PHYSICAL REVIEW D **90**, 125041 (2014)

Soliton defects in one-gap periodic system and exotic supersymmetry

Adrián Arancibia,[1,*] Francisco Correa,[2,3,†] Vít Jakubský,[4,‡] Juan Mateos Guilarte,[5,§] and Mikhail S. Plyushchay[1,∥]

[1]*Departamento de Física, Universidad de Santiago de Chile, Casilla 307 Santiago 2, Chile*
[2]*Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*
[3]*Centro de Estudios Científicos (CECs), Arturo Prat 514 Valdivia, Chile*
[4]*Department of Theoretical Physics, Nuclear Physics Institute, 25068 Řež, Czech Republic*
[5]*Departamento de Física Fundamental and IUFFyM, Universidad de Salamanca, Salamanca E-37008, Spain*

(Received 18 October 2014; published 31 December 2014)

By applying Darboux–Crum transformations to the quantum one-gap Lamé system, we introduce an arbitrary countable number of bound states into forbidden bands. The perturbed potentials are reflectionless and contain two types of soliton defects in the periodic background. The bound states with a finite number of nodes are supported in the lower forbidden band by the periodicity defects of the potential well type, while the pulse-type bound states in the gap have an infinite number of nodes and are trapped by defects of the compression modulations nature. We investigate the exotic nonlinear $\mathcal{N}=4$ supersymmetric structure in such paired Schrödinger systems, which extends an ordinary $\mathcal{N}=2$ supersymmetry and involves two bosonic generators composed from Lax–Novikov integrals of the subsystems. One of the bosonic integrals has a nature of a central charge and allows us to liaise the obtained systems with the stationary equations of the Korteweg–de Vries and modified Korteweg–de Vries hierarchies. This exotic supersymmetry opens the way for the construction of self-consistent condensates based on the Bogoliubov–de Gennes equations and associated with them new solutions to the Gross–Neveu model. They correspond to the kink or kink-antikink defects of the crystalline background in dependence on whether the exotic supersymmetry is unbroken or spontaneously broken.

DOI: 10.1103/PhysRevD.90.125041 PACS numbers: 11.30.Pb, 03.65.-w, 11.10.Lm, 11.30.Na

I. INTRODUCTION

Quantum periodic finite-gap systems find many interesting applications in physics [1–22]. They can be related via the algebro-geometric approach with the integrable Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations [23,24]. The potentials of finite-gap Schrödinger systems correspond to the "snapshots" of the evolving in time generalizations of cnoidal waves solutions to the KdV equation [25]. In a similar way, via the Miura transformation, the scalar Dirac finite-gap potentials can be associated with solutions to the mKdV equation. The infinite-period limit of such potentials corresponds to reflectionless systems [26] and the solitary waves solutions to the KdV and mKdV equations.

Reflectionless second- and first-order quantum systems can be constructed via the Darboux–Crum transformations [27] from the quantum free particle Schrödinger and Dirac systems. The same transformations provide an effective dressing method for construction of Lax–Novikov integrals for these systems. The condition of conservation of them generates the higher-order nonlinear stationary equations for the KdV and mKdV hierarchies [28–31]. This picture also applies for a more general case of Zakharov–Shabat/Ablowitz–Kaup–Newell–Segur hierarchy [32].

It was shown recently in Ref. [31] that the Darboux–Crum transformations yield a possibility to relate reflectionless systems with a different number of bound states in their spectra via a soliton scattering picture. It was also demonstrated that the pairs of reflectionless Schrödinger systems are described not by the ordinary linear or nonlinear $\mathcal{N}=2$ supersymmetry, as this happens in the case of ordinary, nontransparent quantum systems related by a Darboux–Crum transformation. Instead, they are characterized by exotic nonlinear $\mathcal{N}=4$ supersymmetric structure. It is generated by two pairs of the supercharges, which are the 2×2 matrix differential operators of the odd and even orders. In addition, the exotic supersymmetric structure includes two bosonic generators composed from the Lax–Novikov integrals of subsystems, which are differential operators of higher odd order [29,30].

Among all such paired reflectionless Schrödinger systems, there is a special class, in which two lower-order supercharges have the differential order 1. In this case, one of the two bosonic integrals transmutes into the central charge of the exotic nonlinear $\mathcal{N}=4$ superalgebra, while the second bosonic integral generates rotations between the first-order and even-order supercharges. One of the first-order supercharges can be reinterpreted as the Dirac

*adaran.phi@gmail.com
†correa@cecs.cl
‡jakub@ujf.cas.cz
§guilarte@usal.es
∥mikhail.plyushchay@usach.cl

Hamiltonian, which is characterized by its own exotic supersymmetry associated with the central charge of the initial extended Schrödinger system. It is, in fact, the Bogoliubov–de Gennes Hamiltonian, whose potential, being the superpotential of the initial extended Schrödinger system, provides us with self-consistent condensates. The latter supply us, particularly, with kink- and kink-antikink-type solutions for the Gross–Neveu model [30]. A similar picture related to the exotic supersymmetry was also revealed in the pairs of mutually displaced one-gap Lamé systems [22].

A natural question that appears here is whether the Darboux–Crum transformations can be employed to unify the reflectionless and finite-gap properties in the same quantum system. Such a quantum system could be associated with the KdV and mKdV equations, and its potential would correspond to solitary wave solutions propagating in a background of finite-gap, cnoidal-wave-type solutions. The related question then is what happens with the exotic nonlinear supersymmetric structure in such quantum systems.

In this article, we answer the posed questions. To this aim, we apply the Darboux–Crum transformations to the quantum one-gap periodic Lamé system to introduce into its spectrum an arbitrary countable number of bound states in its two, the lowest and the intermediate, forbidden bands. This procedure will provide us the reflectionless nonperiodic one-gap potentials, which will contain two essentially different types of soliton defects in the periodic background. The nature of defects depends on the forbidden band in which they support the bound states. Coherently with this, as it will be shown, the corresponding two types of the bound states possess essentially different properties. We also investigate the exotic nonlinear supersymmetric structure associated with such quantum systems.

Some general mathematical aspects of the theory of the class of the systems we investigate here were discussed in Ref. [33]. The simplest particular examples were considered in Ref. [34]. For the discussion of the problem of defects in a more general context of integrable classical and quantum field theoretical systems, see Refs. [35–37].

