the four momentum of $Z$; i.e., $k^2 = S$ where $\sqrt{S}$ is the center-of-mass energy. Moreover,

$$\begin{aligned} \chi^{\tau^-} &= \frac{i(\not{p}_{\tau^-} + m_\tau)}{p^2_{\tau^-} - m_\tau^2 + im_\tau\Gamma_\tau}, \\ \chi^{\tau^+} &= \frac{i(-\not{p}_{\tau^+} + m_\tau)}{p^2_{\tau^+} - m_\tau^2 + im_\tau\Gamma_\tau}, \end{aligned} \quad (7)$$

are the propagators of leptons $\tau^-$ and $\tau^+$, respectively, where $\Gamma_\tau$ is the total decay width of $\tau$ leptons. In the expression, the weak hadronic currents $J_\alpha^{\pi^-}$, $J_\beta^{\pi^+}$, $J_\alpha^{\rho^-}$, and $J_\beta^{\rho^+}$ in Eq. (5) are as follows:

$$\begin{aligned} J_\alpha^{\pi^-} &= f_\pi V_{ud}q^-_\alpha, \qquad & J_\beta^{\pi^+} &= f_\pi V^*_{ud}q^+_\beta, \\ J_\alpha^{\rho^-} &= f_\rho V_{ud}\varepsilon^*_\alpha, \qquad & J_\beta^{\rho^+} &= f_\rho V^*_{ud}\varepsilon_\beta, \end{aligned} \quad (8)$$

where $f_\pi$ is the decay constant of $\pi$, $f_\rho$ is the decay constant of $\rho$, $V_{ud}$ is the CKM matrix element, and $\varepsilon$ is the polarization vector of $\rho$.

Then we shift to consider the pure leptonic decays of the $\tau$-lepton pair instead of the semileptonic decays, namely, the process

$$\begin{aligned} e^-(p_1) + e^+(p_2) &\to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \\ &\to a(q^-) + \bar{b}(q^+) + \nu_\tau(k_1) + \bar{\nu}_\tau(k_2) \\ &\quad + \bar{\nu}_a(k_3) + \nu_b(k_4) \end{aligned} \tag{9}$$

at Z pole, where $a$ and $b$ are leptons. The corresponding amplitude for the process has the same form as that in Eq. (5), but with different expressions for $J^a_\alpha$ and $J^{\bar{b}}_\beta$,

$$\begin{aligned} J^a_\alpha &= \bar{u}(q^-)\gamma_\alpha(1-\gamma^5)v(k_3), \\ J^{\bar{b}}_\beta &= \bar{u}(k_4)\gamma_\beta(1-\gamma^5)v(q^+). \end{aligned} \tag{10}$$

The differential cross section of processes in Eqs. (4) and (9) is expressed as

$$d\sigma^{a\bar{b}}_1 = \frac{1}{2s}\overline{|M^{a\bar{b}}_1|^2} dLips_{2\to n}. \tag{11}$$

$dLips_{2\to n}$ is the phase space of $2 \to n$ processes, $n = 4$ for the process in Eq. (4), $n = 6$ for the process in Eq. (9). $\overline{|M^{a\bar{b}}_1|^2}$ is the amplitude squared averaging over spins of initial states and summing over spins of final states.

The observable, relating concerned asymmetry, which involves only the unit momenta of the final particles $a$ and $\bar{b}$, may be defined as follows:

$$A^{a\bar{b}}_1 = \frac{\sigma^{a\bar{b}}_1(+) - \sigma^{a\bar{b}}_1(-)}{\sigma^{a\bar{b}}_1(+) + \sigma^{a\bar{b}}_1(-)}, \tag{12}$$

where the cross section $\sigma^{a\bar{b}}_1(+)$ and $\sigma^{a\bar{b}}_1(-)$ are defined as

$$\begin{aligned} \sigma^{a\bar{b}}_1(+) &= \sigma^{a\bar{b}}_1((\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0), \\ \sigma^{a\bar{b}}_1(-) &= \sigma^{a\bar{b}}_1((\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0). \end{aligned}$$

Here $\hat{q}^- = \frac{\vec{q}^-}{|\vec{q}^-|}$ is the flying direction of particles $a$, and $\hat{q}^+ = \frac{\vec{q}^+}{|\vec{q}^+|}$ is the flying direction of particles $\bar{b}$. $\hat{p} = \frac{\vec{p}}{|\vec{p}|}$ is the direction of the incoming $e^-$ beam. The observable $A^{a\bar{b}}_1$ is proportional to the weak dipole moment $d^\tau_w$ when the final charged particles $a$ and $b$ are the same type of particles, such as $\pi^-$ and $\pi^+$ ($\rho^-$ and $\rho^+$, etc.) in the final state. Note that SM also contributes to $A^{a\bar{b}}_1$, when the types of particles $a$ and $b$ are different from each other, such as $\rho^-$ and $\pi^+$ ($\pi^-$ and $\rho^+$, etc.) in the final states.

