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Anomalous magnetoresponse and the Stückelberg axion in holography

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We study the magnetoresponse with nonconserved currents in holography. Nonconserved currents are dual to massive vector fields in anti–de Sitter (AdS). We introduce the mass in a gauge invariant way via the Stückelberg mechanism. In particular we find generalizations of the chiral magnetic effect, the chiral separation effect and the chiral magnetic wave. Since the associated charge is not conserved we need to source it explicitly by a coupling, the generalization of the chemical potential. In this setup we find that in general the anomalous transport phenomena are still realized. The values we find for nonzero mass connect continuously to the values of the anomalous conductivities of the consistent currents, i.e. the proper chiral magnetic effect vanishes for all masses (as it does for the consistent current in the zero mass case) whereas the chiral separation effect is fully present. The generalization of the chiral magnetic wave shows that for small momenta there is no propagating wave but two purely absorptive modes (one of them diffusive). At higher momenta we recover the chiral magnetic wave as a combination of the two absorptive modes. We also study the negative magneto resistivity and find that it grows quadratically with the magnetic field. The chiral magnetic wave and the negative magneto resistivity are manifestations of the chiral magnetic effect that takes place when the (nonconserved) charge is allowed to fluctuate freely in contrast to the case where the charge is fixed by an explicit source. Since the (classical) $U(1)_A$ symmetry of QCD is not at all a symmetry at the quantum level we also argue that using massive vectors in AdS to model the axial singlet current might result in a more realistic holographic model of QCD and should be a good starting point to investigate the dynamics of anomalous transport in the strongly coupled quark gluon plasma.

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I. INTRODUCTION

Anomalies in the quantum theories of chiral fermions belong to the most emblematic properties of relativistic quantum field theory [1,2] (see [3,4] for reviews). They provide stringent consistency conditions on possible gauge interactions but also predict physical processes that would be otherwise highly suppressed such as the decay of the neutral pion into two photons.

Anomalies are not only important for the phenomenology of particle physics but they also are of utmost importance to the theory of quantum many body systems containing chiral fermions. Anomaly cancellation plays a crucial role in the field theoretic understanding of the electroresponse of quantum hall fluids for example. Chiral fermions appear as edge states and the associated anomalies have to be canceled by appropriate anomaly inflow from a gapped bulk reservoir of charge.

More recently the focus has been on ungapped chiral bulk fermions that give rise to new anomaly related transport phenomena in the presence of a magnetic field (chiral magnetic effect [5]) and/or vortices (chiral vortical effect [6,7]). The chiral magnetic effect describes the generation of a current in the presence of a magnetic field,

$$\vec{J} = \sigma_B \vec{B}. \tag{1}$$

The associated chiral magnetic conductivity [8] can be calculated from first principles via a Kubo-type formula,

$$\sigma_B = \lim_{k_z\to 0} \frac{i}{k_z} \langle J_x J_y \rangle (\omega = 0, \vec{k} = k_z \hat{e}_z). \tag{2}$$

Since these effects owe their existence to the presence of (global) anomalies one could expect that their values are universal and independent from interactions. Indeed calculations with free fermions [9,10] give the same result as infinitely strongly coupled theories defined via the AdS/CFT correspondence [11,12]. Furthermore it was shown that the anomalous conductivities are completely determined in hydrodynamics or in effective action approaches [13–15] (with the exception of the gravitational anomaly contribution, whose model independent determination needs additional geometric arguments [16]). Therefore the values of the chiral conductivities related to purely global anomalies are subject to a nonrenormalization theorem akin to the Adler-Bardeen theorem [17].

Chiral conductivities do get renormalized however in the case when the gauge fields appearing in the anomalous divergence of the current are dynamical [17,18]. An example is the singlet $U(1)_A$ current in QCD. Its anomaly is of the form

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$$\partial_\mu J^\mu_A = \epsilon^{\alpha\beta\gamma\delta}\left(\frac{N_c\sum_f q_f^2}{32\pi^2}F_{\alpha\beta}F_{\gamma\delta} + \frac{N_f}{16\pi^2}\mathrm{tr}(G_{\alpha\beta}G_{\gamma\delta}) + \frac{N_cN_f}{96\pi^2}F^5_{\alpha\beta}F^5_{\gamma\delta}\right). \quad (3)$$

Here F is the electromagnetic field strength, G the gluon field strength and F^5 the field strength of an axial gauge field A^5_μ whose only purpose is to sum up insertions of the axial current in correlation functions, i.e. there is no associated kinetic term. N_c and N_f are the numbers of colors and flavors respectively. In this case it has been shown in [17,18] that the vortical conductivity receives two loop corrections whereas later on it has been argued in an effective field theory approach that all chiral conductivities receive higher loop corrections once dynamical gauge fields enter the anomaly equation [19].

It has been argued long ago by 't Hooft that in such a situation one should not think of the classically present $U(1)_A$ symmetry as a symmetry at all on the quantum level [20]. In asymptotically free theories such as QCD there might survive only a discrete subgroup because of instanton contributions. This discrete subgroup can be further broken spontaneously via chiral symmetry breaking but since it was not a symmetry to begin with there is also no associated Goldstone boson, which explains the high mass of the η' meson in QCD. A related fact is that the corresponding triangle diagram receives higher loop corrections via photon-photon or gluon-gluon rescattering. These higher order diagrams lead to a nonvanishing anomalous dimension for the axial current operator J^μ_A. See [21,22] for recent reviews.

These considerations motivate us to study the anomalous magnetoresponse of massive vector fields in holography. Our philosophy is as follows. In quantum field theory we would have to study the path integral

$$Z = \int D\Psi D\bar{\Psi} D\mathcal{A}_q \times \exp\left[i\int d^4x\left(-\frac{1}{2}\mathrm{tr}(G.G) + \bar{\Psi}\not{D}\Psi + \theta\mathcal{O}_A + J.A\right)\right], \quad (4)$$

where $\mathcal{A}_q$ stands collectively for the dynamical gauge fields, G is their field strength tensor and $\mathcal{O}_A$ is the (operator valued) anomaly

$$\mathcal{O}_A = \epsilon^{\alpha\beta\gamma\delta}\left(\frac{N_f}{16\pi^2}\mathrm{tr}(G_{\alpha\beta}G_{\gamma\delta}) + \frac{N_cN_f}{96\pi^2}F_{\alpha\beta}F_{\gamma\delta}\right). \quad (5)$$

Since the anomaly is a quantum operator we need to define a path integral that allows to calculate correlation functions of this anomaly operator. This means that we need to introduce the source field $\theta(x)$ coupling to $\mathcal{O}_A$. For the same reason we also have to include a source for the anomalous current J^μ. This source is the nondynamical gauge field which from now on we denote by A_μ. The covariant derivative in (4) contains only the dynamical gauge fields. The nondynamical gauge fields are coupled with the last term in (4). If we define the effective action $\exp(iW_{\mathrm{eff}}[A,\theta]) = Z$ it is basically guaranteed by construction that this effective action enjoys the gauge symmetry

$$\delta A_\mu = \partial_\mu\lambda, \qquad \delta\theta = -\lambda, \qquad \delta W_{\mathrm{eff}} = 0. \quad (6)$$

We now replace the (strongly coupled) dynamics of the gluon (and fermion) fields, i.e. the path integral over $\mathcal{A}_q$, Ψ and $\bar{\Psi}$ by the dynamics of classical fields propagating in anti–de Sitter (AdS) space. The gravity dual should allow to construct $W_{\mathrm{eff}}[A,\theta]$ as the on-shell action of a field theory in anti–de Sitter space containing a vector field A_μ and a scalar θ obeying the gauge symmetry (6). In addition, as we have argued before, the vector field should source a nonconserved current. Since anti–de Sitter space implies the dual theory to have an additional conformal symmetry the four-dimensional current is nonconserved if and only if its dimension is different from three. This in turn means that the bulk vector field in our AdS theory has to be a massive vector and it is precisely the gauge symmetry (6) that allows the inclusion of a gauge invariant Stückelberg mass in the bulk AdS theory. The anomaly also includes the global part proportional to the field strengths of the nondynamical gauge field; therefore we also need to include a five-dimensional Chern-Simons term in our AdS dual. The relation of the Stückelberg field in holography to the anomaly has been first pointed out in [23] and the necessity to include it in holographic studies of the anomalous transport has very recently also been emphasized in [24].

Moreover, since we have application to the physics of the strongly coupled quark gluon plasma in back of our head, we are lead to study a massive Stückelberg theory with a Chern-Simons term at high temperature, i.e. in the background of an AdS black brane. We make one more simplifying assumption. We do not study any correlation functions including the energy momentum tensor. Therefore we can resort to the so-called probe limit in which we ignore the backreaction for the gauge field theory onto the geometry.

The paper is organized as follows. In Sec. II we define a simple model with one massive vector field. We calculate the (holographically) normalized nonconserved current and compare to the massless case. Then we study the generalization of the chiral magnetic conductivity defined via the Kubo formula (2). We find that the chiral conductivity still exists and in terms of an appropriately defined dimensionless number gets even enhanced compared to the massless case. In the limit of vanishing mass we recover the value of the chiral magnetic conductivity in the consistent current. As is well known this is 2/3 of the standard value most commonly cited (which corresponds to the covariant definition of the current).

We remind the reader of the fact that the chiral magnetic effect in the consistent current for the $U(1)^3$ anomaly of a single Weyl fermion takes the form

$$\vec{J} = \left(\frac{\mu}{4\pi^2} - \frac{A_0}{12\pi^2}\right)\vec{B}, \tag{7}$$

whereas for the triangle anomaly with one axial and two vector-like currents (AVV) of a single Dirac fermion with a vector current preserving regularization it is

$$\vec{J}_V = \left(\frac{\mu_5}{2\pi^2} - \frac{A_0^5}{2\pi^2}\right)\vec{B}, \tag{8}$$

with μ, μ_5 being the (axial) chemical potentials and A_0 and A_0^5 being the background values of the (axial) gauge fields that do not necessarily coincide with the chemical potentials. The customary gauge choice $A_0 = \mu$ and $A_0^5 = \mu_5$ leads to the factor $2/3$ in the $U(1)^3$ case and to a vanishing chiral magnetic effect (CME) in the AVV case.[1] In contrast the chiral separation effect

$$\vec{J}_5 = \frac{\mu_V}{2\pi^2}\vec{B} \tag{9}$$

shows no explicit dependence on the temporal gauge field. This is a consequence of the fact that the vectorlike symmetry is nonanomalous, i.e. μ_V is the chemical potential conjugate to a truly conserved charge.

If one expresses the CME however in terms of the covariant currents the terms depending on the gauge fields are absent. Finally we note that the relation between covariant and consistent currents is $J^\mu_{\rm cov} = J^\mu_{\rm cons} + \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\lambda}A_\nu F_{\rho\lambda}$ for the $U(1)^3$ anomaly and $J^\mu_{{\rm cov},V} = J^\mu_{\rm cons,V} + \frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\lambda}A^5_\nu F_{\rho\lambda}$. In these expressions the currents and Chern-Simons terms all have dimension three.

In Sec. III we consider a massive and a massless vector field in the bulk. Our motivation is that the proper chiral magnetic effect stems from an interplay of vector and axial symmetries. The vector symmetry can be taken as the usual electromagnetic $U(1)$. While the electromagnetic gauge fields are still quantum operators we can assume in the quark gluon plasma context that electromagnetic interactions are weak and to first approximation we might model the vector $U(1)$ as a nondynamical gauge field. Furthermore the vector current of electromagnetic interactions has to be exactly conserved. We compute the chiral magnetic conductivity and the conductivity related to the chiral separation effect. We find that the chiral separation effect is fully realized whereas the chiral magnetic conductivity vanishes. Again we point out that these are the same results that hold for the consistent currents in the case when also the axial current is modeled by a massless vector field. Then we study the chiral magnetic wave [26] and compare our findings to a simple hydrodynamic model in which we include a decay width for the axial charge by hand. We find basically a perfect match between the modes of the phenomenological model and the low lying quasinormal modes of the holographic model. For small momenta we find absence of a propagating wave, whereas for large enough momentum there is indeed a propagating (damped) wave which is the generalization of the chiral magnetic wave. Finally we also study the negative magnetoresistivity induced by the anomaly in a constant magnetic field background. We find by numerical analysis that the negative magnetoresistivity depends quadratically on the magnetic field. The optical conductivity has a Drude peak form whose height is determined by the inverse of the bulk mass. For large magnetic field a gap opens up in the optical conductivity and we also check that the spectral weight gets shifted from the gap region into the peak region such that a sum rule of the form $d(\int d\omega\sigma(\omega))/dB = 0$ holds.

We present our conclusions in Sec. IV, summarize and discuss our results and give some outlook to possible further generalizations of models with holographic Stückelberg axions.

II. HOLOGRAPHIC STÜCKELBERG MECHANISM WITH A U(1) GAUGE FIELD

In this section we consider Maxwell–Chern-Simons theory in the bulk and give a mass to the gauge field via Stückelberg mechanism

$$S = \int d^5x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}(A_\mu - \partial_\mu\theta)(A^\mu - \partial^\mu\theta) + \frac{\kappa}{3}\epsilon^{\mu\alpha\beta\gamma\delta}(A_\mu - \partial_\mu\theta)F_{\alpha\beta}F_{\gamma\delta}\right). \tag{10}$$

The above model provides a mass for the gauge field in a consistent gauge invariant way. Stückelberg terms indeed arise as the holographic realization of dynamical anomalies, as pointed out for the first time in [23] (see also [27] for similar conclusions in the context of AdS/QCD). This has been also emphasized quite recently by the authors of [24] for a class of nonconformal holographic models.