The article is organized as follows. In next section, generic properties of the quantum one-gap periodic Lamé system are summarized, and its infinite-period limit corresponding to the simplest reflectionless Pöschl–Teller model with one bound state is discussed in light of Darboux–Crum transformations. In Sec. III, we consider Darboux translations for Lamé system. We apply Darboux–Crum transformations in Sec. IV to introduce soliton defects into the one-gap Lamé system. The procedure is developed first to generate an arbitrary number of periodicity defects supporting bound states in the lower forbidden band. Then, we do the same for the gap separating the allowed valence and conduction bands. As we shall see, the cases of the even and odd numbers of the bound states in the intermediate forbidden band are characterized by different Darboux–Crum schemes. Finally, we show how to generalize the construction to introduce the bound states in both forbidden bands. We discuss also the application of Darboux–Crum dressing procedure for the construction of the irreducible Lax–Novikov integrals. Section V is devoted to investigation of the exotic nonlinear $\mathcal{N}=4$ supersymmetric structure that appears in the extended Shrödinger systems composed from two arbitrary one-gap systems with periodicity defects. Special attention is given there for the most interesting from the viewpoint of physical applications case when two of the four supercharges are given by the matrix differential operators of the first order. We consider the cases of the unbroken and spontaneously broken exotic supersymmetries and indicate the relation of the obtained systems with the KdV and mKdV equations. The results are summarized in Sec. VI. We point out there further possible research directions for the development of the obtained results and some interesting applications. The Appendix is devoted to a more technical demonstration of a nonsingular nature of the constructed one-gap potentials of a generic form with an arbitrary number of the periodicity defects.

II. ONE-GAP LAMÉ SYSTEM AND ITS INFINITE-PERIOD LIMIT

In this section, we summarize generic properties of the quantum one-gap periodic Lamé system and discuss its infinite-period limit corresponding to the reflectionless Pöschl–Teller model. The Darboux transformations associate the latter system with a free particle and allow us, particularly, to identify its nontrivial Lax–Novikov integral via the dressing procedure. All this will form the basis for application of the method of the Darboux–Crum transformations to introduce two different types of nonperiodic soliton defects into the Lamé system.

A. Spectral properties of one-gap Lamé system

The quantum one-gap Lamé system is described by the Hamiltonian operator

$$H_{0,0} = -\frac{d^2}{dx^2} + V_{0,0}(x),$$
$$V_{0,0}(x) = 2k^2\mathrm{sn}^2x - k^2 = -2\mathrm{dn}^2x + 1 + k'^2, \qquad (2.1)$$

with a periodic potential $V_{0,0}(x) = V_{0,0}(x + 2\mathbf{K})$.[1] The sense of the lower indices introduced here will be clarified

[1] $\mathbf{K} = \mathbf{K}(k)$ is a complete elliptic integral of the first kind corresponding to the modular parameter k, $0 < k < 1$. We also denote $\mathbf{K}' = \mathbf{K}(k')$, where k', $0 < k' < 1$, $k^2 + k'^2 = 1$, is the complementary modular parameter. For the properties of Jacobi elliptic and related functions, see Ref. [38]. For a short summary of the properties we use here, see the Appendix in Ref. [22]. The dependence of these functions on k is not shown explicitly. In the case in which they depend on k' instead of k, we indicate such a dependence explicitly.

in what follows. The eigenstates of $H_{0,0}$ can be found in a closed analytic form for any complex eigenvalue $\mathcal{E}$. Parametrizing the latter in terms of Jacobi's elliptic dn function, $\mathcal{E}(\alpha) = \mathrm{dn}^2\alpha$, we obtain the solutions of the stationary Schrödinger equation $H_{0,0}\Psi^\alpha_\pm = \mathcal{E}(\alpha)\Psi^\alpha_\pm$,

$$\Psi^\alpha_\pm(x) = \frac{\mathrm{H}(x \pm \alpha)}{\Theta(x)} \exp\left[\mp xZ(\alpha)\right]. \tag{2.2}$$

Here, Θ, H, and Z are Jacobi's Theta, Eta, and Zeta functions, while parameter α can take arbitrary complex values. Since the periods of the doubly periodic elliptic function $\mathrm{dn}^2\alpha$ are $2\mathbf{K}$ and $2i\mathbf{K}'$, and it is an even function, without any loss of generality, one can restrict a consideration to a rectangular domain with vertices in 0, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$, and $i\mathbf{K}'$. Hamiltonian (2.1) is a Hermitian operator, and we are interested in the real eigenvalues $\mathcal{E}(\alpha)$.[2] These are provided by further restriction of the values of the parameter α to the borders of the indicated rectangle; see Fig. 1. The horizontal edges correspond to the *lower* and *upper* forbidden zones (lacunas) in the spectrum. The vertical edges correspond, respectively, to the *valence* and *conduction* bands. The necessary information on the bands' structure, including the values of quasimomentum $\kappa(\alpha)$, see below, is summarized in Table I. We supply the parameters β and γ, corresponding to real and imaginary parts of the complex parameter α, with upper index $-/+$ to distinguish whether they correspond to the lower/upper forbidden and allowed bands, respectively.

While the real parameter β^- increases in the open interval $(0, \mathbf{K})$, the energy increases in the lower, semi-infinite forbidden band but decreases in the finite gap separating the allowed bands when β^+ varies in the same interval. In the valence band, the energy increases when the parameter γ^- decreases from $\mathbf{K}'$ to 0; the variation of the parameter γ^+ in the semiopen interval $[0, \mathbf{K}')$ gives the energy monotonically increasing in the semi-infinite conduction band.

Under the shift for the real period $2\mathbf{K}$ of the potential, the eigenstates (2.2) undergo the transformation

$$\Psi^\alpha_\pm(x + 2\mathbf{K}) = \exp\left(\mp i2\mathbf{K}\kappa(\alpha)\right)\Psi^\alpha_\pm(x),$$
$$\text{where } \kappa(\alpha) = \frac{\pi}{2\mathbf{K}} - i\mathrm{Z}(\alpha) \tag{2.3}$$

is the *quasimomentum*, in which the first term is associated with the $2\mathbf{K}$ antiperiodicity of the Eta function, $\mathrm{H}(x + 2\mathbf{K}) = -\mathrm{H}(x)$. The analytical form of the quasimomentum $\kappa(\alpha)$ allows us to determine explicitly when it takes real or complex values and therefore to locate the allowed and forbidden bands. Thus, making use of the

[2]The PT-symmetric generalization [39,40] of (2.1) can also be associated with real values of $\mathcal{E}(\alpha)$; see below.

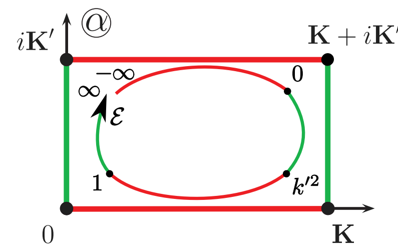

FIG. 1 (color online). Spectrum of the one-gap Lamé system (2.1) as a function of complex parameter α.