It is necessary to consider the $CP$-conjugated processes for Eqs. (4) and (9):

$$\begin{aligned} e^-(p_1) + e^+(p_2) &\to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \\ &\to b(q^-) + \bar{a}(q^+) + \nu_\tau(k_1) + \bar{\nu}_\tau(k_2) \end{aligned}$$

and

$$\begin{aligned} e^-(p_1) + e^+(p_2) &\to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \\ &\to b(q^-) + \bar{a}(q^+) + \nu_\tau(k_1) + \bar{\nu}_\tau(k_2) \\ &\quad + \bar{\nu}_b(k_3) + \nu_a(k_4). \end{aligned} \tag{13}$$

The amplitude for Eq. (13), $M^{b\bar{a}}_1$, can be obtained by the following replacements,

$$M^{b\bar{a}}_1 = M^{a\bar{b}}_1(J^a_\alpha \to J^b_\alpha, J^{\bar{b}}_\beta \to J^{\bar{a}}_\beta). \tag{14}$$

The corresponding differential cross section $d\sigma^{b\bar{a}}_1$ is calculated similarly to $d\sigma^{a\bar{b}}_1$ [see Eq. (11)]. The observable that is analogous to that in Eq. (12) is defined as

$$A^{b\bar{a}}_1 = \frac{\sigma^{b\bar{a}}_1(+) - \sigma^{b\bar{a}}_1(-)}{\sigma^{b\bar{a}}_1(+) + \sigma^{b\bar{a}}_1(-)}, \tag{15}$$

where $\sigma^{b\bar{a}}_1(+)$ and $\sigma^{b\bar{a}}_1(-)$ are expressed as follows

$$\begin{aligned} \sigma^{b\bar{a}}_1(+) &= \sigma^{b\bar{a}}_1((\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0), \\ \sigma^{b\bar{a}}_1(-) &= \sigma^{b\bar{a}}_1((\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0). \end{aligned}$$

It is found that in the sum of $A^{a\bar{b}}_1$ and $A^{b\bar{a}}_1$ the contribution from SM is canceled. Thus, we may define a new general observable as follows:

$$A_1 \equiv \frac{1}{2}(A^{a\bar{b}}_1 + A^{b\bar{a}}_1). \tag{16}$$

It is proportional to the weak dipole moment $d^\tau_w$ and can be written as a linear function of real and imaginary parts of $d^\tau_w$:

$$A_1 = \frac{m_Z}{e}[f_1\mathrm{Re}(d^\tau_w) + g_1\mathrm{Im}(d^\tau_w)], \tag{17}$$

where the slope coefficients $f_1$ and $g_1$ are dimensionless and can be calculated numerically.

Note that these coefficients $f_1$ and $g_1$ for observable $A_1$ are quite small so that it is almost impossible to pick up the signal from backgrounds experimentally. Fortunately, the observable $A_1$ has a character that approximately varies as a trigonometric function $\sin(\cos\theta_{\tau^-})$, so it seems that this property may be applied to the measurement of the weak dipole moment via constructing the observables so as to increase the sensitivity (to enhance the signals). Indeed, the property can be applied to enhance the signals and the precision by introducing some new observables, for which the relevant phase space according to $\cos\theta_{\tau^-} > 0$ and $\cos\theta_{\tau^-} < 0$ is divided into two pieces. As such, the cross sections $\sigma_1^{a\bar{b}}$ for the processes in Eqs. (4) and (9) are divided as below:

$$\sigma_1^{a\bar{b}}(++) = \sigma_1^{a\bar{b}}(\cos\theta_{\tau^-} > 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_1^{a\bar{b}}(+-) = \sigma_1^{a\bar{b}}(\cos\theta_{\tau^-} > 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0),$$
$$\sigma_1^{a\bar{b}}(-+) = \sigma_1^{a\bar{b}}(\cos\theta_{\tau^-} < 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_1^{a\bar{b}}(--) = \sigma_1^{a\bar{b}}(\cos\theta_{\tau^-} < 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0).$$

We define the observables $A_1^{a\bar{b}}(+)$ and $A_1^{a\bar{b}}(-)$:

$$A_1^{a\bar{b}}(+) = \frac{\sigma_1^{a\bar{b}}(++) - \sigma_1^{a\bar{b}}(+-)}{\sigma_1^{a\bar{b}}(++) + \sigma_1^{a\bar{b}}(+-)},$$
$$A_1^{a\bar{b}}(-) = \frac{\sigma_1^{a\bar{b}}(-+) - \sigma_1^{a\bar{b}}(--)}{\sigma_1^{a\bar{b}}(-+) + \sigma_1^{a\bar{b}}(--)}. \tag{18}$$

In a similar way, the observables $A_1^{b\bar{a}}(+)$ and $A_1^{b\bar{a}}(-)$ related to processes in Eq. (13) are also defined:

$$A_1^{b\bar{a}}(+) = \frac{\sigma_1^{b\bar{a}}(++) - \sigma_1^{b\bar{a}}(+-)}{\sigma_1^{b\bar{a}}(++) + \sigma_1^{b\bar{a}}(+-)},$$
$$A_1^{b\bar{a}}(-) = \frac{\sigma_1^{b\bar{a}}(-+) - \sigma_1^{b\bar{a}}(--)}{\sigma_1^{b\bar{a}}(-+) + \sigma_1^{b\bar{a}}(--)}. \tag{19}$$

Here, $\sigma_1^{b\bar{a}}(++)$, $\sigma_1^{b\bar{a}}(+-)$, $\sigma_1^{b\bar{a}}(-+)$, and $\sigma_1^{b\bar{a}}(--)$ are as follows:

$$\sigma_1^{b\bar{a}}(++) = \sigma_1^{b\bar{a}}(\cos\theta_{\tau^-} > 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_1^{b\bar{a}}(+-) = \sigma_1^{b\bar{a}}(\cos\theta_{\tau^-} > 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0),$$
$$\sigma_1^{b\bar{a}}(-+) = \sigma_1^{b\bar{a}}(\cos\theta_{\tau^-} < 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_1^{b\bar{a}}(--) = \sigma_1^{b\bar{a}}(\cos\theta_{\tau^-} < 0, (\hat{q}^+ \times \hat{q}^-)\cdot\hat{p} < 0).$$