As it is well known, in holography we do not have access to the strongly coupled gauge field directly.[2] This implies that the dynamical contribution to the divergence of the

[1] If an axion background is present there is also a term proportional to $\partial_t\theta\vec{B}$. We also emphasize that the anomaly makes the (axial) gauge field an observable precisely via the terms in (7), (8). See e.g. the discussion in [25].

[2] Note however that dynamical gauge fields emerge in the alternative quantization scheme in AdS_4 [28]. They can also be introduced via inclusion of boundary kinetic terms [29].

current enters as a mass term for the gauge field thereby inducing an explicit nonconservation. In contrast the global anomaly is implemented by an explicit Chern-Simons term. This fits the general expectation that the dynamical anomaly cannot be switched off because it is not simply a given by a functional of external fields.

Let us also comment on a crucial difference between model (10) and models of holographic superconductors. Holographic superconductors [30] also give a bulk mass term to the gauge field and they might be written in Stückelberg form as well [31]. The difference is that the Higgs mechanism in the bulk uses a massive scalar field that decays at the boundary and does therefore not change the asymptotic behavior of the gauge field. In our case the mass is constant in the bulk and does therefore change the asymptotic behavior of the vector field as one approaches the boundary of AdS.

We will work in the probe limit with Schwarzschild AdS_5 as background metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2),$$
$$f(r) = \frac{r^2}{L^2} - \frac{r_H^4}{r^2}. \tag{11}$$

As usual we make use of rescaling invariance of the theory to set $r_H = 1$ and $L = 1$, and therefore $\pi T = 1$.

The equations of motion are

$$\nabla_\nu F^{\nu\mu} - m^2(A^\mu - \partial^\mu \theta) + \kappa \epsilon^{\mu\alpha\beta\gamma\rho} F_{\alpha\beta} F_{\gamma\rho} = 0, \tag{12}$$

$$\nabla_\mu (A^\mu - \partial^\mu \theta) = 0. \tag{13}$$

The asymptotic analysis shows that the non-normalizable (N.N.) and the normalizable modes of the gauge field behave as

$$A_{i(\mathrm{N.N.})} \sim A_{i(0)} r^\Delta, \qquad A_{i(\mathrm{N.})} \sim \tilde{A}_{i(0)} r^{-2-\Delta},$$
$$\Delta = -1 + \sqrt{1 + m^2}. \tag{14}$$

Since the mass has to be positive (for the massless case saturates the unitarity bound), there is no possible alternative quantization and the leading term is always to be identified with the N.N. mode. Moreover, there is an upper bound to the value of the mass prescribed by $\Delta = 1$. As we will show via holographic renormalization, the operator dual to the coefficient of the non-normalizable mode is essentially given by the normalizable mode. Its dimension can be found via the following argument. The AdS metric is invariant under the scaling $r \to \lambda r$, $(t, \vec{x}) \to \lambda^{-1}(t, \vec{x})$. Since a gauge field is a one form we have to study the behavior of $A_\mu(r, x) dx^\mu$ under these scalings. One finds then that the normalizable mode has a scaling dimension of

$$\dim(\tilde{A}_{i(0)}) = [J_i] = 3 + \Delta. \tag{15}$$

This implies that if $\Delta > 1$ the dual operator is irrelevant (in the IR) and thus destroys the AdS asymptotics. In the holographic renormalization in Appendix A we find that accordingly the number of counterterms diverges for $\Delta > 1$.[3]

It is clear that the number of counterterms depends on the value of the mass. From now on we will work in the range of masses that minimizes it, namely

$$\Delta < \frac{1}{3} \longleftrightarrow m^2 < \frac{7}{9}. \tag{16}$$

Henceforth we will refer to Δ as the anomalous dimension of the dual current.

The procedure of renormalization for this theory is explained and discussed in detail in Appendix A. The boundary action with the counterterms such that $S_{\mathcal{R}\mathrm{en}} = S + S_{\mathrm{CT}}$ reads

$$S_{\mathrm{CT}} = \int_\partial d^4x \sqrt{-\gamma} \left(\frac{\Delta}{2} B_i B^i - \frac{1}{4(\Delta + 2)} \partial_i B^i \partial_j B^j + \frac{1}{8\Delta} F_{ij} F^{ij} \right), \tag{17}$$

with $B_i \equiv A_i - \partial_i \theta$.

Remarkably, the coupling of the Stückelberg field to the Chern-Simons term in (10) is not optional once the mass is turned on; if one does not add it to the action it appears as a counterterm when holographic renormalization is carried out. More precisely the coefficient in front of $d\theta \wedge F \wedge F$ in (10) is not arbitrary once the mass is turned on. It is fixed to be the negative of the coefficient in front of the Chern-Simons term $A \wedge F \wedge F$, which renders a completely gauge invariant action. If we had not added this term directly from the start it would thus have appeared as a counterterm.

A. The one-point function

From the renormalized action we compute correlators of the dual operators in the boundary theory by means of the usual prescription. In this section we show our results, sticking only to the strictly necessary technical details for the discussion. A detailed discussion of the calculations can be found in Appendix B 1.

Due to the anomalous dimension of the operator the analysis of the one-point function becomes more subtle than in the massless case. In previous works, at zero mass, the leading terms of the expressions were finite. Therefore it made sense to look at the expression for the current expectation value as a functional of the covariant fields before taking the limit $r \to \infty$. This is however not the case when $m \neq 0$, since now all terms are divergent to leading order. Nevertheless, to make comparison with the results at

[3] We thank Ioannis Papadimitriou for pointing this out.

zero mass, we want to look at the result before explicitly taking the limit. In order to do so, we split the unrenormalized one-point function into a term lacking a (subleading) finite contribution (called X below) and terms which do lead to a finite contribution after renormalization,

$$\langle J^i\rangle = \lim_{r\to\infty}\sqrt{-g}\,r^{\Delta}(F^{ir} + r\Delta A^i) + X^i. \tag{18}$$

We see that the contribution arising from the Chern-Simons term in (10) is contained in X^i, which means that it does not contribute explicitly to the current. The renormalized one-point function reads

$$\langle J^i\rangle_{\mathrm{ren}} = 2(1+\Delta)\tilde{A}^i_{(0)}, \tag{19}$$

where $\tilde{A}_{(0)m}$ is the coefficient of the normalizable mode. Let us compare this with the expression for the consistent current that one obtains in absence of mass[4]

$$\begin{aligned}\langle J^i\rangle &= \lim_{r\to\infty}\sqrt{-g}\left(F^{ir} + \frac{4\kappa}{3}\epsilon^{ijkl}A_jF_{kl}\right) + X^i.\\ &\overset{\mathrm{Ren}}{=} 2\tilde{A}^i_{(0)} + \frac{8\kappa}{3}\epsilon^{ijkl}A_{j(0)}\partial_k A_{l(0)}\end{aligned} \tag{20}$$

Here we see that in the massless case ($\Delta = 0$) the Chern-Simons term indeed gives a finite contribution to the current which is explicitly proportional to the sources. It is precisely this term that makes the difference between the covariant and the consistent definition of the current. We remind the reader that in the case of global anomalies one can define a covariant current by demanding that it transforms covariantly under the anomalous gauge transformation. In the AdS/CFT dictionary this covariant current is given by the normalizable mode of the vector field. In contrast the consistent current is defined as the functional derivation of the effective action with respect to the gauge field and in the AdS/CFT correspondence includes the Chern-Simons term in (20).

Equations (18)–(19) establish that we are no longer able to make such a distinction if $m \neq 0$, for there is no explicit finite local contribution of the Chern-Simons term to the current operator. Quite remarkably, all of our results show that (19) corresponds to the consistent current in the zero mass limit. This ultimately implies that in the limit $m \to 0$ the highly nonlocal expression $\tilde{A}_{(0)i}$ gives rise to the two terms in the last line of (20), which include a local term in the external sources. Hence, within the article we will only refer to consistent or covariant currents when analyzing the massless limit.

Another remarkable difference with the massless model is the Ward identity of the current operator. Using the equations of motion we can write the divergence of the current on shell

$$\begin{aligned}\langle \partial_i J^i\rangle &= \lim_{r\to\infty}\sqrt{-g}\,r^{\Delta}\left(m^2\partial^r\theta + r\Delta\partial_i A^i - \frac{\kappa}{3}\epsilon^{ijkl}F_{ij}F_{kl} + \tilde{X}\right),\\ &\overset{\mathrm{Ren}}{=} 2(1+\Delta)\partial_i\tilde{A}^i_{(0)},\end{aligned} \tag{21}$$

where we have extracted the (infinite) Chern-Simons term from (18)[5] because it is convenient for the following discussion. As mentioned before, the fact that the terms in these expressions diverge obscures the interpretation if one does not take the limit $r \to \infty$. Once we take it we find that the Ward identity (21) becomes a tautology since the only term on the right-hand side that gives a finite contribution is determined in the large r expansion directly by the divergence of the normalizable mode of the vector field. Therefore the divergence of the current on shell is unconstrained.

If we now look at what happens when we take the limit $m \to 0$ before we take $r \to \infty$ we see that we recover the expression for the divergence of the consistent current

$$\begin{aligned}\langle \partial_i J^i\rangle &= \lim_{r\to\infty}\sqrt{-g}\left(\frac{\kappa}{3}\epsilon^{ijkl}F_{ij}F_{kl} + \tilde{X}\right)\\ &\overset{\mathrm{Ren}}{=} \sqrt{-g}\frac{\kappa}{3}\epsilon^{ijkl}F_{ij}F_{kl}\end{aligned} \tag{22}$$

contained in the (nonlocal) normalizable mode of the filed. As we will see now the behavior of the conductivity points in the same direction.

B. Two-point functions and anomalous conductivity

Our main interest is to study the effect that the anomalous dimension has on the response of the system in the presence of a magnetic field. As a first step in this direction we compute the anomalous conductivity $J_i = \sigma_{55}B_i$ that is related to a correlator of current operators via the Kubo formula

$$\sigma_{55} = \lim_{k\to 0}\frac{i}{k_z}\langle J_xJ_y\rangle\Big|_{\omega=0}. \tag{23}$$

We emphasize however that B_i does not have the simple interpretation of a magnetic field since its dimension is $2-\Delta$. We want to study the anomalous conductivity in an analogous fashion to [32] and find the dependence of the chiral anomalous conductivity on the source for J^μ. In order to generalize the concept of chemical potential to the

[4]Notice that in the zero mass limit θ becomes a nondynamical field defined at the boundary. The divergence of this field also contributes to the current [25]. In order to keep the discussion simple we chose to take this nondynamical field to vanish since this is the natural value that arises from our background in the zero mass limit.

[5]In other words, $\partial_i X^i = -\frac{\kappa}{3}\epsilon^{ijkl}F_{ij}F_{kl} + \tilde{X}$.

situation at hand we switch on a temporal component of the gauge field in the background $A = \Phi(r)dt$. We choose the axial gauge $A_r = 0$. The equation of motion is

$$\Phi'' + \frac{3}{r}\Phi' - \frac{m^2}{f}\Phi = 0. \tag{24}$$

We solve Eq. (24) numerically,[6] with the following boundary conditions:

$$\phi(r_H) = 0, \qquad \phi(r \to \infty) \sim \mu_5 r^\Delta, \tag{25}$$

with μ_5 being the source. Notice that μ_5 does not correspond to a thermodynamic parameter in our massive model. Rather it should be interpreted as a coupling in the Hamiltonian. Different values of μ_5 correspond therefore to different theories. Different values of a chemical potential correspond only to different filling levels of the low lying fermionic states in the same theory. In the case of an anomalous symmetry one has to distinguish this filling level from the presence of a background constant temporal component of the gauge field [32].

The near horizon analysis shows that we are forced to impose $\Phi(r_H) = 0$. In absence of the mass term the gauge filed is not divergent at the horizon, independently of the finite value it takes at the boundary. This reflects the remnant (recall we work in the axial gauge) gauge freedom that one has in this case: the value of the source can be shifted by a gauge transformation.[7] However the mass term in the equation of motion (e.o.m.) is divergent at the horizon and forces the field to vanish there. Remarkably, this and the asymptotic behavior of the field illustrate the fact that speaking of a chemical potential does no longer make sense. Computing the chemical potential as the integrated radial electric flux in the bulk one obtains

$$\mu = \lim_{r\to\infty} \int_{r_H}^{r} \partial_r A_t dr \to \infty. \tag{26}$$

This can be understood heuristically from the nonconservation of the charge: the energy to introduce and maintain a quantum of charge that is not conserved is infinite.

Since our background is homogeneous in the transverse directions it is easy to see that $\langle \partial_i J^i \rangle = 0$. In particular, the fact that a stationary solution exists implies that it is possible to choose a homogeneous configuration of μ_5 such that it compensates for the decay of the charge that is naturally caused due to the mass term. Namely,

$$\frac{d\rho}{dt} = 0, \tag{27}$$

with ρ being the charge density of the system. We will see that the source necessary to ensure (27) equals the axial chemical potential in the massless limit (recall that only when $m = 0$ we can identify μ_5 with a chemical potential).

