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Euclidean time formulation for the superstring ensembles: Perturbative canonical ensemble with Neveu-Schwarz B-field backgrounds

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We derive the Euclidean time formulation for the equilibrium canonical ensemble of the type IIA and type IIB superstrings, and the spin$(32)/Z_2$ heterotic string. We compactify on $R^8 \times T^2$, and twist by the Neveu-Schwarz sector antisymmetric 2-form B-field potential, spontaneously breaking supersymmetry at low temperatures, while preserving the tachyon-free low-energy gravitational field theory limit. We verify that the super partners of the massless dilaton-graviton multiplet obtain a mass which is linear in the temperature. In addition, we show that the free energy for the superstring canonical ensemble at weak coupling is always strongly convergent in the ultraviolet, high-temperature, regime dominated by the highest mass level number states. We derive the precise form of the exponential suppression as a *linear* power of the mass level, which erases the exponential Hagedorn growth of the degeneracies as the *square root* of mass level number. Finally, we close a gap in previous research giving an unambiguous derivation of the normalization of the one-loop vacuum energy density of the spin$(32)/Z_2$ perturbative heterotic string theory. Invoking the O(32) type IB-heterotic strong-weak duality, we match the normalization of the one loop vacuum energy densities of the T-dual O(32) type IA open and closed string with that of the spin$(32)/Z_2$ heterotic string on $R^9 \times S^1$, for values of the compactification radius, $R_{[O(32)]}$, $R_{IB} \gg \alpha'^{1/2}$, with $R_{IA} < \alpha'^{1/2}$. We show that the type IA thermal solitonic winding spectrum is a simple model for finite temperature pure QCD, transitioning above the critical duality phase transformation temperature to the deconfined ensemble of thermally excited IB gluons.

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I. INTRODUCTION

In this paper we will examine the consequences of T-duality transformations on the Euclidean time coordinate X^0. It is clear that a Wick rotation on the Minkowskian time coordinate maps the noncompact $SO(9,1)$ Lorentz invariant background of a given supersymmetric string theory to the corresponding $SO(10)$ invariant background, with an embedding time coordinate of Euclidean signature. The Euclidean metric $SO(10)$ invariant background arises naturally in any formulation of equilibrium string statistical mechanics in the canonical ensemble, the ensemble characterized by fixed temperature and fixed spatial volume (β, V).

The Polyakov path integral [1] over connected world surfaces can be formulated precisely in a target spacetime of fixed spacetime volume, giving the one-loop string vacuum functional, $\mathcal{W}$, in the $SO(10)$ invariant background. An explicit, first principles, derivation [2] of the Diff $\times$ Weyl invariant sum over connected one-loop string vacuum graphs in the Euclidean target spacetime, providing an expression for the one-loop Helmholtz free energy of the bosonic string canonical ensemble, $F(\beta, V) \equiv -\mathcal{W}/\beta V$, was derived by J. Polchinski in 1986 [2]. Partial results, and conjecture, for the behavior of the superstring canonical and microcanonical ensembles appear in the pioneering paper by J. Atick and E. Witten in 1989 [3]. The challenges in consistent formalism, and the significant physical implications of string statistical mechanics are discussed at length in [3].[1] R. Brandenburger and C. Vafa [5] soon after conjectured a framework for the superstring microcanonical ensemble, pointing to its fascinating consequences for the future of superstring cosmology. There is an extensive literature with various conjectures and partial results on superstring statistical mechanics, and it is beyond the scope of this paper to provide an adequate review [6].

This paper fills in some of the gaps in the derivation of the superstring canonical ensemble's one-loop vacuum energy density at finite temperature in the previous papers [2,3], bringing to completion the derivations we sketched in [7,8]. Our emphasis is on a derivation of the superstring one-loop free energy that *preserves* the world-sheet sWeyl $\times$ sDiff gauge symmetries. In addition, we verify consistency of the low-energy field theory limit of the

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[1] There are some typos in both [2] and [3,4] in the expressions for the string free energy, but they do not take away from the pioneering elegance of these early papers.

superstring free energy with the known properties of the canonical ensemble of finite temperature supergravity and super Yang-Mills gauge theory [4]. Namely, we verify the T^{10} growth of the free energy, and the spontaneous breaking of target space supersymmetry with a mass to the super partners of the massless fields that is linear in the temperature, and *without* the appearance of thermal tachyonic modes. Thus, our result is a derivation of the free energy of the *equilibrium* superstring canonical ensemble.

The appearance of a *tachyonic* mode in the string thermal spectrum would be an indication that the world-sheet conformal field theory is no longer at a fixed point of the two-dimensional renormalization group: the tachyon indicates a relevant flow, and the appearance of thermal tachyons have plagued all previous attempts to describe an equilibrium string canonical ensemble, including [2,3]. An equilibrium statistical mechanics of strings requires that any fixed point belong to a *fixed line* parametrized by inverse temperature β [9]. Target spacetime supersymmetry, and its spontaneous breaking at low temperature along the line of fixed points parametrized by β, introduces new features into the derivation, requiring consistency with both world-sheet and target spacetime gauge symmetries. Remarkably, we will find as a direct consequence of the extended target space T-duality transformations, combined with the weak-strong coupling heterotic-type IB-type IA string dualities, that a physically sensible equilibrium string statistical mechanics in the canonical ensemble exists for all of the six superstring ensembles. The mathematical derivations and results we present in this paper are no longer in complete clash with physics intuition, even if avenues for further research remain.

The Euclidean time formulation for the equilibrium string canonical ensemble occupying a fixed spatial volume, V, describes the ensemble of first quantized single string mass eigenstates in equilibrium with a heat bath at a fixed temperature T. In the absence of a tachyon, or massless tadpoles, the vacuum energy density varies along the line of fixed point conformal field theories of the world-sheet renormalization group; the integrable coupling along this fixed line is simply the inverse temperature β. More precisely, since every ten-dimensional perturbative string theory has at least one flat direction of the superpotential, parametrized by the dilaton, in addition to possible conformally coupled Yang-Mills, and supergravity p-form background gauge potentials, the fixed point belongs in a multidimensional phase space of the world-sheet RG parametrized by any additional target spacetime moduli. Infrared stability of the canonical ensemble of single string mass eigenstates requires the absence of low-temperature tachyonic and massless tadpole modes.

In Sec. II, we note that there is a unique modular invariant expression for the finite temperature one-loop orientable closed type IIA and type IIB string vacuum amplitudes, which both spontaneously breaks target space supersymmetry without the appearance of thermal tachyons. The generic constant background field available in every string theory is the Neveu-Schwarz sector antisymmetric tensor potential which couples to the fundamental closed string [10,11], and in the absence of D-branes and Ramond-Ramond fields, it is the *only* available background gauge field, coupling to the fundamental F string. Hence, we turn on a constant Neveu-Schwarz antisymmetric 2-form potential, $|B_{09}| \equiv \tanh \pi\alpha \simeq \pi\beta_C/\beta$, for small α, that is linear in the temperature for small T, asymptoting to unity at temperatures approaching the string scale. Here, β_C refers to the thermal duality transformation scale, given by the string scale. This step enables the spontaneous breaking of target spacetime supersymmetry, without the presence of a tachyonic mode in the string mass spectrum, and gives a unique result for the one-loop vacuum energy density that both meets all of the two-dimensional gauge symmetries of the type IIA and type IIB superstring theories, in addition to the target spacetime dualities for compactification on a torus, $T^2 \times \mathrm{R}^8$, twisting by the given B_{09}-field.

To understand why it is natural to consider a temperature-dependent background for the NS 2-form gauge potential, recall that finite temperature non-Abelian gauge theory is formulated in Axial gauge, where we set the Euclidean time component of the *vector* potential to zero, $A^0 = 0$, and this gauge choice provides precisely the correct number of propagating modes in finite temperature *gauge* theory, as was discovered in the early papers [12], while preserving a $(D-1)$-dimensional gauge invariance for D-dimensional finite temperature Yang Mills gauge theory. The constant background for the 2-form gauge potential in string theories is the analog of such an axial gauge choice; even in Yang-Mills gauge theory, one could alternatively set A^0 equal to a constant, instead of zero, and the consequence would be to alter the thermal background in which we quantize finite temperature gauge theory, retaining the same number of physical, transverse, propagating modes. Our temperature-dependent constant B-field background achieves the same physics for finite temperature string theory: the number of propagating modes in the antisymmetric 2-form *tensor* gauge potential are truncated to the requisite physical degrees of freedom in a quantum finite temperature *string* theory in D target spacetime dimensions, and the remnant tensor gauge invariance is $(D-2)$ dimensional.

Given our result for the one-loop superstring vacuum energy density, we isolate, in turn, the high-temperature asymptotic behavior at temperatures far above the string scale in Sec. III, and the low-energy supergravity field theoretic limit of finite temperature field theory in Sec. IV. In particular, we verify the T^{10} growth of the field theoretic free energy at temperatures much lower than the string scale. And conversely, we establish the absence of a Hagedorn phase transition in the perturbative type II superstring theories, confirming the strong UV convergence of the one-loop vacuum energy density for either type II superstring canonical ensemble. The corresponding analysis for the closed bosonic string is presented in the Appendix, and forms a useful paradigm for the low- and high-temperature analysis of the type II superstrings in the main text.

Sec. V is devoted to an analysis of the spin$(32)/Z_2$ heterotic string compactified on a twisted torus, $T^2 \times R^8$, where the coordinates of the torus are Euclidean time and space, (X^0, X^9), and with a temperature-dependent B-field background, a constant antisymmetric tensor potential B_{09}. We should clarify that it is natural—and physical, when considering string thermodynamics to allow thermal backgrounds for all of the available gauge potentials that couple to the strings: the NS 2-form gauge potential that couples to the fundamental closed string is the most universal of the background gauge fields, common to all of the superstring theories. In addition, we include the usual variety of Wilson line backgrounds for the vector potential, namely, the Yang-Mills gauge potential, and this plays the role first outlined at some length in [13,14], enabling an approach in the moduli space to distinct enhanced symmetry points with different anomaly-free non-Abelian gauge group. In Secs. V A and V B, we examine the high- and low-temperature regimes of the one-loop vacuum energy density, respectively, demonstrating that the free energy of the spin$(32)/Z_2$ heterotic string theory is finite at the string scale, namely, at what used to be known as the Hagedorn temperature, and thermal duality enables a straightforward procedure to examine the behavior of the ensemble at temperatures above the string scale. The expression for the one-loop vacuum energy density is also strongly convergent in the ultraviolet.

In Sec. VI, we analyze the type IB open and closed unoriented superstring theory compactified on the twisted torus, and focus on the one-loop oriented string graph, which yields the low-energy finite temperature Yang Mills gauge theory limit [4,15], when we set the mass level number to zero, and work in the large radius (low-temperature) limit. This ensemble matches with beautiful accuracy the behavior of a thermal gas of gluons—and upon inclusion of the torus graph, would describe the thermalized gluon-graviton ensemble. At temperatures above the string scale, the dynamics of the type IB string theory is accessible by a thermal duality *transformation* to the type IA string at finite temperature. We show that the finite temperature ground state of this theory is characterized by a tower of thermal winding strings, and appears to be a good match to the confinement phase of non-Abelian gauge theories at low type IA temperatures. Thus, it is plausible to characterize the thermal duality phase transformation as a thermal deconfinement phase transition in the non-Abelian gauge theory. Finally, in Sec. VII, we fill in a remnant gap in our analysis of heterotic string thermodynamics, by deducing the unknown normalization of the one-loop vacuum energy density of the spin$(32)/Z_2$ heterotic string. We obtain this long sought-after normalization constant by invoking a matching calculation of the *weakly coupled* low-energy O(32) type IA and heterotic strings, at large radius for both the spin$(32)/Z_2$ heterotic string and the type IB superstring theories. Sec. VIII provides the conclusions and observations and suggestions for future work.

II. EQUILIBRIUM TYPE II SUPERSTRING CANONICAL ENSEMBLE

In common with the heterotic string, and all closed oriented strings, the ten-dimensional type IIA and type IIB superstrings have a massless NS antisymmetric tensor field, and we will denote this as B_{mn}. Compactifying on a 2-torus, with compact coordinates X^0, X^9, and with embedding metric, G_{mn}, and the antisymmetric 2-form, B_{mn}, in generic Minkowskian signature backgrounds, the Polyakov world-sheet action takes the form [16]:

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g}[g^{ab} G^{mn} \partial_a X_m \partial_b X^n + i\epsilon^{ab} B^{mn} \partial_a X_m \partial_b X^n], \tag{2.1}$$

In the absence of a 2-form background, since the Jacobi theta function $\Theta_{11}(0,\tau)$ vanishes identically due to the zero mode in the (1,1) spin structure sector, the expressions for the IIA and IIB one-loop vacuum energy densities would be identical. In the presence of a background B_{mn} NS 2-form potential, the vacuum energy density of the type IIA and IIB superstrings will differ. Upon compactification on the circle of radius $\beta/2\pi$, the timelike zero mode spectrum is given by [17,18]:

$$p^0_L = \frac{2\pi n^0}{\beta} + \frac{w_9(G^{09} + B^{09})\beta}{2\pi\alpha'},$$
$$p^0_R = \frac{2\pi n^0}{\beta} - \frac{w_9(G^{09} + B^{90})\beta}{2\pi\alpha'}, \tag{2.2}$$

where (n^0, w^0) are, respectively, the momentum and windings about the compact Euclidean time coordinate X^0, shifted by the constant antisymmetric tensor NS 2-form potential, and likewise for (p^9_L, p^9_R). Complexifying the pair of coordinates (X^0, X^9), and with $G^{00} = G^{99} = 1$,

$B^{00} = B^{99} = 0$; $B^{09} = -B^{90} = B$, and $B = |\tanh(\pi\alpha)|$, where $\alpha \equiv (\beta_C/\beta) = \alpha'^{1/2}T$ is dimensionless, and the contribution to the path integral from the (n, w)th sector is therefore

$$\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta^2} + \frac{\alpha' n_9^2}{R_9^2} + \frac{w_9^2(1+\tanh(\pi\alpha))^2\beta^2}{4\pi^2\alpha'} + \frac{w_0^2(1+\tanh(\pi\alpha))^2 R_9^2}{\alpha'}\right)\right] \times \exp\left[-\pi\tau_2(n_0w_9 + n_9w_0) \times 4(1+\tanh(\pi\alpha))\right]. \tag{2.3}$$

Suppressing the oscillator contributions, the thermal ground state energy in the NS-NS sector for physical states satisfying the level matching constraint takes the form

$$(\text{mass})_L^2 = (\text{mass})_R^2 = \frac{4}{\alpha'}\left[-\frac{1}{2} + \frac{1}{2}\left(\frac{4\pi^2\alpha' n_0^2}{\beta^2} + \frac{\alpha' n_9^2}{R_9^2} + \frac{1}{\alpha'}(1+\tanh(\pi\alpha))^2\left[w_9^2\frac{\beta^2}{4\pi^2} + w_0^2R_9^2\right]\right)\right] + \frac{4}{\alpha'}\left[\frac{1}{2}(n_0w_9 + n_9w_0)4(1+\tanh(\pi\alpha))\right]. \tag{2.4}$$

Note that if we take the noncompact zero temperature limit, $R_9 \to \infty$, and $T \to 0$, the B field disappears from the expression for the one-loop vacuum energy, and we recover the supersymmetric ten-dimensional type IIA (IIB) superstring.

Thermal duality transformations are a little more complicated in the presence of a B field, and it is helpful to review the type IIA-type IIB T-duality transformations, also known as the simplest of the mirror maps for generic Calabi-Yau manifolds [19]. The 2-torus is a complex manifold, and in terms of the Euclidean signature Wick rotated complex coordinates, and suppressing the fermionic terms in the world-sheet action with world-sheet (2, 2) supersymmetry, we have the marginal deformed world-sheet action,

$$S = \frac{1}{4\pi\alpha'}\int \{g_{i\bar{j}}(\partial x^i\bar{\partial}x^{\bar{j}} + \partial x^j\bar{\partial}x^{\bar{i}}) - iB_{i\bar{j}}(\partial x^i\bar{\partial}x^{\bar{j}} - \partial x^j\bar{\partial}x^{\bar{i}})\}, \tag{2.5}$$

and we can parametrize the marginal deformations by two real numbers:

$$B = \frac{1}{2}B_{i\bar{j}}dx^i \wedge dx^{\bar{j}}, \quad J = \frac{1}{2}g_{i\bar{j}}dx^i \wedge dx^{\bar{j}}. \tag{2.6}$$

As shown in [11,16], for generic ten-dimensional one-loop scattering amplitudes, the B field does not appear directly in correlation functions, but only in the normalization of the string path integral through the e^{-S} term in the Polyakov path integral. Upon compactification on a suitable manifold, shifts in B appear in the mass level expansions, resulting from twisted Jacobi theta functions, and, in addition, in momentum and winding mode number summations.

We note in passing an important relation between the T dualities of the twisted two-torus, and the simplest example of mirror symmetry in a Calabi-Yau manifold. As elucidated at length in [19], note that shifts of B by an integer, $2\pi n$, leave the action invariant, and correlation functions invariant due to an $SL(2,\mathrm{Z})$ symmetry generated by $\sigma = \frac{1}{4\pi\alpha'}(B + iJ)$. Thus, the moduli space of marginal deformations of this theory are parametrized by the group $SL(2,\mathrm{Z}) \times SL(2,\mathrm{Z})$, where the additional $SL(2,\mathrm{Z})$ describes the complex structure of the torus. In the absence of B, we have a rectangular domain of lengths R_9, R_0, and $\eta = i\frac{R_9}{R_0}$, $\sigma = \frac{i}{\alpha'}R_9R_0$, dividing the complex plane by translations $2\pi R_9$, $2\pi R_0$. Remarkably, the $R_0 \leftrightarrow \alpha'/R_0$ generates the Z_2 mirror map for the 2-torus: the interchange of the $SL(2,\mathrm{Z})$'s; and the additional Z_2 symmetry are generate by complex conjugation, namely, interchange of the two real coordinates, plus a change in the sign of B: inversion in the upper half-plane, $(\sigma,\eta) \leftrightarrow (-\sigma,-\eta)$. Together with the modular group for complex structure, the two Z_2 symmetries generate the full target space modular group. Thus, choosing R_0 to be $\beta/2\pi$, and R_9 the radius of the compact coordinate X^9, the thermal duality transformation that relates finite temperature type IIA and type IIB strings is an Abelian subset of the mirror map for the 2-torus.

In order to understand the stringent constraints on the expression for the one-loop string vacuum amplitude in the finite temperature vacuum imposed by modular invariance, recall that the type II string mass level expansion results from a generic combination of the four, holomorphic and anti–holomorphic, Jacobi theta functions, weighted by, a priori, undetermined phases. Thus, upon compactifying either type II superstring on $R^8 \times T^2$, and twisting by the antisymmetric 2-form potential, $|B_{09}| \equiv \tanh(\pi\alpha)$, the general result for the one-loop vacuum energy density, $\rho \equiv -W/V_8(2\pi R_9\beta)$, of the canonical ensemble of the type II superstrings takes the form

$$
\begin{aligned}
\rho_{II} = &-(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}\cdot(\tau_2)^{-3}[\eta(\tau)\bar{\eta}(\bar{\tau})]^{-6}\times\frac{1}{4}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\bar{\tau}_2\alpha^2}\bar{\eta}(\bar{\tau})}{\bar{\Theta}_{11}(\alpha,\bar{\tau})}\right]\\
&\times\left[\frac{\Theta_{00}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{00}(0,\tau)}{\eta(\tau)}\right)^3-\frac{\Theta_{01}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{01}(0,\tau)}{\eta(\tau)}\right)^3-\frac{\Theta_{10}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{10}(0,\tau)}{\eta(\tau)}\right)^3\right]\\
&\times\left[\frac{\bar{\Theta}_{00}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\eta(\bar{\tau})}\left(\frac{\bar{\Theta}_{00}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3-\frac{\bar{\Theta}_{01}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\bar{\eta}(\bar{\tau})}\left(\frac{\bar{\Theta}_{01}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3-\frac{\bar{\Theta}_{10}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\bar{\eta}(\bar{\tau})}\left(\frac{\bar{\Theta}_{10}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3\right]\\
&\times\sum_{n_i,w_i=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta^2}+\frac{\alpha' n_9^2}{R_9^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta^2}{4\pi^2\alpha'}+\frac{w_0^2(1+\tanh(\pi\alpha))^2R_9^2}{\alpha'}\right)\right]\\
&\times\exp\left[-\pi\tau_2(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\right]\\
&\pm(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}\cdot(\tau_2)^{-3}[\eta(\tau)\bar{\eta}(\bar{\tau})]^{-6}\times\frac{1}{4}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\bar{\tau}_2\alpha^2}\bar{\eta}(\bar{\tau})}{\bar{\Theta}_{11}(\alpha,\bar{\tau})}\right]\\
&\times\left[\frac{\Theta_{11}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{11}(0,\tau)}{\eta(\tau)}\right)^3\right]\left[\frac{\bar{\Theta}_{11}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\bar{\eta}(\bar{\tau})}\left(\frac{\bar{\Theta}_{11}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3\right]\\
&\times\sum_{n_i,w_i=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta^2}+\frac{\alpha' n_9^2}{R_9^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta^2}{4\pi^2\alpha'}+\frac{w_0^2(1+\tanh(\pi\alpha))^2R_9^2}{\alpha'}\right)\right]\\
&\times\exp\left[-\pi\tau_2(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\right]. \qquad (2.7)
\end{aligned}
$$

where the choice of phase, $\mp$, for the (1, 1) spin structure distinguishes the result for the type IIA and type IIB one-loop finite temperature vacuum energy density.