In order to cancel the contribution from SM, we further define new observables as follows:

$$A_1(+) \equiv \frac{1}{2}(A_1^{a\bar{b}}(+) + A_1^{b\bar{a}}(+)),$$
$$A_1(-) \equiv \frac{1}{2}(A_1^{a\bar{b}}(-) + A_1^{b\bar{a}}(-)). \tag{20}$$

Moreover, a new type of observable, which will greatly enhance the signal, is constructed,

$$A_1' \equiv A_1(+) - A_1(-). \tag{21}$$

It depends on the real and imaginary parts of the weak dipole moment $d_w^\tau$ linearly:

$$A_1' = \frac{m_Z}{e}[f_1'\mathrm{Re}(d_w^\tau) + g_1'\mathrm{Im}(d_w^\tau)], \tag{22}$$

where the coefficients $f_1'$ and $g_1'$ are dimensionless and can be evaluated by numerical calculations.

Another new observable, which is expected to be more sensitive to the imaginary part of the weak dipole moment, is defined by a T-even operator $(\hat{q}^+ + \hat{q}^-)\cdot\hat{p}$:

$$A_2^{a\bar{b}} = \frac{\sigma_2^{a\bar{b}}(+) - \sigma_2^{a\bar{b}}(-)}{\sigma_2^{a\bar{b}}(+) + \sigma_2^{a\bar{b}}(-)},$$
$$A_2^{b\bar{a}} = \frac{\sigma_2^{b\bar{a}}(+) - \sigma_2^{b\bar{a}}(-)}{\sigma_2^{b\bar{a}}(+) + \sigma_2^{b\bar{a}}(-)}, \tag{23}$$

where the cross sections $\sigma_2^{a\bar{b}}(+)$, $\sigma_2^{a\bar{b}}(-)$, $\sigma_2^{b\bar{a}}(+)$, and $\sigma_2^{b\bar{a}}(-)$ are defined as

$$\sigma_2^{a\bar{b}}(+) = \sigma_1^{a\bar{b}}((\hat{q}^+ + \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_2^{a\bar{b}}(-) = \sigma_1^{a\bar{b}}((\hat{q}^+ + \hat{q}^-)\cdot\hat{p} < 0),$$
$$\sigma_2^{b\bar{a}}(+) = \sigma_1^{b\bar{a}}((\hat{q}^+ + \hat{q}^-)\cdot\hat{p} > 0),$$
$$\sigma_2^{b\bar{a}}(-) = \sigma_1^{b\bar{a}}((\hat{q}^+ + \hat{q}^-)\cdot\hat{p} < 0).$$

The observable $A_2^{a\bar{b}}$ has the similar properties as $A_1^{a\bar{b}}$; i.e., the contribution from SM is canceled when $a$ and $b$ are the same type of particles, while when the type of particle $a$ is different from that of particle $b$, SM does contribute some to this observable. Whereas if we add $A_2^{a\bar{b}}$ and $A_2^{b\bar{a}}$ together, the contribution from SM is canceled. The new observable is proportional to the weak dipole moment $d_w^\tau$ and can be expressed as a linear function of its real and imaginary parts:

$$A_2 \equiv \frac{1}{2}(A_2^{a\bar{b}} + A_2^{b\bar{a}})$$
$$= \frac{m_Z}{e}[f_2\mathrm{Re}(d_w^\tau) + g_2\mathrm{Im}(d_w^\tau)]. \tag{24}$$

The slope coefficients $f_2$ and $g_2$ are dimensionless and can be evaluated from the theoretical calculation.

We also consider processes

$$e^-(p_1) + e^+(p_2) \to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+})$$
$$\to a(q^-) + \nu_\tau(k_1) + \tau^+(p_{\tau^+})$$

and

$$e^-(p_1) + e^+(p_2) \to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+})$$
$$\to a(q^-) + \nu_\tau(k_1) + \bar{\nu}_a(k_3) + \tau^+(p_{\tau^+}) \tag{25}$$

at Z pole. Here the particle $a$ denotes a meson in the first process, while in the second process, it denotes a proper lepton. The diagrams for the first process in Eq. (25) and its $CP$-conjugated process are put in Fig. 2. The amplitudes for the processes in Eq. (25) can be written as

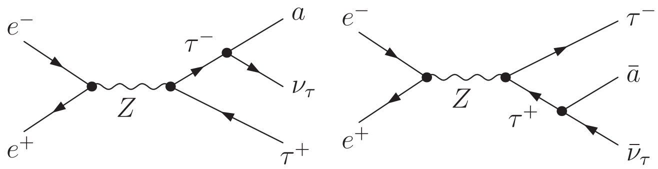


FIG. 2. Feynman diagrams for processes $e^-e^+ \to a\nu_\tau\tau^+$ (left) and $e^-e^+ \to \tau^-\bar{a}\bar{\nu}_\tau$ (right).

$$M_3^a = ie\bar{v}(p_2)\gamma^\mu(V_Z + \gamma^5 A_Z)u(p_1)\chi^Z_{\mu\nu} \times \frac{G}{\sqrt{2}}\bar{u}(k_1)\gamma^\alpha(1-\gamma^5)J^a_\alpha\chi^{\tau^-} \times ie\left(\gamma^\nu V_Z + \gamma^\nu\gamma_5 A_Z + \frac{d_w^\tau}{e}k_\delta\sigma^{\delta\nu}\gamma_5\right)v(p_{\tau^+}), \quad (26)$$

and the expression here for the weak hadronic current $J^a_\alpha$ can be found in Eqs. (8) or (10).