Once we have built the background we can proceed to switch on perturbations on top of it in order to compute the two-point function (23). To linear order in the external source $\tilde{A}^i_{(0)} \approx \tilde{A}^i_{(0)} + \delta\tilde{a}^i_{(0)}$. From (19) we have

$$\langle J^n J^m \rangle = 2(1+\Delta)\eta^{ml}\frac{\delta\tilde{a}_{(0)l}}{\delta a_{(0)n}}. \tag{28}$$

Here $\tilde{a}_{(0)m}$ is the coefficient of the normalizable mode of the perturbation. We compute the above expression numerically. Again we leave technical details for Appendix B 2 because the analysis is tedious and it is based on standard techniques. We show the result in Fig. 1. A comment is in order here regarding the temperature dependence on the plots. Dimensional analysis of the correlator $[\langle JJ \rangle] = 6 + 2\Delta$ implies that the conductivity now has dimension $[\sigma] = 1 + 2\Delta$. This in turn causes the physical conductivity to have a temperature dependence $\sigma \sim T^{3\Delta}$. As usual, from numerics we can only plot dimensionless quantities $\sigma/(\pi T)^{3\Delta}$ and $\mu_5/(\pi T)^{1-\Delta}$.

The plot on the left panel of Fig. 1 shows the dependence of the conductivity with the source for different values of the mass. Despite the fact that the slope changes the behavior is always linear in the dimensionless source parameter. The plot on the right panel shows the conductivity coefficient vs the anomalous dimension of the current Δ. Remarkably the conductivity gets enhanced by the presence of the mass term in the bulk. In addition to this enhancement the plot shows another feature that deserves a comment. In the limit $\Delta(m) \to 0$ the conductivity goes to the numerical value $5.333 \sim \frac{16}{3}$. Let us now look at the analytic solution for zero mass shown in (2.25) of [32],[8]

$$\langle J^i_5 J^j_5 \rangle = -4i\tilde{\kappa}k(3\mu_5 - \alpha)\epsilon_{ij}, \tag{29}$$

where μ_5 here is the thermodynamic chemical potential, α is the source, i.e. the boundary value of the temporal component of the gauge field and $\tilde{\kappa} = \frac{2\kappa}{3}$ in our convention. If one chooses the gauge $\alpha = \mu_5$ then one obtains

$$\langle J^i_5 J^j_5 \rangle = -8i\tilde{\kappa}k\mu_5\epsilon_{ij}. \tag{30}$$

[6]The analytic solution can be worked out in terms of hypergeometric functions. Since we need to resort to numerical methods later on, when studying fluctuations around the background, we found it more convenient to apply purely numerical methods also for the background.

[7]Gauge transformations that are nonzero at the horizon are not true gauge transformations but global transformations.

[8]This model contained two massless vector fields in the bulk, one modeling the conserved vector and the other the anomalous axial symmetry. It is clear that the result obtained in our model with one massive vector should be compared in the zero mass limit to the axial vector sector of the model in [32].

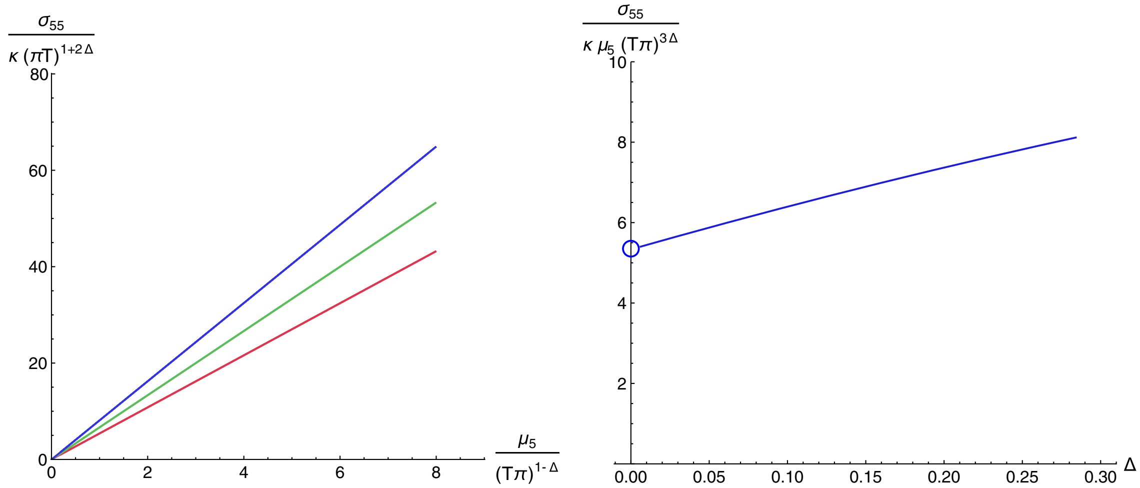

FIG. 1 (color online). Left: Plot of the conductivity versus the source for $m^2 = 1/2$ (upper line), $m^2 = 1/4$ (middle line), $m^2 = 0.01$ (lower line). Right: Plot of the conductivity coefficient as a function of the anomalous dimension; the circle stands for the asymptotic value in the limit $\Delta \to 0$.

In our numerical results we have absorbed the Chern-Simons coupling in the definition of the external B-field (or equivalently set it to one in the fluctuation equations). Taking into account the difference in the normalizations of the Chern-Simons couplings in [32] we can extract from (29) the numerical value $\sigma_{55} = 16/3$ which coincides with the $m \to 0$ limit in our case. We conclude that our result matches the analytic formula only if we identify $\alpha = \mu_5$, as mentioned right after Eq. (27). This is also consistent with what we found in the expression for the current.

The fact that we found a time-independent background solution obscures the nonconservation of the charge. The best way to shed light onto the explicit decay of charge is by considering a trivial background (in particular, all the sources vanish) and look at the spectrum of quasinormal modes. In the massless case the lowest quasinormal mode (QNM) shows a diffusion-type behavior, namely $\omega = -iDk^2$. This diffusive mode has to develop a gap when $m \neq 0$ due to the nonconservation of the charge. Technical details on how to compute QNM can be found in [33]. Indeed we find that the lowest QNM is no longer massless. The gap Γ depends on the value of the bulk mass as depicted in Fig. 2.

This indicates that the charge is no longer conserved. Furthermore a simple phenomenological model including only the dynamics of the lowest quasinormal mode suggests that the nonconservation can be modeled by writing $\partial_\mu J^\mu = -\frac{1}{\tau} J^0$, where τ is the inverse of the gap Γ of the lowest quasinormal mode. Indeed, such a phenomenological decay law together with Fick's law $\vec{J} = -D\vec{\nabla} J^0$ suggests a gapped pseudodiffusive mode $\omega + i/\tau + iDk^2 = 0$ which indeed is what we find from the QNM spectrum (see the next section).

III. THE STÜCKELBERG $U(1) \times U(1)$ MODEL

In this section we introduce an extra unbroken Abelian symmetry in the bulk. This allows us to switch on an "honest" external magnetic field in the dual theory and therefore study not only the axial conductivity but the chiral magnetic conductivity and the chiral separation conductivity as well. In addition we will be able to study the effect of the mass on the chiral magnetic wave and on the electric conductivity. The Lagrangian reads

$$\mathcal{L} = \Bigg(-\frac{1}{4} F^2 - \frac{1}{4} H^2 - \frac{m^2}{2} (A_\mu - \partial_\mu \theta)(A^\mu - \partial^\mu \theta) + \frac{\kappa}{2} \epsilon^{\mu\alpha\beta\gamma\delta} (A_\mu - \partial_\mu \theta)(F_{\alpha\beta} F_{\gamma\delta} + 3 H_{\alpha\beta} H_{\gamma\delta}) \Bigg), \qquad (31)$$

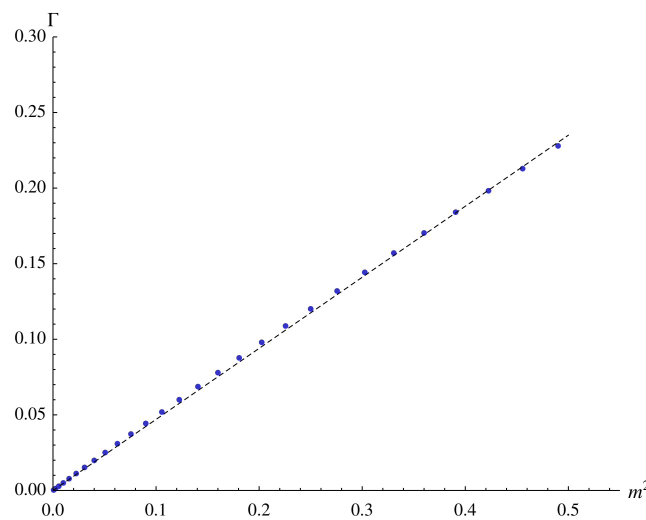

FIG. 2 (color online). The gap Γ versus m^2. The black line corresponds to a linear fit.

where $F = dA$ and $H = dV$. The new dynamical U(1) in the bulk is massless and couples to the Chern-Simons term in the usual way. As in the previous section we work in the probe limit with Schwarzschild AdS_5 as background metric. The scalar field transforms nontrivially only under the massive $U(1)$. From now on we will refer to the massless $U(1)$ as vector V_μ and to the massive $U(1)$ as axial A_μ. The equations of motion for the gauge fields are

$$\nabla_\mu F^{\mu\nu} - m^2(A^\nu - \partial^\nu\theta) + \frac{3\kappa}{2}\epsilon^{\nu\alpha\beta\gamma\rho}(F_{\alpha\beta}F_{\gamma\rho} + H_{\alpha\beta}H_{\gamma\rho}) = 0, \tag{32}$$

$$\nabla_\nu H^{\nu\mu} + 3\kappa\epsilon^{\mu\alpha\beta\gamma\rho}F_{\alpha\beta}H_{\gamma\rho} = 0. \tag{33}$$

The equation of motion of the scalar remains unchanged [see Eq. (13)]. Non-normalizable and normalizable modes of the axial gauge field have the same asymptotics for large r as the gauge field in the $U(1)$ model. The vector field shows the same behavior at infinity as usual,

$$V_{i(\mathrm{N.N.})} \sim V_{i(0)}r^0, \quad V_{i(\mathrm{N.})} \sim \tilde{V}_{i(0)}r^{-2}. \tag{34}$$

The holographic renormalization of this model is discussed in Appendix A 2. The result is the following boundary term:

$$S_{\mathrm{CT}} = \int_\partial d^4x\sqrt{-\gamma}\left(\frac{\Delta}{2}B_iB^i - \frac{1}{4(\Delta+2)}\partial_iB^i\partial_jB^j + \frac{1}{8\Delta}F_{ij}F^{ij} + \frac{1}{8}H_{ij}H^{ij}\log r^2\right), \tag{35}$$

with $B_i = A_i - \partial_i\theta$. There are two differences from the result in the previous model, on the one hand, the appearance of the usual $\sim$log term for the vector gauge field. On the other, the role of the coupling of the Stückelberg field to the Chern-Simons term in (31) is different because now we have two independent couplings $d\theta \wedge F \wedge F$ and $d\theta \wedge H \wedge H$. The former is mandatory, as in the $U(1)$ model. The latter however is optional since it is a finite boundary term.[9] We have chosen to include it. As we will see, this will not affect the results in our concrete background, but it is potentially useful for other models since it cancels possible finite contributions to the vector current stemming from the Stückelberg field.

A. One-point functions

First we compute the one-point functions of the gauge fields. The technical details of the calculations can be found in Appendix C 1. As in (18) we hide all terms that do not contain any finite contribution in vectors X^i and Y^i, obtaining

$$\langle J_V^i\rangle = \lim_{r\to\infty}\sqrt{-g}\,(H^{ir} + 6\kappa\epsilon^{ijkl}(A_j - \partial_j\theta)H_{kl}) + X^i, \tag{36}$$

$$\langle J_A^i\rangle = \lim_{r\to\infty}\sqrt{-g}\,r^\Delta(F^{ir} + r\Delta A^i) + Y^i. \tag{37}$$

The axial current behaves as in the previous model. Recall that the leading term in the asymptotic expansion of the axial gauge field diverges and so does the Chern-Simons term in (36). Nevertheless, contrary to the axial current, this term has a subleading finite part (which is the reason why we do not include it in X^i). Looking at the complete expansion for the axial gauge field (B3), we see that this finite contribution is proportional to the source of θ instead of the source of the gauge field. This is of course different from what one finds in the massless case. In addition, it is here where we see the effect that the coupling $d\theta \wedge H \wedge H$ has. It cancels this finite contribution proportional to the source dual to the Stückelberg field. As mentioned before, this cancellation comes from the choice we made in the action and can be removed at will.

We can now look at the Ward identities. Substituting the e.o.m. in the divergence of the current we find

$$\langle\partial_iJ_V^i\rangle_{\mathrm{Ren}} = 0, \quad \langle\partial_iJ_A^i\rangle_{\mathrm{Ren}} = (2+2\Delta)\partial_i\tilde{A}_{i(0)}. \tag{38}$$

The vector current is conserved as in the massless case. The result for the axial current is the same as in the previous model: its divergence is unconstrained reflecting the fact that it is a nonconserved current.

B. Two-point functions and anomalous conductivities

The presence of an extra $U(1)$ allows us to obtain the following anomalous conductivities from Kubo formulas [32,35]:

$$\sigma_{\mathrm{CME}} = \lim_{k\to0}\frac{i\epsilon_{ij}}{2k}\langle J^iJ^j\rangle(\omega=0,k), \tag{39}$$

$$\sigma_{\mathrm{CSE}} = \lim_{k\to0}\frac{i\epsilon_{ij}}{2k}\langle J_5^iJ^j\rangle(\omega=0,k), \tag{40}$$

$$\sigma_{55} = \lim_{k\to0}\frac{i\epsilon_{ij}}{2k}\langle J_5^iJ_5^j\rangle(\omega=0,k). \tag{41}$$

In order to study these we have to switch on a source for both axial and vector charges. Since the vector charge is conserved at the boundary it is possible to define a nondivergent chemical potential for it. In fact, since the vector charge is conserved we do not need to source it by a constant V_0 at the boundary. Formally V_0 is just a pure gauge and therefore does not enter any physical observables. It is however a convenient and standard choice to reflect the presence of the chemical potential in the vector

[9] At zero mass this coupling corresponds to the axion term discussed in [34].

sector by choosing $V_0 = \mu$ at the boundary and $V_0 = 0$ at the horizon. In this case the difference of potentials at the boundary and the horizon is the energy needed to introduce one unit of charge into the ensemble. This is a finite quantity and by definition the chemical potential μ.