properties of Jacobi's Zeta function, one finds that in the lower forbidden band the quasimomentum takes pure imaginary values, $\kappa(\beta^- + i\mathbf{K}') = -i\mathrm{z}(\beta^-)$, $\mathrm{z}(\beta^-) = \frac{d}{d\beta^-}\log\mathrm{H}(\beta^-)$. In accordance with this, the quasimomentum varies in the complex plane along the imaginary axis so that $\kappa \to -i\infty$ for $\beta^- \to 0$, $\mathcal{E} \to -\infty$, and $\kappa \to 0$ when $\beta^- \to \mathbf{K}$, $\mathcal{E} \to 0$. The amplitude of the wave functions (2.2) in this band increases exponentially in one of the two directions on the real axis x, and eigenfunctions $\Psi^{\alpha=\beta^-+i\mathbf{K}'}_\pm(x)$ correspond therefore to nonphysical states. In the valence band, the quasimomentum takes real values, $\kappa(\mathbf{K} + i\gamma^-) = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^-}{\mathbf{K}'}) - \frac{d}{d\gamma^-}\log\Theta(\gamma^- + \mathbf{K}'|k')$, where it increases monotonically from $\kappa = 0$ ($\mathcal{E} = 0$) to $\kappa = \frac{\pi}{2\mathbf{K}}$ ($\mathcal{E} = k'^2$). The wave functions (2.2) inside the valence band correspond to the two linearly independent Bloch states. In the intermediate energy gap, the quasimomentum is complex valued, $\kappa(\beta^+) = \frac{\pi}{2\mathbf{K}} - i\mathrm{Z}(\beta^+)$. In accordance with the relation $\frac{d}{d\beta}\mathrm{Z}(\beta) = \mathrm{dn}^2\beta - \frac{\mathbf{E}}{\mathbf{K}}$, where $\mathbf{E}$ is the complete elliptic integral of the second kind, and $k'^2 < \frac{\mathbf{E}}{\mathbf{K}} < 1$, the imaginary part in $\kappa(\beta^+)$ varies monotonically in the interval $\beta^+ \in (0, \beta_*]$, $0 < \mathrm{Z} \le \mathrm{Z}(\beta_*)$, where β_* corresponds to the equality $\mathrm{dn}^2\beta_* = \frac{\mathbf{E}}{\mathbf{K}}$ and then decreases monotonically approaching the zero value in the interval $\beta^+ \in (\beta_*, \mathbf{K})$. In the conduction band, like in the valence band, the quasimomentum takes real values, $\kappa(i\gamma^+) = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^+}{\mathbf{K}'}) - \frac{d}{d\gamma^+}\log\mathrm{H}(\gamma^+ + \mathbf{K}'|k')$. It increases here monotonically from $\frac{\pi}{2\mathbf{K}}$ ($\mathcal{E} = 1$) to $+\infty$ ($\mathcal{E} \to \infty$). Inside this band, for any value of the energy, the two wave functions (2.2) correspond to the two linearly independent physical Bloch states.

The properties of a periodic quantum system are effectively reflected by the discriminant $\mathcal{D}(\mathcal{E})$ (Lyapunov function) of the corresponding stationary Schrödinger equation, which is defined as a trace of the monodromy matrix representing the operator of the translation for the period of the potential [23,41–43]. Its form $\mathcal{D}(\mathcal{E}) = 2\cos(2\mathbf{K}\kappa(\mathcal{E}))$ for the one-gap Lamé system (2.1) is shown on Fig. 2. In the lower prohibited zone and in the valence band, the explicit analytic form is given, respectively, by $\mathcal{D}(\mathcal{E}(\beta^- + i\mathbf{K}')) = 2\cosh(2\mathbf{K}\mathrm{z}(\beta^-))$ and $\mathcal{D}(\mathcal{E}(\mathbf{K} + i\gamma^-)) = 2\cos(2\mathbf{K}\kappa(\gamma^-|k'))$. In the energy gap separating the valence and conduction bands, it reduces to

TABLE I. Bands and their characteristics. Here $z(\beta^-) = Z(\beta^-) + \mathrm{cn}\,\beta^- \mathrm{ds}\,\beta^-$, $\kappa(\gamma^\pm|k') = \frac{\pi}{2\mathbf{K}}(1 - \frac{\gamma^\pm}{\mathbf{K}'}) - Z(\gamma^\pm|k') + f_\pm$, $f_- = k'^2 \mathrm{sn}(\gamma^-|k')\mathrm{cd}(\gamma^-|k')$, and $f_+ = \mathrm{sn}(\gamma^+|k')\mathrm{dc}(\gamma^+|k')$.

Band	$\alpha = \beta + i\gamma$	$\mathcal{E}(\alpha)$	$\kappa(\alpha)$
Lower forbidden	$\beta \equiv \beta^- \in (0, \mathbf{K}),\ \gamma = \mathbf{K}'$	$(-\infty, 0) \ni \mathcal{E} = -\mathrm{cs}^2\beta^-$	$-iz(\beta^-)$
Valence	$\beta = \mathbf{K},\ \gamma \equiv \gamma^- \in [0, \mathbf{K}']$	$[0, k'^2] \ni \mathcal{E} = k'^2 \mathrm{cd}^2(\gamma^-|k')$	$\kappa(\gamma^-|k')$
Upper forbidden (gap)	$\beta \equiv \beta^+ \in (0, \mathbf{K}),\ \gamma = 0$	$(k'^2, 1) \ni \mathcal{E} = \mathrm{dn}^2\beta^+$	$\frac{\pi}{2\mathbf{K}} - iZ(\beta^+)$
Conduction	$\beta = 0,\ \gamma \equiv \gamma^+ \in [0, \mathbf{K}')$	$[0, +\infty) \ni \mathcal{E} = \mathrm{dc}^2(\gamma^+|k')$	$\kappa(\gamma^+|k')$

$\mathcal{D}(\mathcal{E}(\beta^+)) = -2\cosh(2\mathbf{K}Z(\beta^+))$. The minimum of the curve at $\mathcal{E} = \mathrm{dn}^2\beta_* = \frac{\mathbf{E}}{\mathbf{K}}$ corresponds to the maximum value $Z(\beta_*) > 0$ of the Zeta function. In the conduction band, we have $\mathcal{D}(\mathcal{E}(\gamma^+)) = 2\cos(2\mathbf{K}\kappa(\gamma^+|k'))$. The infinite number of oscillations of the curve between -2 and $+2$ extrema values of the $\mathcal{D}(\mathcal{E})$ is associated in this band with the zero of $\mathrm{cn}(\gamma^+|k')$ at $\gamma^+ = \mathbf{K}'$ appearing in the denominator of the function f_+ in the structure of $\kappa(\gamma^+|k')$; see Table I.

At the edges of the valence and conduction bands, where $|\mathcal{D}| = 2$, $\frac{d\mathcal{D}}{d\mathcal{E}} \neq 0$, the two wave functions (2.2) reduce, up to numerical factors, to the same periodic, $\psi_1 = \mathrm{dn}\,x$ ($\mathcal{E} = 0$), and antiperiodic, $\psi_2 = \mathrm{cn}\,x$ ($\mathcal{E} = k'^2$) and $\psi_3 = \mathrm{sn}\,x$ ($\mathcal{E} = 1$), eigenstates. The second, linear independent eigenfunctions at the edges of the valence and conduction bands are given by $\Psi_i(x) = \psi_i(x)\mathcal{I}_i$, $i = 1, 2, 3$, where $\mathcal{I}_i(x) = \int dx/\psi_i^2(x)$ are expressed in terms of the incomplete elliptic integral of the second kind, $\mathrm{E}(x) = \int_0^x \mathrm{dn}^2 x dx$: $\mathcal{I}_1(x) = \frac{1}{k'^2}\mathrm{E}(x + \mathbf{K})$, $\mathcal{I}_2(x) = x - \frac{1}{k'^2}\mathrm{E}(x + \mathbf{K} + i\mathbf{K}')$, $\mathcal{I}_3(x) = x - \mathrm{E}(x + i\mathbf{K}')$. The functions $\Psi_i(x)$ are not bounded on the real line and correspond to nonphysical eigenstates of the Lamé Hamiltonian operator. They also can be obtained from the states (2.2) by differentiation in α. Namely, derivatives of the functions $\Psi_+^\alpha(x)$ in α at $\alpha = 0$ and $\alpha = \mathbf{K}$ give some linear combinations of the functions $\psi_i(x)$ and $\Psi_i(x)$ with $i = 3$ and $i = 2$, respectively, while the derivative of the function (2.6) in parameter β^- at $\beta^- = \mathbf{K}$ gives a linear combination of $\psi_1(x)$ and $\Psi_1(x)$.