The corresponding $CP$-conjugated processes to Eq. (25)

$$e^-(p_1) + e^+(p_2) \to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \to \tau^-(p_{\tau^-}) + \bar{a}(q^+) + \bar{\nu}_\tau(k_2)$$

and

$$e^-(p_1) + e^+(p_2) \to \tau^-(p_{\tau^-}) + \tau^+(p_{\tau^+}) \to \tau^-(p_{\tau^-}) + \bar{a}(q^+) + \bar{\nu}_\tau(k_2) + \nu_a(k_4) \quad (27)$$

are also investigated. The relevant amplitude can be expressed

$$M_3^{\bar{a}} = ie\bar{v}(p_2)\gamma^\mu(V_Z + \gamma^5 A_Z)u(p_1)\chi^Z_{\mu\nu}\bar{u}(p_{\tau^-}) \times ie\left(\gamma^\nu V_Z + \gamma^\nu\gamma_5 A_Z + \frac{d_w^\tau}{e}k_\delta\sigma^{\delta\nu}\gamma_5\right) \times \chi^{\tau^+}\frac{G}{\sqrt{2}}\gamma^\beta(1-\gamma^5)v(k_2)J^{\bar{a}}_\beta, \quad (28)$$

where the definition for $J^{\bar{a}}_\beta$ is that in Eqs. (8) or (10). The differential cross sections $d\sigma_3^a$ and $d\sigma_3^{\bar{a}}$ can be calculated in terms of the matrix element $M_3^a$ and $M_3^{\bar{a}}$, respectively.

One more new observable, which depends on both the unit momenta of $\tau^-$ and $\tau^+$ and their decay products, can be defined:

$$A_3^a = \frac{\sigma_3^a(+) - \sigma_3^a(-)}{\sigma_3^a(+) + \sigma_3^a(-)}, \quad A_3^{\bar{a}} = \frac{\sigma_3^{\bar{a}}(+) - \sigma_3^{\bar{a}}(-)}{\sigma_3^{\bar{a}}(+) + \sigma_3^{\bar{a}}(-)}, \quad (29)$$

$$\sigma_3^a(+) = \sigma_3^a((\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- > 0),$$
$$\sigma_3^a(-) = \sigma_3^a((\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- < 0),$$
$$\sigma_3^{\bar{a}}(+) = \sigma_3^{\bar{a}}((\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ > 0),$$
$$\sigma_3^{\bar{a}}(-) = \sigma_3^{\bar{a}}((\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ < 0). \quad (30)$$

Here $\hat{p}_{\tau^-} = \frac{\vec{p}_{\tau^-}}{|\vec{p}_{\tau^-}|}$ is the unit momentum of $\tau^-$. Note that SM does contribute some to the observable $A_3^a$ and $A_3^{\bar{a}}$; however, according to the $CP$ property of processes in Eqs. (25) and (27), one may conclude that the contribution from SM in $A_3^a + A_3^{\bar{a}}$ is canceled, and similar discussions can be found in Ref. [16]. Thus, the new general observable can be defined, and again it depends on the real and imaginary parts of weak dipole moment $d_w^\tau$ linearly:

$$A_3 \equiv \frac{1}{2}(A_3^a + A_3^{\bar{a}}) = \frac{m_Z}{e}[f_3\mathrm{Re}(d_w^\tau) + g_3\mathrm{Im}(d_w^\tau)]. \quad (31)$$

Here $f_3$ and $g_3$ are dimensionless coefficients, which can be determined numerically from theoretical calculations.

Note that these slope coefficients $f_3, g_3$ for observable $A_3$ are so small that it is nearly impossible to make a difference between the signal and the background, similar to the case of observable $A_1$. Following the technique dealing with $A_1$, we separate the phase space into two parts: $\cos\theta_{\tau^-} > 0$ and $\cos\theta_{\tau^-} < 0$. To investigate the observable $A_3^a$ by dividing the phase space, the total cross sections $\sigma_3^a$ for processes in Eq. (25) are then divided into four parts,

$$\sigma_3^a(++) = \sigma_3^a(\cos\theta_{\tau^-} > 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- > 0),$$
$$\sigma_3^a(+-) = \sigma_3^a(\cos\theta_{\tau^-} > 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- < 0),$$
$$\sigma_3^a(-+) = \sigma_3^a(\cos\theta_{\tau^-} < 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- > 0),$$
$$\sigma_3^a(--) = \sigma_3^a(\cos\theta_{\tau^-} < 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^- < 0).$$

Observables $A_3^a(+)$ and $A_3^a(-)$ are defined in terms of the division $\cos\theta_{\tau^-} > 0$ and $\cos\theta_{\tau^-} < 0$, respectively

$$A_3^a(+) = \frac{\sigma_3^a(++) - \sigma_3^a(+-)}{\sigma_3^a(++) + \sigma_3^a(+-)}, \quad A_3^a(-) = \frac{\sigma_3^a(-+) - \sigma_3^a(--)}{\sigma_3^a(-+) + \sigma_3^a(--)}. \quad (32)$$

Similarly observables $A_3^{\bar{a}}(+)$ and $A_3^{\bar{a}}(-)$ relating to the processes in Eq. (27) are defined

$$A_3^{\bar{a}}(+) = \frac{\sigma_3^{\bar{a}}(++) - \sigma_3^{\bar{a}}(+-)}{\sigma_3^{\bar{a}}(++) + \sigma_3^{\bar{a}}(+-)}, \quad A_3^{\bar{a}}(-) = \frac{\sigma_3^{\bar{a}}(-+) - \sigma_3^{\bar{a}}(--)}{\sigma_3^{\bar{a}}(-+) + \sigma_3^{\bar{a}}(--)}. \quad (33)$$