We choose the twist in the argument of the Jacobi theta functions: $\pi\alpha \equiv \pi(\beta_C/\beta) = \pi\alpha'^{1/2}T$, vanishing as we approach the zero temperature supersymmetric vacuum. In addition, as $T \to T_C$, the characteristic of the Jacobi theta functions approaches π, reversing the sign of Θ_{01} and Θ_{10}, and leaving Θ_{00} with the same sign. This implies that as $\tanh(\pi\alpha)$ runs from $[0,\pi]$, the expression above interpolates smoothly from the vanishing one-loop vacuum energy density of the supersymmetric type IIA string at zero temperature, and a combination of Jacobi theta functions that superficially resembles that of the *tachyonic* and nonsupersymmetric type 0A string's vacuum energy density, *except* that the would-be thermal tachyon's mass has been shifted up to mass level zero, at $\pi\alpha = \pi$, and likewise for the type IIB string. In addition, the spacetime fermions at mass level zero in the type IIA superstring, acquire a mass which is *linear* in the temperature, as a consequence of the cross term for momentum and winding modes, which is linear in the temperature-dependent B field. Finally, in the high-temperature regime, note that the function $\tanh(\pi\alpha) \to +1$ asymptotically, as $\beta \to \beta_C$, so that the one-loop vacuum energy density depends on temperature only through the explicit temperature dependence of the winding mode summation, and the B-field background is of primary significance in the low-temperature behavior of the Jacobi theta functions below T_C.

In addition, we find that modular invariance has simultaneously forced the unique combination of world-sheet fermionic spin structures displayed in the expression above, up to shifts of the twisting parameter α by an integer, corresponding to an element of the $SL(2,Z)$ target space duality group [19]. A modular invariant expression for the one-loop vacuum energy density at finite temperature was absent in all previous analyses of the type II superstring finite temperature vacuum energy density [3,8], and the correct result for the type IIB one-loop vacuum energy density at finite temperature was essential prior to the analogous analysis for the type IB superstring. This is due to the absence of a sharp world-sheet self-consistency criterion such as one-loop modular invariance for open string amplitudes. With the results derived in this section, we shall also be able to invoke self-consistency between the type IIB and type IB one loop amplitudes using the orientifold projection to arrive at the expression for the one-loop vacuum energy density of the type IB superstring [18].

III. HIGH T CONVERGENCE OF STRING VACUUM ENERGY DENSITY

In order to perform the world-sheet modular integral, as for the closed bosonic string theory analyzed in the Appendix, we make a change of variable, $y = 1/\tau_2$, and calculate the result in the noncompact $R_9 \to \infty$ limit for both the IIA and IIB strings, distinguished by the distinct mass level degeneracy functions, namely, the $f_m^{\mathrm{IIA(IIB)}}(\alpha)$, in the presence of the nonvanishing contribution of the Ramond-Ramond sector, and with the correct measure for moduli with ten noncompact target spacetime coordinates, and corresponding world-sheet superconformal algebras [10],

$$\rho_{\mathrm{IIA(IIB)}} = -(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty} f_m^{\mathrm{IIA(IIB)}}(\alpha)\sum_{n,w=-\infty}^{\infty}\int_{-1/2}^{+1/2} d\tau_1 \times \int_0^{1/\sqrt{1-\tau_1^2}} dy y^{9/2} e^{-2\pi m/y} \times e^{-2\pi\left(\frac{4\pi^2\alpha' n_0^2}{\beta^2}+\frac{\alpha' n_9^2}{R_9^2}+\frac{1}{\alpha'}(1+\tanh(\pi\alpha))^2\left[w_9^2\frac{\beta^2}{4\pi^2}+w_0^2R_9^2\right]\right)\frac{1}{y}} \times e^{-2\pi(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\frac{1}{y}}, \quad (3.1)$$

where $|B|$ is the non-negative constant background field in the Neveu-Schwarz 2-form potential, parametrized as $\tanh(\pi\alpha)$, with domain (0, 1). Here, we have parametrized $\alpha = T/T_C$. At low temperatures, the $f_m(\alpha)$ are B-field-dependent degeneracy functions, unlike the integer degeneracies at zero field: $b_m^{\mathrm{IIA(IIB)}}$, at mass level number m of the IIA or IIB string mass level expansion. Each has a polynomial dependence on the hyperbolic functions of α. Namely, the functions, $f_m(\alpha)$, are implicit *polynomial* functions of the B field, with argument given as follows: $f_m(\alpha) \equiv f_m(1+\tanh(\pi\alpha))$. In the high-temperature limit —since the hyperbolic tangent asymptotes to unity, as $T \to T_C$, the polynomials become pure numbers: $f_m(\alpha) \to f_m(2)$, as at zero temperature, where $f_m(0) \equiv b_m$, the zero temperature degeneracy at string mass level m. The partition function is therefore finite at temperatures approaching the string scale. Recall that it is type IIA–type IIB thermal duality, or, $R_0 \leftrightarrow 1/R_0$ duality, is one of the two Z_2 maps that generate the $SL(2,Z)\times SL(2,Z)$ target space duality group of the torus [19], thereby also covering the temperature range above T_C by means of a thermal duality transformation contained within the T-duality group of the torus (X^0, X^9). The expression for the partition function given here is also manifestly invariant under the other Z_2 map, which interchanges $R_9 \leftrightarrow \beta/2\pi$, while changing the sign of B_{09}, which leaves $|B| = \tanh(\pi\alpha)$ unchanged.

Our next step is to perform the modular integrals explicitly, substituting the high-temperature UV asymptotic expansion for the Whittaker functions that arise upon integrating over the world- sheet modulus $y = 1/\tau_2$, which maps the domain of the integral to $[0, u]$, where we have parametrized $u = (1-\tau_1^2)^{-1/2}$. The integral over y can be recognized as an integral representation[2] of the Whittaker function, $\mathcal{W}_{-13/4,9/4}(A/u)$ (GR 3.471.2) [20,21],

$$\int_0^u dy y^{\nu-1} e^{-\frac{A}{y}} = A^{\frac{\nu-1}{2}} u^{\frac{\nu+1}{2}} e^{-\frac{A}{2u}} W_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u), \quad \nu = 11/2. \quad (3.2)$$

Notice that the argument of the exponential in the integrand above is *always* large everywhere in the τ_1 domain in the high energy domain $m \gg 0$ so that the Whittaker function is an exact integral representation of the modular integral, valid for the full range of values of the temperature, namely, $0 \le \beta \le \infty$. By inspection, it is also evident that the integral is bounded for all values of the target spacetime moduli, namely, R_9, β, and $|B|$, as ensured by $SL(2,Z)$ duality of the moduli space of the 2-torus [19]. Hence, prior to performing the τ_1 integration, we can substitute the asymptotic expansion for the Whittaker function (GR 9.227) [20,21],

$$\mathcal{W}_{(\lambda,\mu)}(z) \sim e^{-z/2} z^{\lambda}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}z^{-k}\left[\mu^2-\left(\lambda-\frac{1}{2}\right)^2\right] \times \left[\mu^2-\left(\lambda-\frac{3}{2}\right)^2\right]\cdots\left[\mu^2-\left(\lambda-k+\frac{1}{2}\right)^2\right]\right), \quad (3.3)$$

where $z = A/u$. We have also expanded u in the convergent power series expansion valid for $\tau_1 < 1$. To proceed, we note that since the hyperbolic tangent asymptotes to unity for large temperatures of order, and above the string scale, the degeneracy functions asymptote to their explicit values, $f_m^{(\mathrm{II})}(1+\pi)$, and the mass level expansion is corrected by numerical factors:

$$I(m) = \frac{1}{4}\sum_{w=-\infty}^{\infty} A^{-1}\int_{-1/2}^{+1/2} d\tau_1 (1-\tau_1^2)^{-11/8} \times e^{-\frac{1}{2}A(1-\tau_1^2)^{1/2}}\mathcal{W}_{-\frac{13}{4},\frac{9}{4}}(A[1-\tau_1^2]^{1/2}). \quad (3.4)$$

The term-by-term integrals over τ_1 can be evaluated by substituting the asymptotic expansion for the Whittaker function valid at large mass level number,

$$\begin{aligned} I(m,w) &\equiv \frac{1}{4}A^{-1}\left[\int_0^{2/\sqrt{3}} - \int_0^1\right] du u^{\frac{11}{4}-3}(1-1/u^2)^{-1/2} e^{-\frac{1}{2}A/u}\mathcal{W}_{-\frac{13}{4},\frac{9}{4}}(A/u) \\ &= \frac{1}{4}A^{-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A^{-k}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \times \frac{1}{k!}(-1)^{k+1}C_k\left[\int_0^{2/\sqrt{3}} - \int_0^1\right] du u^{\frac{3}{4}-1-2r+k} e^{-A/u}, \\ &= \frac{1}{4}A^{-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A^{-k}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \times \frac{1}{k!}(-1)^{k+1}C_k \\ &\quad \times [\mathcal{W}_{(-7-4k+8r)/8,(3+4k-8r)/8}(\sqrt{3}A/2) - \mathcal{W}_{(-7-4k+8r)/8,(3+4k-8r)/8}(A)], \\ C_0 &= -1, \qquad C_k \equiv \left[\left(\frac{13}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2-\left(\frac{9}{4}\right)^2\right], \end{aligned} \quad (3.5)$$

[2]As in the Appendix, in what follows, (GR #) denotes the corresponding equation number for a mathematical identity from the text [20].

for all $k \geq 1$. Iterating the substitution of the asymptotic expansion for large argument of the Whittaker functions, we find an explicit exact result for the finite temperature vacuum energy density of the type II strings:

$$\rho_{\mathrm{IIA(IIB)}} = -(4\pi^2\alpha')^{-5}\frac{1}{4}\sum_{m=0}^{\infty} f_m^{\mathrm{IIA(IIB)}}(1+\pi)\sum_{n=-\infty}^{\infty}\sum_{w=-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)]$$
$$\times\sum_{k=0}^{\infty}A^{-1-k}\frac{1}{k!}(-1)^k C_k\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j C_{(j,k,r)}[(\sqrt{3}A/2)^{(-7-4k-4j+8r)/8}e^{-\sqrt{3}A/4} - (A)^{(-7-4k-4j+8r)/8}e^{-A/2}],$$
$$C_0 = -1, \qquad C_k \equiv \left[\left(\frac{13}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right],$$
$$C_{(0,0,0)} = -1, \qquad C_{(j,k,r)} \equiv \left[\left(\frac{11+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]\left[\left(\frac{19+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]\cdots$$
$$\times\left[\left(\frac{3+4j+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]. \tag{3.6}$$

The result is $O(e^{-A})$, providing the exponential suppression as a *linear* power of mass level number m. Thus, following an analytic evaluation of the integrals over world-sheet moduli, (τ_1, τ_2), we find that the numerical correction, $I(m)$, to the degeneracies, b_m, in the mass level expansion, is an exponential suppression which is *linear as a function of mass level number m*. The exponent A at arbitrary mass level m is given by

$$e^{-A} = \exp\left[-2\pi\left(m + \left\{\frac{2\pi\alpha' n_0^2}{\beta^2} + \frac{\alpha' n_9^2}{2\pi R_9^2} + \frac{1}{2\pi\alpha'}2^2\left[w_9^2\frac{\beta^2}{4\pi^2} + w_0^2 R_9^2\right]\right\}\right)\right]\exp\left[-2\pi(n_0 w_9 + n_9 w_0)4(2)\right]. \tag{3.7}$$

Thus, the $O(e^{-m})$ term arising from the modular integral erases the $O(e^{\sqrt{m}})$ growth of the degeneracies, $b_m^{(\mathrm{II})}$ at large mass level number, m [6,22]. Notice that while the numerical factors in the exponent function differ from the low-temperature form, this expression—and consequently, the partition function—are still manifestly invariant under both of the Z_2 maps that generate the target space duality group of the torus.

As a consequence, the *convergence* of the type II string vacuum energy density in the ultraviolet regime, namely, at high mass level numbers, is extremely rapid, and for *all* values of the target spacetime moduli: (R_9, β), an exponential suppression with increasing m:

$$\rho_{\mathrm{IIA(IIB)}} = -(4\pi^2\alpha')^{-5}\frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{w=-\infty}^{\infty} f_m^{\mathrm{IIA(IIB)}}(1+\pi)\sum_{r=0}^{\infty}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)]$$
$$\times\sum_{k=0}^{\infty}A^{-(15+12k)/8}\frac{1}{k!}(-1)^k C_k\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j C_{(j,k,r)}[(\sqrt{3}/2)^{(-7-4k-4j+8r)/8}e^{-\sqrt{3}A/4} - e^{-A/2}]A^{r-j/2}. \tag{3.8}$$

It is instructive to examine the exponent A for the leading target spacetime moduli-dependent corrections in the noncompact limit $R_9 \to \infty$, and at temperatures well above the string scale. In this limit the spatial momentum and spatial winding mode numbers dominate the summations in the vacuum energy density,

$$\rho_{\mathrm{IIA(IIB)}} \simeq -(4\pi^2\alpha')^{-5}\frac{1}{4}\sum_{m=0}^{\infty}\sum_{n_9=-\infty}^{\infty}\sum_{w_9=-\infty}^{\infty} f_m^{\mathrm{IIA(IIB)}}(1+\pi)$$
$$\times\sum_{r=0}^{\infty}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)]A^r$$
$$\times\sum_{k=0}^{\infty}A^{-(15+12k)/8}\frac{1}{k!}C_k\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^{j+k}C_{(j,k,r)}$$
$$\times[(\sqrt{3}/2)^{(-7-4k-4j+8r)/8}e^{-\sqrt{3}A/4} - e^{-A/2}]A^{-\frac{j}{2}} \tag{3.9}$$

where, in this limit, we can approximate the exponent, A, as the variable:

$$A \simeq 2\pi m\left(1 + \frac{\alpha' n_9^2}{2\pi m R_9^2} + \frac{2^2 w_9^2}{4\pi^2 m\alpha' T^2}\right). \tag{3.10}$$

Expressing the series as the sum of two like-sign infinite series, it is apparent that successive terms in each are suppressed by a factor of $1/m$, in addition to the overall exponential suppression. This will lead to very rapid convergence. This is also true for the summation over m: there is a well-known *square root* exponential growth as a function of mass level m of the degeneracies, $b_m^{(\mathrm{IIA-IIB})}$, at large mass level numbers [6,7], but rapid convergence of the free energy is driven by the variable A, which provides an *exponential suppression linear as a function of mass level number.* Note that in the high-temperature large radius approximation given by Eq. (3.10), the target space duality invariance is no longer manifest, since we have explicitly ignored thermal duality and $R_9 \leftrightarrow 1/R_9$ duality, by taking the limits $T, R_9 \to \infty$.

IV. LOW T BEHAVIOR IN THE FIELD THEORETIC SUPERGRAVITY LIMIT

We have shown that the presence of the background antisymmetric 2-form potential gives a stable, tachyon-free, and massless tadpole-free ensemble of thermal gravitons at temperatures below the string scale, when we explicitly truncate the mass level expansion to the $m = 0$ term, and perform the modular integral preserving modular invariance, a part of the world-sheet gauge invariance. The finite temperature graviton ensemble arises from the infinite summation over thermal momentum modes, namely, the tower of Matsubara thermal excitations, and the exponent, A, takes the simple form: $A \simeq \frac{4\pi^2\alpha' n_0^2}{\beta^2} + (n_0 w_9 + n_9 w_0)\frac{4(1+\tanh(\pi\alpha))}{\alpha'}$. Namely, we have the expected sum over Matsubara modes, in addition to the B-field (linear) temperature-dependent addition, which is subleading at low temperatures. As a simple check, we verify the expected T^{10} growth with low temperatures, of the vacuum energy density truncating to the, mass level zero, low-energy supergravity field theoretic limit. As before, by truncating the exponent function A as appropriate for the zero-temperature-large-radius limit, the expression—and consequently, the result for the field theoretic one-loop vacuum energy density—will no longer be invariant under the target space dualities.

It is rather easy to give the result of the τ_1 integration in explicit closed form, following a change of variable, $x^2 = 1 - \tau_1^2$, $|\tau_1| \le 1/2$, $|x| \le 1$, and use of the power series representation of the Whittaker function, with $\nu = 5$, as is appropriate in the low-temperature regime, $\beta \to \infty$:

$$I(m) = \frac{1}{4}\sum_{n_0=-\infty}^{\infty} A^{\frac{\nu-1}{2}}\left[\int_0^1 - \int_0^{\sqrt{3}/2}\right] dx x^{-\frac{\nu+1}{2}+1}(1-x^2)^{-1/2} \times e^{-\frac{1}{2}Ax}\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(Ax), \tag{4.1}$$

and substitute the power series representation for the Whittaker function [GR 9.237] [21], valid for $\beta \to \infty$, including the complete infinite sum over thermal momentum modes, n_0, and taking the m $= 0$, massless field limit, namely, expanding about the origin $A = 0$. The integrand takes the standard special function form of an infinite power series, plus logarithmic, plus finite polynomial correction [21]:

$$\mathcal{W}_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-\mu-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)} \times \left\{\sum_{k=0}^{\infty}\frac{\Gamma(\mu+k-\lambda+\frac{1}{2})}{k!(2\mu+k)!} z^k \left[\Psi(k+1) + \Psi(2\mu+k+1) - \Psi\left(\mu+k-\lambda+\frac{1}{2}\right) - \ln z\right] + (-z)^{-2\mu}\sum_{k=0}^{2\mu-1}\left[\frac{\Gamma(2\mu-k)\Gamma(k-\mu-\lambda+\frac{1}{2})}{k!}(-z)^k\right]\right\}, \quad \text{where } z = Ax, \quad \lambda = -\frac{\nu+1}{2}, \quad \mu = \frac{\nu}{2}. \tag{4.2}$$

We now carry out these steps systematically. The infinite summation on thermal momentum mode number, n_0, can be carried out explicitly, expressible in terms of Riemann zeta functions. We restrict to the target spacetime bosonic fields at mass level $m = 0$. Note that it is only their target spacetime super partners that receive a mass linear in temperature. Thus,

$$\begin{aligned}
\rho_0^{\mathrm{IIA(IIB)}}(\beta) &= -\frac{1}{4}\cdot(4\pi^2\alpha')^{-\nu} b_0^{\mathrm{IIA(IIB)}} \sum_{n_0=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx x^{-\frac{\nu+1}{2}+1}(1-x^2)^{-1/2} A^{\frac{\nu-1}{2}} \exp\left[-\frac{A}{2}x\right]\mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(Ax) \\
&= -\frac{1}{4}\cdot(4\pi^2\alpha')^{-\nu} f_0^{\mathrm{IIA(IIB)}}(\pi\alpha'^{1/2}T)\sum_{n_0=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx x(1-x^2)^{-1/2} A^{\nu}\frac{(-1)^{\nu}e^{-Ax}}{\Gamma(1)\Gamma(\nu+1)} \\
&\quad\times\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\nu+1+k)}{k!(\nu+k)!}(Ax)^k[\Psi(k+1) - \ln(Ax)] + (-Ax)^{-\nu}\sum_{k=0}^{\nu-1}\left[\frac{\Gamma(\nu-k)\Gamma(k+1)}{k!}(-Ax)^k\right]\right\} \\
&= -2\cdot(4\pi^2\alpha')^{-\nu}\frac{(-1)^{\nu} f_0^{\mathrm{IIA(IIB)}}(\pi\alpha'^{1/2}T)}{(\nu)!}\sum_{n_0=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx(1-x^2)^{-1/2} \\
&\quad\times\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\times A^{\nu+k}x^{k+1}\exp[-Ax][\Psi(k+1) - \ln A - \ln x] + \sum_{k=0}^{\nu-1}(-1)^{k-\nu}\Gamma(\nu-k)\exp[-Ax]A^k x^{k-\nu+1}\right\}.
\end{aligned} \tag{4.3}$$

Substituting the Taylor expansions for $\ln(x)$, and the power series expansion from the change in variable, τ_1, to $x \equiv 1/u$, as in the previous section, the result for the vacuum energy density is obtained by making use of the two elementary integrals,

$$\int_{\sqrt{3}/2}^{1} dx x^{\alpha+2r-1} e^{-Ax} = A^{-\alpha-2r}[\gamma(\alpha+2r, A) - \gamma(\alpha+2j, \sqrt{3}A/2)], \tag{4.4}$$

and

$$\sum_{r=0}^{\infty}(-1)^{2r+1}\frac{1}{j!}[(1/2)(3/2)\cdots(r-1/2)]\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}\int_{\sqrt{3}/2}^{1} dx x^{\alpha+2r-1}(x-1)^{l}e^{-Ax}. \tag{4.5}$$

Performing the integrals over x gives the following result for the vacuum energy density:

$$\begin{aligned}\rho_0^{(\mathrm{II})} = {} & -2\cdot(4\pi^2\alpha')^{-\nu} b_0^{(\mathrm{IIA-IIB})}\sum_{n_0=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{k!}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \\ & \times\frac{(-1)^{\nu}}{(\nu)!}\Bigg\{A^{\nu-2-2r}[\gamma(k+2r+2,A)-\gamma(k+2r+2,\sqrt{3}A/2)][\Psi(k+1)-\ln A] \\ & +\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}\frac{l!}{j!(l-j)!}A^{\nu-2-2r-j}[\gamma(k+2r+j+2,A)-\gamma(k+2r+j+2,\sqrt{3}A/2)]\Bigg\} \\ & -2\cdot(4\pi^2\alpha')^{-\nu}\frac{b_0^{(\mathrm{IIA-IIB})}}{(\nu)!}\sum_{n_0=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\nu-1}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \\ & \times(-1)^{k}\Gamma(\nu-2-k)A^{\nu-2-2r}[\gamma(k+2r-\nu+2,A)-\gamma(k+2r-\nu+2,\sqrt{3}A/2)],\end{aligned} \tag{4.6}$$

where $A \simeq \frac{8\pi^3\alpha' n_0^2}{\beta^2} + (n_0 w_9 + n_9 w_0)4(1+\tanh(\pi\alpha))$ in the massless field limit and at large spatial radius, and low temperatures.