We want to see how the dependence of the conductivities on the source and/or chemical potential is affected by the mass. Our background consists of the nontrivial temporal components of both gauge fields. It is static and homogeneous in the dual theory so the bulk fields only depend on the radial coordinate (again we work in the axial gauge $A_r = 0, V_r = 0$),

$$\theta(r) = 0, \qquad A = \phi(r)dt, \qquad V = \chi(r)dt. \tag{42}$$

The equations to solve are

$$\phi'' + \frac{3}{r}\phi' - \frac{m^2}{f}\phi = 0, \tag{43}$$

$$\chi'' + \frac{3}{r}\chi' = 0. \tag{44}$$

The boundary conditions for the gauge fields at infinity $\phi(r \to \infty) = \mu_A r^{\Delta}$; $\chi(r \to \infty) = \mu_V$ determine the value of the sources. As usual, (44) has the analytic solution

$$\chi(r) = \mu_V - \frac{\mu_V}{r^2}. \tag{45}$$

Expanding the action to second order in the perturbations and differentiating with respect to the sources we obtain the concrete expressions for the renormalized correlators

$$\langle J_i^V J_j^V \rangle_{\mathrm{Ren}} = 2\eta_{mj} \frac{\delta \tilde{v}_{i(0)}}{\delta v_{m(0)}}, \tag{46}$$

$$\langle J_i^A J_j^A \rangle_{\mathrm{Ren}} = (2 + 2\Delta)\eta_{mj} \frac{\delta \tilde{a}_{i(0)}}{\delta a_{m(0)}}, \tag{47}$$

$$\langle J_i^A J_j^V \rangle_{\mathrm{Ren}} = 2\eta_{mj} \frac{\delta \tilde{v}_{i(0)}}{\delta a_{m(0)}} = (2 + 2\Delta)\eta_{mj} \frac{\delta \tilde{a}_{i(0)}}{\delta v_{m(0)}}. \tag{48}$$

We compute the above correlators numerically. For a detailed explanation see Appendix C 2. In the following we comment on the outcome.

Axial conductivity: the conductivity σ_{55} related to the correlator of two axial currents behaves identically to Sec. II B. Hence, we refer the reader to Fig. 1 and the corresponding discussion.

Chiral separation conductivity: we show the result in Fig. 3. In the plot on the lhs we show the behavior of the conductivity with the vector chemical potential μ. We find that there is no dependence on the source μ_5 for any value of the mass/anomalous dimension. As in the axial conductivity we observe an enhancement with increasing mass. In addition, in the massless limit the conductivity approaches the value $\sigma_{\mathrm{CSE}} \approx 12$ in numerical units. Again this is in agreement with the analytic solution for $m = 0$ [32]. Notice that for this conductivity even in $m = 0$ there is no dependence on the value of the vector field at the horizon (the source).

Chiral magnetic conductivity: the CME vanishes in our background. This is in perfect agreement with all the findings so far. As it happened with the rest of anomalous conductivities, in the massless limit the CMC approaches the value that one obtains for the consistent currents. We believe that the fact that it vanishes even in the massive case

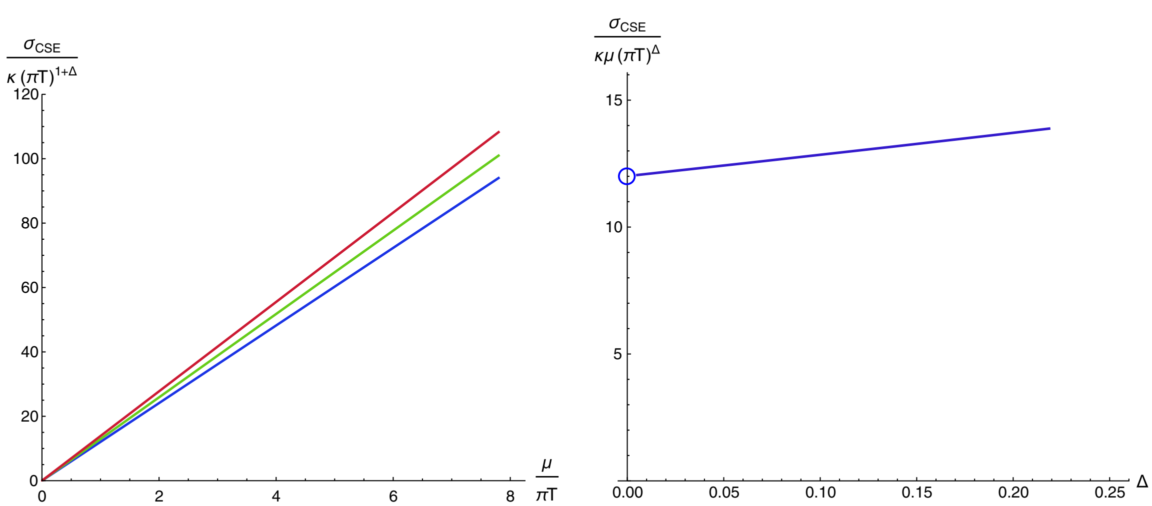

FIG. 3 (color online). Left: Plot of the chiral separation conductivity (CSC) versus the chemical potential μ for: $m^2 = 1/2$ (upper line), $m^2 = 1/3$ (middle line), $m^2 = 0$ (lower line). This conductivity is independent of the axial source μ_5. Right: Plot of the CSC coefficient as a function of the anomalous dimension $\Delta = \sqrt{m^2 + 1} - 1$.

is a consequence of the presence of the source μ_5. The necessary source to achieve a stationary solution for any value of m is such that it forces the anomalous response of J^i_V to B^i_V to vanish, very much as it occurs at zero mass. This does however not imply that the chiral magnetic effect does not exist in this model. As we will see in the following, if we allow the axial charge to fluctuate freely (as opposed to fixing its value via a source term) the chiral magnetic effect is realized. In particular it gives rise to a (generalization of the) chiral magnetic wave and to a negative magnetoresistivity, both of which can be understood as a manifestation of the chiral magnetic effect.

C. The chiral magnetic wave

We start by reviewing the essential features in the case when also the axial current is a canonical dimension three current. The chiral magnetic wave (CMW) is a collective massless excitation that arises form the coupling of vector and axial density waves in the presence of a magnetic field [26]. In addition, this mode can only appear in the spectrum if there is an underlying axial anomaly. The dispersion relation for this mode corresponds to a damped sound wave

$$\omega(k) = \pm v_\chi k - iDk^2, \tag{49}$$

although it is related to transport of electric and axial charge. This mode can be thought of as a combination of the CME and the chiral separation effect (CSE). The vector charge and the axial charge oscillate one into the other giving rise to a propagating wave. This wave mode is present even in the absence of net axial or vector charge. The CMW is expected to play an important role in the experimental confirmations of anomaly induced transport effects. It has been argued in the case of heavy ion collisions that the CMW induces a quadrupole moment in the electric charge distribution of the final state hadrons [36,37].

Let us analyze how this propagating mode is affected by the Stückelberg mechanism in the bulk. Before we proceed to study holographic numerical results we can perform a purely hydrodynamic computation as follows. As we have already shown in the previous section, the presence of the mass term for the axial vector field leads to a nonvanishing, purely imaginary gap for the lowest quasinormal mode. We will include this gap as a decay constant for axial charge. Consider thus a model with axial and vector symmetries. Under the assumption of the existence of a AVV anomaly in the system, the constitutive relations for the current in the presence of a background magnetic field B read

$$j^x_V = \frac{\kappa\rho_A B}{\chi_A} - D\partial_x\rho_V, \qquad j^x_A = \frac{\kappa\rho_V B}{\chi_V} - D\partial_x\rho_A, \tag{50}$$

with D being the diffusion constant and κ the anomaly coefficient. We assume CME and CSE to be present. They are expressed in terms of charge densities and the susceptibilities χ_A, χ_V [26]. On the other hand we have the (non)conservation equations

$$\partial_\mu j^\mu_V = 0, \qquad \partial_\mu j^\mu_A = -\Gamma\rho_A, \tag{51}$$

where $\Gamma(m)$ is the charge dissipation induced by the coupling to the underlying gauge anomaly.[10] From here we get the coupled equations

$$\omega\rho_V + \frac{k\kappa\rho_A B}{\chi_A} + ik^2 D\rho_V = 0, \tag{52}$$

$$(\omega + i\Gamma)\rho_A + \frac{k\kappa\rho_V B}{\chi_V} + ik^2 D\rho_A = 0. \tag{53}$$

Assuming now that the equations are linearly dependent we get

$$\omega_\pm = -\frac{i\Gamma}{2} - iDk^2 \pm \sqrt{\frac{B^2k^2\kappa^2}{\chi_A\chi_V} - \frac{\Gamma^2}{4}}. \tag{54}$$

The mode associated to ω_+ is massless and expected to arise due to the fact that the vector symmetry is conserved. It basically represents the diffusion law for the conserved vector charge. The ω_- mode is gapped, i.e. $\omega_-(k=0) = -i\Gamma$. Both combine at a critical value for the momentum $k_C(\Gamma, B, \chi_{(V,A)})$ that makes the term inside the square root vanish.

(i) If $4B^2\kappa^2k^2 > \chi_A\chi_V\Gamma^2$ the square root is real and we obtain a contribution linear in k (which boils down to the well-known linear dispersion relation of the chiral magnetic wave in the limit $\Gamma = 0$).

(ii) If $4B^2\kappa^2k^2 < \chi_A\chi_V\Gamma^2$ the square root contribution is completely contained in the imaginary part of the frequency.

In summary, we see that

$$k_C = \frac{\chi_A\chi_V\Gamma^2}{4B^2\kappa^2}. \tag{55}$$

For $k > k_C$ we get a propagating mode whose dispersion relation approximates the one of the CMW.[11] On the contrary, if $k < k_C$, there is no real part of the frequency (i.e. no chiral magnetic wave); one of the modes remains massless and the other develops a gap Γ.

With this phenomenological model in mind we look for these modes in our holographic model. In order to find the CMW we look at the QNM spectrum in presence of a constant magnetic field B in the z-direction. Since the CMW is present at zero axial and vector charge densities,

[10] We also assume vanishing external electric field and therefore there is no $\vec{E}.\vec{B}$ term present in the equation for the axial current.

[11] Observe that for $k \gg k_c$ the slope $\Re(\omega)/k$ is the same as in the case $\Gamma = 0$.

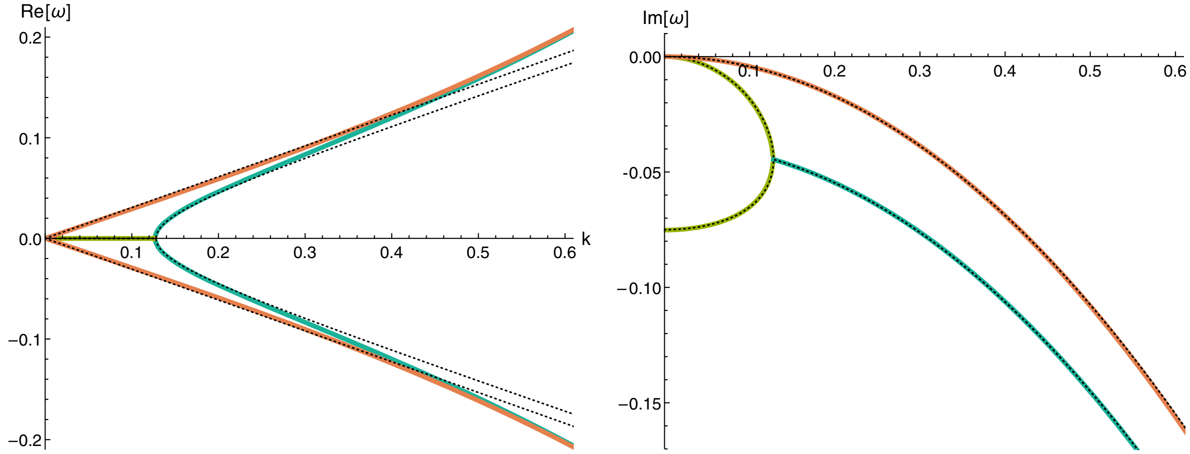

FIG. 4 (color online). Real and imaginary parts of the frequency of the lowest QNM as a function of k. Solid lines correspond to numerical data with $\kappa B = 0.05$ and two different values of the mass: $m^2 = 0$ (left: highest and lowest lines, right: highest line) and $m^2 = 0.15$ ($\Delta = 0.08$) (left, right: others). The massive case is given two different colors to highlight the regimes $k < k_c$ (green) and $k > k_c$ (blue). Dashed lines correspond to the analytic formula (54). The massless case shows the behavior of the CMW. With a nonvanishing mass such a behavior is recovered for $k > k_c$.

we do not switch on any chemical potential in the background. The only nonzero field in our ansatz for the background is $A_x = By$. It is easy to check that such an ansatz satisfies the equations of motion trivially. Subsequently we study the perturbations, with momentum k aligned with the magnetic field. Applying the determinant method of [33] we are able to obtain the dispersion relation of the CMW as depicted in Fig. 4; we show the dispersion relation of the lowest QNMs for both $m = 0$ (orange) and $m > 0$ (green, blue) in presence of B. On top of this we plot (dashed lines) the predictions of the phenomenological model (54).