For any value of the parameter α, under the parity reflection, $Pf(x) = f(-x)$, the states (2.2) satisfy the relation

$$P\Psi_\pm^\alpha(x) = -\Psi_\mp^\alpha(x). \tag{2.4}$$

The properties of the wave functions (2.2) in corresponding bands under the T, $Tf(x) = f^*(x)$, and the composed PT operations [39,40] are shown in Table II.

Notice that in the lower forbidden band

$$\Psi_\pm^{\beta^- + i\mathbf{K}'}(x) = \pm i q^{-1/4} \exp\left(-i\frac{\pi\beta^-}{2\mathbf{K}}\right) F(\pm x; \beta^-), \tag{2.5}$$

where

$$F(x; \beta^-) = \frac{\Theta(x + \beta^-)}{\Theta(x)} \exp(-x z(\beta^-)) \tag{2.6}$$

is a real-valued function of x, which takes positive values, $F(x; \beta^-) > 0$. Here, $q = \exp(-\pi\mathbf{K}'/\mathbf{K})$ is Jacobi's nome, and we used the relation $\mathrm{H}(x + i\mathbf{K}') = iq^{-1/4}\exp(-i\frac{\pi x}{2\mathbf{K}})\Theta(x)$. In this band, one can employ alternatively the real functions $F(x; \beta^-)$ and $F(-x; \beta^-) = PF(x; \beta^-)$ as two linear independent solutions.

The operator PT distinguishes whether the function (2.2) belongs to the forbidden or allowed band. When it corresponds to the physical Bloch state, it is also the eigenfunction of the PT. In contrast, the functions (2.2) from the forbidden bands cease to be eigenstates of the PT operator. Instead, certain linear combinations of the two states (2.2) with the opposite sign of the quasimomentum have to be taken to create the eigenstates of the PT operator in the forbidden bands.

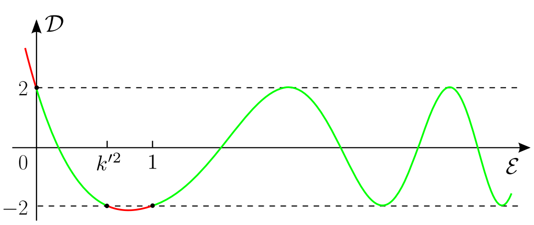

FIG. 2 (color online). The discriminant $\mathcal{D}(\mathcal{E})$ of the one-gap Lamé system. The scale is linear in energy for $\mathcal{E} < 1$, while for $\mathcal{E} > 1$ a logarithmic scale is used here. The parts shown in red correspond to the lower ($\mathcal{E} < 0$) and to the upper ($k'^2 < \mathcal{E} < 1$) forbidden bands.

B. Infinite period limit: Reflectionless Pöschl–Teller system and Darboux transformations

Before we pass to the discussion of the introduction of the periodicity defects, corresponding to solitons, into the

TABLE II. Properties of the eigenfunctions under the T and PT operations. Here, $c = \exp(i\frac{\pi\beta^-}{\mathbf{K}})$.

Band	$\Psi_\pm^\alpha(x)$	$T\Psi_\pm^\alpha(x)$	$PT\Psi_\pm^\alpha(x)$
Lower forbidden	$\Psi_\pm^{\beta^- + iK'}(x)$	$-c\Psi_\pm^{\beta^- + iK'}(x)$	$c\Psi_\mp^{\beta^- + iK'}(x)$
Valence	$\Psi_\pm^{K + i\gamma^-}(x)$	$-\Psi_\mp^{K + i\gamma^-}(x)$	$\Psi_\pm^{K + i\gamma^-}(x)$
Upper forbidden (gap)	$\Psi_\pm^{\beta^+}(x)$	$\Psi_\pm^{\beta^+}(x)$	$-\Psi_\mp^{\beta^+}(x)$
Conduction	$\Psi_\pm^{i\gamma^+}(x)$	$\Psi_\mp^{i\gamma^+}(x)$	$-\Psi_\pm^{i\gamma^+}(x)$

spectrum of the one-gap Lamé system, we consider briefly the analogous procedure for the infinite-period limit case. The picture in such a limit case is more simple and transparent, and it is useful to bear it in mind when we generalize the method to the very Lamé system.

In the infinite-period limit $\mathbf{K}\to\infty$, which is equivalent to any of the three limits $k\to 1$, $k'\to 0$, or $\mathbf{K}'\to\pi/2$, operator (2.1) transforms into the Hamiltonian of the reflectionless Pöschl–Teller system

$$H_1=-\frac{d^2}{dx^2}+V_1(x),\qquad V_1(x)=-\frac{2}{\cosh^2 x}+1. \tag{2.7}$$

In this limit, the valence band shrinks into one discrete energy level $\mathcal{E}=0$. The wave functions (2.2) of the valence band with $\alpha=\mathbf{K}+i\gamma^-$, $\gamma^-\in[0,\mathbf{K}']$ transform into the unique bound state described by the normalizable wave function $\Psi_{\mathcal{E}=0}(x)=\mathrm{sech}x$. The conduction band, parametrized by $\alpha=i\gamma^+$, $\gamma^+\in[0,\mathbf{K}')$, transforms into the scattering part of the spectrum of the system (2.7). In the limit, we have $\gamma^+\in[0,\frac{\pi}{2})$. Introducing the notation $\tan\gamma^+=\mathrm{k}$, $0\leq\mathrm{k}<\infty$, we find that the rescaled wave functions $q^{-1/4}\Psi_{\mp}^{\alpha=i\gamma^+}(x)$ of the conduction band transform, up to an inessential constant multiplicative factor, into the wave functions

$$\Psi_{\pm}^{\mathrm{k}}(x)=(\pm i\mathrm{k}-\tanh x)e^{\pm i\mathrm{k}x}. \tag{2.8}$$

Corresponding energy $\mathcal{E}=\mathrm{dn}^2(i\gamma^+|k)=\mathrm{dc}^2(\gamma^+|k')$ transforms in the limit $k'\to 0$ into $1/\cos^2\gamma^+=1+\mathrm{k}^2$, which is the eigenvalue of the eigenstates (2.8) of the Pöschl–Teller Hamiltonian (2.7). The nondegenerate state $\Psi^0=\tanh x$ ($\mathrm{k}=0$) corresponds here to the state of energy $\mathcal{E}=1$ described by $\mathrm{sn}x$ at the edge of the conduction band of the Lamé system (2.1).