$\sigma_3^{\bar{a}}(++)$, $\sigma_3^{\bar{a}}(+-)$, $\sigma_3^{\bar{a}}(-+)$, and $\sigma_3^{\bar{a}}(--)$ are listed in the following:

$$\sigma_3^{\bar{a}}(++) = \sigma_3^{\bar{a}}(\cos\theta_{\tau^-} > 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ > 0),$$
$$\sigma_3^{\bar{a}}(+-) = \sigma_3^{\bar{a}}(\cos\theta_{\tau^-} > 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ < 0),$$
$$\sigma_3^{\bar{a}}(-+) = \sigma_3^{\bar{a}}(\cos\theta_{\tau^-} < 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ > 0),$$
$$\sigma_3^{\bar{a}}(--) = \sigma_3^{\bar{a}}(\cos\theta_{\tau^-} < 0, (\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^+ < 0).$$

To cancel the contribution from SM, the general observables are defined as follows:

$$A_3(+) \equiv \frac{1}{2}(A_3^a(+) + A_3^{\bar{a}}(+)),$$
$$A_3(-) \equiv \frac{1}{2}(A_3^a(-) + A_3^{\bar{a}}(-)). \quad (34)$$

A new type of observable that will greatly enhance the signal is constructed,

$$A_3' \equiv A_3(+) - A_3(-), \quad (35)$$

which can be expressed as a linear function as the real and imaginary parts of the weak dipole moment $d_w^\tau$,

$$A_3' = \frac{m_Z}{e}[f_3'\mathrm{Re}(d_w^\tau) + g_3'\mathrm{Im}(d_w^\tau)]; \quad (36)$$

the dimensionless coefficients $f_3'$ and $g_3'$ can be determined numerically.

In this section, three types of new observables $A_1'$, $A_2$, and $A_3'$ are constructed via operators $(\hat{q}^+\times\hat{q}^-)\cdot\hat{p}$, $(\hat{q}^+ + \hat{q}^-)\cdot\hat{p}$ and $(\hat{p}\times\hat{p}_{\tau^-})\cdot\hat{q}^\pm$, respectively. These new observables are found to be proportional to the real and imaginary parts of the weak dipole moment $d_w^\tau$, with slope coefficients dependent on the SM coupling of gauge boson $Z$ to leptons, the kinematics of processes, and the decay modes of $\tau$ leptons. These coefficients can be calculated numerically for different decay products of $\tau^-$ and $\tau^+$. Once these newly constructed observables are measured, the real and imaginary parts of the weak dipole moment $d_w^\tau$ can be extracted by solving the equations in Eqs. (22), (24), and (36).

## III. THE OBSERVABLES EMPLOYED BY LEP-I FOR MEASURING $d_w^\tau$

To determine the weak dipole moment $d_w^\tau$ at $Z$ pole, several observables and methods were employed at LEP-I. In this section, some of the observables used at LEP-I are reviewed briefly.

In Refs. [11–13], the $CP$-odd tensor observables, which are constructed by the momenta or unit momenta of the decay products of $\tau^+$ and $\tau^-$,

$$T_{ij} = (\vec{q}^+ - \vec{q}^-)_i(\vec{q}^+\times\vec{q}^-)_j + (i\to j)$$

or

$$\hat{T}_{ij} = (\hat{q}^+ - \hat{q}^-)_i\frac{(\hat{q}^+\times\hat{q}^-)_j}{|\hat{q}^+\times\hat{q}^-|} + (i\to j) \quad (37)$$

are studied carefully. Here $i, j = 1, 2, 3$ are Cartesian vector indices. Expectation values for them are proportional to the real part of the weak dipole moment $d_w^\tau$ as

$$\langle T_{ij}\rangle = \frac{m_Z}{e}cs_{ij}\mathrm{Re}(d_w^\tau), \qquad \langle\hat{T}_{ij}\rangle = \frac{m_Z}{e}\hat{c}s_{ij}\mathrm{Re}(d_w^\tau). \quad (38)$$

Here $s_{ij}$ is the tensor polarization of the intermediate $Z$ state, which can be written as $\mathrm{diag}(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3})$ if the beam direction is identified with the three-axis. The coefficients $c$ and $\hat{c}$ are calculated precisely in Ref [13]. Note that we have checked the results; i.e., the same values for these coefficients as those in Ref. [13] are obtained. By measuring observables in Eq. (37), an up-bound for the real part of the weak dipole moment $d_w^\tau$ was obtained by OPAL and ALEPH Collaborations at LEP-I [3–5]. Note that the advantage of this method is that only the flying directions of $\tau$ decay products are enough for the measurement; namely, any information about the momenta and spins of $\tau$ leptons is not needed.

In Ref. [6], the so-called optimal T-even and T-odd observables, $\hat{Q}^+$ and $\hat{Q}^-$, are employed to measure the $CP$-violating effects in process $Z\to\tau^-\tau^+$ by OPAL collaboration. They are defined as follows:

$$\hat{Q}^+ = \frac{M_{CP}^{\mathrm{Im}}}{M_{\mathrm{SM}}}, \qquad \hat{Q}^- = \frac{M_{CP}^{\mathrm{Re}}}{M_{\mathrm{SM}}}, \quad (39)$$