The numerical results are in perfect agreement with the analytic analysis. We observe the appearance of a critical momentum k_C, induced by the mass term. Below this momentum the chiral magnetic wave is not really wavelike (i.e. $\Re[\omega(k)] = 0$ for $k < k_C$); the two modes decouple, giving rise to a diffusive mode and gapped purely imaginary mode. Such a spectrum is what one would expect to find in the model if there was no CMW, that is, the unbroken vector charge exhibits diffusive behavior, with a massless mode protected by the symmetry, whereas the analogous mode for broken axial U(1) symmetry develops a gap Γ. This gap is proportional to the mass and gets diminished the stronger the magnetic field. Above the critical momentum the two modes fuse again, giving rise to the expected behavior of the CMW. Since the CMW is a propagating oscillation between axial and vector charge we see that for small momentum the decay of the axial charge dominates, i.e. the axial charge decays before it can oscillate back into vector charge. The strength of the mixing of the charges is proportional to the momentum. This mixing becomes large enough and the oscillation fast enough to allow the buildup of a propagating (damped) wave at large enough momentum.

We show the behavior of the gap Γ with the mass for different values of the magnetic field in Fig. 5. We find that the gap goes as $\sim m^2$ and that it is inversely proportional to the strength of the magnetic field.

D. Negative magnetoresistivity

As a last step we study how the electric conductivity is affected by the mass. In the absence of mass the CMW induces perfect (i.e. infinite) DC conductivities for both the electric and the axial conductivities along the magnetic field. However, from the QNM analysis of the previous section we know that this cannot hold anymore. We expect a finite conductivity but with a strong Drude-like peak at

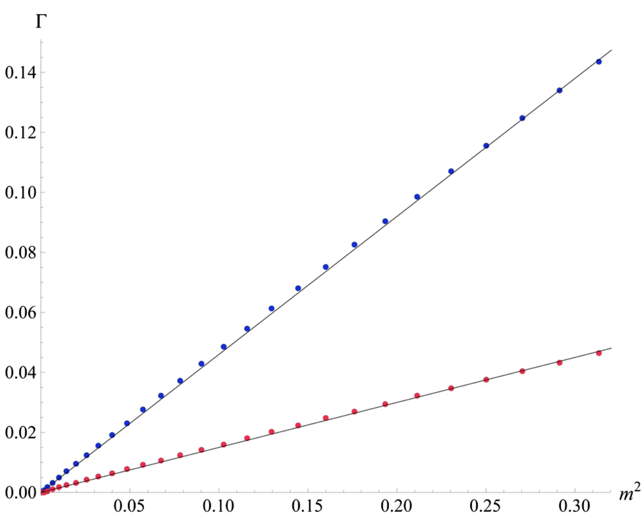

FIG. 5 (color online). The gap Γ versus m^2 for different values of the magnetic field $\kappa B = 0.01$ (upper line) and $\kappa B = 0.5$ (lower line). Black lines correspond to linear fits.

zero frequency. As we will see this is indeed what is happening.

In order to analyze the longitudinal conductivity along the magnetic field we switch on perturbations on top of the background that we used to study the CMW, namely, an external magnetic field pointing in the z-direction. The electric conductivity along the magnetic field can be extracted from the correlator

$$\sigma_{||} = \lim_{\omega\to 0} \frac{i}{2\omega} \langle J^z J^z \rangle(\omega, k = 0). \tag{56}$$

Since this conductivity is obtained at zero momentum we can assume spatial homogeneity for the perturbations. The coupled equations of motion can be found in Appendix D. The analysis of the two-point function reveals that for this configuration of the background the correlator we want to compute has the usual expression

$$\langle J^z J^z \rangle_{\mathrm{Ren}} = \lim_{r\to\infty} r^3 \partial_r \mathbb{H} + \omega^2 \log(r). \tag{57}$$

We solve the equations numerically with infalling boundary conditions and build the bulk-to boundary propagator (BBP). Our results are shown in Figs. 6–8.

The well-known Kramers-Kronig relations imply that a pole in the imaginary part of the conductivity at zero frequency signals the presence of a delta function peak in the real part, i.e. and infinite DC conductivity. As soon as we turn the mass on, we observe that the DC conductivity is not a delta function anymore (see Figs. 6–7). This fact has important consequences in Ohm's law for an anomalous system with an explicit breaking term. It has been first pointed out that the axial anomaly induced a large DC conductivity in a magnetic field (or a negative magnetoresistivity) in [38]. More recent studies of this phenomenon are [39,40]. In these studies Weyl fermions of opposite chirality appear as the effective electronic excitations at low energies in a crystal (Weyl semimetal). The associated axial symmetry is however only an approximate one since the electronic quasiparticles can be scattered from one Weyl cone into another. The associated scattering rate is called the intervalley scattering rate τ_i. It turns out that the conductivity in these Weyl semimetals is indeed proportional to the intervalley scattering rate. Our findings are in complete analogy; the inverse of the gap in Fig. 5 plays the role of the intervalley scattering time leading to a finite but strongly peaked DC magnetoconductivity.

By numerical analysis we find the dependence of the DC conductivity on m, κ and B. Results are shown in Fig. 8. We can approximate it by

$$D \approx 72 \frac{\kappa^2 B^2}{m^2}. \tag{58}$$

Since in Fig. 5 we found that the gap is proportional to m^2 we indeed see that the DC conductivity scales linearly with the inverse of the gap as expected. We also find that it depends quadratically on the magnetic field. Again this is the expected result at least for small magnetic fields. For larger magnetic fields the weak coupling analysis shows however a linear dependence on the magnetic field that can be traced back to the fact that all fermionic quasiparticles are in the lowest Landau level.

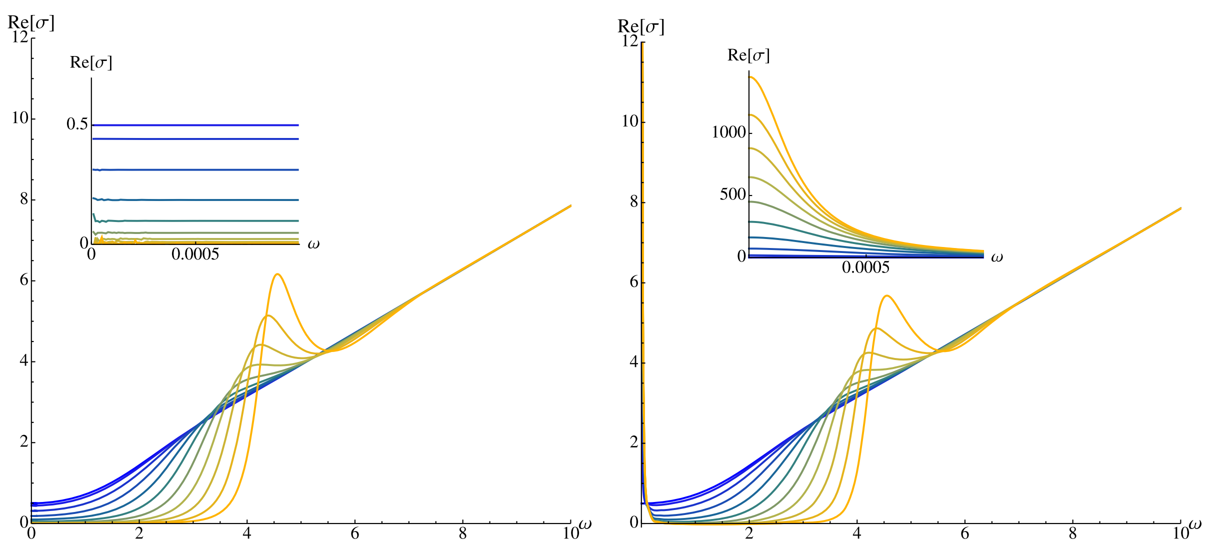

FIG. 6 (color online). Real part of the conductivity in the longitudinal sector for $\Delta = 0$ (left) and $\Delta = 0.1$ (right). Different colors correspond to different values of the magnetic field B, from $\kappa B = 0$ (blue) to $\kappa B = 0.5$ (yellow). The behavior of the conductivity at high frequencies is qualitatively the same for both values of Δ. The conductivity shows a Drude peak as soon as the mass (Δ) is switched on whereas it has delta function peak centered at $\omega = 0$ for zero mass.

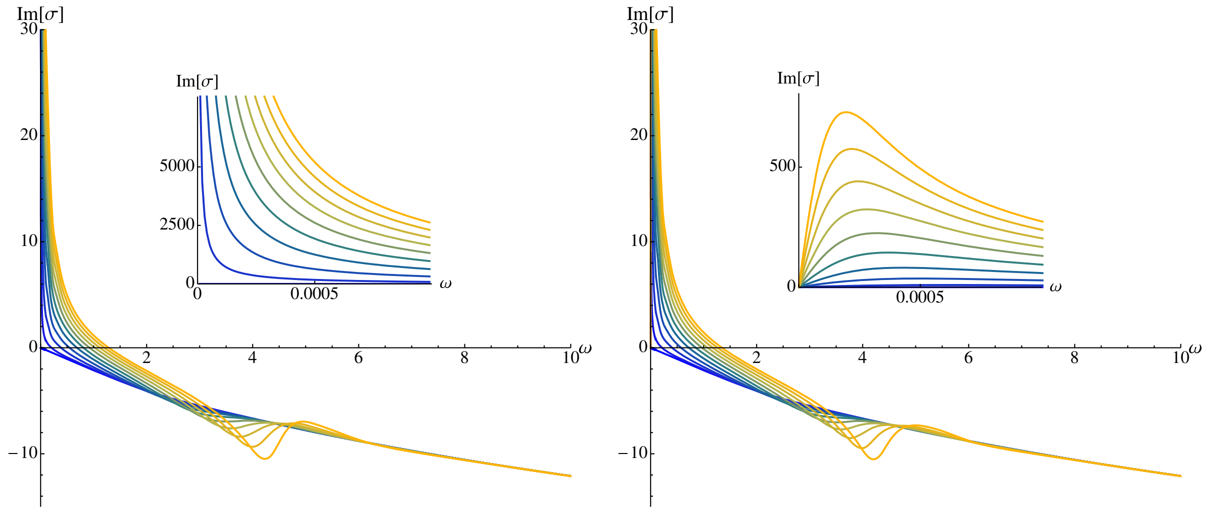

FIG. 7 (color online). Imaginary part of the conductivity in the longitudinal sector for $\Delta = 0$ (left) and $\Delta = 0.1$ (right). Different colors correspond to different values of the magnetic field B, from $\kappa B = 0$ (blue) to $\kappa B = 0.5$ (yellow). In agreement with the real part in Fig. 6 the zero frequency behavior shows a pole only when the mass is absent, signaling the presence of a delta function in the real part. As soon as the mass is switched on $\Im[\sigma]$ vanishes at the origin.

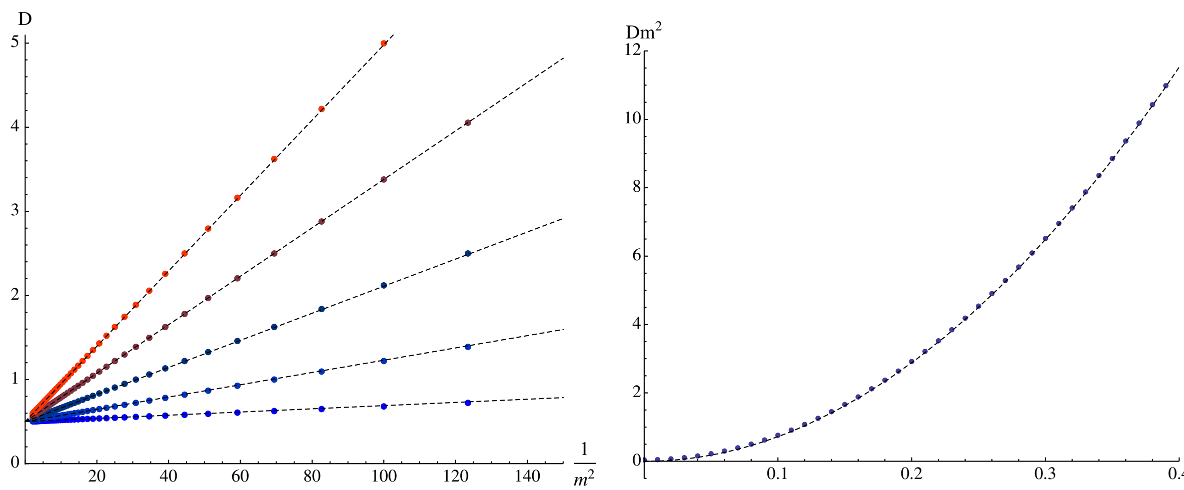

FIG. 8 (color online). Left: We show the value of the highest point of the Drude peak (D) against $\frac{1}{m^2}$ for several values of κB from $\kappa B = 0.005$ (lower line) to $\kappa B = 0.25$ (upper line). Dashed lines correspond to linear fits. Right: Dependence of the slopes in the lhs plot as a function of κB. The dashed line corresponds to a quadratic fit; we find the coefficient to be ≈ 72.

We found that our results show a kind of instability for too large magnetic field such that we were not able to see this cross over to linear behavior. This might be an artifact of the probe limit or a genuine instability of the theory at high magnetic fields (similar to the Chern-Simons term induced instabilities in an electric field found in [41]). We leave this issue for further investigation.