The scattering states (2.8) can be presented in the form $\Psi_{\pm}^{\mathrm{k}}(x)=A_\varphi e^{\pm ikx}$ in terms of the first-order differential operator

$$A_\varphi=\varphi(x)\frac{d}{dx}\frac{1}{\varphi(x)}=\frac{d}{dx}-\tanh x,\qquad \varphi(x)=\cosh x. \tag{2.9}$$

Operator A_φ together with the Hermitian conjugate $A_\varphi^\dagger$ intertwine the reflectionless system (2.7) with the free particle Hamiltonian shifted for an additive constant,

$$H_0=-\frac{d^2}{dx^2}+1, \tag{2.10}$$

and provide the factorization of both:

$$A_\varphi A_\varphi^\dagger=H_1,\qquad A_\varphi^\dagger A_\varphi=H_0,$$
$$A_\varphi H_0=H_1A_\varphi,\qquad A_\varphi^\dagger H_1=H_0A_\varphi^\dagger. \tag{2.11}$$

Relations (2.11) correspond to the Darboux transformations that relate the free particle system with the reflectionless Pöschl–Teller system. The alternative form to express the same relation between the systems corresponds to the equality

$$H_1=H_0-2\frac{d^2}{dx^2}\log\varphi(x). \tag{2.12}$$

The wave function $\varphi(x)=\cosh x$ is a nodeless nonphysical eigenstate of the free particle H_0, and the operator A_φ produces an almost isospectral mapping of all the physical and nonphysical states of H_0, except $\varphi(x)$, $A_\varphi\varphi(x)=0$, into corresponding states of the system H_1. The only physical bound state $\Psi_{\mathcal{E}=0}(x)=\mathrm{sech}x$ of H_1 of zero energy, for which there is no bound state analog in the physical spectrum of H_0, is obtained by applying the operator A_φ to the wave function $\tilde{\varphi}(x)=\varphi(x)\int\frac{dx}{\varphi^2(x)}$. This is the nonphysical eigenstate of (2.10) of the same zero eigenvalue as $\varphi(x)$. It reduces here just to the derivative of the latter, $\tilde{\varphi}(x)=\sinh x=\varphi'(x)$. Analogously, the application of the operator $A_\varphi^\dagger$ to the eigenstates of H_1 in correspondence with the last relation in (2.11) produces the eigenstates of H_0. The unique bound state $\Psi_{\mathcal{E}=0}(x)=\mathrm{sech}x$ of H_1 is the zero mode of the first-order operator $A_\varphi^\dagger$.

The free particle system (2.10) has a nontrivial integral $p=-i\frac{d}{dx}$. It distinguishes the plane waves $e^{\pm i\mathrm{k}x}$, which are the eigenstates of H_0 of the same energy, and detects a unique nondegenerate state $\Psi_{\mathcal{E}=1}(x)=1$ corresponding to $\mathrm{k}=0$ by annihilating it. In correspondence with the last two relations in (2.11) and the described picture of the mapping associated with the Darboux transformations, one finds that the operator

$$\mathcal{P}=-iA_\varphi\frac{d}{dx}A_\varphi^\dagger \tag{2.13}$$

is the Hermitian integral for the reflectionless system H_1. We refer to this as the dressing procedure. Similarly to p, this operator distinguishes the eigenstates (2.8), being analogs of the plane wave states for the free particle, $\mathcal{P}\Psi_{\pm}^{\mathrm{k}}(x)=\pm\mathrm{k}(1+\mathrm{k}^2)\Psi_{\pm}^{\mathrm{k}}(x)$. It annihilates the lowest nondegenerate state $\Psi^0(x)=\tanh x$ in the scattering sector, and the bound state[3] $\Psi_{\mathcal{E}=0}(x)=\mathrm{sech}x$. Integral (2.13) satisfies the Burchnall–Chaundy relation [45]

$$\mathcal{P}^2=H_1^2(H_1-1). \tag{2.14}$$

Since the free particle has the integral $p=-i\frac{d}{dx}$, the H_0 and the Pöschl–Teller Hamiltonian (2.7) can be intertwined

[3] Being the third-order differential operator, (2.13) also turns into zero the state $\varphi(x)=\cosh x$, which is a nonphysical eigenstate of the free particle Hamiltonian (2.10) [44].

not only by the first-order operator (2.9) and its conjugate $A_\varphi^\dagger$ but also by the second-order operators

$$B_\varphi = A_\varphi \frac{d}{dx} \quad \text{and} \quad B_\varphi^\dagger. \tag{2.15}$$

The first- and second-order intertwining operators together with the integrals p and $\mathcal{P}$ of the systems H_0 and H_1 constitute the building blocks of the exotic centrally extended $\mathcal{N}=4$ nonlinear supersymmetry of the system described by the 2×2 matrix Hamiltonian $\mathcal{H} = \mathrm{diag}(H_0, H_1)$ [31].

Suppose now that we want to construct another reflectionless system proceeding from the Pöschl–Teller system (2.7) by means of a new Darboux transformation, or a composition of them, that corresponds to the Darboux–Crum transformation. There are three different ways to do this. First, one can construct a reflectionless system with an additional, second bound state lying below the unique, zero energy bound state of the system (2.7). Another case corresponds to the situation in which we want to introduce a bound state with the energy level lying between the zero energy level of the already existing bound state and the edge of the scattering sector of energy $\mathcal{E}=1$. At last, one can construct a reflectionless system completely isospectral to the system (2.7) but with the displaced potential ("soliton center"). Having at hands the building blocks corresponding to the described three possibilities, by the appropriate generalization of the procedure, we can construct a reflectionless system with an arbitrary number of bound states and arbitrary positions of the corresponding soliton centers [29,30].

The first situation is realized by the construction in a way similar to (2.9) of the Darboux generator on the basis of the nodeless function

$$\varphi_1(x;\kappa_1,\tau_1) = A_\varphi \sinh\kappa_1(x+\tau_1), \tag{2.16}$$

where $\kappa_1 > 1$ and τ_1 is an arbitrary real parameter. The function $\varphi_1(x;\kappa_1,\tau_1)$ is the nonphysical eigenstate of (2.7) with energy $1-\kappa_1^2$, and τ_1 is associated with the center (phase) of the second soliton (the first soliton is characterized by $\tau_0 = 0$ and the amplitude $\kappa_0 = 1$) in the potential of the system

$$H_2 = H_1 - 2\frac{d^2}{dx^2}\log\varphi_1(x) \tag{2.17}$$

with two bound states; cf. (2.12). Note that alternatively H_2 can be presented in terms of the second-order Darboux–Crum transformation applied to the free particle, $H_2 = H_0 - 2\frac{d^2}{dx^2}\log\mathbb{W}(x)$, where $\mathbb{W}(x)$ is the Wronskian of the two nonphysical states of the free particle, $\varphi = \cosh x$ and $\phi = \sinh\kappa_1(x+\tau_1)$, $\mathbb{W}(x) = W(\varphi,\phi) = \varphi\phi' - \varphi'\phi$.