where

$$M_{CP}^{\mathrm{Im}} = (\hat{p}_{\tau^-}\cdot\hat{p})[(\hat{p}_{\tau^-}\cdot\hat{s}_+)(\hat{p}\cdot\hat{s}_-) - (\hat{p}_{\tau^-}\cdot\hat{s}_-)(\hat{p}\cdot\hat{s}_+)],$$
$$M_{CP}^{\mathrm{Re}} = (\hat{p}_{\tau^-}\cdot\hat{p})(\hat{p}_{\tau^-}\times(\hat{s}_+ - \hat{s}_-))\cdot\hat{p},$$
$$M_{\mathrm{SM}} = 1 + (\hat{p}_{\tau^-}\cdot\hat{p})^2 + \hat{s}_+\cdot\hat{s}_-(1 - (\hat{p}_{\tau^-}\cdot\hat{p})^2) - 2(\hat{p}\cdot\hat{s}_+)(\hat{p}\cdot\hat{s}_-) + 2(\hat{p}_{\tau^-}\cdot\hat{p}) \times[(\hat{p}_{\tau^-}\cdot\hat{s}_+)(\hat{p}\cdot\hat{s}_-) + (\hat{p}_{\tau^-}\cdot\hat{s}_-)(\hat{p}\cdot\hat{s}_+)].$$

Here $\hat{s}_+$ and $\hat{s}_-$ are spin vectors of $\tau^+$ and $\tau^-$ leptons in their respective rest frame, respectively.[1] Note that only the unit momenta and spins of $\tau^-$ and $\tau^+$ leptons are required in order to obtain the average values for operators in Eq. (39). Theoretically, the expectation values for these T-even and T-odd operators are proportional to the imaginary part and real part of $d_w^\tau$, respectively;

$$\langle\hat{Q}^+\rangle = \frac{m_Z}{e}g_{\mathrm{LEP}}\mathrm{Im}(d_w^\tau),$$
$$\langle\hat{Q}^-\rangle = \frac{m_Z}{e}f_{\mathrm{LEP}}\mathrm{Re}(d_w^\tau). \quad (40)$$

[1]Note that the definition of T-even and T-odd operators in Sec. 2 of Ref. [6] is exchanged in comparison with the definition in Eq. (39). We suspect that there are misprints in Ref. [6].

The values of the coefficients $f_{\rm LEP}$ and $g_{\rm LEP}$ corresponding to the different decay modes of $\tau^-$ and $\tau^+$ leptons were obtained by simulation and are presented in Table 4 of Ref. [6]. Note that the advantage of the chosen observables is that both the real and imaginary parts of the weak dipole moment $d_w^\tau$ can be determined, while only real part of $d_w^\tau$ is measured by employing operators in Eq. (37). Moreover, choices of the observables in Eq. (39) can optimize the signal-to-background ratio in the measurements [6].

In Ref. [14], the following observables relating to $\tau$ decay products were defined

$$A_{\rm sc}^a = \frac{\sigma_{\rm sc}^a(+) - \sigma_{\rm sc}^a(-)}{\sigma_{\rm sc}^a(+) + \sigma_{\rm sc}^a(-)},$$
$$A_{\rm sc}^{\bar a} = \frac{\sigma_{\rm sc}^{\bar a}(+) - \sigma_{\rm sc}^{\bar a}(-)}{\sigma_{\rm sc}^{\bar a}(+) + \sigma_{\rm sc}^{\bar a}(-)}, \qquad (41)$$

where the cross section $\sigma_{\rm sc}^a(+)$, $\sigma_{\rm sc}^a(-)$, $\sigma_{\rm sc}^{\bar a}(+)$, and $\sigma_{\rm sc}^{\bar a}(-)$ expressed as

$$\sigma_{\rm sc}^a(+) = \left[\int_0^1 d(\cos\theta_{\tau^-})\int_0^\pi d\Phi_a + \int_{-1}^0 d(\cos\theta_{\tau^-})\int_\pi^{2\pi} d\Phi_a\right]\frac{d\sigma_3^a}{d(\cos\theta_{\tau^-})d\Phi_a},$$
$$\sigma_{\rm sc}^a(-) = \left[\int_0^1 d(\cos\theta_{\tau^-})\int_\pi^{2\pi} d\Phi_a + \int_{-1}^0 d(\cos\theta_{\tau^-})\int_0^{\pi} d\Phi_a\right]\frac{d\sigma_3^a}{d(\cos\theta_{\tau^-})d\Phi_a},$$
$$\sigma_{\rm sc}^{\bar a}(+) = \left[\int_0^1 d(\cos\theta_{\tau^-})\int_0^\pi d\Phi_{\bar a} + \int_{-1}^0 d(\cos\theta_{\tau^-})\int_\pi^{2\pi} d\Phi_{\bar a}\right]\frac{d\sigma_3^{\bar a}}{d(\cos\theta_{\tau^-})d\Phi_{\bar a}},$$
$$\sigma_{\rm sc}^{\bar a}(-) = \left[\int_0^1 d(\cos\theta_{\tau^-})\int_\pi^{2\pi} d\Phi_{\bar a} + \int_{-1}^0 d(\cos\theta_{\tau^-})\int_0^{\pi} d\Phi_{\bar a}\right]\frac{d\sigma_3^{\bar a}}{d(\cos\theta_{\tau^-})d\Phi_{\bar a}}. \qquad (42)$$

Here $\Phi_a$ and $\Phi_{\bar a}$ are defined in Fig. 3 in the Appendix; $d\sigma_3^a$ and $d\sigma_3^{\bar a}$ are differential cross sections calculated from the processes in Eqs. (25) and (27), respectively. Both the observables $A_{\rm sc}^a$ and $A_{\rm sc}^{\bar a}$ are free from the backgrounds due to SM and to satisfy the relation [14]

$$A_{\rm sc}^a = A_{\rm sc}^{\bar a} \propto {\rm Re}(d_w^\tau).$$

A genuine $CP$-violating observable is defined

$$A_{\rm sc}^{CP} = \frac{1}{2}(A_{\rm sc}^a + A_{\rm sc}^{\bar a}), \qquad (43)$$

which can be expressed as the linear function of the real part of the weak dipole moment $d_w^\tau$,

$$A_{\rm sc}^{CP} = \frac{m_Z}{e} f_{\rm sc}{\rm Re}(d_w^\tau). \qquad (44)$$

Here $f_{\rm sc}$ is the dimensionless coefficients that depends on different decay modes of $\tau$ leptons and can be calculated numerically. Note that the OPAL and L3 Collaborations did employ this observable to measure the weak dipole moment of the $\tau$ lepton at LEP-I [7,8].