Substituting the power series expansion for the incomplete gamma functions [GR 8.354.1],

$$\gamma(q,A) = \sum_{s=0}^{\infty}\frac{(-1)^s A^{q+s}}{s!(q+s)}, \tag{4.7}$$

gives the simpler result:

$$\begin{aligned}\rho_0^{\mathrm{IIA(IIB)}}(\beta) = {} & -\frac{1}{4}\cdot(4\pi^2\alpha')^{-\nu}\frac{(-1)^{\nu} b_0^{\mathrm{IIA(IIB)}}}{(\nu)!}\sum_{n_0=-\infty}^{\infty}A^{\nu}\sum_{r=0}^{\infty}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \\ & \times\sum_{s=0}^{\infty}\frac{(-1)^s A^s}{s!}\Bigg\{\sum_{k=0}^{\infty}\frac{A^k}{k!}\left[\frac{1}{k+2r+2+s}-\frac{(\sqrt{3}/2)^{k+2r+2+s}}{k+2r+2+s}\right]\{\Psi(k+1)-\ln A\} \\ & +\sum_{k=0}^{\infty}\frac{A^k}{k!}\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}\left(\frac{l!}{j!(l-j)!}\right)\left[\frac{1}{k+2r+j+2+s}-\frac{(\sqrt{3}/2)^{k+2r+j+2+s}}{k+2r+j+2+s}\right] \\ & -\sum_{k=0}^{\nu-1}(-1)^k\Gamma(\nu-k)\left[\frac{1}{k+2r-\nu+2+s}-\frac{(\sqrt{3}/2)^{k+2r-\nu+2+s}}{k+2r-\nu+2+s}\right]\Bigg\}.\end{aligned} \tag{4.8}$$

Substituting in this expression for $A \simeq \frac{8\pi^3\alpha' n_0^2}{\beta^2} + (n_0 w_9 + n_9 w_0)4(1+\tanh(\pi\alpha))$, the infinite summations over the thermal momentum modes, n_0, can be recognized as the Riemann zeta function $\zeta(z,q)$ [GR9.531] [23] and its derivative,

$$\sum_{n=0}^{\infty}(n+q)^{-z} \equiv \zeta(z,q), \qquad \zeta(-n,0) = -\frac{B_{n-1}}{n+1}, \qquad \zeta'(z) \equiv \sum_{n=1}^{\infty} n^{-z}\ln n. \tag{4.9}$$

This last substitution gives the explicit result,

$$\rho_0^{\mathrm{IIA(IIB)}}(\beta) = -(4\pi^2\alpha')^{-\nu}\frac{(-1)^\nu b_0^{\mathrm{IIA(IIB)}}}{\nu!}(8\pi^3\alpha' T^2)^\nu \sum_{r=0}^{\infty}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)]$$
$$\times \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}(8\pi^3\alpha')^{k+s}\beta^{-2k-2s}\left[\frac{1-(\sqrt{3}/2)^{k+2r+2+s}}{k+2r+2+s}\right]\right.$$
$$\times[\zeta(-2\nu-2k-2s;0)\Psi(k+1)-\zeta'(-2\nu-2k-2s)\ln[2\pi\alpha'\beta^{-2}]]$$
$$+\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}\left(\frac{l!}{j!(l-j)!}\right)\zeta(-2\nu-2k-2s;0)$$
$$\left.\times(2\pi\alpha')^k\beta^{-2k}\left[\frac{1-(\sqrt{3}/2)^{k+2r+j+2+s}}{k+2r+2+s}\right]-\sum_{p=0}^{\nu-1}(-1)^p\Gamma(\nu-p)\left[\frac{1-(\sqrt{3}/2)^{p+2r-\nu+2+s}}{p+2r-\nu+2+s}\right]\right\}. \tag{4.10}$$

Keeping the leading terms in powers of $(\alpha'^{-1/2}T)\ll 1$, we extract the large radius, and low-temperature, low-energy supergravity field theory limit of the type IIA(IIB) one-loop vacuum energy density, with $\nu=5$:

$$\lim_{\beta\to\infty}\rho^{\mathrm{IIA(IIB)}}$$
$$= -(4\pi^2\alpha')^{-5}(8\pi^3\alpha')^5 T^{10}\frac{b_0^{\mathrm{IIA(IIB)}}}{5!}$$
$$\times\left\{\frac{1}{8}\zeta(-10;0)(1+\Psi(1)-\zeta'(-10)\ln[2\pi\alpha' T^2])\right.$$
$$\left.+\sum_{p=0}^{4}(-1)^p\Gamma(5-p)\left[\frac{1}{p-3}-\frac{(\sqrt{3}/2)^{p-3}}{p-3}\right]\right\}. \tag{4.11}$$

V. EQUILIBRIUM SPIN$(32)/Z_2$ STRING CANONICAL ENSEMBLE

Our starting point in this section is the generating functional of connected one-loop heterotic string vacuum graphs, $W(\beta)=\ln Z(\beta)$, where $Z(\beta)$ is the canonical partition function. $W(\beta)$ can be derived from first principles by the manifestly Weyl × diffeomorphism invariant Polyakov string path integral quantization on target spacetime, $R^8\times T^2$, with toroidal radii R_H and $\beta_H/2\pi$, respectively [1,2]. The Helmholtz free energy, $F(\beta)$, and the vacuum energy density, $\rho(\beta)$, can be directly inferred from $W(\beta)$.

Generic points in the moduli space of the 2-torus can be reached by group elements that preserve the Z_2 subgroups of the target space duality group of the perturbative spin$(32)/Z_2$ heterotic string compactified on a 2-torus: $R_H\to\alpha'/R_H$, $\beta_H\to 4\pi^2\alpha'/\beta_H$. In the presence of a background Neveu-Schwarz sector antisymmetric 2-form potential, B_{09}, it is convenient to complexify the pair of coordinates (X^0,X^9), and with $G^{00}=G^{99}=1$, $B^{00}=B^{99}=0$; $B^{09}=-B^{90}=B$, and $|B_{09}|=\tanh(\pi\alpha)$, the contribution to the path integral from the (n,w)th sector is

$$\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{\alpha' n_9^2}{R_9^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta^2}{4\pi^2\alpha'}\right.\right.$$
$$\left.\left.+\frac{w_0^2(1+\tanh(\pi\alpha))^2R_9^2}{\alpha'}\right)\right]$$
$$\times\exp\left[-\pi\tau_2\frac{4}{\alpha'}\left[\frac{1}{2}(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\right]\right]. \tag{5.1}$$

Notice that the pure thermal momentum and twisted thermal winding states, $(n^0,0)$ and $(0,w^9)$, are potential thermal tachyons that enter into the general expression for the string mass level expansion. The T-duality transformations are a little more complicated in the presence of constant background $|B_{09}|$, $\mathbf{A_9}$ NS-sector gauge potentials. Adding to our discussion of the type II superstrings following [19], we parametrize the marginal deformations by three real numbers:

$$B=\frac{1}{2}B_{i\bar{j}}dx^i\wedge dx^{\bar{j}},\qquad J=\frac{1}{2}g_{i\bar{j}}dx^i\wedge dx^{\bar{j}},$$
$$A=\frac{1}{2}A_i^I dx^i q_I,\qquad I=1,\ldots,16. \tag{5.2}$$

The B field appears both in the normalization of the string path integral through the e^{-S} term in the Polyakov path integral [11,16], also resulting in a nonvanishing characteristic for the Jacobi theta functions, and, together with the shift due to the Wilson line, $\mathbf{q}\cdot\mathbf{A_9}$, in the Lorentzian self-dual (17, 1)-dimensional lattice momentum summations.

The generating functional of heterotic connected one-loop vacuum string graphs in the equilibrium finite temperature vacuum will be given by compactification on $R^8\times T^2$, with radii R_H, $\beta_H/2\pi$, $\mathbf{A}$ is the Wilson line wrapping the spatial coordinate of radius R^H, $\mathbf{q}$ is a vector in the Lorentzian self-dual lattice $\Gamma^{(17,1)}$,

$$
\begin{aligned}
\mathcal{W}_H = {}& \mathcal{N}\beta_{\rm H}2\pi R_9 L^9 (4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\left\{\frac{d^2\tau}{4\tau_2^2}\tau_2^{-4}[\eta(\tau)\bar{\eta}(\bar{\tau})]^{-6}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\bar{\tau}_2\alpha^2}\bar{\eta}(\bar{\tau})}{\bar{\Theta}_{11}(\alpha,\bar{\tau})}\right]\right\}\\
&\times\left[\frac{\bar{\Theta}_{00}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\eta(\bar{\tau})}\left(\frac{\bar{\Theta}_{00}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3-\frac{\bar{\Theta}_{01}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\bar{\eta}(\bar{\tau})}\left(\frac{\bar{\Theta}_{01}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3-\frac{\bar{\Theta}_{10}(\alpha,\bar{\tau})}{e^{\pi\tau_2\alpha^2}\bar{\eta}(\bar{\tau})}\left(\frac{\bar{\Theta}_{10}(0,\bar{\tau})}{\bar{\eta}(\bar{\tau})}\right)^3\right]\\
&\times\sum_{n_0,w_0=-\infty}^{\infty}\sum_{n_9,w_9=-\infty}^{\infty}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta_H^2}{4\pi^2\alpha'}\right)\right]\\
&\times\exp\left[-\pi\tau_2\frac{4}{\alpha'}\left(\frac{1}{2}(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\right)\right]\times[\eta(\tau)]^{-16}\\
&\times\sum_{\mathbf{k}\in\Gamma^{(17,1)}}\exp\left[-\pi\tau_2\alpha'\left\{\left(\frac{n_9}{R_H}+\frac{w_0(1+\tanh(\pi\alpha))R_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A_9}-\frac{w_9}{R_{\rm H}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2\right\}\right]\\
&\times\exp\left[-\pi\tau_2\alpha'\left\{\left(\frac{n_9}{R_H}-\frac{w_0(1+\tanh(\pi\alpha))R_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A_9}-\frac{w_9}{R_{\rm H}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2\right\}\right]\exp\left[-\pi\tau_2\{(\mathbf{q}-w_9R_H\mathbf{A}_9)^2\}\right], \qquad (5.3)
\end{aligned}
$$

where $\mathcal{N}$ is the hitherto unknown normalization of the one-loop heterotic string vacuum energy density, which will be computed in Sec. VII of this paper. The 16-dimensional Euclidean even integral self-dual lattice contained in $\Gamma^{(17,1)}$ is that for the affine Lie group $\mathrm{spin}(32)/\mathrm{Z}_2$.

As for the closed bosonic string analyzed in the Appendix, a change of variable to $y = 1/\tau_2$, and with $i = 0, 9$, expresses the τ_2 integral in the form

$$
\rho^{(\rm H)}(\beta_H) = -\mathcal{N}(4\pi^2\alpha')^{-5}\cdot\sum_{m=0}^{\infty}f_m^{(\rm H)}(\alpha)\sum_{n_i=-\infty}^{+\infty}\sum_{w_i=-\infty}^{+\infty}\sum_{\mathbf{q}\in\Gamma^{(17,1)}}\int_{-1/2}^{+1/2}d\tau_1\int_0^{1/\sqrt{1-\tau_1^2}}\frac{1}{4}dy y^{9/2}e^{-\pi D/y}, \qquad (5.4)
$$

where the exponent D equals

$$
\begin{aligned}
D \equiv {}& 2m\pi+\left[\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta_H^2}{4\pi^2\alpha'}\right]+\frac{4}{\alpha'}\left[\frac{1}{2}(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))\right]\\
&+\pi\alpha'\left\{\left(\frac{n_9}{R_H}+\frac{w_0(1+\tanh(\pi\alpha))R_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A_9}-\frac{w_9}{R_{\rm H}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2\right.\\
&\left.+\left(\frac{n_9}{R_H}-\frac{w_0(1+\tanh(\pi\alpha))R_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A_9}-\frac{w_9}{R_{\rm H}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2+(\mathbf{q}-w_9R_H\mathbf{A}_9)^2\right\}. \qquad (5.5)
\end{aligned}
$$

The integral over τ_2 can be recognized as an integral representation of the Whittaker function, $\mathcal{W}_{\nu,\lambda}(z)$:

$$
\int_0^u dx x^{\nu-1}e^{-\frac{\pi D}{x}} = (\pi D)^{\frac{\nu-1}{2}}u^{\frac{\nu+1}{2}}e^{-\frac{\pi D}{2u}}W_{-\frac{\nu+1}{2},\frac{\nu}{2}}(\pi D/u), \quad \nu = 11/2, \qquad (5.6)
$$

where we have set $u = (1-\tau_1^2)^{-1/2}$.

A. High T convergence of $\mathbf{spin(32)/Z_2}$ energy density

The high mass level number, UV asymptotics of the $\mathrm{spin}(32)/\mathrm{Z}_2$ heterotic string can be inferred by substituting the asymptotic expansion for the Whittaker function [20,21]:

$$
\mathcal{W}_{(\kappa,\lambda)}(z)\sim e^{-z/2}z^{\kappa}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}z^{-k}\left[\lambda^2-\left(\kappa-\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(\kappa-\frac{3}{2}\right)^2\right]\cdots\left[\lambda^2-\left(\kappa-k+\frac{1}{2}\right)^2\right]\right). \qquad (5.7)
$$

To proceed, note the degeneracies, $b_m^{(\rm H)}$, in the heterotic string mass level expansion are corrected by numerical factors given by the integral

$$
I(m) = \frac{1}{4}\sum_{w=-\infty}^{\infty}(\pi D)^{-1}\int_{-1/2}^{+1/2}d\tau_1(1-\tau_1^2)^{-3/2}e^{-\frac{1}{2}\pi D(1-\tau_1^2)^{1/2}}\mathcal{W}_{-\frac{13}{4},\frac{11}{4}}(\pi D[1-\tau_1^2]^{1/2}). \qquad (5.8)
$$

The term-by-term integrals over τ_1 can be evaluated by substituting the asymptotic expansion for the Whittaker function valid at large mass level number,

$$\begin{aligned} I(m,w) &\equiv \frac{1}{4}(\pi D)^{-1}\left[\int_0^{2/\sqrt{3}} - \int_0^1\right] du u^{\frac{11}{4}-3}(1-1/u^2)^{-1/2} e^{-\frac{1}{2}\pi D/u}\mathcal{W}_{-\frac{11}{4},\frac{9}{4}}(\pi D/u) \\ &= \frac{1}{4}(\pi D)^{-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}(\pi D)^{-k}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)]\times\frac{1}{k!}(-1)^{k+1}C_k \\ &\quad\times[\mathcal{W}_{(-7-4k+8r)/8,(3+4k-8r)/8}(\pi D\sqrt{3}/2) - \mathcal{W}_{(-7-4k+8r)/8,(3+4k-8r)/8}(\pi D)], \\ C_0 = -1, \qquad & C_k \equiv \left[\left(\frac{13}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right], \end{aligned} \tag{5.9}$$

for all $k \geq 1$.

Iterating the substitution of the asymptotic expansion for large argument of the Whittaker functions, as in Sec III, we find an explicit exact result for the finite temperature vacuum energy density of the heterotic strings:

$$\begin{aligned} \rho^{(\mathrm{H})}(\beta_H) &= -\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty} f_m^{(\mathrm{H})}[\infty]\sum_{n_i=-\infty}^{+\infty}\sum_{w_i=-\infty}^{+\infty}\sum_{\mathbf{k}\in\Gamma^{(17,1)}}\sum_{r=0}^{\infty}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)] \\ &\quad\times\sum_{k=0}^{\infty}(\pi D)^{-1-k}\frac{1}{k!}(-1)^k C_k\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j C_{(j,k,r)}[(\sqrt{3}\pi D/2)^{(-7-4k-4j+8r)/8}e^{-\pi D\sqrt{3}/4} - (\pi D)^{(-7-4k-4j+8r)/8}e^{-\pi D/2}]. \\ C_0 &= -1, \qquad C_k \equiv \left[\left(\frac{13}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2 - \left(\frac{9}{4}\right)^2\right]. \\ C_{(0,0,0)} &= -1, \qquad C_{(j,k,r)} \equiv \left[\left(\frac{11+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]\left[\left(\frac{19+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]\cdots \\ &\quad\times\left[\left(\frac{3+4j+4k-8r}{8}\right)^2 - \left(\frac{3+4k-8r}{8}\right)^2\right]. \end{aligned} \tag{5.10}$$

This result should be compared with the analogous result for the perturbative type II superstring derived in Eq. (3.6). As in that case, the reader might be concerned whether the summation over k is convergent: the numerical coefficients, $C_k > 1$, grow with increasing k, and successive terms in the series have alternating sign. However, for large mass level number, m, the succeeding terms in the summation are suppressed due to the negative powers of m. Expressing the series as the sum of two like-sign infinite series, it is apparent that successive terms in each are suppressed by a factor of $1/m$, in addition to the overall exponential suppression. This will lead to very rapid convergence.

The result is $O(e^{-D})$, providing the exponential suppression as a *linear* power of mass level number m. Thus, following an analytic evaluation of the integrals over world-sheet moduli, (τ_1, τ_2), we find that the numerical correction, $I(m)$, to the asymptotic degeneracies, $f_m^{\mathrm{H}}[\infty]$, in the mass level expansion, is an exponential suppression of the precise form,

$$\begin{aligned} D \equiv 2m\pi &+ \left[\frac{4\pi^2\alpha' n_0^2}{\beta_H^2} + \frac{w_9^2 2^2\beta_H^2}{4\pi^2\alpha'}\right] + \frac{4}{\alpha'}\left[\frac{1}{2}(n_0 w_9 + n_9 w_0)4(1+\tanh(\pi\alpha))\right] \\ &+ \pi\alpha'\left\{\left(\frac{n_9}{R_H} + \frac{2w_0 R_H}{\alpha'} - \mathbf{q}\cdot\mathbf{A_9} - \frac{w_9}{R_{\mathrm{H}}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2 + \left(\frac{n_9}{R_H} - \frac{2w_0 R_H}{\alpha'} - \mathbf{q}\cdot\mathbf{A_9} - \frac{w_9}{R_{\mathrm{H}}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2 + (\mathbf{q} - w_9 R_H\mathbf{A_9})^2\right\}, \end{aligned} \tag{5.11}$$

which can be compared with Eq. (3.7) for the type IIA(IIB) superstrings. The result is $O(e^{-m})$ which will erase the $O(e^{\sqrt{m}})$ growth of the asymptotic degeneracies, $f^{(\mathrm{H})}(2)$ at large mass level number, m, [6,22], and for temperatures at, and above, the string scale.