To obtain a reflectionless system with an additional bound state inside the energy interval (0, 1), which separates the bound state level of the system (2.7) with the continuous part of the spectrum, one can apply to (2.7) the Darboux–Crum transformation generated by the two nonphysical states $\phi_1(x;\kappa_1,\tau_1) = A_\varphi\cosh\kappa_1(x+\tau_1)$ and $\phi_2(x;\kappa_2,\tau_2) = A_\varphi\sinh\kappa_2(x+\tau_2)$. If we restrict the parameters $\kappa_{1,2}$ by the condition $0<\kappa_1<\kappa_2<1$, the corresponding Wronskian $\mathbb{W}(x) = W(\phi_1,\phi_2)$ has no zeros. This produces a system with a regular reflectionless potential

$$V_3(x) = V_1(x) - 2\frac{d^2}{dx^2}\log\mathbb{W}(x), \tag{2.18}$$

which has three bound states with energies $1-\kappa_1^2$, $1-\kappa_2^2$, and 0. Sending then one of the two translation parameters, τ_2 or τ_1, to any of the limits $+\infty$ or $-\infty$, we get a reflectionless system with two bound states of energies $1-\kappa_1^2$ and 0 when we send $|\tau_2|\to\infty$, or with energies $1-\kappa_2^2$ and 0 when $|\tau_1|\to\infty$. The indicated limit changes the translation parameters of the remaining added soliton as well as of the initial one with $\kappa_0 = 1$ and $\tau_0 = 0$ in correspondence with the picture of soliton scattering; see Ref. [31].

There is another possibility to introduce one additional bound state into the spectrum of the system (2.7) with the energy inside the interval (0, 1). One can apply to (2.7) a Darboux transformation constructed on the basis of its nonphysical state $\phi(x;\kappa,\tau) = A_\varphi\sinh\kappa(x+\tau)$, $0<\kappa<1$. This will produce a singular system. Shifting then $\tau\to\tau + i\frac{\pi}{2\kappa}(1-\kappa)$ and $x\to x + i\frac{\pi}{2}$, we get a regular reflectionless system with two bound states with energies $1-\kappa^2$ and 0.

Finally, to produce a system completely isospectral to the system (2.7), one can apply to the latter the Darboux transformation based on the function [31] $f(x;\kappa) = A_\varphi\exp(\kappa x)$, where $\kappa > 1$. In the present simplest case of H_1, this will give us the shifted system (2.7), in which the argument of the potential x changes for[4] $x+\lambda$, where $\lambda = \frac{1}{2}\log\frac{\kappa-1}{\kappa+1}$.

In all three indicated cases, the corresponding extended system $\mathcal{H} = \mathrm{diag}(H_1,\tilde{H})$ will be described by the exotic centrally extended nonlinear $\mathcal{N}=4$ supersymmetry [29–31]. Such reflectionless systems will correspond to the $k\to 1$ limit of the systems obtained from the one-gap Lamé system by introducing into it the periodicity defects by means of the appropriate Darboux(–Crum) transformation.

[4]In the case of a reflectionless system with $n>1$ bound states, the isospectral deformation of the potential, which can be generated by applying the appropriate Darboux–Crum transformation, corresponds to a "snapshot" of the evolved n-soliton solution of the Korteweg–de Vries equation; see Refs. [29–31]. In that case, like in the case of Lamé system with periodicity defects we consider below, the form of the isospectrally deformed potential is different from the original one.

In the subsequent sections, we describe how to introduce such periodicity defects and discuss the associated exotic nonlinear supersymmetric structure.

III. DARBOUX TRANSLATIONS OF THE LAMÉ SYSTEM

Assume that we have a system described by a Hamiltonian operator of the most general form $H = -\frac{d^2}{dx^2} + U(x)$ and that $\psi(x)$ is its an arbitrary physical, or nonphysical eigenstate, $H\psi = \mathcal{E}\psi$. As in (2.9), we define the first-order operators

$$A_\psi = \psi \frac{d}{dx}\frac{1}{\psi} = \frac{d}{dx} + \Delta(x), \qquad \Delta(x) = -\frac{d}{dx}\log\psi(x), \tag{3.1}$$

and

$$A^\sharp_\psi = -\frac{1}{\psi}\frac{d}{dx}\psi = -\frac{d}{dx} + \Delta(x). \tag{3.2}$$

If $\psi(x)$ is a real valued function modulo a possible complex multiplicative constant, then the operators A_ψ and $A^\sharp_\psi$ are mutually conjugate, $A^\sharp_\psi = A^\dagger_\psi$. Another, linear independent eigenstate of H of the same eigenvalue $\mathcal{E}$ is given by $\tilde{\psi}(x) = \psi(x)\int dx/\psi^2(x)$. The action of the operator A_ψ on this eigenstate produces a kernel of the operator $A^\sharp_\psi$, $A_\psi\tilde{\psi}(x) = 1/\psi(x)$. The second-order operator $A^\sharp_\psi A_\psi = -\frac{d^2}{dx^2} + \Delta^2(x) - \Delta'(x)$ has exactly the same kernel, spanned by $\psi(x)$ and $\tilde{\psi}(x)$, as the second-order differential operator $H - \mathcal{E}$, and therefore $A^\sharp_\psi A_\psi = H - \mathcal{E}$, and $\Delta^2(x) - \Delta'(x) = U(x) - \mathcal{E}$.

Consider now the operator $A_\psi A^\sharp_\psi = -\frac{d^2}{dx^2} + \Delta^2(x) + \Delta'(x) = A^\sharp_\psi A_\psi + 2\Delta'(x) \equiv \tilde{H} - \mathcal{E}$. The wave function $1/\psi(x)$ is the eigenstate of the Schrödinger Hamiltonian operator $\tilde{H}$ of eigenvalue $\mathcal{E}$. Another, linear independent eigenstate of $\tilde{H}$ of the same eigenvalue $\mathcal{E}$ is $\frac{1}{\psi(x)}\int\psi^2(x)dx$. The latter is mapped by the operator $A^\sharp_\psi$ into the state $\psi(x)$ being the zero mode of A_ψ.

Let us return now to the Lamé system (2.1). Its eigenstates $\Psi^\alpha_+(x)$ obey the property

$$\Psi^\alpha_+(-x-\alpha-i\mathbf{K}') = -\Psi^\alpha_-(x+\alpha+i\mathbf{K}') = \frac{\mathcal{C}(\alpha)}{\Psi^\alpha_+(x)}, \tag{3.3}$$

where $\mathcal{C}(\alpha) = -\exp\left(\alpha(\mathrm{Z}(\alpha) + i\frac{\pi}{2\mathbf{K}}) + i\mathbf{K}'\mathrm{Z}(\alpha)\right)$. Taking $\psi(x) = \Psi^\alpha_+(x)$ in (3.1), we obtain the factorization for the one-gap Lamé Hamiltonian,

$$A^\sharp_{\Psi^\alpha_+}A_{\Psi^\alpha_+} = H_{0,0}(x) - \mathcal{E}(\alpha). \tag{3.4}$$

Making use of the relation (3.3), we find then that

$$A_{\Psi^\alpha_+}A^\sharp_{\Psi^\alpha_+} = H_{0,0}(x+\alpha+i\mathbf{K}') - \mathcal{E}(\alpha). \tag{3.5}$$

As the Darboux-partner of the Lamé Hamiltonian $H_{0,0}(x)$, we obtain therefore the translated Hamiltonian operator $H_{0,0}(x+\alpha+i\mathbf{K}')$.