Finally we note that we have checked that the sum rule is fulfilled for several values of the mass and the magnetic field. This sum rule takes the form $\frac{d}{dB}\int \Re(\sigma(\omega))d\omega = 0$. The sum rule implies that the peak is built up by shifting spectral weight from higher frequencies towards $\omega = 0$. In fact this is precisely what can be seen in Fig. 6 where it is evident that the region of intermediate frequencies gets depleted and correspondingly a gap in the magneto-optical conductivity opens up as the magnetic field strength is increased. Note that this gap is present still in the region where we found quadratic scaling (58).

IV. CONCLUSIONS

We have studied anomaly related transport phenomena in a bottom-up holographic model with massive vector fields and Stückelberg axion. One of our motivations was that the dynamical part of the axial anomaly, i.e. the gluonic contribution, is dual to the dynamics of axions in holography. Its precisely this axion that can be used in the bulk Stückelberg mechanism to give mass to the bulk gauge field. The operator dual to this massive gauge field is a nonconserved current and this nonconservation is manifest in the fact that we did not find a constraint on the divergence of the current. Throughout the paper, we have restricted ourselves to the probe approximation.

Equipped with the above model for an anomalous massive $U(1)$ gauge field, in Sec. II we have studied carefully the form of the current one-point function, showing that the well-known Bardeen-Zumino polynomial does not exist if the mass $m \neq 0$. The resulting form of the (holographically renormalized) current tends to the consistent definition in the massless limit. Moreover, as described by (21) the divergence of such a current is not constrained. Moving to the two-point functions, the anomalous conductivity σ_{55} has been computed using its definition via a Kubo formula. We find that its value corresponds to the one associated to the consistent current in the zero mass limit. We also showed that the QNM spectrum has a gap in contrast to the massless case in which there exists a hydrodynamic diffusion mode. We stress that the nonconserved current J^i is not a hydrodynamic degree of freedom because of this gap. Furthermore the parameter μ_5 is not a chemical potential but a coupling constant. Nevertheless we think it would be an interesting exercise to work out constitutive relations for the nonconserved current extending the well-established methods of the fluid/gravity correspondence [42] to this case.

In Sec. III we implemented the interplay between nonconserved axial and conserved vector currents. We also studied a wider set of anomalous conductivities using Kubo formulas (39)–(41) We found that (as expected) the axial conductivity is identical to the case with only one axial gauge field. The chiral separation conductivity is independent of the source μ_5, behaves linearly with μ and increases with the mass, as depicted in Fig. 3. Finally, the chiral magnetic conductivity vanishes for all the values of m that we have studied; we interpret this fact as an effect of the source that ensures that the background is time independent. As $m \to 0$, all the conductivities approach the value corresponding to consistent definition of the currents. Subsection III C is devoted to the study of the CMW [26] in the presence of mass. First, we perform an analytic analysis of the modes in a phenomenological model that implements the axial nonconservation via a relaxation term [see Eq. (51)]. This model predicts that a propagating wavelike mode can build up only for large enough momentum. Indeed we find from our quasinormal mode analysis that the model can be fitted very well to the QNM spectrum and that indeed a propagating chiral magnetic wave is absent for small momenta.

Finally we have also studied the negative magnetoresistivity and showed that a sum rule holds for the magneto-optical conductivity. The strength of the DC conductivity is proportional to the square of the magnetic field and inverse proportional to the gap. This is in agreement with weak coupling considerations for small magnetic fields and an intervalley scattering relaxation time for axial charge. Unfortunately we were not able to see the expected crossover to linear behavior in the magnetic field because our numerics indicated a possible instability at large B-field. Whether this is an artifact of the probe limit (which assumes negligible backreaction of the gauge field on the geometry) or a genuine instability we leave to further investigation.

This brings us to possible generalizations of the present work. First we would like to mention that the usage of Stückelberg axions in the context of holographic studies of anomaly induced transport has recently also been suggested in [24]. Our model is a first step in this direction and following [24] one might improve it by giving up conformal symmetry by working directly with the model of [23] or a suitable simplification thereof. Another rather straightforward generalization would be to take the backreaction onto the geometry into account. This opens the way to study also the generalizations of the chiral vortical effect and one could also include the mixed gauge gravitational anomaly. As we emphasized the nonconserved currents do not strictly belong to the set of hydrodynamic variables. But one can easily imagine a situation in which the gap of the lowest quasinormal mode in the massive vector sector is much smaller than the separation to the higher QNMs. In this case it would make sense to include these modes in the gauge/gravity correspondence and work out the constitutive relations. A very interesting question arises in connection to the possibility of defining covariant or consistent currents in the massless case. We found that the nonconserved current goes over into the consistent current in the zero mass limit. Is is possible to generalize the notion of covariant current to the massive case? It is also known that the consistent currents are not unique but can be redefined by adding finite counterterms (the Bardeen counterterms). It is precisely this choice of counterterms that allows us to shift the anomaly completely into the axial sector (even when a mixed gravitational anomaly is present). It would certainly be interesting to include the gravitational anomaly and to see how the Bardeen-Zumino terms arise in the zero mass limit.

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APPENDIX A: HOLOGRAPHIC RENORMALIZATION

1. U(1) model

In order to renormalize the theory shown in (10) we follow the procedure in [43]. Within this approach the renormalization procedure consists of an expansion of the canonical momenta and the on-shell action λ in eigenfunctions of the dilatation operator. This operator can be obtained taking the asymptotic leading term of the radial derivative

$$\partial_r = \int d^d x\left(\dot\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + \dot A_i\frac{\delta}{\delta A_i} + \dot\theta\frac{\delta}{\delta\theta}\right) \sim \int d^d x\left(2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + \Delta A_i\frac{\delta}{\delta A_i} + O(e^{-r})\right), \quad (A1)$$

$$\delta_D = \int d^d x\left(2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + \Delta A_i\frac{\delta}{\delta A_i}\right). \quad (A2)$$

Notice that this operator is not gauge invariant. Nevertheless $S_{\mathrm{C.T}}$. must be gauge invariant since the bulk Lagrangian is invariant too. Therefore $S_{\mathrm{C.T}}$. must be expressible as a functional of $B_i \equiv A_i - \partial_i\theta$. We will see that this is indeed the case even though we expand the on-shell action in eigenfunctions of the (non–gauge invariant) dilatation operator δ_D. We choose the axial gauge $A_r = 0$. Recall that

$$\Delta(\Delta+2) = m^2. \quad (A3)$$

Our notation for the eigenfunctions of the dilatation operator reads

$$\delta_D X_{(a)} = -aX_{(a)}, \qquad \delta_D X_{(4)} = -4X_{(4)} - 2\tilde X_{(4)}. \quad (A4)$$

All our results were obtained in the probe limit and therefore, for simplicity, we adapt the renormalization procedure to this limit. This implies that we will use the e.o.m. for the fields, instead of the Hamiltonian constraint in Einstein equations, to determine the eigenfunctions of the dilatation operator the canonical momenta are expanded in. In addition we set the extrinsic curvature $K_{ij} \equiv \dot\gamma_{ij} = 2\gamma_{ij}$, which in our setup is enough for the boundary analysis. The matter e.o.m., written in terms of $E_i \equiv \dot A_i$ and $\Pi \equiv \dot\theta$, are

$$\dot E_i + 2E_i - m^2(A_i - \partial_i\theta) + \partial^j F_{ji} + 2\kappa\epsilon^{irjkl}E_j F_{kl} = 0, \quad (A5)$$

$$\dot\Pi + 4\Pi - \partial^i(A_i - \partial_i\theta) = 0, \quad (A6)$$

$$\Pi = \frac{1}{m^2}(\partial^i E_i - \kappa\epsilon^{rijkl}F_{ij}F_{kl}). \quad (A7)$$

With (A2) and the e.o.m. we can determine the explicit form of the different terms in the expansions

$$E_i = E_{i(-\Delta)} + E_{i(0)} + E_{i(2-2\Delta)} + E_{i(2-\Delta)} + E_{i(2)} + \cdots, \quad (A8)$$

$$\Pi = \Pi_{i(2-\Delta)} + \Pi_{i(2)} + \cdots, \quad (A9)$$

$$E_{i(-\Delta)} = \Delta A_i, \quad (A10)$$

$$E_{i(0)} = -\Delta\partial_i\theta, \quad (A11)$$

$$\Pi_{(2-\Delta)} = \frac{1}{(\Delta+2)}\partial_i A^i, \quad (A12)$$

$$\Pi_{(2)} = \frac{-1}{(\Delta+2)}\Box\theta. \quad (A13)$$

Other terms like $E_{i(2-2\Delta)}$ are nonzero but as we will see they do not contribute to the counterterms. We can determine the expressions for the higher order operators needed to expand the radial derivative:

$$\partial_r = \delta_D + \delta_{(\Delta)} + \delta_{(2-2\Delta)} + \delta_{(2-\Delta)} + \delta_{(2)} + \delta_{(2+\Delta)} + \cdots, \quad (A14)$$

$$\delta_{(\Delta)} = \int d^d x' E_{i(0)}(x')\frac{\delta}{\delta A_i(x')}, \quad (A15)$$

$$\delta_{(2-\Delta)} = \int d^d x'\left(E_{i(2-2\Delta)}(x')\frac{\delta}{\delta A_i(x')} + \Pi_{(2-\Delta)}(x')\frac{\delta}{\delta\theta(x')}\right), \quad (A16)$$

$$\delta_{(2)} = \int d^d x'\left(E_{i(2-\Delta)}(x')\frac{\delta}{\delta A_i(x')} + \Pi_{(2)}(x')\frac{\delta}{\delta\theta(x')}\right), \quad (A17)$$

$$\delta_{(2+\Delta)} = \int d^d x' E_{i(2)}(x')\frac{\delta}{\delta A_i(x')}. \quad (A18)$$

Once we have these we just need the equation for the on-shell action

$$\dot\lambda + \lambda - \mathcal{L}_m = 0, \quad (A19)$$

$$\dot\lambda + 4\lambda + \frac{1}{2}E_i E^i + \frac{m^2}{2}\Pi^2 + \frac{m^2}{2}(A_i A^i - 2A_i\partial^i\theta + \partial_i\theta\partial^i\theta) + \frac{1}{4}F_{ij}F^{ij} + \frac{4\kappa}{3}(A_i - \partial_i\theta)E_j F_{kl}\epsilon^{irjkl} - \frac{\kappa}{3}\Pi F_{ij}F_{kl}\epsilon^{irjkl} = 0. \quad (A20)$$

Using this equation and the expansion of the radial derivative we can determine the terms of the eigenfunction expansion of the on-shell action

$$\lambda = \lambda_{(0)} + \lambda_{(2-2\Delta)} + \lambda_{(2-\Delta)} + \lambda_{(2)} + \lambda_{(4-4\Delta)} + \lambda_{(4-3\Delta)} + \lambda_{(4-2\Delta)} + \lambda_{(4-\Delta)} + \lambda_{(4)} + \tilde{\lambda}_{(4)} \log e^{2r} + \cdots. \tag{A21}$$

It is important to remark that depending on the value of $0 \leq \Delta \leq 1$ new terms may appear in this expansion. For example, the next possible term in this expansion is $\lambda_{(6-6\Delta)}$. Therefore, as in the rest of the paper, we restrict our analysis to

$$6 - 6\Delta > 4 \to \Delta < \frac{1}{3}. \tag{A22}$$

Furthermore, for a large enough mass ($\Delta = 1$) the number of possible counterterms becomes infinite. This is to be expected since for such a value of the mass the operator dual to the gauge field becomes marginal.

We are now ready to proceed solving (A20) order by order in dilatation weight

$$\lambda_{(0)} = 0, \tag{A23}$$

$$\lambda_{(2-2\Delta)} = \frac{-\Delta}{2} A_i A^i, \tag{A24}$$

$$\lambda_{(2-\Delta)} = \Delta \partial_i \theta A^i, \tag{A25}$$

$$\lambda_{(2)} = \frac{-\Delta}{2} \partial_i \theta \partial^i \theta. \tag{A26}$$

At this point one can see that these first terms of the on shell (O.S.) action expansion can be rearranged in terms of the B_i field:

$$\lambda_{(2-2\Delta)} + \lambda_{(2-\Delta)} + \lambda_{(2)} = -\frac{\Delta}{2} B_i B^i. \tag{A27}$$

It is a nice check to find all the terms explicitly and then rearrange them like this although, as mentioned before, this is to be expected. Moreover we can use this in our advantage: once one obtains a counterterm which is only proportional to A_i the following terms can be determined by just imposing that λ has to be gauge invariant.

$$\lambda_{(4-4\Delta)} = 0, \qquad \lambda_{(4-3\Delta)} = 0. \tag{A28}$$

Let us analyze the following term with some detail:

$$\dot{\lambda}|_{(4-2\Delta)} + 4\lambda_{(4-2\Delta)} + E_{i(-\Delta)} E^i_{(2-\Delta)} + \frac{m^2}{2} \Pi^2_{(2-\Delta)} + \frac{1}{4} F_{ij} F^{ij} + \frac{4\kappa}{3} \epsilon^{rijkl} F_{jk} (E_{i(0)} A_l - E_{i(-\Delta)} \partial_l \theta) = 0, \tag{A29}$$

$$(\delta_D + 4)\lambda_{(4-2\Delta)} + \delta_{(2-\Delta)} \lambda_{(2-\Delta)} + \delta_{(2)} \lambda_{(2-2\Delta)} + E_{i(-\Delta)} E^i_{(2-\Delta)} + \frac{m^2}{2} \Pi^2_{(2-\Delta)} + \frac{1}{4} F_{ij} F^{ij} = 0, \tag{A30}$$

$$\lambda_{(4-2\Delta)} = \frac{1}{4(\Delta + 2)} \partial_i A^i \partial^j A_j - \frac{1}{8\Delta} F_{ij} F^{ij}. \tag{A31}$$

It is remarkable that the term proportional to κ vanishes due to the contraction of a symmetric $(E_{i(0)} A_l - E_{i(-\Delta)} \partial_l \theta)$ and an antisymmetric ϵ^{rijkl} tensor. Here we see the importance of the coupling of the Stückelberg filed to the Chern-Simons term. If we had not added it, at this point we would have found an extra counterterm of the form $-\theta \wedge F \wedge F$. In this expression we have neglected total derivatives. From this last equation we can infer the following two orders by imposing gauge invariance. So in terms of the gauge invariant field B_i the counterterm reads

$$\lambda_{4-2\Delta} + \lambda_{4-\Delta} + \lambda_4^* = \frac{1}{4(\Delta + 2)} \partial_i B^i \partial_j B^j - \frac{1}{8\Delta} F_{ij} F^{ij}. \tag{A32}$$

Note that we cannot determine $\lambda_{(4)}$ with just the boundary analysis. λ_4^* is just a part of λ_4 which is imposed by gauge invariance and can be obtained from the asymptotics.