Thus, the *convergence* of the free energy in the ultraviolet, namely, at high mass level numbers, and for *all* values of the target spacetime moduli: $(R_9, \beta_H; \tanh(\pi\alpha))$, is extremely rapid, an exponential suppression with increasing m:

$$\rho_{\mathrm{H}} \simeq -\mathcal{N}(4\pi^2\alpha')^{-5}\frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{w=-\infty}^{\infty} f_m^{\mathrm{H}}(\alpha)\sum_{r=0}^{\infty}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)]$$
$$\times\sum_{k=0}^{\infty}(\pi D)^{-(15+12k)/8}\frac{1}{k!}(-1)^k C_k\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j C_{(j,k,r)}$$
$$\times[(\sqrt{3}/2)^{(-7-4k-4j+8r)/8}e^{-\pi D\sqrt{3}/2}-e^{-\pi D}](\pi D)^{j+r}. \tag{5.12}$$

The high-temperature behavior at high mass level numbers, and large spatial radius, is dominated by the twisted thermal windings, $w_9^2\beta_H^2$, and the spatial momentum modes. As with the type IIA(IIB) superstrings, since $\tanh(\pi\alpha)$ asymptotes to $+1$, the appearance of the B field is absent in the high-temperature limit. In the large radius limit, $R_9\to\infty$, the $n_0=w_0=0$ sector dominates, and the spin$(32)/Z_2$ exponent asymptotes to

$$D\simeq 2m\pi+\frac{w_9^2 2^2\beta_H^2}{4\pi^2\alpha'}+2\pi\alpha'\left(\frac{n_9}{R_H}\right)^2+\pi\alpha'\left(\mathbf{q}\cdot\mathbf{A_9}+\frac{w_9}{R_{\mathrm{H}}}\mathbf{A_9}\cdot\mathbf{A_9}\right)^2+(\mathbf{q}-w_9R_H\mathbf{A}_9)^2. \tag{5.13}$$

Recall the square root exponential growth as a function of mass level m of the degeneracies, $b_m^{(H)}$, at large mass level numbers derived in [6]. However, the rapid convergence of the free energy is driven by the exponent, D, which provides an exponential suppression *linear* as a function of mass level number. As shown for the type IIA and IIB strings in the previous section, our explicit analytic integration of the heterotic closed string world-sheet moduli has pinned down the precise mathematical form of the convergence in the ultraviolet: as fast as an exponential superimposed on the power law suppression of the degeneracies at high mass levels.

B. Low T supergravity spin$(32)/Z_2$ gauge field theory limit

As for the type IIA(IIB) superstring, we now examine the noncompact $R_H\to\infty$, $\beta_H\to\infty$ ten-dimensioanl low-energy field theory limit, where the contributions from all winding modes are suppressed, specializing to the massless low-energy field theory limit of the string mass spectrum, m = 0. The power series representation of the Whittaker function then takes the general form of an infinite power series, plus logarithmic, plus finite polynomial correction [20,21], and we have set x = 1/u:

$$\mathcal{W}_{\lambda,\mu}(z)=\frac{(-1)^{2\mu}z^{\mu+\frac{1}{2}}e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-\mu-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)}\times\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\mu+k-\lambda+\frac{1}{2})}{k!(2\mu+k)!}\right.$$
$$\times z^k\left[\Psi(k+1)+\Psi(2\mu+k+1)-\Psi\left(\mu+k-\lambda+\frac{1}{2}\right)-\ln z\right]$$
$$\left.+(-z)^{-2\mu}\sum_{k=0}^{2\mu-1}\left[\frac{\Gamma(2\mu-k)\Gamma(k-\mu-\lambda+\frac{1}{2})}{k!}(-z)^k\right]\right\},$$
$$\text{where } z=D(1-\tau_1^2)^{1/2},\ \lambda=-3,\ \mu=5/2. \tag{5.14}$$

We perform the change of variable from τ_1 to x as shown in the Appendix, also substituting the power series representation for the Whittaker function, and the Taylor series expansions, so that every term by term integral over the variable x is an incomplete gamma function, as in Sec. IV.

Substitution in the expression above gives the following result for the 496 massless gauge bosons of the spin$(32)/Z_2$ heterotic string [10,24], or by setting $\mathbf{q}=(1^8\cdots,0^8)$, we have the 240 gauge bosons for $SO(16)\times SO(16)$, in addition to the massless bosonic fields of the chiral ten-dimensional N = 1 supergravity multiplet: the so-called Neveu-Schwarz sector of the type I–heterotic supergravity, without the Ramond-Ramond antisymmetric tensor potentials, and the $(\mathbf{1},\mathbf{28},\mathbf{35})$ irreps of the transverse SO(8) rotation subgroup of the Lorentz group, respectively, the scalar dilaton, symmetric, and antisymmetric, rank-two tensor fields:

$$\mathcal{W}^{[\mathrm{O}(32)]}_{\mathrm{m=w_i=0;n_9=0}}(T_H) = \mathcal{N}\cdot 2\cdot \beta_H \mathcal{V}_9(4\pi^2\alpha')^{-5} f_0^{(\mathrm{H})}(2)\sum_{\mathbf{q}}\sum_{n_i,w_i=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx x^{-2}(1-x^2)^{-1/2}D^2\exp\left[-\frac{D}{2}x\right]\mathcal{W}_{-3,5/2}(Dx)$$
$$= \mathcal{N}\cdot 2\cdot(2\pi R_H)\mathcal{V}_9(4\pi^2\alpha')^{-5} f_0^{(\mathrm{H})}(2)\sum_{q_I}\sum_{n_0=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx x(1-x^2)^{-1/2}D^5\frac{(-1)^5 e^{-Dx}}{\Gamma(1)\Gamma(6)}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{\Gamma(6+k)}{k!(5+k)!}(Dx)^k[\Psi(k+1)+\Psi(6+k)-\Psi(6+k)-\ln(Dx)]\right.$$
$$\left.+(-Dx)^{-5}\sum_{k=0}^{4}\left[\frac{\Gamma(5-k)\Gamma(k+1)}{k!}(-Dx)^k\right]\right\}$$
$$= \mathcal{N}\cdot 2\cdot \beta_H\mathcal{V}_9(4\pi^2\alpha')^{-5}\frac{(-1)^5 f_0^{(\mathrm{H})}(2)}{5!}\sum_{\mathbf{q}}\sum_{n_0=-\infty}^{\infty}\int_{\sqrt{3}/2}^{1} dx(1-x^2)^{-1/2}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\times D^{5+k}x^{k+1}\exp[-Dx][\Psi(k+1)-\ln D-\ln x]+\sum_{k=0}^{4}(-1)^{k-5}\Gamma(5-k)\exp[-Dx]D^k x^{k-4}\right\}. \tag{5.15}$$

Substituting the Taylor expansions for the logarithm, and performing the integrals over x, gives the following result for the vacuum energy density:

$$\rho_0^{[\mathrm{O}(32)]}(R_H) = -\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{(-1)^5 f_0^{(\mathrm{H})}(2)}{5!}\sum_{q_I}\sum_{n_i,w_i=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{k!}C_{(r)}^{(-1/2)}$$
$$\times\left\{D^{3-2r}[\gamma(k+2r+2,D)-\gamma(k+2r+2,\sqrt{3}D/2)][\Psi(k+1)-\ln D]\right.$$
$$\left.+\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}C_{(j)}^{(l)}D^{3-2r-j}[\gamma(k+2r+j+2,D)-\gamma(k+2r+j+2,\sqrt{3}D/2)]\right\}$$
$$-\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{f_0^{(\mathrm{H})}(2)}{5!}\sum_{q_I}\sum_{n_0=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{4}C_{(r)}^{(-1/2)}(-1)^k\Gamma(5-k)$$
$$\times D^{3-2r}[\gamma(k+2r-3,D)-\gamma(k+2r-3,\sqrt{3}D/2)]. \tag{5.16}$$

Setting $\mathbf{A}$ to zero, restricting to the spin$(32)/\mathrm{Z}_2$ group alone. In the $m=0$ low-energy finite temperature supergravity-SO(32) Yang Mill gauge field theory limit, we have

$$D\simeq\frac{4\pi^2\alpha' n_0^2}{\beta_H^2}+(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))+\alpha'\left(\frac{n_9}{R_H}\right)^2, \tag{5.17}$$

where we present the dominant terms in the exponent, at large spatial radius, and low temperatures.

As a final simplification, we substitute the power series representation for the incomplete gamma function [20,21], which gives the simpler result:

$$\rho_0^{[\mathrm{O}(32)]}(\beta_H) = -\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{(-1)^5 f_0^{(\mathrm{H})}(2)}{5!}\sum_{n_i,w_i=-\infty}^{\infty}D^5\sum_{r=0}^{\infty}C_{(r)}^{(-1/2)}\sum_{s=0}^{\infty}\frac{(-1)^s D^s}{s!}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{D^k}{k!}\left[\frac{1}{k+2r+2+s}-\frac{(\sqrt{3}/2)^{k+2r+2+s}}{k+2r+2+s}\right]\{\Psi(k+1)-\ln D\}\right.$$
$$+\sum_{k=0}^{\infty}\frac{D^k}{k!}\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}C_{(j)}^{(l)}\left[\frac{1}{k+2r+j+2+s}-\frac{(\sqrt{3}/2)^{k+2r+j+2+s}}{k+2r+j+2+s}\right]$$
$$\left.-\sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1}{k+2r-3+s}-\frac{(\sqrt{3}/2)^{k+2r-3+s}}{k+2r-3+s}\right]\right\}. \tag{5.18}$$

Completing the square for the terms in n_0, w_9, n_9, w_0, the infinite summations over the n_0 and n_9 momentum modes can be expressed as the Riemann zeta function $\zeta(z, q)$ [21] and its derivative:

$$\sum_{n=0}^{\infty}(n+q)^{-z} \equiv \zeta(z,q), \quad \zeta(-n,0) = -\frac{B_{n-1}}{n+1}, \\ \zeta'(s) \equiv \sum_{n=1}^{\infty} n^{-s}\ln n. \tag{5.19}$$

$$\rho_0^{(\mathrm{H})} = \mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{f_0^{(\mathrm{H})}(2)}{5!}(4\pi^2\alpha')^5 T_H^{10} \\ \times\left\{\left[\frac{1}{8}\zeta(-10;0)(1+\Psi(1)-\zeta'(-10)\ln[\alpha'\beta_H^{-2}])\right]\right. \\ \left.-\sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1-(\sqrt{3}/2)^{k-3}}{k-3}\right]\right\}. \tag{5.20}$$

We note in passing that it can be shown [10,13] that the leading power law dependence on radius in the large radius limit, $D^5 \simeq T_H^{-10}$, holds for the generic Wilson line, and the generic nine-dimensional non-Abelian gauge group obtained in Wilson line circle compactifications.

VI. TYPE IB O(32) GLUON ENSEMBLE IN NONCOMMUTATIVE SPACETIME

Recall that the expression for the ten-dimensional open and closed unoriented type IB string one-loop vacuum energy density contains four terms, summing, respectively, world sheets with the topology of a torus, annulus, Mobius strip, and Klein bottle [10,18]. One of the remarkable features of an open and closed oriented string ensemble in an embedding noncommutative target space time—due to the external antisymmetric tensor potential—is that the open and closed string sectors of the theory perceive distinct target spacetime metrics, as was noted by Seiberg and Witten in 1999 [25]. Namely, the open string states with (0, 9) excitations have a mass spectrum that scales as integer multiples of an effective, B-field-dependent, string tension, as was pointed out by Novak and I [16]. Open string states with (0, 9) polarizations can therefore probe distance scales *shorter* than the fundamental string scale, $\alpha'^{1/2}$, and the corresponding masses are *heavier* than if they had been measured with respect to the fundamental string spacetime metric [25].

In most of this section, we will therefore focus on the open oriented type IA-IB string sector, in the presence of the B-field background, which will show is a fascinating model for gluons and gauge solitons of the O(32) Yang Mills gauge theory in a noncommutative target spacetime [16,25]. The contribution of the type IB torus to the one-loop vacuum energy density on $R^8 \times T^2$, summing oriented world sheets with the topology of a torus, and upon twisting the compactified torus, T^2, with a constant background 2-form NS potential $|B_{09}| = \tanh(\pi\alpha)$, is as was derived in Sec. II for the type IIB superstring,

$$\rho_{\mathrm{tor}}^{(\mathrm{IB})} = -(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}\cdot(\tau_2)^{-3}[\eta(\tau)\bar\eta(\bar\tau)]^{-6}\times\frac{1}{4}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\bar\tau_2\alpha^2}\bar\eta(\bar\tau)}{\bar\Theta_{11}(\alpha,\bar\tau)}\right] \\ \times\left[\frac{\Theta_{00}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{00}(0,\tau)}{\eta(\tau)}\right)^3 - \frac{\Theta_{01}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{01}(0,\tau)}{\eta(\tau)}\right)^3 - \frac{\Theta_{10}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{10}(0,\tau)}{\eta(\tau)}\right)^3\right] \\ \times\left[\frac{\bar\Theta_{00}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\eta(\bar\tau)}\left(\frac{\bar\Theta_{00}(0,\bar\tau)}{\bar\eta(\bar\tau)}\right)^3 - \frac{\bar\Theta_{01}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta(\bar\tau)}\left(\frac{\bar\Theta_{01}(0,\bar\tau)}{\bar\eta(\bar\tau)}\right)^3 - \frac{\bar\Theta_{10}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta(\bar\tau)}\left(\frac{\bar\Theta_{10}(0,\bar\tau)}{\bar\eta(\bar\tau)}\right)^3\right] \\ \times\sum_{n,w}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_{\mathrm{IB}}^2}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta_{\mathrm{IB}}^2}{4\pi^2\alpha'}+\frac{w_0^2(1+\tanh(\pi\alpha))^2R_{\mathrm{IB}}^2}{\alpha'}\right)\right] \\ +\exp[-\pi\tau_2(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))] - (4\pi^2\alpha')^{-4} \\ \times\int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}\cdot(\tau_2)^{-3}[\eta(\tau)\bar\eta(\bar\tau)]^{-6}\times\frac{1}{4}\left[\frac{e^{\pi\tau_2\alpha^2}\eta(\tau)}{\Theta_{11}(\alpha,\tau)}\right]\left[\frac{e^{\pi\bar\tau_2\alpha^2}\bar\eta(\bar\tau)}{\bar\Theta_{11}(\alpha,\bar\tau)}\right] \\ \times\left[\frac{\Theta_{11}(\alpha,\tau)}{e^{\pi\tau_2\alpha^2}\eta(\tau)}\left(\frac{\Theta_{11}(0,\tau)}{\eta(\tau)}\right)^3\right]\left[\frac{\bar\Theta_{11}(\alpha,\bar\tau)}{e^{\pi\tau_2\alpha^2}\bar\eta(\bar\tau)}\left(\frac{\bar\Theta_{11}(0,\bar\tau)}{\bar\eta(\bar\tau)}\right)^3\right] \\ \times\sum_{n,w}\exp\left[-\pi\tau_2\left(\frac{4\pi^2\alpha' n_0^2}{\beta_{\mathrm{IB}}^2}+\frac{\alpha' n_9^2}{R_{\mathrm{IB}}^2}+\frac{w_9^2(1+\tanh(\pi\alpha))^2\beta_{\mathrm{IB}}^2}{4\pi^2\alpha'}+\frac{w_0^2(1+\tanh(\pi\alpha))^2R_{\mathrm{IB}}^2}{\alpha'}\right)\right] \\ \times\exp[-\pi\tau_2(n_0w_9+n_9w_0)4(1+\tanh(\pi\alpha))], \tag{6.1}$$

where the choice of phase, $(+)$, for the (11) spin structure, relative to the (00) spin structure, distinguishes the result for the type IB torus graph at finite temperature. The twist in the argument of the Jacobi theta functions: $|\tanh(\pi\alpha)| \equiv |B|$, and $\alpha \equiv (\beta_C/\beta) = \alpha'^{1/2}T$, is linear in the temperature, measured in units of the inverse string scale. For very low temperatures, note that the B field itself grows linearly with temperature, parametrizing the marginal deformation from the zero temperature supersymmetric vacuum. The expression above is valid for all values of $1/\beta$, a target spacetime modulus, with $|B|$ asymptoting to unity at temperatures approaching the string scale. A thermal duality transformation gives the corresponding behavior of the finite temperature type IA open and closed superstring.

Pure type IIB closed superstrings have no Yang-Mills gauge fields, but spin one gauge fields can exist in the world volume of D-branes in the type IB, and the T_9-dual type IA, open and closed unoriented superstring theories [10,26]. The thermal modes of the open oriented string sector, namely, the annulus graph's contribution to the finite temperature one-loop type IB vacuum energy density, contains the tower of field-theoretic Matsubara states with thermal momenta: $p_0 = 2n_0\pi/\beta_{\mathrm{IB}}$, where $n \in Z$:

$$\sum_{n_0,n_9=-\infty}^{\infty} \exp\left[-\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2}(n_9 - R_{\mathrm{IB}}\mathbf{q}\cdot\mathbf{A_9})^2 t\right]. \quad (6.2)$$

Thus, at finite temperature, the normalized open oriented type IB superstring vacuum functional summing one-loop graphs with the topology of an annulus, and in the presence of an external B field[3] is given by the expression

$$\begin{aligned}\mathcal{W}_{\mathrm{ann}}^{(\mathrm{IB})} &= V_{10}(8\pi^2\alpha')^{-5}(1+|B_{09}|)\int_0^\infty \frac{dt}{2t}\cdot(2t)^{-5}\eta(it)^{-6} \times \sum_{n_i=-\infty}^{\infty}\exp\left[-t\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2}(n_9 - R_{\mathrm{IB}}\mathbf{q}\cdot\mathbf{A_9})^2 t\right]\\ &\quad\times\left[\frac{\Theta_{00}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{00}(0,it)}{\eta(it)}\right)^3 - \frac{\Theta_{01}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{01}(0,it)}{\eta(it)}\right)^3 - \frac{\Theta_{10}(\alpha,\tau)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{10}(0,it)}{\eta(it)}\right)^3\right]\\ &\quad - (8\pi^2\alpha')^{-5}(1+|B_{09}|)\int_0^\infty\frac{dt}{2t}\cdot(2t)^{-4}[\eta(it)]^{-6}\frac{1}{4}\left[\frac{e^{i\pi t\alpha^2}\eta(it)}{\Theta_{11}(\alpha,it)}\right]\left[\frac{\Theta_{11}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)}\left(\frac{\Theta_{11}(0,it)}{\eta(it)}\right)^3\right]\\ &\quad\times\sum_{n_i=-\infty}^{\infty}\exp\left[-t\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2}(n_9 - R_{\mathrm{IB}}\mathbf{q}\cdot\mathbf{A_9})^2 t\right]. \end{aligned} \quad (6.3)$$

As explained in [16], the field-dependent normalization is most simply interpreted as the effective field-dependent string tension of the fundamental *closed* oriented string, $\tau_F^2 = 4\pi\alpha'$, which is twice the square of the fundamental oriented open string tension. Thus, from Eq. (20) of [16], we have

$$\begin{aligned}\tau_{\mathrm{eff}}^{-(p+1)/2} &= (2\pi\alpha'_{\mathrm{eff}})^{-(p+1)/2}\\ &\equiv [(1+\tanh(\pi\alpha))(2\pi\alpha')^{-(p+1)/2}],\end{aligned} \quad (6.4)$$

where $p+1$ is the number of noncompact spacetime coordinates—equal to the critical dimension, ten, for the type IB superstring at very low temperatures, when β_{IB} is large—vs nine noncompact coordinates at temperatures approaching, and of order, the string scale. Notice that the effective string tension is mainly a useful notion at temperatures far below the string scale, since the function $\tanh(\pi\alpha)$ asymptotes to unity at high temperatures of order the string scale, $\alpha'^{-1/2}$. As noted above, the effective tension has also been referred to in [25] as string states perceiving the open string "metric", as opposed to the closed string "metric", since the string mass spectrum is, respectively, measured in integer multiples of the effective, or the fundamental closed, string tension.

What is the role of the *unoriented* open and closed string sectors of the finite temperature type IB superstring, respectively, the Mobius strip and Klein bottle graphs? This has been clarified by many authors [10,18], and we have nothing new to add to the discussion at finite temperature, namely, in a temperature-dependent B-field background. The massless tadpole cancellation in the NS-NS and R-R sectors holds as in the zero temperature vacuum, and the main goal of the orientation projection is to determine the anomaly-free gauge group explicitly, and with a choice of phase, we pick $O(2N)$, with $2N = 32$. Since our main interest is in the Yang Mills gauge fields sector of the type IB O(32) string, namely, the oriented massless open string modes and their thermal excitations, we now focus attention in what follows on the annulus graph alone.

In passing, we note that the expression for the one-loop (Helmholtz) free energy of the T_9-dual type IA canonical ensemble, is deduced by performing a T-duality transformation on R_{IB}. Notice, that due to the antisymmetric

[3]The B-field-dependent normalization for the open oriented bosonic string one-loop amplitude was determined by Novak and I in Eq. (16) of [16]. See also the earlier work [11].

tensor potential, it is the T_9-duality which brings the thermal winding modes into the type IA string mass spectrum, and in addition to the tower of Matsubara thermal momentum modes:

$$\sum_{n_0,w_0=-\infty}^{\infty} \exp\left[-\frac{8\pi^3\alpha' n_0^2}{\beta_{\rm IB}^2} - 8\pi^3\alpha'\left(\frac{w_0 R_{\rm IA}}{\alpha'}[1+|B_{09}|] -(\mathbf{q}\cdot\mathbf{A_9} - w_0 R_{\rm IA}\mathbf{A_9}\cdot\mathbf{A_9})\right)^2\right]. \quad (6.5)$$

The T_9 duality has mapped this to a type IA O(32) string state with 32 coincident D8-branes on one of the two O8 planes at either end of an *interval* of size $R_{\rm IA}$. Thus, the Euclidean time formulation of the type IA string has compactification on $\mathbf{R}^8 \times S^1 \times S^1/Z_2$. The T_9 duality transformation enables one to examine the small distance behavior of the finite temperature type IB superstring, at spatial radii smaller than the string scale; this can be useful in the approach to certain enhanced gauge symmetry points in the moduli space of eight-dimensional and nine-dimensional compactifications of the type IB superstring [27]. It should be noted that the cancelation of dilation tadpoles in the *two-loop* type I open oriented superstring amplitude will be determined by the specific choices of non-Abelian gauge group.