The latest bound on the real part and imaginary part of $d_w^\tau$ was determined by the ALPEH Collaboration in 2003 [9]. This measurement is based on the method of maximum likelihood fit to the date, taking into account the differential cross section with different spins of $\tau$ leptons, including spin correlation. The differential cross section of $e^-e^+ \to \tau^-\tau^+$ can be expressed as [9,15]

$$\frac{d\sigma}{d\cos\theta_{\tau^-}}(\hat s_-, \hat s_+) = R_{00} + \sum_{\mu=1,3} R_{\mu 0} s_-^\mu + \sum_{\nu=1,3} R_{0\nu} s_+^\nu + \sum_{\mu,\nu=1,3} R_{\mu\nu} s_-^\mu s_+^\nu. \qquad (45)$$

Here $R_{\mu\nu}$ terms are functions of fermion couplings to gauge boson $Z$ and the $\theta_{\tau^-}$. The contribution from SM is separated from those from non-SM by defining

$$(R_{\mu\nu})_\pm = R_{\mu\nu} \pm R_{\nu\mu}.$$

Note that both the real and imaginary parts of weak dipole moment $d_w^\tau$ can be determined by measuring $(R_{\mu\nu})_\pm$.

## IV. NUMERICAL RESULTS AND DISCUSSIONS

In this paper, we highlight the measurements of $d_w^\tau$, the weak dipole moment of $\tau$-lepton, at an $e^+e^-$ collider that runs at $Z$-boson pole (a $Z$ factory). In contrast to low-energy colliders such as B factory and BEPC, such a high-energy collider at $\sqrt{S} \simeq m_Z$ has an additional advantage besides the $Z$-boson resonance effect for measuring the weak dipole moment. Namely, at such a high-energy collider, the produced $\tau$ and $\bar\tau$ gain a great momentum, i.e., the great Lorentz boost ($\gamma \simeq 25.66$); correspondingly, statistically, the produced $\tau$ ($\bar\tau$) leptons may travel a few *mm* before they decay [17] so that the flying directions of the produced $\tau$ and $\bar\tau$ leptons at a $Z$ factory can be reconstructed quite well with a fine vertex detector by precisely measuring the charged track between the colliding point of $e^+e^-$ and the decay vertices of the $\tau$ and $\bar\tau$ leptons, respectively. The so-called three-dimensional method developed in Ref. [18] is based on the determination of flying directions of the produced $\tau$ and $\bar{\tau}$ leptons, and it was widely employed by experimental groups at LEP-I, for example, the measurements of the weak dipole moment of $\tau$ lepton [6–9], as well as the lifetime and polarization of the produced $\tau$ lepton [17,19]. Considering the advantage further that the flying directions of the produced $\tau$ and $\bar{\tau}$ leptons may be constructed at a $Z$ factory, here we propose new observables as in Eqs. (21), (24), and (35), and two of them, i.e., observables defined in Eqs. (21) and (35), relate to the unit momentum of the produced $\tau$ lepton deeply.

The amplitudes for the production and decays of $\tau$ lepton pair at $e^-e^+$ colliders at $Z$ pole are calculated using the package FeynArts [20] and FormCalc [21,22]. Newly constructed observables defined in Eqs. (21), (24), and (35) are calculated, and their dimensionless coefficients $f_1'$ and $g_1'$, $f_2$ and $g_2$, as well as $f_3'$ and $g_3'$ are listed in Table I. In order to compare the sensitivity of the observables for measuring the weak dipole moment $d_w^\tau$, coefficients of those observables employed at LEP-I are also listed in Table I.

First, let us consider the observable $\hat{T}_{33}$ defined in Eq. (37). Here the values of coefficients $\hat{c}$ for the decay modes of $\tau$ lepton are taken from Table VI of Ref. [13]. From Eq. (38), one can see clearly that $\hat{c}s_{33}$ describes the sensitivity of the expectation values for the operator $\hat{T}_{33}$ to the weak dipole moment $d_w^\tau$. Hence, we define

$$\hat{c}_{33} \equiv \hat{c}s_{33} = \frac{1}{3}\hat{c}$$

and list their values for the concerned decay modes of $\tau$ lepton in Table I. Obviously, these numerical relation holds

$$|f_1'| > |f_3'| > |\hat{c}_{33}|.$$

Note that the first term and the second term in Eq. (37) are exactly the same for $i = j = 3$, and hence, there exists a constant of 2 in the definition of $\hat{T}_{ij}$. From Table I, it is concluded that our newly constructed observables $A_1'$ and $A_3'$ are much more sensitive to the real part of $d_w^\tau$ than the observable defined by Eq. (37).