In the case of the lower prohibited band, the wave function $\Psi^{\beta^-+i\mathbf{K}'}_+(x)$ reduces to the real function $F(x;\beta)$ modulo a constant multiplier, see Eqs. (2.5) and (2.6), and we have $A_{\Psi^\alpha_+} = A_F$, $A^\sharp_{\Psi^\alpha_+} = A^\dagger_F$. The property $\mathrm{dn}(x+2i\mathbf{K}') = -\mathrm{dn}x$ gives us then in (3.5) the same Hermitian Lamé Hamiltonian operator but shifted for the real distance β^-, $0<\beta^-<\mathbf{K}$, $H_{0,0}(x+\alpha+i\mathbf{K}') = H_{0,0}(x+\beta^-)$. The obtained Darboux transformations, supersymmetry, and physics associated with them were studied in diverse aspects in Ref. [22]. Note here that the real function $F(x;\beta^-)$ takes positive values for all x, blows up exponentially when $x\to-\infty$, and tends to zero for $x\to+\infty$. The limit case $\beta^- = \mathbf{K}$ corresponds to a translation for the half of the period of Lamé Hamiltonian. It is produced on the basis of the ground state $\psi(x) = \mathrm{dn}x$ [19]. The obtained Darboux transformations are analogous to the translation transformations in the case of the Pöschl–Teller system (2.7) with one bound state, which are constructed on the basis of the exponentlike nonphysical eigenstates $\psi = A_\varphi \exp\kappa x$, $\kappa>1$, of H_1.

In the forbidden band separating the allowed bands, the eigenfunction $\Psi^{\beta^+}_+(x)$ takes real values, but it has an infinite number of zeros at the points $-\beta^+ + 2n\mathbf{K}$, $n\in\mathbb{Z}$. In this case, relation (3.4) gives us the factorization of the Lamé Hamiltonian $H_{0,0}(x)$ in terms of the singular mutually conjugate Darboux generators. The alternative product (3.5) of these first-order differential operators produces the Hermitian operator $H_{0,0}(x+\beta^+ + i\mathbf{K}')$ with the singular Treibich–Verdier potential [46]

$$V_{0,0}(x+\beta^+ + i\mathbf{K}') = \frac{2}{\mathrm{sn}^2(x+\beta^+)} - k^2, \tag{3.6}$$

where we have taken into account the identity $\mathrm{sn}(x+i\mathbf{K}') = 1/k\,\mathrm{sn}\,x$. The limiting case $\beta^+ = 0$ corresponds to the singular Darboux transformation constructed on the basis of the eigenfunction $\psi(x) = \mathrm{sn}x$ at the edge of the conduction band. Another limit case $\beta^+ = \mathbf{K}$ gives rise to the singular transformation based on the eigenfunction $\psi(x) = \mathrm{cn}x$ at the edge of the valence band, for which the Treibich–Verdier potential reduces to

$$V_{0,0}(x+\mathbf{K}+i\mathbf{K}') = 2\mathrm{dc}^2x - k^2, \tag{3.7}$$

where we have employed the identity $\mathrm{sn}(x+\mathbf{K}+i\mathbf{K}') = \mathrm{dn}x/k\mathrm{cn}x$.

Inside the valence band, the eigenstate $\Psi_+^{\mathbf{K}+i\gamma^-}(x)$ takes nonzero but complex values. The Darboux partner (3.5) reduces in this case to the nonsingular PT-symmetric Hamiltonian with the potential

$$V_{0,0}(x+\alpha+i\mathbf{K}') = 2\mathrm{dc}^2(x+i\gamma^-) - k^2. \tag{3.8}$$

The edge value $\gamma^- = \mathbf{K}'$ corresponds here to the regular Hermitian Lamé Hamiltonian operator shifted for the half-period, $H_{0,0}(x+\mathbf{K})$. Another edge value $\gamma^- = 0$ gives the singular Hermitian Treibich–Verdier Hamiltonian (3.7) obtained on the basis of the edge state $\psi(x) = \mathrm{cn}x$.

At last, inside the conduction band, the Hamiltonian in (3.5) reduces to the regular PT-symmetric operator with the potential

$$V_{0,0}(x+i\gamma^+ + i\mathbf{K}') = \frac{2}{\mathrm{sn}^2(x+i\gamma^+)} - k^2. \tag{3.9}$$

The edge case $\gamma^+ = 0$ reduces to the singular Treibich–Verdier potential generated via the choice $\psi(x) = \mathrm{sn}x$.

The described first-order Darboux transformations can also be considered for the values of the parameter α lying inside the rectangle in Fig. 1. In this case, the partner Hamiltonian will be nonsingular with the potential taking complex values, which, however, will be neither a Hermitian nor PT-symmetric operator. Indeed, under Hermitian conjugation, the shifted Hamiltonian operator from (3.5) transforms as $(H_{0,0}(x+\alpha+i\mathbf{K}'))^\dagger = H_{0,0}(x+\alpha^*+i\mathbf{K}')$, where we have taken into account the pure imaginary period $2i\mathbf{K}'$ of the potential $V_{0,0}(x)$. Analogously, we have $PT(H_{0,0}(x+\alpha+i\mathbf{K}')) = H_{0,0}(x-\alpha^*+i\mathbf{K}')$, where the even nature of the potential has additionally been taken into account. The shifted Hamiltonian is therefore Hermitian if $\alpha-\alpha^* = 2n\mathbf{K}+2im\mathbf{K}'$, $n, m \in \mathbb{Z}$, while it is PT symmetric when $\alpha+\alpha^* = 2n\mathbf{K}+2im\mathbf{K}'$. For the α region shown in Fig. 1, the first condition is satisfied only on the upper and lower horizontal edges of the rectangle, which correspond to the prohibited zones in the spectrum, while the second relation takes place only on the vertical edges corresponding to the allowed valence and conduction bands.

Below, we shall see that the higher-order Darboux–Crum transformation corresponding to a composition of the Darboux transformations, each of which generates the translated Lamé system of the form (3.5), produces the Lamé system with a shift of the argument equal to the sum of individual translations.

IV. LAMÉ SYSTEM DEFORMED BY NONPERIODIC, SOLITON DEFECTS

In this section, we show how to introduce the reflectionless, soliton (nonperiodic) defects into the one-gap Lamé system.