A. Low T behavior of O(32) IB thermal gluon ensemble

As shown for the low-energy supergravity-Yang Mills field theoretic limit of the $\mathrm{spin}(32/Z)_2$ heterotic string ensemble, it is helpful to verify that the noncompact limit of the open oriented one-loop type IB string amplitude, truncated to the mass level zero, and at temperatures very far below the string scale, displays the expected T^{10} growth of the O(32) thermal gluon ensemble.[4]

The normalized one-loop vacuum energy density takes the form

$$\rho_{\rm IB}(\beta_{\rm IB}, R_{\rm IB}; \alpha, A_9) = -(8\pi^2\alpha')^{-5}(1+|B_{09}|)\int_0^\infty \frac{dt}{2t}(2t)^{-5}\sum_{m=0}^{\infty} f_m^{(\rm IB)}(\alpha) \times \sum_{n_i,w_i=-\infty}^{\infty} \exp\left[-t\left\{\pi m + \frac{8\pi^3\alpha' n_0^2}{\beta_{\rm IB}^2} - \frac{8\pi^3\alpha'}{R_{\rm IB}^2}(n_9 - R_{\rm IB}\mathbf{q}\cdot\mathbf{A_9})^2\right\}\right] \times \exp\left[-t(n_0 w_9 + n_9 w_0)4(1+\tanh(\pi\alpha))\right], \quad (6.6)$$

where $\mathbf{q} = (0^{32})$ for the O(32) string, and $\mathbf{q} = (1^{16}, 0^{16})$ for the $O(16)\times O(16)$ type IB superstring in eight noncompact spacetime dimensions.

Next, we split the range of integration, making a change of variable $2t = 1 + x$ in the integral. We recognize the first term as an integral representation of the Whittaker function, $\mathcal{W}_{-3,5/2}(z)$ [20],

$$\int_0^\infty (1+x)^{-\nu} e^{-zx} dx = z^{\frac{\nu}{2}-1} e^{-z/2} \mathcal{W}_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(z) z \equiv A, \quad \nu = 6, \quad \mathcal{W}_{\lambda,-\mu} = \mathcal{W}_{\lambda,\mu}, \quad (6.7)$$

and the exponent A is, respectively, defined as

$$A_{\rm IB} \equiv \frac{1}{2}\pi m + \frac{4\pi^3\alpha' n_0^2}{\beta_{\rm IB}^2} - \frac{4\pi^3\alpha'}{R_{\rm IB}^2}(n_9 - R_{\rm IB}\mathbf{q}\cdot\mathbf{A_9})^2. \quad (6.8)$$

In the noncompact limit, $R_9 \to \infty$, and for adequately low temperatures far beneath the string scale, and at mass level zero, the exponent function, A, can be approximated by the thermal term. The following analysis of the power series representation of the special functions holds in this limit alone, as in the corresponding discussions in Sec. IV, and Sec. VB, for the closed superstrings.

The remaining integral over the variable $-x$ in the domain [0, 1], is subtracted from the Whittaker function above, and it gives the first of the three confluent hypergeometric series in two variables [20]:

$$\int_0^1 x^{\mu-1}(1-x)^{-\lambda}(1-yx)^{-\rho} e^{-xA} dx = B(\mu,\lambda) \times \Phi_1(\mu,\rho,\lambda+\mu; A, y), \quad \mu = \lambda = 1,\ \rho = 5,\ y = 1. \quad (6.9)$$

This function is related to the first of the convergent Appel series in two variables, $F_1(a,b,b',c;A,y)$, when $|A| < 1$, as is true in the mass level number zero, low temperatures far below the string scale, and large radius limit. The Appel function, F_1, by definition, equals Φ_1, when $b' = 0$ [20],

[4]There is one significant difference in how we obtain the low-energy field theory limit in contrast with [4], since we perform the modular integral in the expression for the one-loop string amplitude, and set mass level number to zero, as opposed to evaluating the amplitude at a fixed value of the annulus' modulus, $t = 1$, as was proposed in [4]. The point is that such a procedure violates world-sheet modular invariance, and it is unnecessary to break the world-sheet gauge symmetries while extracting specific kinematic regimes of the theory

$$F_1(a,b,b',c;A,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n A^m y^n}{(c)_{m+n}m!n!}\bigg|_{b'=0} = \Phi_1(a,b,c;A,y) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m A^m y^n}{(c)_{m+n}m!n!}, \tag{6.10}$$

and Φ_1 can thus be related to the familiar hypergeometric function, $F(a,b,c;A)$, when $y=1$, reducing to a hypergeometric series in one variable,

$$F_1(a,b,b',c;A,y) = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m A^m}{(c)_m m!}F(a+m,b';c+m;y). \tag{6.11}$$

Thus, setting $b'=0$, and $y=1$, we obtain the simplified result [20]

$$\Phi_1(a,b,c;A,1) = F(a,b,c;A), \qquad F(a+m,0,c+m;1)=1, \qquad a=1, c=2, b=5, y=1. \tag{6.12}$$

Recall that the hypergeometric function is convergent in the region inside the unit circle, $|A|\le 1$, and the series representation above is the expansion about the origin [20].

We likewise substitute the power series expansion for the Whittaker function about $A=0$ [20]:

$$\mathcal{W}_{\lambda,\mu}(A) = \frac{(-1)^{2\mu}A^{\mu+\frac{1}{2}}e^{-\frac{1}{2}A}}{\Gamma(\frac{1}{2}-\mu-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)} \times \left\{\sum_{k=0}^{\infty}\frac{\Gamma(\mu+k-\lambda+\frac{1}{2})}{k!(2\mu+k)!}A^k\left[\Psi(k+1)+\Psi(2\mu+k+1)-\Psi\left(\mu+k-\lambda+\frac{1}{2}\right)-\ln A\right]\right.$$
$$\left. + (-A)^{-2\mu}\sum_{k=0}^{2\mu-1}\left[\frac{\Gamma(2\mu-k)\Gamma(k-\mu-\lambda+\frac{1}{2})}{k!}(-A)^k\right]\right\}, \qquad \text{where } \lambda=-\frac{\nu}{2},\ \mu=\frac{(1-\nu)}{2},\ \nu\equiv 6. \tag{6.13}$$

This gives the following expression for the finite temperature vacuum energy density of the canonical ensemble of thermal gluons in the noncompact limit, and at temperatures far below the string scale,

$$\rho_0(T_{\rm IB}) = -(8\pi^2\alpha')^{-5}(1+|B_{09}|)f_0^{[\rm IB]}(\alpha)\frac{(-1)^5}{5!}2(4\pi^3\alpha')^5T_{\rm IB}^{10}\sum_{n_0=1}^{\infty}e^{-4\pi^3\alpha' n_0^2T_{\rm IB}^2}n_0^{10}$$
$$\times\left\{\sum_{k=0}^{\infty}\frac{1}{(5+k)!}A^k[\Psi(k+1)-\ln A]+\sum_{k=0}^{4}\Gamma(5-k)(-1)^kA^{k-5}\right\}$$
$$-(8\pi^2\alpha')^{-5}(1+|B_{09}|)f_0^{[\rm IB]}(\alpha)\sum_{n_0=0}^{\infty}e^{-4\pi^3\alpha' n_0^2T_{\rm IB}^2}\sum_{l=0}^{\infty}\frac{(1)_l(5)_l}{(2)_l l!}(4\pi^3\alpha' n_0^2T_{\rm IB}^2)^l. \tag{6.14}$$

Substituting the Taylor expansion for the exponentials, valid in the vicinity of $T_{\rm IB}=0$,

$$\sum_{n_0=1}^{\infty}e^{-4\pi^3\alpha' n_0^2T_{\rm IB}^2}n_0^{10} \equiv \sum_{j=0}^{\infty}\sum_{n_0=1}^{\infty}n_0^{10+2j}\frac{(-1)^j}{j!}(4\pi^3\alpha' T_{\rm IB}^2)^j$$
$$=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(4\pi^3\alpha' T_{\rm IB}^2)^j\zeta(-10-2j,0), \tag{6.15}$$

the result can be recognized as an infinite summation over Riemann zeta functions. Comparing with the low-energy supergravity-Yang Mills field theory limit of the heterotic spin$(32)/Z_2$ string vacuum energy density at finite temperature in Eq. (5.21), we find the subleading, and convergent, finite corrections to the expected $(\alpha'^{1/2}T)^{10}$ growth:

$$\rho_0(T_{\rm IB}) \simeq -(8\pi^2\alpha')^{-5}(1+|B_{09}|)f_0^{[\rm IB]}(\alpha)\frac{(-1)^5}{5!}(4\pi^3\alpha')^5T_{\rm IB}^{10}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(4\pi^3\alpha' T_{\rm IB}^2)^j$$
$$\times\left\{\frac{1}{(5)!}[\zeta(-10-2j,0)(\Psi(1)-\ln[4\pi^3\alpha' T_{\rm IB}^2])+\zeta'(-10-2j)]+\Gamma(5)(4\pi^3\alpha' T_{\rm IB}^2)^{-5}\zeta(-10,0)\right\}$$
$$-(8\pi^2\alpha')^{-5}(1+|B_{09}|)f_0^{[\rm IB]}(\alpha)\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^j}{j!}\zeta(-2j-2l,0)\frac{(1)_l(5)_l}{(2)_l l!}(4\pi^3\alpha' T_{\rm IB}^2)^{l+j}. \tag{6.16}$$

For completeness, we give the explicit form of the field-dependent degeneracy function $f_0^{\rm IB}(\alpha)$, at level $m = 0$—similar analyses can be found in the references [10,16,23],

$$\frac{1}{2{\rm Sinh}(\frac{1}{2}[\pi\tanh(\pi\alpha)])}\times[2(2{\rm Cosh}(2\pi[\tanh(\pi\alpha)])+6) + 16{\rm Cosh}(\pi[\tanh(\pi\alpha)])], \tag{6.17}$$

and it is easy to verify that the $\alpha \simeq 0$ limit, gives the zero temperature, target spacetime supersymmetric, result for the degeneracies of target spacetime bosonic minus target spacetime fermionic states: $\frac{1}{2}[16 - 16]$.

B. High T asymptotics of type IB O(32) gluon ensemble

Let us now examine the high temperature regime of the type IB open and oriented string ensemble at temperatures far above the string scale. The high temperature behavior for the ten-dimensional O(32) type IB string canonical ensemble is given by the *low* temperature behavior of the thermal dual type IA canonical ensemble: thermal momentum modes in the type IB string are mapped to thermal winding modes in the T_0-dual type IA string, and due to the twisting by the constant background $|B_{09}|$ field at low type IA temperatures, we must perform *both* the T_9 and the T_0 duality transformations. Thus, we examine the large $\beta_{[{\rm IA}]}$ regime, in the limit $R_{[{\rm IA}]} \gg \alpha'^{1/2}$, and consider the high IB temperature asymptotics at mass level zero.

In other words, we search for a signal of a thermal phase transition[5] in the high temperature type IB thermal gluon ensemble, at temperatures approaching—and above–the string scale, T_C, the type IB thermal momentum modes are transformed to type IA winding soliton strings, at IB temperatures far above the string scale. Recall, also, that the $|B_{09}|$ field asymptotes to unity as T approaches T_C, so that the exponent function will be dominated by the spatial winding number w_9 term, where we note that the general background could extend to a timelike Wilson line background $\mathbf{A}_0$:

$$A_{[{\rm IA}]} \equiv 2\pi m + \frac{8\pi^3}{\alpha'}(w_9^2\beta_{[{\rm IA}]}^2 + w_0^2 R_{\rm IA}^2)[1+|\tanh(\pi\alpha)|]^2 - 8\pi^3\alpha'\left\{\left(\frac{w_0 R_{\rm IA}}{\alpha'} + \frac{w_9\beta_{[{\rm IA}]}}{2\pi\alpha'}\right)[1+\tanh(\pi\alpha)] - (\mathbf{q}\cdot\mathbf{A_9} - w_0 R_{\rm IA}\mathbf{A_9}\cdot\mathbf{A_9})^2\right\}. \tag{6.18}$$

As in Sec. III, and Sec. VA, we begin by setting $y = 1/t$, and expanding in integer powers of $e^{-\pi t}$, which yields the open oriented string mass level expansion, with coefficients given by the $|B|$ field-dependent degeneracies, $f_m^{({\rm IB})}(\alpha)$. We set $m = 0$. This gives the modular integral in the T_9, T_0 duality transformed form,

$$\rho_0^{({\rm I})} = -(8\pi^2\alpha')^{-5}\int_0^\infty dy y^{11/2} \times \sum_{w_9=-\infty}^{+\infty} f_0^{({\rm I})}\exp\left[-\frac{8\pi^3}{\alpha'}(2w_9^2\beta_{[{\rm IA}]}^2)\frac{1}{y}\right]. \tag{6.19}$$

Next, we split the range of integration, making a change of variable $2t = 1 + x$ in the modular integral. We recognize the first term as an integral representation of the Whittaker function, $\mathcal{W}_{-11/4,-9/4}(A)$ [20],

$$\int_0^\infty (1+x)^{-\nu}e^{-Ax}dx = A^{\frac{\nu}{2}-1}e^{-A/2}\mathcal{W}_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(A)$$
$$\nu = 11/2, \quad \mathcal{W}_{\lambda,-\mu} = \mathcal{W}_{\lambda,\mu}, \tag{6.20}$$

and in the noncompact limit, $R_{\rm IA} \to \infty$, and for adequately high type IB temperatures far above the string scale, the duality transformed exponent function, A, is dominated by the type IA spatial winding modes. In this limit, we will substitute the asymptotic expansion for large argument of the Whittaker function, as in Sec. III, and Sec. VA, for the closed superstrings:

$$\mathcal{W}_{(\kappa,\lambda)}(A) \sim e^{-A/2}A^\kappa\left(1 + \sum_{k=1}^\infty \frac{1}{k!}A^{-k}\left[\lambda^2 - \left(\kappa - \frac{1}{2}\right)^2\right] \times\left[\lambda^2 - \left(\kappa - \frac{3}{2}\right)^2\right]\cdots\left[\lambda^2 - \left(\kappa - k + \frac{1}{2}\right)^2\right]\right). \tag{6.21}$$

The remaining integral over the variable $-x$ in the domain [0, 1], is subtracted from the Whittaker function above, and it gives the first of the three confluent hypergeometric function in two variables [20]:

$$\int_0^1 x^{\mu-1}(1-x)^{-\lambda}(1-\beta x)^{-\rho}e^{-xA}dx = B(\mu,\lambda)\Phi_1(\mu,\rho,\lambda+\mu;A,\beta),$$
$$\mu = \beta = 1,\ \rho + \lambda = 11/2. \tag{6.22}$$

As in Sec. VIA, we can identify the Φ function with the ordinary hypergeometric function, $F(\mu,\rho,\lambda+\mu;A)$, but we must now employ its analytic continuation to the region *outside* the unit circle $A = 1$ (GR 9.154.1). We have $\mu = 1$, and choose $\rho = 5 = \mu + n$, $n = 4$, $\lambda = \frac{1}{2}$, which gives

[5]More precisely, this is a phase *transformation*, since either side of the phase boundary at T_C is a weakly coupled open and closed unoriented superstring theory.

$$F(\mu,\mu+n;\lambda+\mu;A)=\frac{\Gamma(\lambda+\mu)}{\Gamma(\mu)\Gamma(\mu+n)}\frac{1}{\pi}\sin\pi(\lambda)\left\{(-A)^{-\mu-n}\sum_{k=0}^{\infty}\frac{\Gamma(\mu+n+k)\Gamma(1-\lambda+n+k)}{\Gamma(k+1)\Gamma(k+n+1)}g(k)A^{-k}\right.$$
$$\left.+\sum_{k=0}^{n-1}\frac{\Gamma(\mu+k)\Gamma(1-\lambda+k)\Gamma(n-k)}{\Gamma(k+1)}(-A)^{-\mu-k}\right\},$$
$$\text{where } g(k)=\ln(-A)+\pi\cot(\pi\lambda)+\psi(k+1)+\psi(k+n+1)-\psi(\mu+k+n)-\psi(1-\lambda+k+n). \quad (6.23)$$

Prior to making these substitutions, let us pause to compare our derivation in this section with that of the UV asymptotics of the type IIA(IIB) and heterotic closed superstrings, where we considered the asymptotic limit of large closed string mass level number, in addition to high temperatures far above the string scale. The point is that when more energy is available to the closed string ensemble—in contact with a heat bath at temperatures high above the string scale, the higher mass level number modes of excitation of the single closed string are easily excited, in addition to the thermal momentum and thermal winding number modes. The open string ensemble responds differently to an influx of heat energy: since higher open string mass level numbers correspond to longer open strings, it is thermodynamically preferable for a single long open string to split into massless thermal gluons, excited to various thermal momentum number modes. The splitting transition is forbidden in a theory of pure closed strings, so that an influx of heat energy can lead to the excitation of the higher mass level string modes. In an open and closed string theory, instead, the condensate of highly energetic thermal gluons—zero length open IB strings, undergoes a phase transition[6] at temperatures far above the string scale to the type IA thermal winding modes: these are solitonic winding number strings, from the perspective of the open string ensemble—and they wrap the spatial compact dimension—of large radius $R_{\rm IA}$.

To emphasize the differences in the result for the open and closed type IA(IB) string, we present the formal result, including all open string mass levels, as in previous Sec. III, and Sec. VA, prior to the truncation to mass level number zero:

$$\rho^{(\rm IA)}=-(8\pi^2\alpha')^{-5}[1+\tanh(\pi\alpha)]\sum_{m=0}^{\infty}f_m^{(\rm IA)}(\alpha)\sum_{w_9=-\infty}^{+\infty}\left[m\pi+\frac{8\pi^3w_9^2}{\alpha' T_{\rm IA}^2}\right]^{-1}\exp\left[-\pi m-\frac{16\pi^3w_9^2}{\alpha' T_{\rm IA}^2}\right]\left\{1+\sum_{k=1}^{\infty}\frac{1}{k!}\right.$$
$$\left.\times\left[m\pi+\frac{8\pi^3w_9^2}{\alpha' T_{\rm IA}^2}\right]^{-k}\left\{\left[\left(\frac{13}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\right\}\right\}$$
$$-(8\pi^2\alpha')^{-5}(1+\tanh\pi\alpha)\sum_{m=0}^{\infty}f_m^{(\rm IA)}(\alpha)\sum_{w_9=-\infty}^{+\infty}B\left(1,\frac{1}{2}\right)\left[\frac{\Gamma(3/2)}{\pi\Gamma(5)}\right]$$
$$\times\left\{\sum_{k=0}^{+\infty}(-1)^5A^{-5-k}g(k)\frac{\Gamma(5+k)\Gamma(\frac{11}{2}+k)}{\Gamma(k+1)\Gamma(k+5)}+\sum_{k=0}^{2}(-1)^{1+k}A^{-1-k}\Gamma\left(\frac{1}{2}+k\right)\Gamma(4-k)\right\},$$
$$\text{where } g(k)=\ln(-A)+\psi(k+1)-\psi(k+11/2),\quad \text{and}\quad A\equiv m\pi+\frac{8\pi^3w_9^2}{\alpha' T_{\rm IA}^2}. \quad (6.24)$$

Prior to performing the infinite summations over spatial windings, notice that for high open string mass levels, where the degeneracy f_m is known to grow as fast as the exponential of the square root of mass level number, the first asymptotic expansion, which arises from the Whittaker function, smoothly erases the Hagedorn growth, with the exponential suppression, linear in the mass level number. This is precisely as for the type IIA(IIB) and heterotic closed oriented superstrings, where the fundamental domain of the modular group of the world-sheet torus eliminates the troublesome $\tau_2\to 0$ limit [2,10,18].

This is not the case for the annulus graph of the open and closed type IB(IA) superstring. The second asymptotic expansion, which arises from the hypergeometric function analytically continued to large argument, does not provide an exponential suppression in the mass level number. Thus, in order to proceed, we henceforth set the mass level

[6]We emphasize once more that the thermal duality transformation may, or may not, suggest a dynamical phase transition. We should point out that, in addition to the splitting transition [10,18], a theory of open and closed strings permits open-closed string conversion [10,18]. Since the IA winding modes are solitonic closed strings, the precise dynamics that might underlie the thermal duality transformation, while suggestive, is not transparent.

number to zero, and restrict our considerations in this paper to the interesting possibility of a phase transition in the level zero thermal gluon ensemble, that can be examined by analytic methods.