TABLE I. Values for slope coefficients $f_1'$, $g_1'$, $f_2$, $g_2$, $f_3'$, $g_3'$, $\hat{c}_{33}$, $f_{\rm sc}$, $f_{\rm LEP}$, and $g_{\rm LEP}$. Here "×" means that there is no definition for the corresponding observables.

| | $\pi^-\pi^+$ | $\rho^-\rho^+$ | $\pi^-\rho^+$ | $e^-e^+$ |
|---|---|---|---|---|
| $f_1'$ | 0.938 | 0.464 | 0.789 | −0.322 |
| $g_1'$ | −0.0039 | −0.0016 | −0.0073 | 0.0068 |
| $f_2$ | 0.020 | 0.002 | 0.006 | −0.002 |
| $g_2$ | 0.2394 | 0.1129 | 0.2000 | −0.0545 |
| $f_3'$ | −0.815 | −0.366 | × | 0.272 |
| $g_3'$ | 0.0048 | 0.0023 | × | −0.0016 |
| $\hat{c}_{33}$ | −0.667 | −0.304 | −0.513 | 0.209 |
| $f_{\rm sc}$ | −0.408 | −0.183 | × | 0.136 |
| $f_{\rm LEP}$ | 0.201 | 0.211 | 0.204 | −0.056 |
| $g_{\rm LEP}$ | −0.0450 | −0.0490 | −0.0265 | −0.0046 |

Second, let us highlight the observable $A_{\rm sc}^{CP}$ defined in Eq. (43), which was employed by OPAL and L3 Collaborations at LEP-I [7,8]. It is proportional to the real part of weak dipole moment $d_w^\tau$, and the corresponding coefficient $f_{\rm sc}$ is calculated and listed in Table I. Furthermore, the so-called optimal T-even and T-odd operators in Eq. (39) that are used to measure the $CP$-violating effects in process $Z \to \tau^-\tau^+$ by the OPAL Collaboration [6] are also concerned here. Their expectation values are proportional to the real part and imaginary part of $d_w^\tau$, respectively. Values for their coefficients $f_{\rm LEP}$ and $g_{\rm LEP}$ are taken from Table 4 of Ref. [6] and are listed in Table I, too.

From Table I, one can draw conclusions as follows. Generally, the newly constructed observables depend on both the real and imaginary parts of the weak dipole moment $d_w^\tau$ linearly. For the observables defined by T-odd operators, such as $A_1'$ and $A_3'$, their slope coefficients satisfy these relations:

$$|f_1'| \gg |g_1'|, \qquad |f_3'| \gg |g_3'|$$

and

$$|f_1'| > |f_3'| > |f_{\rm sc}| > |f_{\rm LEP}|.$$

Especially for $\pi^-, \pi^+$ final states, we have

$$|f_1'| \gtrsim |f_3'| \simeq 2|f_{\rm sc}| \simeq 4|f_{\rm LEP}|.$$

It means that the observables $A_1'$ and $A_3'$ are much more sensitive to the $CP$-violated weak dipole moment than the observables employed at LEP-I. Although these two observables are constructed further by dividing the phase space relating to the direction of the produced $\tau$ lepton into two pieces, their superiority will not be reduced, especially at future colliders with high luminosity and equipping advanced detectors with very fine vertex detector so as to resolute the direction of the produced $\tau$ lepton. In principle, when two of the new observables are measured, then with the coefficients listed in Table I, the real part and imaginary part of the weak dipole $d_w^\tau$ can be determined separately.

The relations in the following hold for observable $A_2$, which is defined via the T-even operator,

$$|f_2| \ll |g_2|, \qquad |g_2| > |g_{\rm LEP}|$$

(see Table I). For $\pi^-, \pi^+$ final states, we have

$$|g_2| \simeq 5|g_{\rm LEP}|.$$

This implies that the observable $A_2$ depends on the imaginary part of $d_w^\tau$ much more strongly than the real part. Moreover, it is superior to the observable employed by LEP-I for measurement of the imaginary part of the weak dipole moment $d_w^\tau$.

In this paper, the new observables, constructed in terms of $CP(T)$-odd operators with $\tau$-lepton phase space being divided, are suggested for measuring the weak dipole moment $d_w^\tau$ at an $e^+e^-$ collider running at Z pole.

Then by precisely numerical calculations, it is shown that the new observables are more sensitive in measuring $d_w^\tau$ than those applied by LEP-I. They are proportional to the real and imaginary parts of $d_w^\tau$, and the slope coefficients are much larger than those employed at LEP-I. It implies that the effects caused by the weak dipole moment $d_w^\tau$ are enhanced greatly once these new observables are employed; i.e., the signals for the weak dipole moment $d_w^\tau$ are amplified at a $Z$ factory in certain senses in comparison with the previous observables.

Therefore, it is expected that once these new constructed observables are employed in future measurements or by reanalyzing the LEP-I data, an improved up-bound for the real and imaginary parts of the weak dipole moment $d_w^\tau$ will be reached.

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## APPENDIX: THE RELEVANT MOMENTA IN THE REFERENCE FRAME

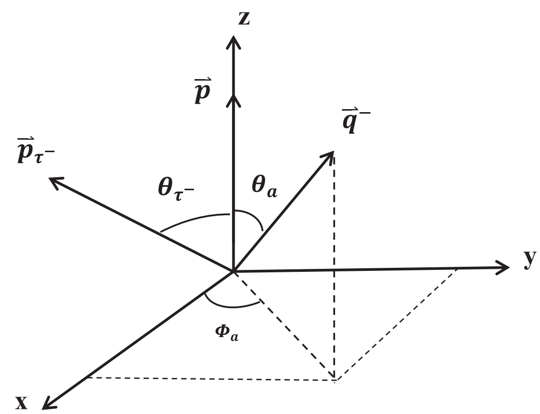


FIG. 3. The reference frame employed in this paper.


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