We only lack the $\sim$log term, which is obtained by evaluating the equation to fourth order

$$\tilde{\lambda}_{(4)} = 0. \tag{A33}$$

Thus, the $S_{\rm CT}$ reads

$$S_{\rm CT} = \int_\partial d^d x \sqrt{-\gamma} \left(\frac{\Delta}{2} B_i B^i - \frac{1}{4(\Delta + 2)} \partial_i B^i \partial_j B^j + \frac{1}{8\Delta} F_{ij} F^{ij} \right). \tag{A34}$$

2. U(1) × U(1) model

Few things change if we introduce a second gauge field (nonmassive, nonanomalous in the boundary). The asymptotic behavior remains unchanged. Specially, the vector gauge field behaves as it usually does and thus it does not contribute to the dilatation operator.

$$\delta_D = \int d^d x \left(2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} + \Delta A_i \frac{\delta}{\delta A_i} + O(e^{-r}) \right). \tag{A35}$$

The equation of the O.S. action has to be modified:

$$\dot{\lambda} + 4\lambda + \frac{1}{2} E_i E^i + \frac{1}{2} \Sigma_i \Sigma^i + \frac{m^2}{2} \Pi^2 + \frac{m^2}{2} (A_i A^i - 2A_i \partial^i \theta + \partial_i \theta \partial^i \theta) + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{4} H_{ij} H^{ij} + 2\kappa (A_i - \partial_i \theta)(E_j F_{kl} + 3\Sigma_j H_{kl}) \epsilon^{irjkl} - \frac{\kappa}{2} \Pi (F_{ij} F_{kl} + 3H_{ij} H_{kl}) \epsilon^{irjkl} = 0, \tag{A36}$$

where Σ and H_{ij} are the momentum[12] and the field strength of the vector gauge field. It is not difficult to realize that the only term proportional to V_i that will contribute to the divergent part of λ is the kinetic term $H_{ij}H^{ij}$. Since this is of order 4, it will only contribute to the logarithmic term and therefore

$$S_{\mathrm{CT}} = \int_\partial d^d x \sqrt{-\gamma} \left(\frac{\Delta}{2} B_i B^i - \frac{1}{4(\Delta+2)} \partial_i B^i \partial_j B^j + \frac{1}{8\Delta} F_{ij} F^{ij} + \frac{1}{8} H_{ij} H^{ij} \log e^{2r} \right). \tag{A37}$$

APPENDIX B: CORRELATORS IN THE U(1) MODEL

1. One-point function

In order to derive the one-point function of the (non-conserved) vector operator dual to the gauge field we write fields as background plus perturbations,

$$\mathcal{A}_\mu = A_\mu + a_\mu, \qquad \theta = \theta + \phi; \tag{B1}$$

We expand the renormalized action to first order in the perturbations

$$\begin{aligned} S^{(1)}_{\mathcal{R}} = & \int dr d^4 x \sqrt{-g} [a_\mu (\nabla_\nu F^{\nu\mu} - m^2 (A^\mu - \partial^\mu \theta) + \kappa \epsilon^{\mu\alpha\beta\gamma\rho} F_{\alpha\beta} F_{\gamma\rho}) - \phi \nabla_\mu (A^\mu - \partial^\mu \theta)] \\ & + \int_\partial d^4 x \sqrt{-g} \left[a_i \left(F^{ir} + \frac{4}{3} \kappa (A_j - \partial_j \theta) F_{kl} \epsilon^{rijkl} \right) - \phi (F_{ij} F_{kl} \epsilon^{rijkl} + m^2 (A^r - \partial^r \theta)) \right] \\ & + \int_\partial d^4 x \sqrt{-\gamma} a_i \left(\Delta (A^i - \partial^i \theta) + \frac{1}{2(\Delta+2)} \partial^i (\partial_j A^j - \Box \theta) - \frac{1}{2\Delta} \partial_j F^{ji} \right) \\ & + \int_\partial d^4 x \sqrt{-\gamma} \phi \left(\Delta (\partial_j A^j - \Box \theta) + \frac{1}{2(\Delta+2)} \Box (\partial_j A^j - \Box \theta) \right). \end{aligned} \tag{B2}$$

The bulk integral contains the e.o.m. for the background fields. The second line shows the boundary term that arises from the unrenormalized action S whereas the third and fourth lines contain the expansion of the counterterm action S_{CT}. By inspection of the equations of motion one finds that the most general asymptotic expansion of the fields reads

$$A_\mu \sim \sum_{i=0}^{\infty} A_{\mu(i)} r^{\Delta - i} + \sum_{i=0}^{\infty} \tilde{A}_{\mu(i)} r^{-2-\Delta-i} + \sum_{i=0}^{\infty} \tilde{\theta}_{\mu(i)} r^{-i} + \sum_{n>1, i\geq 2(n-1)}^{\infty} \omega_{\mu(n,i)} r^{n\Delta - i} + \sum_{n>1, i\geq 3n}^{\infty} \tilde{\omega}_{\mu(n,i)} r^{-n\Delta - i} + \sum_{i\geq 4} A_{L(i)} r^{(-i)} \log(r), \tag{B3}$$

$$\theta \sim \sum_i \theta_{(i)} r^{(-i)} + \sum_{n\geq 1, i\geq 2n} \tilde{\Psi}_{(n,i)} r^{(n\Delta - i)} + \sum_{n\geq 1, i\geq 3n+2} \tilde{\Psi}_{(-n,i)} r^{(-n\Delta - i)} + \sum_{i\geq 4} \theta_{L(i)} r^{(-i)} \log(r), \tag{B4}$$

[12]As we did with the axial field we define $\Sigma_i \equiv \dot{V}_i$.

with $\Delta = \sqrt{1+m} - 1$ that is bounded to be $\Delta < 1$. The coefficient of the leading (non-normalizable mode) term $A_{(0)x}$ is to be identified with the source of the dual operator. $\tilde{A}_{(0)x}$ is the coefficient of the normalizable mode. $\omega_{(n,i)}, \tilde{\omega}_{(n,i)}$ arise due to the nonlinearities of the e.o.m. and can be expressed as functionals of the sources of the other components of the gauge field $A_{(0)y\neq x}$. Finally, the $\tilde{\theta}$ and A_L terms arise from the coupling to the Stückelberg field and are functionals of the source of θ; the logarithmic terms are subleading with respect to the normalizable mode, contrary to what happens in the massless case. In the expansion for θ we find the $\theta_{(i)}$ coefficients that contain both the non-normalizable $i = 0$ and the normalizable $i = 4$ mode. The $\Psi, \tilde{\Psi}$ terms appear due to the coupling to the gauge field.

From the boundary term of the O.S. action one can obtain the one-point function of the dual operator J^i.

As usual, it is convenient to group all the fields in a vector of the appropriately normalized fields[13] $\psi = (r^{-\Delta} a_i, \phi)$ and express them as the (matrix valued) BBP times a vector $\psi_{(0)}$ made of the value of the sources

$$\psi_I(r) = F_{IJ}(r)\psi_{J(0)}, \qquad F(\Lambda) = \mathbb{I}. \tag{B5}$$

Moreover, it will be useful to separate the BBP matrix in a rectangular matrix $\mathbb{F}$ and a vector $\mathbb{G}$ such that

$$F = \begin{pmatrix} \mathbb{F} \\ --- \\ \mathbb{G} \end{pmatrix}, \qquad a_I = r^{\Delta}\mathbb{F}_{IJ}\psi_{J(0)}, \qquad \phi = \mathbb{G}_J\psi_{J(0)}. \tag{B6}$$

In terms of these the expectation value of the current reads

$$\begin{aligned}\langle J^m\rangle = {} & \lim_{r\to\infty}\sqrt{-g}\left(r^{\Delta}\mathbb{F}_{im}\left(F^{ir} + \frac{4\kappa}{3}\epsilon^{ijkl}(A_j - \partial_j\theta)F_{kl}\right) - \mathbb{G}_m(F_{ij}F_{kl}\epsilon^{rijkl} + m^2(A^r - \partial^r\theta)\right)\\ & + \lim_{r\to\infty}\sqrt{-\gamma}\, r^{\Delta}\mathbb{F}_{im}\left(\Delta(A^i - \partial^i\theta) + \frac{1}{2(\Delta+2)}\partial^i(\partial_j A^j - \Box\theta) - \frac{1}{2\Delta}\partial_j F^{ji}\right)\\ & + \lim_{r\to\infty}\sqrt{-\gamma}\,\mathbb{G}_m\left(\Delta(\partial_j A^j - \Box\theta) + \frac{1}{2(\Delta+2)}\Box(\partial_j A^j - \Box\theta)\right).\end{aligned} \tag{B7}$$

The above expression is quite messy and needs some inspection. In the massless case [32] all terms proportional to $\mathbb{F}_{i\neq m}, \mathbb{G}_m$ vanish in the $r\to\infty$ limit and therefore are not explicitly written in the literature. When the mass is present, however, all terms in the expression are divergent. This is easy to check given the expansions (B3). To have a better understanding of the properties of the current it is convenient to collect the terms that do not contain finite contributions as shown in the main text (18).

2. Two-point functions

Equation (28) is the correct expression for the correlator $\langle J_y J_z\rangle$. However, one usually does not have an analytic solution for the e.o.m. and therefore one has to construct the BBP numerically. This implies that we are interested in (28) expressed as a linear combination of the BBP and its derivatives. In principle one can derive this combination directly from the O.S. action to second order in perturbations but this might be rather tedious. A simpler strategy is to look at the asymptotic expansions for the perturbations and then invert the series to find the expression of the normalizable mode as a combination of $\mathbb{F}, \dot{\mathbb{F}}$. The anomalous conductivity (23) is a good opportunity to perform an explicit example.

First we switch on perturbations for all fields with momentum k aligned to the x direction and frequency ω: $\delta\theta = \sigma(r)e^{-i\omega t + ikx}$ and $\delta A_\mu = a_\mu(r)e^{-i\omega t+ikx}$. The linearized e.o.m. for these perturbations naturally separate in decoupled sectors; since we are interested in the correlator $\langle J_y J_z\rangle$ we just look at

$$f a_y'' + \left(f' + \frac{f}{r}\right)a_y' + \left(-m^2 + \frac{\omega^2}{f} - \frac{k^2}{r^2}\right)a_y - \frac{8ik\kappa\phi' a_z}{r} = 0 \tag{B8}$$

$$f a_z'' + \left(f' + \frac{f}{r}\right)a_z' + \left(-m^2 + \frac{\omega^2}{f} - \frac{k^2}{r^2}\right)a_z + \frac{8ik\kappa\phi' a_y}{r} = 0, \tag{B9}$$

which decouple from the other equations.