The infinite summations over spatial windings w_9 in the annulus graph can be readily performed at mass level zero, resulting in the familiar zeta functions, and their derivatives:

$$\rho_0^{(\mathrm{IA})} = -(8\pi^2\alpha')^{-5}(1+\tanh\pi\alpha) f_0^{(\mathrm{IA})}(\alpha)\left(\frac{8\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right)^{-1}\sum_{j=0}^{\infty}\frac{1}{j!}\left[-\frac{16\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right]^j \left\{\zeta(-2j,0)+\sum_{k=1}^{\infty}\zeta(-2j-2k,0)\frac{1}{k!}\left[\frac{8\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right]^{-k}\right.$$
$$\left.\times\left\{\left[\left(\frac{13}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\left[\left(\frac{17}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\cdots\left[\left(\frac{9+4k}{4}\right)^2-\left(\frac{9}{4}\right)^2\right]\right\}\right\}$$
$$-(8\pi^2\alpha')^{-5}(1+\tanh\pi\alpha) f_0^{(\mathrm{IA})}(\alpha) B\left(1,\frac{1}{2}\right)\left[\frac{\Gamma(3/2)}{\pi\Gamma(5)}\right]\left\{\sum_{k=0}^{+\infty}(-1)^5\left(\frac{8\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right)^{-5-k}\frac{\Gamma(5+k)\Gamma(\frac{11}{2}+k)}{\Gamma(k+1)\Gamma(k+5)}\right.$$
$$\times\left[\ln\left(-\frac{8\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right)2\zeta'(-10-2k)+(\psi(k+1)-\psi(k+11/2))\zeta(-10-2k,0)\right]$$
$$\left.+\sum_{k=0}^{2}(-1)^{1+k}\zeta(-2-2k,0)\left(\frac{8\pi^3}{\alpha' T_{\mathrm{IA}}^2}\right)^{-1-k}\Gamma\left(\frac{1}{2}+k\right)\Gamma(4-k)\right\}. \tag{6.25}$$

We should emphasize that T_C is in no sense an ultimate temperature beyond which the type IB string canonical ensemble breaks down, nor is there a novel non-stringlike phase above T_C. The type IB theory above T_C has a benign string theoretic description as the T_0-dual type IA string. Thus, on either side of the phase boundary we have a weakly coupled and self-consistent perturbative open and closed unoriented string theory; except, that the tower of thermal IB open string momentum modes has been transformed to a tower of *solitonic* IA winding number modes. It is uncanny that the appropriate match with the known behavior of finite temperature QCD appears to be the type IA open and closed string theory: at low type IA temperatures, there is a phase with confined tubes of flux in the thermal winding mode spectrum. As T_{IA} approaches the string scale, the type IA ensemble appears to *deconfine*, giving the ensemble of thermally excited gluons zero length, open oriented, IB strings. Whether this observation of a *suggestive* dynamics underlying the type IA to IB thermal duality transformation is a genuine signal of the QCD deconfinement phase transition remains to be studied in the future.

What should we make of the term in the annulus graph summing all mass levels in the open and closed type IA or type IB string that grows as the exponential of the square root of the open string mass level number, with only a power law suppression from the analytically continued confluent hypergeometric function? That question remains open to further investigation. Our main point in the analysis given in this paper is to distinguish the possible signal of a deconfinement transition in the gluon ensemble, evidenced by the stringy thermal duality transformation from winding IA strings to thermally excited IB gluons—from the Hagedorn transition, or Hagedorn divergence, of the open string annulus graph on its own—and even with the addition of the torus, and the unoriented one-loop string graphs. These well-known additions will not alter the new physics we have discovered. As we have noted—the crucial missing elements are a study of the open string splitting process, and the open-to-closed oriented string conversion process. They are likely to make a significant physical distinction in the full analysis of the one-loop dynamics of the type IB (IA) open and closed unoriented string canonical ensemble. To the best of our knowledge, previous authors who have pointed to applications of the "Hagedorn transition" have made no reference to this crucial distinction between the physics of the closed oriented—and that of the open and closed unoriented—superstrings, and this fundamental distinction deserves further study.

VII. NORMALIZATION OF THE HETEROTIC ONE-LOOP AMPLITUDE FROM DUALITIES

We begin by recapitulating the known target spacetime and strong-weak coupling dualities relating the ten-dimensional type IB-type IA O(32) unoriented superstring theories and the $\mathrm{spin}(32)/Z_2$ heterotic string theory, compactified on a twisted torus, $T^2\times R^8$. We shall thereby determine the unknown normalization, $\mathcal{N}$, of the $\mathrm{spin}(32)/Z_2$ perturbative heterotic string one loop vacuum amplitude.

As was pointed out in [28] as far back as 1986—unlike the normalization of the bosonic and type IIA and IIB superstring one-loop amplitudes—the normalization of the one-loop heterotic string vacuum amplitude *cannot* be computed by the methodology outlined in the papers [1,2]. Nevertheless, in terms of this unknown normalization, $\mathcal{N}$, the analysis in [28] showed that by invoking a combination of factorization theorems and the bootstrap method, namely, matching the low- and high-energy asymptotics of heterotic closed string scattering amplitudes at generic order in the genus expansion, the entire tower of normalizations of the multiloop heterotic string amplitudes can be deduced. To complete this very elegant argument, however, the first step, namely, the normalization of the one-loop vacuum energy density of the heterotic string has to be deduced by independent means. This is the problem we will address in this section, showing that the strong-weak coupling heterotic-type IB-type IA dualities [27] in the low-energy supergravity field theory limit enables a matching of the normalizations of the respective one-loop vacuum energy densities, thereby giving an unambiguous means to compute the hitherto unknown normalization of the spin$(32)/Z_2$ heterotic string one-loop vacuum amplitude.

Consider the torus-compactified ten-dimensional heterotic spin$(32)/Z_2$ and type IB O(32) string theories in the large spatial radius limit, $R_{[O(32)]} \simeq R_{IB} \gg \alpha'^{1/2}$, twisting by a near-vanishing B field, namely, $|B| = \tanh(\pi\alpha)$ approaches zero. Equivalently, given our considerations in the previous sections of this paper, we consider the low temperature limit of either superstring theory. We begin with the target space T_9-duality transformations for either string, and the type IB–heterotic strong-weak coupling relation [10,27],

$$g_{O(32)} = \frac{1}{g_{IB}}; \qquad g_{IA} = g_{IB}\left(\frac{\alpha'}{R_{IB}^2}\right),$$
$$g_{IA} = \frac{1}{g_{[O(32)]}}\left(\frac{\alpha'}{R_{[O(32)]}^2}\right), \tag{7.1}$$

and where the second relation follows from the T_9-duality transformation relating the type IB and type IA superstring theories. In the large radius limit we are considering, both g_{IA} and $g_{[O(32)]}$ take values at weak coupling, while the type IB O(32) string theory is strongly coupled. The addition of a spatial Wilson line background enables an analogous determination for the $O(16) \times O(16)$ string theory [13,27].

Consider extending the standard strong-weak heterotic-type IB duality map [27] to twisted torus compactifications, and the basic T-duality relations, to generalized target space T-duality transformations in the presence of the background fields of the Neveu-Schwarz (NS) sector: G_{mn}, B_{mn}, and A^I_m, the color index $I = 1, \ldots, 16$, labeling the components of the gauge lattice, and m, $n = 0, 1, \ldots, 9$ label embedding target spacetime coordinates [11,16,23]. Unlike the heterotic strings, the unoriented type IB and type IA strings do not have the massless NS antisymmetric tensor field. However, the *constant* mode of the antisymmetric NS 2-form survives the orientifold projection, and we denote this as the B field, B_{mn}. Under a generalized T_m, T-duality transformation of the coordinate X^m_{IB} in the type IB string, the background fields transform as follows:

$$R^m_{IB} \to \frac{\alpha'}{R^m_{IA}}, \qquad n^m \to w_n(G^{mn} + B^{mn}), \qquad A^m_{IB} \leftrightarrow A^m_{IA},$$
$$q_I \to q_I - w_n R A^n_I, \tag{7.2}$$

where the index I labels the location of the Ith D8-brane along the interval S^1/Z_2 of size R^m_{IA}, in the background of the type IA string compactified on the interval X^m_{IA}. It is easy to deduce the T_9-duality transformation of a general Wilson line background in the type IB string; it is mapped to the slightly more complicated winding mode background of the type IA string as follows:

$$\text{type IB}: \; p^m_{IB} = \frac{n^m}{R_{IB}} - q_I A^m_I.$$
$$\text{type IA}: \; p^m_{IA} = \frac{w_n R_{IA}}{\alpha'}(G^{mn} + B^{mn}) - (q_I - w_n R_{IA} A^n_I) A^m_I. \tag{7.3}$$

Note, in particular, that it is *possible* to have both string couplings $(g_{IA}, g_{[O(32)]}) \ll 1$, namely, at weak coupling, when both of the compactification radii, $(R_{IB}, R_{[O(32)]}) \gg \alpha'^{1/2}$. Namely, this is the regime where the type IA string background approaches sub-string-scale–size interval separating the O8 planes: $R_{IA} \ll \alpha'^{1/2}$. The type IB O(32) string is becoming strongly coupled, but we still have a weak coupling description in the T_9-dual type IA string. Our goal is to apply this weak coupling relation linking the two superstring theories, only one of which, namely the heterotic string, is at large radius, to match the normalizations of their respective one-loop vacuum amplitudes. String theory is unusual from the perspective of quantum field theories, since the string scale, $\alpha'^{1/2}$, does not designate the small distance cutoff below which the computations in the theory become invalid. On the contrary, using dualities, we are able to compute amplitudes at arbitrarily short distances much below the string scale. This is a consequence of the exact renormalizability of the theory, which has been described elsewhere [2,16,23,29].

The one-loop connected vacuum functional for the twisted torus-compactified spin $(32)/Z_2$ heterotic string theory in generic Wilson line background can be deduced as follows: we begin with the results of the Polyakov path integral quantization of the closed bosonic string theory, and of the oriented closed type II superstring theories, and deduce the existence of an anomaly-free and ultraviolet finite ten-dimensional string theory with gauge group spin $(32)/Z_2$, such that the world-sheet superconformal field theory preserves the two-dimensional $N = (1, 0)$ supersymmetric diffeomorphism × Weyl gauge invariances [1,2,24]. This remarkable, but unusual, chiral string theory is based on the ten-dimensional Lorentz invariant, chiral supergravity with sixteen spacetime supercharges, coupled to an anomaly-free and ultraviolet finite, supersymmetric gauge theory. In addition, the result for the one-loop vacuum energy density needs to manifestly invariant under the target space duality group of the twisted torus, and include the target space duality transformations that accompany Wilson line backgrounds with a requisite shift in the vector potential, thereby preserving ten-dimensional Yang Mills gauge invariance [13,14]. It was a truly remarkable achievement to be able to deduce an expression for the one-loop vacuum energy density that meets all of these tests for consistency [24]. Thus, by symmetry considerations alone, in addition to some clever application of integral self-dual lattices, the authors of [24] succeeded in deducing the the well-known standard result for the one-loop heterotic string vacuum amplitude, apart from its unknown normalization $\mathcal{N}$:

$$\mathcal{W}(R_{\rm H};\mathbf{A}) = (2\pi R_{\rm H})L^9\mathcal{N}(4\pi^2\alpha')^{-5}\int_{\mathcal{F}}\left\{\frac{d^2\tau}{4\tau_2^2}\tau_2^{-4}[\eta(\tau)\bar\eta(\bar\tau)]^{-8}\right\}\left[\left(\frac{\Theta_{00}(\tau)}{\eta(\tau)}\right)^4-\left(\frac{\Theta_{01}(\tau)}{\eta(\tau)}\right)^4-\left(\frac{\Theta_{10}(\tau)}{\eta(\tau)}\right)^4\right][\bar\eta(\bar\tau)]^{-16}$$
$$\times\sum_{\mathbf{q}\in\Gamma^{(16,0)},n,w} e^{-\pi\tau_2\alpha'\{(\frac{n}{R_H}+\frac{wR_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A}-\frac{w}{R_{\rm H}}\mathbf{A}^2)^2+(\frac{n}{R_H}-\frac{wR_H}{\alpha'}-\mathbf{q}\cdot\mathbf{A}-\frac{w}{R_{\rm H}}\mathbf{A}^2)^2\}}e^{-\pi\tau_2\{(\mathbf{q}-wR_H\mathbf{A})^2\}}. \quad (7.4)$$

Here $\mathbf{q}$ is any sixteen component vector in the gauge lattice, contained within the integral Lorentzian self-dual lattice, $\Gamma^{(18,2)}$, describing the moduli space of the torus-compactified spin$(32)/Z_2$ heterotic string in the background of a Wilson line $\mathbf{A}$, with its variety of enhanced symmetry points. Recall that the normalization of the one-loop vacuum energy density, $\rho = -\mathcal{W}/V_{10}$, is free of any ambiguity associated with the choice of regularization of the embedding target spacetime volume since the analogous choice can be made for either superstring theory [2,10]. The heterotic string partition function has been written as the product of Jacobi theta functions, and an infinite summation over the vectors in a (18,2)-dimensional Lorentzian integral self-dual lattice [14]. The properties of this compactification lattice can be deduced by requiring a target space duality invariant extension of one of the two allowed sixteen-dimensional even integral Euclidean self-dual gauge lattices, $\Gamma^{\rm spin(32)}/Z_2$. As usual, the one-loop heterotic string amplitude is an integral of the heterotic string partition function, integrating over the domain of integration for the complex world-sheet moduli, $(\tau,\bar\tau)$, namely, the fundamental domain, $\mathcal{F}$, of the modular group of the torus [2,10].

The first step towards the goal of determining the unknown normalization $\mathcal{N}$ of the one-loop spin$(32)/Z_2$ heterotic string vacuum energy density will be to explicitly perform an analytical integration over the world-sheet moduli of the torus in the modular integral, yielding a result given solely in terms of the bare string tension, $1/4\pi\alpha'$, the target spacetime moduli, namely, the radius of the target circle, R_H, and a possible Wilson line background, $\mathbf{A}$, in addition to the convergent infinite series that arises from the mass level summation over the degeneracies of the heterotic string mass spectrum, mass level by mass level, and, finally, a numerical factor that follows from performing the modular integral.

Term by term in the heterotic string mass level expansion, the numerical degeneracies are the result of summing spacetime bosonic and spacetime fermonic modes, which of necessity contribute to the string partition function with opposite sign. Hence, for the target spacetime supersymmetric string theory, the partition function will vanish. We therefore restrict our modular integration to the integrand which includes the mass level expansion summing the degeneracies of target spacetime spacetime bosonic modes alone. The result for the torus-compactified heterotic string one-loop vacuum energy density takes the form[7]

$$\rho^{[{\rm O}(32)]}_{\rm bos}(R_H,\mathbf{A}) = -\mathcal{N}(4\pi^2\alpha')^{-5}\sum_{m=0}^{\infty}b_m^{({\rm H})}\sum_{n=-\infty}^{\infty}\sum_{w=-\infty}^{\infty}2\int_0^{1/2}d\tau_1 e^{2\pi i n w\tau_1}D^2(1-\tau_1^2)^{-3/2}\exp\left[-\frac{1}{2}D(1-\tau_1^2)^{1/2}\right]$$
$$\times\mathcal{W}_{-3,5/2}(D(1-\tau_1^2)^{1/2}). \quad (7.5)$$

The notation is as follows. At any mass level, m, in the heterotic string mass level expansion with corresponding target spacetime boson degeneracy $b_m^{({\rm H})}$, we recall that the integral over the modulus, τ_2, can be recognized as a standard integral representation of the Whittaker function, $\mathcal{W}_{-3,5/2}(Dx)$, where we have made the usual change of variable, $x^2 = 1-\tau_1^2$, $|\tau_1| \le 1/2$, $|x| \le 1$, and the variable D, is the resulting argument in the exponential, $e^{-D\tau_2}$, which appears in the integrand

[7]Compare with the discussion prior to Eq. (4.1) for the type IIA(IIB) superstring.

upon deriving the mass level expansion. This is as in Eq. (5) of the Appendix, but with the parameter, ν, in that equation set to half the critical target spacetime dimension of the *heterotic* string, and the exponent function, A, replaced by the exponent function for the heterotic string, namely, D:

$$D \equiv 2m\pi + \pi\alpha' \left[\left(\frac{n}{R_H} - \frac{wR_H}{\alpha'} - \mathbf{q} \cdot \mathbf{A} - wR_{\mathrm{H}}\mathbf{A}^2 \right)^2 + \left(\frac{n}{R_H} + \frac{wR_H}{\alpha'} - \mathbf{q} \cdot \mathbf{A} - wR_{\mathrm{H}}\mathbf{A}^2 \right)^2 \right] - \pi[(-2wR_{\mathrm{H}}\mathbf{q} \cdot \mathbf{A} + w^2 R_H^2 \mathbf{A}^2)]. \tag{7.6}$$

The next step is to tackle the integral over the world-sheet modular parameter τ_1. For the purposes of this section, where our goal is to use the field theoretic limit of the 10D heterotic-type IB duality to deduce the unknown normalization, $\mathcal{N}$, we directly give the result in the noncompact $R_H \to \infty$ limit, where the contributions from the winding modes are suppressed, also specializing to the massless low-energy field theory limit of the spin$(32)/Z_2$ heterotic string, setting the mass level number, $m = 0$. The power series representation of the Whittaker function takes the general form of an infinite power series, plus logarithmic, plus finite polynomial correction [20,21], and we have substituted the usual change of variable from τ_1 to x, as defined below:

$$\begin{aligned} \mathcal{W}_{\lambda,\mu}(x) = \frac{(-1)^{2\mu} x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x}}{\Gamma(\frac{1}{2}-\mu-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)} \times \Bigg\{ \sum_{k=0}^{\infty} \frac{\Gamma(\mu+k-\lambda+\frac{1}{2})}{k!(2\mu+k)!} \\ \times x^k \left[\Psi(k+1) + \Psi(2\mu+k+1) - \Psi\left(\mu+k-\lambda+\frac{1}{2}\right) - \ln z \right] \\ + (-x)^{-2\mu} \sum_{k=0}^{2\mu-1} \left[\frac{\Gamma(2\mu-k)\Gamma(k-\mu-\lambda+\frac{1}{2})}{k!} (-x)^k \right] \Bigg\}, \\ \text{where } x = D(1-\tau_1^2)^{1/2}, \quad \lambda = -3, \quad \mu = 5/2. \end{aligned} \tag{7.7}$$

Expressing the Whittaker function as a convergent power series expansion for small argument, and likewise substituting Taylor series expansions for the additional standard mathematical functions appearing in the modular integral, gives the following result:

$$\begin{aligned} \mathcal{W}^{[\mathrm{O}(32)]}_{\mathrm{m}=w=0}(R_H) &= \mathcal{N} \cdot 2 \cdot (2\pi R_H)\mathcal{V}_9(4\pi^2\alpha')^{-5} b_0^{(\mathrm{IB})} \sum_{\mathbf{q}} \sum_{n=-\infty}^{\infty} \int_{\sqrt{3}/2}^{1} dx x^{-2}(1-x^2)^{-1/2} D^2 \exp\left[-\frac{D}{2}x\right] \mathcal{W}_{-3,5/2}(Dx) \\ &= \mathcal{N} \cdot 2 \cdot (2\pi R_H)\mathcal{V}_9(4\pi^2\alpha')^{-5} b_0^{(\mathrm{H})} \sum_{q_I} \sum_{n=-\infty}^{\infty} \int_{\sqrt{3}/2}^{1} dx (1-x^2)^{-1/2} D^5 \frac{(-1)^5 e^{-Dx}}{\Gamma(1)\Gamma(6)} \\ &\times \Bigg\{ \sum_{k=0}^{\infty} \frac{\Gamma(6+k)}{k!(5+k)!} (Dx)^k [\Psi(k+1) + \Psi(6+k) - \Psi(6+k) - \ln(Dx)] \\ &+ (-Dx)^{-5} \sum_{k=0}^{4} \left[\frac{\Gamma(5-k)\Gamma(k+1)}{k!} (-Dx)^k \right] \Bigg\} \\ &= \mathcal{N} \cdot 2 \cdot (2\pi R_H)\mathcal{V}_9(4\pi^2\alpha')^{-5} \frac{(-1)^5 b_0^{(\mathrm{H})}}{5!} \sum_{\mathbf{q}} \sum_{n=-\infty}^{\infty} \int_{\sqrt{3}/2}^{1} dx (1-x^2)^{-1/2} \\ &\times \Bigg\{ \sum_{k=0}^{\infty} \frac{1}{k!} \times D^{5+k} x^{k+1} \exp[-Dx][\Psi(k+1) - \ln D - \ln x] + \sum_{k=0}^{4} (-1)^{k-5} \Gamma(5-k) \exp[-Dx] D^k x^{k-4} \Bigg\}. \end{aligned} \tag{7.8}$$

Substituting the Taylor expansions for the logarithm function, and performing the integral over x, gives the following result for the vacuum energy density,

$$\rho_0^{[\mathrm{O}(32)]}(R_H) = -\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{(-1)^5 b_0^{(\mathrm{H})}}{5!}\sum_{q_I}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{k!}C_{(r)}^{(-1/2)}$$
$$\times\Bigg\{D^{3-2r}[\gamma(k+2r+2,D)-\gamma(k+2r+2,\sqrt{3}D/2)][\Psi(k+1)-\ln D]$$
$$+\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}C_{(j)}^{(l)}D^{3-2r-j}[\gamma(k+2r+j+2,D)-\gamma(k+2r+j+2,\sqrt{3}D/2)]\Bigg\}$$
$$-\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{b_0^{(\mathrm{H})}}{5!}\sum_{q_I}\sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{4}C_{(r)}^{(-1/2)}(-1)^k\Gamma(5-k)$$
$$\times D^{3-2r}[\gamma(k+2r-3,D)-\gamma(k+2r-3,\sqrt{3}D/2)], \tag{7.9}$$

where we recall that $D=2\pi[\alpha'(n/R_H-q_iA_I)^2]$, in the limit where mass level number $m=0$, and the spatial winding modes, w, are dropped from the expression string vacuum energy density, a consequence of our having specialized to the large radius and low temperature limit of the one-loop vacuum amplitude.

As a final simplification, we substitute the power series representation for the incomplete gamma function [20,21], also setting $\mathbf{A}$ to zero for the $\mathrm{spin}(32)/\mathrm{Z}_2$ gauge group:

$$\gamma(q,D)=\sum_{s=0}^{\infty}\frac{(-1)^s D^{q+s}}{s!(q+s)},\qquad D=2\pi\alpha' R_H^{-2}n^2. \tag{7.10}$$

Substitution above gives the simpler result:

$$\rho_0^{[\mathrm{O}(32)]}(R_H) = -\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{(-1)^5 b_0^{(\mathrm{H})}}{5!}\sum_{n=-\infty}^{\infty}D^5\sum_{r=0}^{\infty}C_{(r)}^{(-1/2)}\sum_{s=0}^{\infty}\frac{(-1)^s D^s}{s!}$$
$$\times\Bigg\{\sum_{k=0}^{\infty}\frac{D^k}{k!}\left[\frac{1}{k+2r+2+s}-\frac{(\sqrt{3}/2)^{k+2r+2+s}}{k+2r+2+s}\right]\{\Psi(k+1)-\ln D\}$$
$$+\sum_{k=0}^{\infty}\frac{D^k}{k!}\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}C_{(j)}^{(l)}\left[\frac{1}{k+2r+j+2+s}-\frac{(\sqrt{3}/2)^{k+2r+j+2+s}}{k+2r+j+2+s}\right]$$
$$-\sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1}{k+2r-3+s}-\frac{(\sqrt{3}/2)^{k+2r-3+s}}{k+2r-3+s}\right]\Bigg\}. \tag{7.11}$$

The infinite summations over the momentum modes can be recognized as the Riemann zeta function $\zeta(z,q)$ [21] and its derivative:

$$\sum_{n=0}^{\infty}(n+q)^{-z}\equiv\zeta(z,q),\qquad \zeta(-n,0)=-\frac{B_{n-1}}{n+1},\qquad \zeta'(s)\equiv\sum_{n=1}^{\infty}n^{-s}\ln n, \tag{7.12}$$

$$\rho^{[\mathrm{O}(32)]}(R_H) = -\mathcal{N}\cdot 2(4\pi^2\alpha')^{-5}\frac{(-1)^5 b_0^{(\mathrm{H})}}{5!}(2\pi\alpha')^5 R_H^{-10}$$
$$\times\sum_{r=0}^{\infty}C_{(r)}^{(-1/2)}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\Bigg\{\sum_{k=0}^{\infty}\frac{1}{k!}(2\pi\alpha')^{k+s}R_H^{-2k-2s}\left[\frac{1-(\sqrt{3}/2)^{k+2r+2+s}}{k+2r+2+s}\right]$$
$$\times\{\zeta(-10-2k-2s;0)\Psi(k+1)-\zeta'(-10-2k-2s)\ln[2\pi\alpha' R_H^{-2}]\}$$
$$+\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=1}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l+1}}{l}C_{(j)}^{(l)}\zeta(-10-2k-2s;0)(2\pi\alpha')^k R_H^{-2k}\left[\frac{1-(\sqrt{3}/2)^{k+2r+j+2+s}}{k+2r+j+2+s}\right]$$
$$-\sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1-(\sqrt{3}/2)^{k+2r-3+s}}{k+2r-3+s}\right]\Bigg\}. \tag{7.13}$$

Keeping the leading terms in inverse powers of R_H, we extract the large radius and low-temperature limit of the expression

$$\rho_0^{(\mathrm{H})} = \mathcal{N} \cdot 2(4\pi^2\alpha')^{-5} \frac{b_0^{(\mathrm{H})}}{5!} (2\pi\alpha')^5 R_H^{-10} \left\{ \left[\frac{1}{8}\zeta(-10;0)\{1+\Psi(1)-\zeta'(-10)\ln[2\pi\alpha' R_H^{-2}]\} \right] - \sum_{k=0}^{4} (-1)^k \Gamma(5-k) \left[\frac{1-(\sqrt{3}/2)^{k-3}}{k-3} \right] \right\}. \tag{7.14}$$

We note in passing that the leading power law dependence on radius in the large radius limit: $D^5 \simeq R_H^{-10}$, holds for the generic Wilson line, and the generic nine-dimensional non-Abelian gauge group obtained in Wilson line circle compactifications [13,14].

Recall [10,24] the well-known degeneracy of target spacetime bosonic modes in the spin$(32)/Z_2$ heterotic string, and restricting to only the Yang-Mills sector, since our comparison is with the low-energy limit of the open and oriented one-loop amplitude of the type IB O(32) superstring, we have 496 gauge bosons for SO(32), with vanishing Wilson line background, $\mathbf{q} = (0, \ldots, 0)$, or, 240 gauge bosons for non-Abelian gauge group $\mathrm{SO}(16) \times \mathrm{SO}(16)$, when in a Wilson line background, parametrized by the gauge lattice vector, $\mathbf{q} = (1^8, 0^8)$. We therefore truncate b_0^{H}, to summing only the 496 gauge bosons at the spin$(32)/Z_2$ enhanced symmetry point, where we recall that $496 = \frac{1}{2}32(31)$, the number of massless gauge bosons in the type IB superstring with orthogonal group O(32). Hence, there is no ambiguity in comparing the normalizations of the one-loop string vacuum energy densities since we identify the corresponding massless bosonic modes in either string theory.

Thus, we compactify the type IB string on $R^8 \times T^2$, of radius R_{IB}, with inverse temperature, β_{IB}, and twisted by a constant temperature-dependent antisymmetric 2-form potential in the Neveu-Schwarz sector, $|B_{09}| \equiv \tanh(\pi\alpha) \simeq \pi\alpha = \pi(\beta_C/\beta_{\mathrm{IB}})$. The thermal modes of the type IB open string sector vacuum energy density contains the tower of Matsubara states with thermal momentum: $p_0 = 2n\pi/\beta$, where $n \in Z$:

$$\sum_{n_0,n_9=-\infty}^{\infty} \exp\left[-\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2} n_9^2 t \right]. \tag{7.15}$$

Thus, with 32 Chan-Paton factors, and at finite temperature, the open and oriented type IB string graphs are given by the expression

$$\begin{aligned} \rho_{\mathrm{ann}}^{(\mathrm{IB})} = & -(8\pi^2\alpha')^{-5}(1+\tanh\pi\alpha) \int_0^\infty \frac{dt}{2t} \cdot (2t)^{-4} \eta(it)^{-6} \sum_{n_i=-\infty}^{\infty} \exp\left[-t\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2} n_9^2 t \right] \\ & \times \left[\frac{\Theta_{00}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)} \left(\frac{\Theta_{00}(0,it)}{\eta(it)} \right)^3 - \frac{\Theta_{01}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)} \left(\frac{\Theta_{01}(0,it)}{\eta(it)} \right)^3 - \frac{\Theta_{10}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)} \left(\frac{\Theta_{10}(0,it)}{\eta(it)} \right)^3 \right] \\ & - (8\pi^2\alpha')^{-5}(1+\tanh\pi\alpha) \int_0^\infty \frac{dt}{2t} \cdot (2t)^{-4} [\eta(it)]^{-6} \frac{1}{4} \left[\frac{e^{i\pi t\alpha^2}\eta(it)}{\Theta_{11}(\alpha,it)} \right] \left[\frac{\Theta_{11}(\alpha,it)}{e^{i\pi t\alpha^2}\eta(it)} \left(\frac{\Theta_{11}(0,it)}{\eta(it)} \right)^3 \right] \\ & \times \sum_{n_i=-\infty}^{\infty} \exp\left[-t\frac{8\pi^3\alpha' n_0^2}{\beta_{\mathrm{IB}}^2} - \frac{8\pi^3\alpha'}{R_{\mathrm{IB}}^2} n_9^2 t \right]. \end{aligned} \tag{7.16}$$

Since our main interest in this paper is in the gauge sector of the type IB O(32) string, and at near-zero temperature, we now focus attention on the mass level zero limit of the annulus graph alone, dropping all thermal excitations, and keeping only the spatial momentum modes.

Taking the massless limit $m = 0$ of the twisted torus-compactified unoriented open and closed type IB string, it is helpful to make a change of variable, prior to performing the modular integral. Summing on momentum modes:

$$\begin{aligned} I(m)|_{\mathrm{m=0}} & \equiv 496 \sum_{n=0}^{\infty} \int_0^\infty dt t^{-5-1} e^{-\frac{2\pi\alpha'}{R_{\mathrm{IB}}^2}(n)^2 t} \\ & = 496 \left(\frac{\pi\alpha'}{R_{\mathrm{IB}}^2} \right)^5 \left[\sum_{n=1}^{\infty} n^{10} \right] \Gamma(-5) \\ & = 496 \left(\frac{2\pi\alpha'}{R_{\mathrm{IB}}^2} \right)^5 \zeta(-10,0)\Gamma(-5). \end{aligned} \tag{7.17}$$

The infinite sum over spatial momentum modes can be recognized as a Riemann zeta function. The result takes the form

$$\rho_0^{(\mathrm{IB})}(R_{\mathrm{IB}}) = -(8\pi^2\alpha')^{-5}496R_{\mathrm{IB}}^{-10}\zeta(-10,0)\Gamma(-5). \quad (7.18)$$

Here R_{IB} denotes the ten-dimensional spacetime volume, and the result holds in the noncompact target spacetime supersymmetric limit, for $R_{\mathrm{IB}} \gg \alpha'^{1/2}$. Our goal is to match the normalizations in the regime where the T-dual type IA string interval radius, R_{IA}, is small, and both the type IA and heterotic strings are at weak coupling. A T_9 duality transformation maps the circle compactified type IB string with 32 coincident D9-branes to the type IA string on an interval of size R_{IA}, with orientifold planes at either end, and 32 coincident D8-branes on one of the O8 planes. The low- energy massless O(32) gauge theory limit gives the result

$$\rho_0^{(\mathrm{IA})}(R_{\mathrm{IA}}) = 496[(8\pi^2\alpha')^{-5}R_{\mathrm{IA}}^{10}\alpha'^{-10}]\zeta(-10,0)\Gamma(-5). \quad (7.19)$$

Comparison with the result for the heterotic string vacuum energy density above, enables us to determine the unknown normalization of the one loop vacuum energy density of the spin$(32)/\mathrm{Z}_2$ heterotic string, $\mathcal{N}$. As mentioned earlier, the overall phase matches, and is determined by modular invariance in the heterotic string theory—a property which extends to arbitrary order in the string loop expansion [28]. Our interest here is in the numerical normalization of the heterotic string one-loop vacuum energy densities. We find the simple result :

$$\mathcal{N} = -2^{-7}\frac{1}{5}[\Gamma(6)\Gamma(6)][R_H^{10}R_{\mathrm{IA}}^{10}]\alpha'^{-10}\left\{\frac{1}{8}\zeta(-10;0)(1+\mathbf{\Psi}(1)-\zeta'(-10)\ln[2\pi\alpha' R_H^{-2}]) - \sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1-(\sqrt{3}/2)^{k-3}}{k-3}\right]\right\}^{-1}. \quad (7.20)$$

Note that since the string vacuum functional is dimensionless, the powers of $R_{\mathrm{IB}} \equiv \alpha'/R_{\mathrm{IA}}$ and R_{H}, combine as expected to give, in natural units, the dimensionless factor, $[G_{\mathrm{IB}}^5 G_{\mathrm{H}}^5]$, the product of the embedding target space metrics, and our final result for the heterotic, type IB, string, large radius limit is

$$\mathcal{N}_{\mathrm{Spin}(32)/\mathrm{Z}_2} = 2^{-10}\left[\frac{1}{5}(5!)^2\right][G_H^5 G_{\mathrm{IB}}^5]\left\{\boldsymbol{\zeta}(-10;0)(1+\mathbf{\Psi}(1)-\boldsymbol{\zeta}'(-10)\ln[2\pi G_H]) - \sum_{k=0}^{4}(-1)^k\Gamma(5-k)\left[\frac{1-(\sqrt{3}/2)^{k-3}}{k-3}\right]\right\}^{-1}. \quad (7.21)$$

Thus, the long-absent [28] unambiguous normalization of the one-loop vacuum amplitude of the spin$(32)/\mathrm{Z}_2$ heterotic string has been obtained by invoking a low-energy matching calculation at zero temperature to the O(32) type IB string, a beautiful application of the strong/weak duality analysis in [27].

Remarkably, we can apply this same logic to the $E_8 \times E_8$ heterotic string. Recall that the relation between the ratio of gauge coupling and compactification radius differs for the closed heterotic string theories and the open and closed type IB and type IA strings [10]; further, that any ambiguity in the normalization due to the volume of the twisted-torus compactification radii, since we only invoke dimensionless ratios in deriving our duality relations. Thus, using the T-duality relations to substitute for the coupling and spatial compactification radius of the T_9-dual $E_8 \times E_8$ heterotic string, we have the result

$$\frac{2\pi R_{[\mathrm{O}(32)]}}{g_{[\mathrm{O}(32)]}^2} = \frac{2\pi R_{[\mathrm{E}_8\times\mathrm{E}_8]}}{g_{[\mathrm{E}_8\times\mathrm{E}_8]}^2}, \qquad g_{\mathrm{IA}} = \frac{1}{g_{[\mathrm{E}_8\times\mathrm{E}_8]}}\left(\frac{R_{[\mathrm{E}_8\times\mathrm{E}_8]}^3}{\alpha'^{3/2}}\right), \quad (7.22)$$

and we could compare the weakly coupled $E_8 \times E_8$ heterotic string, with a *strongly coupled* type IA string, in the background of 32 D8–D0-branes, with O8 planes separated by a sub-string-scale–sized interval [30]. We shall save the full analysis for a future work.

VIII. CONCLUSIONS

Perturbative superstring theory as formulated in the world-sheet formalism inherently includes a description of the background; thus, for finite-temperature string theory, the "heat-bath" representing the embedding target space of fixed spatial volume and fixed inverse temperature is forced upon us, together with any background potentials and fluxes in the Neveu-Schwarz and Ramond-Ramond sectors,[8] in addition to the embedding target spacetime metric. Therefore, we are ordinarily restricted to the canonical ensemble in formulating finite temperature

[8] In this paper, we have not as yet included any Ramond-Ramond sector background antisymmetric tensor potentials, or fluxes [26], since they have not become necessary for the closed oriented superstring canonical ensembles. In the case of the type IA(IB)–heterotic–M theory $E_8 \times E_8$ open and closed unoriented string ensemble, we anticipate inclusion of a discussion of all of the Ramond-Ramond constant background fields, an investigation postponed for the future.

perturbative string theory when using the world-sheet superconformal field theory framework. It is thus heartening that perturbative type II superstring amplitudes at higher genus have been the subject of an intensive revisit [31], enabling the extension of our work to higher orders in string perturbation theory, and possibly, the analysis of the all–orders high–temperature asymptotics of the superstring vacuum energy density.

We should caution the reader that while an immense literature exists on proposals for microcanonical and grand canonical formulations for weakly coupled strings, the conceptual basis of these analyses is as yet uncertain. Some of the pitfalls have been described in [3,5,32]. One could argue that the string/M microcanonical ensemble is *essential* for a formulation of string quantum cosmology, or the statistical mechanics of the Universe: the Universe is, by definition, an isolated closed system [33], and it is meaningless to invoke the canonical ensemble of the "fundamental" degrees of freedom. On the other hand, many questions in the standard model of cosmology ought to be answerable within the framework of the perturbative superstring canonical ensemble.

A second important observation pertains to the expected Jeans instability of a gravitating ensemble, and formulation of the thermodynamic limit in the presence of gravitational interactions. Consider an ensemble of pointlike strings of total mass, M, and Schwarschild radius, R_S, and with one-loop vacuum energy density $\rho = F(\beta)/R_S^{D-1}$. Recall that the Newtonian gravitational coupling, $G_N \simeq g^2$, where g is the fundamental closed string coupling. Since $M \sim \rho R_S^{D-1}$, we have the relation

$$R_S \gg \left(\frac{1}{g^2\rho(\beta)}\right)^{1/2}. \tag{8.1}$$

This would seem to suggest that our considerations in this paper are limited to weak heterotic string coupling, as is precisely compatible with our understanding of the weakly coupled perturbative type IB O(32) thermalized graviton-gluon ensemble. For large-loop-size gravitating heterotic solitonic strings, we might expect a Jeans instability as was proposed for any gravitating gauge ensemble by D. Gross, L. Yaffe, and [32]. Compatibility with the strong-weak dualities of the heterotic-type IB-type IA strings is essential to circumvent a direct clash with the Jeans instability at strong heterotic string coupling, or in M theory at finite size R_{10}. Nevertheless, as was shown in Sec. VI B and Sec. VI C of this paper, there is a thermal-dual type IA ensemble of closed thermal winding strings at low type IA temperatures. Our results therefore suggests a thermal phase *transformation* of the thermal gluon-graviton ensemble at type IB temperatures far above the string scale, even if the phase transition at T_C is not explicitly described. Such a thermal phase transition does *not* occur in the canonical ensembles of the closed superstring theories, which have a finite one-loop vacuum energy density at the string scale temperature, and are thermal dual to weakly coupled closed string canonical ensembles, exactly alike in their statistical mechanics and thermodynamics properties. Thus, we have given evidence for two remarkable dynamical phenomena in the fundamental string canonical ensemble in a twisted torus and with generic background fields: the Hagedorn divergence occurs only for the thermalized gauge-graviton ensemble, whose normalized one-loop vacuum energy density has been derived from that of the normalized, finite temperature, open oriented type IB superstring canonical ensemble, and the low-energy limit of the weakly coupled finite temperature perturbative type IA (fundamental) string ensemble is an ensemble of long (solitonic) thermal winding strings, which appear to be stable at low temperatures and weak type IA string coupling. Nevertheless, the type IA superstring being an open and closed unoriented superstring theory, the thermal (fundamental) string excitations of the (solitonic) winding strings could lead to a splitting transition at high temperatures, and strong coupling. Thus, the low-energy limit of the type IA superstring at finite temperature appears to match the physics of non-Abelian gauge theories: confinement at low type IA temperatures, and a *plausible* thermal phase transition that is identical to the deconfinement transition at the string scale, leading to a type IB *transformed* phase of thermalized gluons and gravitons. Our results appear to give credence to the long sought-after interpretation of the "Hagedorn phase transition" in the type IA open and closed string theory that can be identified with the deconfinement, or "long" string, phase transition in string theory [3,5,6].

Our results also give a resolution to the puzzling instability of flat spacetime pointed to in [32]; we have shown that background fields—which are always, in any case, generated by thermal fluctuations of the vacuum, and present in the loop-renormalized string tension, the renormalized fundamental string coupling constant, and the renormalized target spacetime moduli that describe the *stable* thermal vacuum of the superstring theories at finite temperature—can smooth infrared instabilities. This is very encouraging, and further strong evidence in favor of the physical and mathematical self-consistency of string/M theory, an explanation for the bewildering multiplicity of superstring theories, and of the plethora of vacua. There is only one theory, and the plethora of solutions to the string equations of motion simply enable computations at weak string coupling and large (target space moduli) radi, or low temperature, that describe different kinematical regimes of the theory, where specific dynamical phenomena can occur.