The asymptotic analysis of Eqs. (B8)–(B9) reveals that close to the boundary the perturbations behave as

[13]Since the gauge field diverges at the boundary precisely as $\sim r^{\Delta}$, this choice for the normalization allows us to have a finite BBP and to collect the sources of the dual theory in $\psi_{(0)}$.

$$a_i(r\to\infty)\sim a_{(0)i}\left(r^{\Delta}-\frac{k^2}{4\Delta}r^{\Delta-2}\right)+a_{(0)j}\epsilon_{ij}\frac{8\mu k\kappa i}{3(\Delta-2)}r^{2\Delta-2}+\frac{\tilde{a}_i}{r^{2+\Delta}},\qquad(\text{B10})$$

where $\tilde{a}_i$ is the normalizable mode of the perturbation. In principle it has a complicated dependence on the sources but in the linear response regime we can write

$$\tilde{a}_i=\rho a_{i(0)}+\tilde{\rho}a_{j(0)}\longrightarrow\frac{\delta\tilde{a}_i}{\delta a_{(0)j}}=\tilde{\rho}.\qquad(\text{B11})$$

That allows us to write (B10) as

$$a_i(r\to\infty)\sim a_{(0)i}\left(r^{\Delta}-\frac{k^2}{4\Delta}r^{\Delta-2}+\frac{\rho}{r^{\Delta+2}}\right)+a_{(0)j}\epsilon_{ij}\left(\frac{8\mu k\kappa i}{3(\Delta-2)}r^{2\Delta-2}+\frac{\tilde{\rho}}{r^{2+\Delta}}\right),\qquad(\text{B12})$$

which is more useful to make the connection to the BBP matrix

$$\mathbb{F}=\begin{pmatrix}b(r) & c_+(r)\\ c_-(r) & d(r)\end{pmatrix},\qquad(\text{B13})$$

with[14]

$$b(r)=d(r)\sim 1-\frac{k^2}{4\Delta r^2}+\frac{\rho}{r^{2+2\Delta}},\qquad c_\pm\sim\pm\left(\frac{8\mu k\kappa i}{3(\Delta-2)}r^{\Delta-2}+\frac{\tilde{\rho}}{r^{2+2\Delta}}\right).\qquad(\text{B14})$$

At this point we can invert the series to the order of the normalizable mode. In our concrete case we have

$$\tilde{\rho}=\lim_{r\to\infty}r^{2+2\Delta}\frac{(2-\Delta)c(r)+rc'(r)}{-3\Delta}.\qquad(\text{B15})$$

So the last thing to do is to numerically construct the BBP imposing infalling boundary conditions at the horizon and compute the latter formula. For a detailed explanation on how to numerically construct the BBP we refer the reader to [33]. Due to how we numerically construct $\mathbb{F}$, one may find some issues when computing $\lim_{r\to\infty}c(r)$ so we rather use an alternative expression involving only derivatives of $c(r)$. One can easily derive

$$\tilde{\rho}=\lim_{r\to\infty}r^{3+2\Delta}\frac{(3-\Delta)c'(r)+rc''(r)}{6\Delta(\Delta+1)}.\qquad(\text{B16})$$

This expression combined with Eqs. (28) and (23) leads finally to an expression for the conductivity

$$\sigma_{55}=\lim_{k\to 0}\frac{i}{k_x}\lim_{k\to\infty}r^{3+2\Delta}\frac{(3-\Delta)c'(r)+rc''(r)}{6\Delta}\bigg|\omega=0.\qquad(\text{B17})$$

APPENDIX C: CORRELATORS IN THE $U(1)\times U(1)$ MODEL

1. One-point functions

First of all we expand the action to first order in perturbations

$$\begin{aligned}S^{(1)}_{\mathcal{R}}=&\int drd^4x\sqrt{-g}\left[a_\mu\left(\nabla_\nu F^{\nu\mu}-m^2(A^\mu-\partial^\mu\theta)+\frac{3\kappa}{2}\epsilon^{\mu\alpha\beta\gamma\rho}(F_{\alpha\beta}F_{\gamma\rho}+H_{\alpha\beta}H_{\gamma\rho})\right)\right]\\&+\int drd^4x\sqrt{-g}[v_\mu(\nabla_\nu H^{\nu\mu}+3\kappa\epsilon^{\mu\alpha\beta\gamma\rho}F_{\alpha\beta}H_{\gamma\rho})-\phi\nabla_\mu(A^\mu-\partial^\mu\theta)]\\&+\int_\partial d^4x\sqrt{-g}[a_i(F^{ir}+2\kappa(A_j-\partial_j\theta)F_{kl}\epsilon^{rijkl})]\\&+\int_\partial d^4x\sqrt{-g}[v_i(H^{ir}+6\kappa(A_j-\partial_j\theta)H_{kl}\epsilon^{rijkl})-\phi(F_{ij}F_{kl}\epsilon^{rijkl}+m^2(A^r-\partial^r\theta))]\\&+\int_\partial d^4x\sqrt{-\gamma}a_i\left(\Delta(A^i-\partial^i\theta)+\frac{1}{2(\Delta+2)}\partial^i(\partial_jA^j-\Box\theta)-\frac{1}{2\Delta}\partial_jF^{ji}\right)\\&+\int_\partial d^4x\sqrt{-\gamma}v_i\left(-\frac{1}{2}\partial_jH^{ji}\log(r)\right)\int_\partial d^4x\sqrt{-\gamma}\phi\left(\Delta(\partial_jA^j-\Box\theta)+\frac{1}{2(\Delta+2)}\Box(\partial_jA^j-\Box\theta)\right).\end{aligned}\qquad(\text{C1})$$

[14]Here we make some abuse of language when we refer to the block in $\mathbb{F}$ that affects a_x, a_y as $\mathbb{F}$. The true $\mathbb{F}$ is actually a 4×5 matrix as explained in (B6).

From the e.o.m. we find that the expansions for the scalar and the massive gauge field remain qualitatively unchanged up to the normalizable mode with respect to what we found in the $U(1)$ model. The expansion for the vector field is

$$V_\mu = \sum_i V_{\mu(i)} r^{-i} + \sum_{i\geq 2} \tilde{V}_{\mu(i)} r^{-i} \log(r) + \sum_{n,i\geq n+1} \Lambda_{\mu(n,i)} r^{n\Delta - i}, \tag{C2}$$

Where the $\sim\Lambda$ terms appear due to the mixing with the axial gauge field via Chern Simons. As in the previous case it is convenient to define the BBP with the fields normalized $(r^{-\Delta}a_i, v_i, \phi)$ so that we can impose

$$\psi_I(0) \equiv \begin{pmatrix} a_{t(0)} \\ \vdots \\ v_{t(0)} \\ \vdots \\ \phi_{(0)} \end{pmatrix}, \qquad F(\Lambda) = \mathbb{I}. \tag{C3}$$

I is useful to divide the BBP in two rectangular matrices $\mathbb{F}, \mathbb{H}$ and a vector $\mathbb{G}$,

$$F = \begin{pmatrix} \mathbb{F} \\ ------ \\ \mathbb{H} \\ ------ \\ \mathbb{G} \end{pmatrix}, \qquad a_I = r^\Delta \mathbb{F}_{IJ} \psi_{J(0)},$$

$$v_I = \mathbb{H}_{IJ} \psi_{J(0)}, \qquad \phi = \mathbb{G}_J \psi_{J(0)}. \tag{C4}$$

From this one can derive the renormalized one-point functions. The expressions can be found in the main text in (36).

2. Two-point functions

In order to obtain the two-point functions in (39)–(41) we switch on perturbations with momentum aligned to the z-direction $\delta\theta = \sigma(r)e^{-i\omega t + ikz}$, $\delta A_\mu = a_\mu(r)e^{-i\omega t+ikz}$ and $\delta V_\mu = v_\mu(r)e^{-i\omega t+ikz}$ on top of our background (42). The equations decouple and in the sector we are interested in we are left to four coupled equations for a_x, a_y, v_x, v_y.

$$a_y'' + \left(\frac{f'}{f} + \frac{1}{r}\right) a_y' + \left(\frac{\omega^2}{f^2} - \frac{k^2}{r^2 f} - \frac{m^2}{f}\right) a_y - \frac{12ik\kappa\phi'}{fr} a_z - \frac{12ik\kappa\chi'}{fr} v_z = 0, \tag{C5}$$

$$a_z'' + \left(\frac{f'}{f} + \frac{1}{r}\right) a_z' + \left(\frac{\omega^2}{f^2} - \frac{k^2}{r^2 f} - \frac{m^2}{f}\right) a_z + \frac{12ik\kappa\phi'}{fr} a_y + \frac{12ik\kappa\chi'}{fr} v_y = 0, \tag{C6}$$

$$v_y'' + \left(\frac{f'}{f} + \frac{1}{r}\right) v_y' + \left(\frac{\omega^2}{f^2} - \frac{k^2}{r^2 f}\right) v_y - \frac{12ik\kappa\chi'}{fr} a_z - \frac{12ik\kappa\phi'}{fr} v_z = 0, \tag{C7}$$

$$v_z'' + \left(\frac{f'}{f} + \frac{1}{r}\right) v_z' + \left(\frac{\omega^2}{f^2} - \frac{k^2}{r^2 f}\right) v_z + \frac{12ik\kappa\chi'}{fr} a_y + \frac{12ik\kappa\phi'}{fr} v_y = 0. \tag{C8}$$

The asymptotic analysis of these equations allows to write the near boundary expansion

$$a_i(r\to\infty) \sim a_{(0)i}(r^\Delta + Mr^{\Delta-2}) + a_{(0)j}\epsilon_{ij}\tilde{M}r^{2\Delta-2} + \frac{\tilde{a}_i}{r^{\Delta+2}}, \tag{C9}$$

$$v_i(r\to\infty) \sim v_{(0)i}(1) + v_{(0)j}\epsilon_{ij}(\tilde{M}r^{\Delta-2}) + \frac{\tilde{v}_i}{r^2}, \tag{C10}$$

where M and $\tilde{M}$ are functions of k, κ, A_t', V_t'. In the linear response limit the normalizable modes $\tilde{a}_i\tilde{v}_i$ can only depend linearly on the sources; therefore we may rewrite the expansions

$$a_i(r\to\infty) \sim a_{(0)i}\left(r^\Delta + Mr^{\Delta-2} + \frac{\rho}{r^{2+\Delta}}\right) + a_{(0)j}\epsilon_{ij}\left(\tilde{M}r^{2\Delta-2} + \frac{\tilde{\rho}}{r^{2+\Delta}}\right) + v_{(0)i}\frac{\tilde{\tilde{\rho}}}{r^{2+\Delta}} + v_{(0)j}\epsilon_{ij}\frac{\tilde{\tilde{\tilde{\rho}}}}{r^{2+\Delta}}, \tag{C11}$$

$$v_i(r\to\infty) \sim v_{(0)i}\left(1 + \frac{\eta}{r^2}\right) + v_{(0)j}\epsilon_{ij}\left(\tilde{M}r^{\Delta-2} + \frac{\tilde{\eta}}{r^2}\right) + a_{(0)i}\frac{\tilde{\tilde{\eta}}}{r^2} + a_{(0)j}\epsilon_{ij}\frac{\tilde{\tilde{\tilde{\eta}}}}{r^2}. \tag{C12}$$

This allows us to write

$$\langle J_i^V J_j^V \rangle = 2\frac{\delta\tilde{v}_i}{\delta v_{j(0)}} = 2\tilde{\eta}_i, \tag{C13}$$

$$\langle J_i^A J_j^A \rangle = (2+2\Delta)\frac{\delta\tilde{a}_i}{\delta a_{j(0)}} = (2+2\Delta)\tilde{\rho}_i, \tag{C14}$$

$$\langle J_i^A J_j^V \rangle = 2\frac{\delta\tilde{v}_i}{\delta a_{j(0)}} = 2\tilde{\tilde{\eta}}_i. \tag{C15}$$

Now we perform the same analysis as in (B 2), seeking the correct expression of these correlators as a linear combination of the BBP and its derivatives in order to compute the conductivities numerically. We find

$$2\tilde{\eta}_i = \lim_{r\to\infty} -r^3 p'(r), \tag{C16}$$

$$(2+2\Delta)\tilde{\rho}_i = \lim_{r\to\infty} r^{3+2\Delta}\frac{(3-\Delta)b'(r)+rb''(r)}{3\Delta}, \tag{C17}$$

$$2\tilde{\tilde{\eta}}_i = \lim_{r\to\infty} -r^{3+2\Delta}\frac{v'(r)}{\Delta+1}, \tag{C18}$$

where $p(r)$, $b(r)$ and $v(r)$ are the functions that appear in the matrix valued BBP

$$\begin{pmatrix} r^{-\Delta}a_y(r) \\ r^{-\Delta}a_z(r) \\ v_y(r) \\ v_z(r) \end{pmatrix} = \begin{pmatrix} a(r) & b(r) & c(r) & d(r) \\ i(r) & j(r) & k(r) & l(r) \\ m(r) & n(r) & o(r) & p(r) \\ u(r) & v(r) & w(r) & y(r) \end{pmatrix}\begin{pmatrix} a_{y(0)} \\ a_{z(0)} \\ v_{y(0)} \\ v_{z(0)} \end{pmatrix}. \tag{C19}$$

APPENDIX D: $U(1)\times U(1)$ MODEL: PERTURBATIONS FOR THE CMW

In order to compute the QNM spectrum and the electric conductivities with a constant and homogeneous background magnetic field we switch on perturbations with momentum k aligned to the magnetic field and frequency ω. The decoupled sector of equations we are interested in reads

$$a_t'' + \frac{3}{r}at' - \left(\frac{k^2}{fr^2}+\frac{m^2}{f}\right)a_t - \frac{\omega k}{fr^2}a_z + \frac{12\kappa B}{r^3}v_z' + \frac{i\omega m^2}{f}\eta = 0, \tag{D1}$$

$$v_t'' + \frac{3}{r}vt' - \frac{k^2}{fr^2}v_t - \frac{\omega k}{fr^2}v_z + \frac{12\kappa B}{r^3}a_z' = 0 \tag{D2}$$

$$a_z'' + \left(\frac{f'}{f}+\frac{1}{r}\right)a_z{}' + \left(\frac{\omega^2}{f^2}-\frac{m^2}{f}\right)a_z + \frac{\omega k}{f^2}a_t + \frac{12\kappa B}{fr}v_t' - \frac{ikm^2}{f}\eta = 0, \tag{D3}$$

$$v_z'' + \left(\frac{f'}{f}+\frac{1}{r}\right)v_z' + \frac{\omega^2}{f^2}v_z + \frac{\omega k}{f^2}v_t + \frac{12\kappa B}{fr}a_t' = 0, \tag{D4}$$

$$\eta'' + \left(\frac{3}{r}+\frac{f'}{f}\right)\eta' + \left(\frac{\omega^2}{f^2}-\frac{k^2}{f}\right)\eta + \frac{i\omega}{f^2}a_t + \frac{ik}{fr}a_z = 0, \tag{D5}$$

with a, v, η being the perturbations for the axial, vector and Stückelberg fields respectively and f the blackening factor of the metric. There are also two constraints:

$$\omega a_t' + \frac{kf}{r^2}a_z' + \frac{12\kappa B}{r^3}(\omega v_z + kv_t) - im^2 f\eta' = 0, \tag{D6}$$

$$\omega v_t' + \frac{kf}{r^2}v_z' + \frac{12\kappa B}{r^3}(\omega a_z + ka_t) = 0. \tag{D7}$$

The equations for the electric conductivity can be obtained turning off the momentum.

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