The inclusion of higher order corrections from multiloop superstring perturbation theory become essential to address the perturbative evidence for a Jeans instability for strings of macroscopic size explicitly. It is heartening that the small blemishes that had remained in the impressive framework for higher-loop superstring amplitudes developed from the mid-1980's onwards [28,34], have been recently addressed in a seminal series of outstanding papers by E. Witten, and collaborators [31]. Thus, there is genuine hope that a more explicit two-loop analysis of the type II superstring vacuum energy density can be performed, which would shed light on the stability of nontachyonic, and dilation tadpole-free, nonsupersymmetric vacua, in the presence of background fields, the most fundamental of which is the Neveu-Schwarz antisymmetric 2-form gauge potential. The physical consistency of nonsupersymmetric, and nontachyonic, type II and heterotic string asymmetric orbifold compactifications has been tested, long well-known to string theorists in the late eighties, and most often explored in the free fermionic constructions. The recent developments in [31] have led to a most elegant calculation of the two-loop vacuum energy of the $Z_2 \times Z_2$ Calabi-Yau orbifold compactifications on the product of three 2-tori by D'Hoker and Phong [35], giving the nonvanishing two-loop vacuum energy density for the $\mathrm{spin}(32)/Z_2$ heterotic string theory, with unbroken gauge symmetry $SO(26) \times SU(3) \times U(1)$. In these models, spacetime supersymmetry is broken by two-loop corrections via a Fayet-Ilioupoulos D term, while both the one-loop amplitudes, and the tree-level mass spectrum, has unbroken supersymmetry. It would be interesting to extend these results to compactification on twisted tori, twisting by the background 2-form B field, and with the temperature-dependent background fields, which likely provide a class of interesting models for supersymmetry breaking in string/M theory, in addition to shedding light on the systematic multiloop renormalization [29] of the string tension, the string coupling, and additional target spacetime moduli; the masses and couplings in heterotic string theory. My own view is that until the status of observational supersymmetric particle physics is more clearly understood—such as, clues as to the scale of supersymmetry breaking and the mass of the lightest superpartner—the more serious theoretical issues of understanding *why* there is a string multiverse [30,36,37], and what is the correct physical interpretation of the various dualities that relate disparate regions of the space of string vacua, are of greater urgency to resolve. I believe this is a fundamental question, hinting at the existence of a more efficient mathematical reformulation of string/M theory [38–40]. Nevertheless, it would be a great breakthrough if a compelling phenomenological model for spacetime supersymmetry breaking could be constructed within this class of relatively accessible Calabi-Yau compactifications, and given that the role of fluxes in moduli stabilization has by now been extensively examined [41], establishing that a sensible model for spacetime supersymmetry breaking is likely viable. More generally, we have in addition both the NS five-branes, and the full set of allowed background p-form fields in the Ramond-Ramond sector of the type IIA, type IIB, and type IA-IB superstring theories, as became clear with the discovery of D-branes [5,26,36,39,41], which is further stimulus for work on the open questions regarding the proper physical interpretation of the generalized electric-magnetic duality group of string/M theory [40]. It should be noted that our results are evidence for phase transitions in the moduli space of string theories, not unlike the classic topology-changing phase transitions discovered in [38,39], approached from the world-sheet perspective, and the explicit computation of superstring one-loop amplitudes. It would be fascinating to combine these apparently distinct perspectives since they bear a close resemblance, as noted in the outset where we pointed out the similarity of the thermal duality relation to a mirror symmetry transformation [19].

In closing, we mention that the target spacetime duality symmetries of heterotic string backgrounds with compact space, *and compact time*, in both Euclidean and Minkowskian signature, had been studied in depth in a far-reaching work in [42]. Backgrounds with Euclidean time signature are more readily amenable to first quantization by standard principles of world-sheet superconformal field theory, and have significant physics implications for finite temperature string and gauge theory, for early Universe cosmology, and for the statistical mechanics of the single string canonical ensemble. Eventually, we would like a formalism for nonequilibrium string statistical mechanics, which can directly probe the question of phase transitions. A framework for string quantization in Minkowskian spacetime is a necessary prerequisite, and nonequilibrium string/M statistical mechanics is an outstanding open problem which we hope is inspiration to fundamental physics theorists!

Of immediate interest, it is likely that we can address the following question in a future work. A derivation of the one-loop free energy of the type IA $E_8 \times E_8$ canonical ensemble, is required to be compatible with the S-duality relations for type IA/M theory. Consider the classical moduli space of the type IA string on $R^8 \times S^1 \times S^1/Z_2$ and compare with the heterotic string, namely, M theory on $S^1/Z_2 \times S^1$, where the radius of the S^1 is $R_{[E_8 \times E_8]}$, and the interval is $R_{10} \equiv g\alpha'^{1/2}$, with g the fundamental closed string coupling, $g_{[E_8 \times E_8]}$. In terms of the 11-dimensional Planck length, $M_{11} = g^{-1/3}\alpha'^{1/2}$, and R_{10}, the strong weak duality map yields the following result:

$$g_{\mathrm{IA}} = \frac{1}{(M_{11}R_{10})^{3/2}}(R^3_{[E_8 \times E_8]}(M_{11}^{9/2}R_{10}^{3/2})) = R^3_{[E_8 \times E_8]}M_{11}^3. \tag{8.2}$$

Substituting for $R_{[E_8 \times E_8]} = \beta_{[E_8 \times E_8]}/2\pi$, we see that it is possible to go above the string scale, probing the high-temperature regime up to the 11-dimensional Planck scale, $\alpha'^{-1/2} < T < M_{11}$, with weak g_{IA}, and independent of R_{10}. This relation is clear evidence that the type IA $E_8 \times E_8$ string admits a weak coupling analysis with O8 planes, 32 D8-branes, and 32 D0-branes [30]. We postpone a discussion of the one-loop free energy of the type IA $E_8 \times E_8$ superstring for future work.

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APPENDIX: CLOSED BOSONIC STRING VACUUM ENERGY DENISTY

The one-loop vacuum amplitude for the 26-dimensional closed bosonic string is given by an ordinary integral over the complex modulus, τ, parametrizing the conformally inequivalent classes of closed Riemann surfaces with the topology of a torus [2]. We will show that the two (real) integrals over moduli, (τ_1, τ_2), of the genus one Riemann surfaces summed in the one-loop string vacuum amplitude can be carried out explicitly, and in closed form, by the procedure of term-by-term integration of the closed string level expansion. The integration domain in the complex τ plane is the fundamental domain, $\mathcal{F}$, of the group of modular transformations of the torus [2]: $-\frac{1}{2} \le \tau_1 \le \frac{1}{2}$, $|\tau| \ge 1$. The result will be an expression for the bosonic string one-loop vacuum amplitude, expressed in terms of the degeneracies, b_m, at successive levels in the world-sheet conformal field theory mass level expansion, where m denotes level number, $m = 0, \ldots, \infty$. These are weighted, term-by-term in the level expansion, by numerical factors arising from the modular integrals. Our goal in this Appendix is to calculate the precise ultraviolet asymptotic closed string level expansion by performing an explicit analytic term-by-term integration over the moduli (τ_1, τ_2).

The leading tachyonic contribution in the vacuum amplitude diverges as $\tau_2 \to \infty$. This vacuum instability is absent in the infrared finite and stable, target spacetime supersymmetric type II superstrings discussed in the main text; we will simply excise this term from the infinite sum in the illustrative calculation. Since our interest is in the high temperature (short distance) regime, we can drop the summation on thermal momentum modes; keeping only windings. We make a change of variable, $y = 1/\tau_2$, expanding in powers of $(q\bar{q})^{1/2} = e^{-2\pi\tau_2}$. The asymptotic expansion in level number, m, must be manipulated with care since we interchange the order in which we perform the modular integrals and level summation. Notice that for each term in the infinite summation, the integrand is manifestly finite for finite values of $b_m^{(\mathrm{bos})}$, with no divergences everywhere in the fundamental domain. However, it is a well-known fact that the degeneracies grow as the exponential of the square root of the mass level number, m, at large m, a phenomenon known as the Hagedorn growth of the world-sheet conformal field theory partition function [6]. Thus, it is necessary to examine the numerical correction to the coefficients in the string mass level expansion which arise from the modular integration in the expression for the string one loop vacuum energy density.

We begin with the well-known result for the one-loop vacuum energy density of the circle compactified 26D closed bosonic string theory derived in [2,10,17], on a circle of radius $R = \beta/2\pi$:

$$\rho_{\mathrm{bos}} = -(4\pi^2\alpha')^{-26/2} \int_{-1/2}^{+1/2} d\tau_1 \int_{\sqrt{1-\tau_1^2}}^{\infty} \frac{d^2\tau}{4\tau_2^2} \tau_2^{-12} |\eta(\tau)\bar{\eta}(\bar{\tau})|^{-24}$$
$$\times \sum_{n=-\infty}^{+\infty} \sum_{w=-\infty}^{+\infty} \exp\left[-2\pi\tau_2\left(\frac{4\pi^2\alpha' n^2}{\beta^2} + \frac{w^2\beta^2}{4\pi^2\alpha'}\right) + 2\pi i n w \tau_1\right]$$
$$= -(4\pi^2\alpha')^{-13} \frac{1}{4} \int_{-1/2}^{+1/2} d\tau_1 \int_0^{1/\sqrt{1-\tau_1^2}} dy \sum_{n=-\infty}^{+\infty} \sum_{w=-\infty}^{+\infty} \sum_{m=0}^{\infty} b_m^{(\mathrm{bos})} y^{\nu-1} e^{-2\pi m y} e^{-2\pi\tau_2(\frac{4\pi^2\alpha' n^2}{\beta^2} y + 2\pi \frac{w^2\beta^2}{4\pi^2\alpha'} \frac{1}{y}) + 2\pi i n w \tau_1}. \quad \text{(A1)}$$

Note that $b_m^{(\mathrm{bos})}$ denotes the degeneracy at level m in the string mass level expansion. Upon interchanging the order of mass level summation, with the modular integrals, we see upon inspection that the degeneracies will be corrected by term-by-term numerical coefficients arising from the integrals. Note that we compute these coefficients in each momentum or winding mode sector of the one loop vacuum amplitude, taking the noncompact (low temperature) or small radius (high temperature) approximation in the last step prior to obtaining an analytic result, where the variable y denotes, respectively, τ_2, or its inverse, for the small radius UV asymptotics [10]. The second equality has been written for the asymptotic UV limit of the closed bosonic string, with $\nu = 13$, and $y = 1/\tau_2$. In the main text, this step is carried out for the type II superstring theories in the noncompact target spacetime supersymmetric 10D limit. In this Appendix, we explain how either limit of the one loop string vacuum energy density can be analyzed, and then specialize instead to the small radius, or high temperature, behavior of the closed bosonic string theory.

Let us denote the numerical correction by $I(m)$, an infinite summation over thermal winding and momentum modes. The integral over τ_2 can be recognized as an integral representation[9] of the Whittaker function, $\mathcal{W}_{\nu,\lambda}(x)$ (GR 3.471.2) [20,21]:

$$\int_0^u dx x^{\nu-1} e^{-\frac{A}{x}} = A^{\frac{\nu-1}{2}} u^{\frac{\nu+1}{2}} e^{-\frac{A}{2u}} W_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A/u), \qquad \text{(A2)}$$

where we have set $u = (1-\tau_1^2)^{-1/2}$, and A, ν are given by:

$$A = 2\pi m\left(1 + \frac{2\pi\alpha' n^2}{m\beta^2} + \frac{w^2\beta^2}{4m\pi^2\alpha'}\right), \quad \nu = 13. \qquad \text{(A3)}$$

To proceed, note the degeneracies, $b_m^{(\mathrm{bos})}$, in the bosonic string mass level expansion are corrected by numerical factors:

$$I(m;n,w) = \frac{1}{4}\sum_{w=-\infty}^{\infty} A^{\frac{\nu-1}{2}} \int_{-1/2}^{+1/2} d\tau_1 (1-\tau_1^2)^{-7/2} e^{-\frac{1}{2}A(1-\tau_1^2)^{1/2}} \mathcal{W}_{-\frac{\nu+1}{2},\frac{\nu}{2}}(A[1-\tau_1^2]^{1/2}). \qquad \text{(A4)}$$

The term-by-term integrals over τ_1 will be evaluated by substituting the appropriate, namely, power series or asymptotic, expansion for the Whittaker function, valid in the limit of small or large argument. Namely, for the closed bosonic string, with $\nu = 13$, and changing variables from τ_1 to u, we have:

$$I(m) \equiv \frac{1}{4} A^6 \left[\int_0^{\sqrt{3}/2} - \int_0^1\right] du u^{7-3} (1-1/u^2)^{-1/2} e^{-\frac{1}{2}A/u} \mathcal{W}_{-\frac{14}{2},\frac{13}{2}}(A/u). \qquad \text{(A5)}$$

The term by term integrals over u of a convergent series representation of the Whittaker function can be performed explicitly as follows. Set z equal to the argument of the Whittaker function. Then, each of the term by term integrals over the new variable $z = Ax$ will be found to take the general form, for some integer j, and fraction κ:

$$\sum_{r=0}^{\infty} C_{(r)}^{(-1/2)} \int_{\sqrt{3}/2}^{1} dx x^{-1+2r+j+\kappa} e^{-Ax},$$
$$C_{(r)}^{(-1/2)} = (-1)^r \frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)]. \qquad \text{(A6)}$$

The $C_{(r)}^{(-1/2)}$ are the binomial coefficients for the power series expansion, $(1-x^2)^{-1/2}$, with $|x|^2 \le 1$. We can then use the elementary integrals, for a given fraction κ, and a given integer part j:

$$\int_{\sqrt{3}/2}^{1} dx x^{\kappa+2r+j-1} e^{-Ax}$$
$$= A^{-\kappa-2r-j}[\gamma(\kappa+j+2r, A) - \gamma(\kappa+j+2r, \sqrt{3}A/2)]. \qquad \text{(A7)}$$

The large radius IR asymptotics of the string mass spectrum can be inferred from the convergent power series representation for the Whittaker function which takes the general form of an infinite power series, plus a logarithmic term, plus a finite polynomial correction (GR 9.237.1) [20]:

$$\mathcal{W}_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-\mu-\lambda)\Gamma(\frac{1}{2}+\mu-\lambda)} \times \left\{\sum_{k=0}^{\infty} \frac{\Gamma(\mu+k-\lambda+\frac{1}{2})}{k!(2\mu+k)!} z^k \left[\Psi(k+1) + \Psi(2\mu+k+1) - \Psi\left(\mu+k-\lambda+\frac{1}{2}\right) - \ln z\right]\right.$$
$$\left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \left[\frac{\Gamma(2\mu-k)\Gamma(k-\mu-\lambda+\frac{1}{2})}{k!}(-z)^k\right]\right\}. \qquad \text{(A8)}$$

[9]In what follows, (GR #) denotes the corresponding equation number for a mathematical identity from the text [20].

Note that, as above, $\ln(x)$ can be expanded in a Taylor series about $|x|=1$, where the $C^{(l)}_{(j)}$ are the binomial coefficients with integer l, j:

$$\ln x \equiv \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l}(x-1)^l = \sum_{l=1}^{\infty}\sum_{j=0}^{l} C^{(l)}_{(j)} \frac{(-1)^{l+j+1}}{l} x^j. \tag{A9}$$

We will not carry out these power series substitutions explicitly for the closed bosonic string, since the IR limit is tachyonic [2,10]. However, a similar manipulation is used in the main text for the massless low-energy supergravity field theory limit of the 10D tachyon-free type IIA and type IIB superstrings.

On the other hand, the small radius UV asymptotics of the string mass spectrum can be inferred from the asymptotic expansion for the Whittaker function (GR 9.227) [21]:

$$\mathcal{W}_{(\kappa,\lambda)}(z) \sim e^{-z/2} z^{\kappa}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}z^{-k}\left[\lambda^2-\left(\kappa-\frac{1}{2}\right)^2\right]\left[\lambda^2-\left(\kappa-\frac{3}{2}\right)^2\right]\cdots\left[\lambda^2-\left(\kappa-k+\frac{1}{2}\right)^2\right]\right), \tag{A10}$$

where $z \simeq 2\pi m$ with corrections of $O(4\pi^2 R^2/m\alpha')$. Substituting the asymptotic expansion for large argument of the Whittaker function in Eq. (1.30), and solving for the integrals over z in Eqs. (1.31), (1.32) gives:

$$\begin{aligned} I(m) &= \frac{1}{4}A^{-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A^{-k}(-1)^{2r+1}\frac{1}{r!}[(1/2)(3/2)\cdots(r-1/2)] \\ &\quad\times \frac{1}{k!}(-1)^{k+1}C_k[\mathcal{W}_{(-13-k+2r)/2,(12+k-2r)/2}(A\sqrt{3}/2)-\mathcal{W}_{(-13-k+2r)/2,(12+k-2r)/2}(A)], \\ C_0 &= -1, \quad C_k \equiv \left[\left(\frac{15}{2}\right)^2-\left(\frac{13}{2}\right)^2\right]\left[\left(\frac{17}{2}\right)^2-\left(\frac{13}{2}\right)^2\right]\cdots\left[\left(\frac{13+2k}{2}\right)^2-\left(\frac{13}{2}\right)^2\right], \end{aligned} \tag{A11}$$

Thus, we find confirmation by explicit computation of the dramatic reduction in the degrees of freedom in string theory at high temperatures [3,10]; the result is $O(e^{-A})$, providing the exponential suppression as a *linear* power of mass level number m. More explicitly, the numerical correction, $I(m)$, to the degeneracies, $b_m^{(\mathrm{bos})}$, in the closed bosonic string mass level expansion, is an exponential suppression of the precise form: $e^{-(2\pi m + w^2\beta^2/2\pi\alpha')}$, which erases the $O(e^{\sqrt{m}})$ growth of the degeneracies, $b_m^{(\mathrm{bos})}$ at large m [6].

The convergence of the free energy in the ultraviolet, namely, at high mass level numbers and high energies is extremely rapid, an exponential suppression:

$$\begin{aligned} \rho_{\mathrm{bosonic}} &= -(4\pi^2\alpha')^{-13}\frac{1}{4}\sum_{m=0}^{\infty} b_m^{(\mathrm{bos})}\sum_{w=-\infty}^{\infty} A^{-1}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} A^{-k}\frac{1}{r!}[(-1/2)(-3/2)\cdots(-r+1/2)] \\ &\quad\times\frac{1}{k!}(-1)^k C_k[(\sqrt{3}A/2)^{(-13-k+2r)/2}e^{-A\sqrt{3}/2}-(A)^{(-13-k+2r)/2}e^{-A}] \\ &= -(4\pi^2\alpha')^{-13}\sum_{m=0}^{\infty} b_m^{(\mathrm{bos})}\sum_{w=-\infty}^{\infty}\sum_{k=0}^{\infty}\left[\frac{1}{k!}(-1)^k C_k\right]\left[2\pi m\left(1+\frac{w^2\beta^2}{4m\pi^2\alpha'}\right)\right]^{-(15+3k)/2} \\ &\quad\times\left\{\frac{a^{-(15+3k)/2}}{\sqrt{1-(\frac{2}{\sqrt{3}A})^2}}e^{-\pi m\sqrt{3}(1+\frac{w^2\beta^2}{4m\pi^2\alpha'})}-\frac{1}{\sqrt{1-(\frac{1}{A})^2}}e^{-2\pi m(1+\frac{w^2\beta^2}{4m\pi^2\alpha'})}\right\}, \end{aligned} \tag{A12}$$

where $a=\sqrt{3}/2$. At first sight, the reader might be concerned whether the summation over k is convergent: the numerical coefficients, $C_k > 1$, grow with increasing k, and successive terms in the series have alternating sign. However, for large mass level number, m, the succeeding terms in the summation are suppressed due to the negative powers of m. Expressing the series as the sum of two like-sign infinite series, it is apparent that successive terms in each are suppressed by a factor of $1/m$, in addition to the overall exponential suppression. This will lead to very rapid convergence. This is also true for the summation over m: there is a well-known *square root exponential growth as a function of mass level m* of the degeneracies, b_m, at large mass level numbers [6,7], but rapid convergence of the free energy is driven by the variable A, which provides an *exponential suppression linear as a function of mass level* *number*! Our explicit analytic integration of the closed string world-sheet moduli has pinned down the precise mathematical form: the convergence in the ultraviolet is as fast as an exponential superimposed on the power law suppression of the degeneracies at high mass levels.

Our result point to the exact renormalization properties of any 2d Weyl × diffeomorphism invariant critical string theory. Since the 26-dimensional closed bosonic string has neither supersymmetry, nor is it free of vacuum instabilities, it is nice to have an explicit analytic derivation establishing that the closed bosonic string mass level expansion is nevertheless ultraviolet finite, and convergent. This conclusion was implicit in the final expression for the one-loop string vacuum amplitude derived as a diffeomorphism × Weyl invariant path integral [2], extended and reviewed by us in [16,29]. We note that the original demonstration of the significance of the one-loop modular transformations dates to the 26D Virasoro-Shapiro model [17]. The Weyl × diffeomorphism invariant measure of the integral for 2d quantum gravity conformally coupled to 26 scalars, is finite everywhere in the fundamental domain of the modular group of the torus, a property that can also be deduced at any order in the string loop expansion [28].

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