A. Lower forbidden band

The real-valued eigenfunction $F(x;\beta^-)$ in the lower prohibited band has the modulated exponentlike behavior. Let us take a linear combination of the two eigenfunctions of the same eigenvalue,

$$\mathcal{F}_\pm(x;\beta^-,C) = CF(x;\beta^-) \pm \frac{1}{C}F(-x;\beta^-), \tag{4.1}$$

where $\mathbf{K} > \beta^- > 0$ and a real parameter C is restricted by the condition $C > 0$. These states have the properties $\mathcal{F}_\pm(-x;\beta^-,C^{-1}) = \pm\mathcal{F}_\pm(x;\beta^-,C)$. The function $\mathcal{F}_+(x;\beta^-,C)$ takes strictly positive values and blows up exponentially in the limits $x\to\pm\infty$. The function $\mathcal{F}_-(x;\beta^-,C)$, on the other hand, tends exponentially to $+\infty$ and $-\infty$ when x tends to $-\infty$ and $+\infty$, respectively, and has a unique zero whose position depends on the values of the parameters β^- and C. The form of the functions $\mathcal{F}_\pm(x;\beta^-,C)$ is shown in Fig. 3.

Construct now the first-order operator

$$A_{0,1} = \mathcal{F}_+(1)\frac{d}{dx}\frac{1}{\mathcal{F}_+(1)} = \frac{d}{dx} - \frac{d}{dx}\log\mathcal{F}_+(1), \tag{4.2}$$

where $\mathcal{F}_+(1) = \mathcal{F}_+(x;\beta_1^-,C_1)$. We have $A_{0,1}^\dagger A_{0,1} = H_{0,0}-\varepsilon_1^-$, and $A_{0,1}A_{0,1}^\dagger = H_{0,1}-\varepsilon_1^-$, where $\varepsilon_1^- \equiv \mathcal{E}(\beta_1^- + i\mathbf{K}') = -\mathrm{cn}^2\beta_1^-/\mathrm{sn}^2\beta_1^- < 0$,

$$H_{0,1} = H_{0,0} - 2\frac{d^2}{dx^2}(\log\mathcal{F}_+(1)) = -\frac{d^2}{dx^2} + V_{0,1}(x), \tag{4.3}$$

$$V_{0,1}(x) = 1 + k'^2 - 2\frac{\mathbf{E}}{\mathbf{K}} - 2\frac{d^2}{dx^2}(\log\chi_{0,1}^{\beta_1^-}(x;C_1)), \tag{4.4}$$

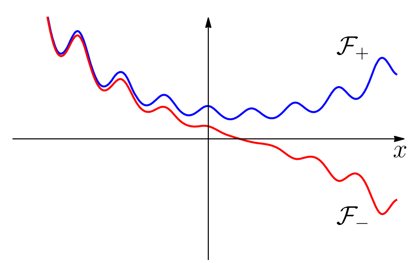

FIG. 3 (color online). At $C=1$, $\mathcal{F}_+(x;\beta^-,C)$ is an even function, while $\mathcal{F}_-(x;\beta^-,C)$ is odd. The symmetry of nonphysical eigenfunctions $\mathcal{F}_\pm(x;\beta^-,C)$ of $H_{0,0}$ is broken for $C\neq 1$. Here, the case $C>1$ is shown. With C increasing, the minimum of $\mathcal{F}_+(x;\beta^-,C)>0$ and zero of $\mathcal{F}_-(x;\beta^-,C)$ are displaced to the right. A similar situation occurs when $0<C<1$ but with a displacement to the negative coordinate axis. In fact, the form of the functions for $0<C<1$ is obtained from that for $C>1$ via the relation $\mathcal{F}_\pm(x;\beta^-,C) = \pm\mathcal{F}_\pm(-x;\beta^-,C^{-1})$.

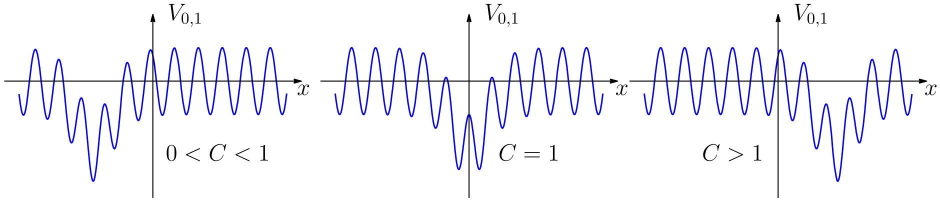

FIG. 4 (color online). Potential with a one-soliton defect that supports a bound state in the lower forbidden band. The soliton is broader when the energy of the bound state is closer to zero, and a greater number of oscillations are observable within it. The depth (amplitude) of the soliton, on the other hand, increases when the negative energy of the bound state is deeper. The sequence of the pictures illustrates the propagation of the soliton in the periodic background of the Lamé potential.

$$\chi_{0,1}^{\beta_1^-}(x;C_1) = C_1\Theta(x+\beta_1^-)\exp(-xz(\beta_1^-)) + \frac{1}{C_1}\Theta(x-\beta_1^-)\exp(xz(\beta_1^-)). \tag{4.5}$$

The $\Theta(x)$ function appearing in the denominator of $\mathcal{F}_+(x)$, see Eq. (2.6), cancels the nontrivial potential term $-2\mathrm{dn}^2x$ in the Lamé Hamiltonian $H_{0,0}$ via the equality $\frac{d^2}{dx^2}(\log\Theta(x)) = \mathrm{dn}^2x - \frac{\mathbf{E}}{\mathbf{K}}$, that results in the nonperiodic potential (4.3), (4.4); see Fig. 4. By the Darboux construction, the system $H_{0,1}$ has the same spectrum as the one-gap Lamé system except that it possesses an additional discrete level of energy ε_1^-. This is the eigenvalue of the bound state described by the normalizable nodeless wave function

$$\Psi_{0,1}^{;1}(x;\beta_1^-,C_1) = \frac{1}{\mathcal{F}_+(x;\beta_1^-,C_1)} \tag{4.6}$$

shown in Fig. 5, which is a zero mode of the operator $A_{0,1}^\dagger$. The nonzero lower index in the Hamiltonian and potential reflects here the property that the system possesses one bound state in the lower forbidden band.

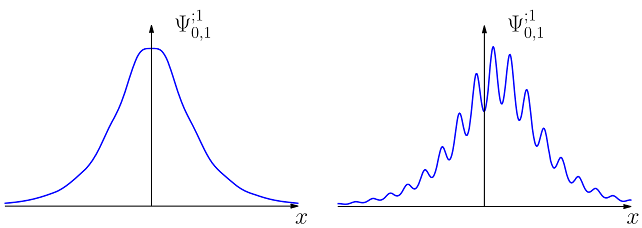

FIG. 5 (color online). The bound state eigenfunction of the system $H_{0,1}$. The state on the left corresponds to the potential $V_{0,1}$ with $C = 1$ in the central picture in Fig. 4. The state on the right, with $C > 1$, has energy closer to zero: when the energy modulus is lower, the state is broader, and the oscillations in it are well notable. By varying the parameter C, the soliton defect in the potential is displaced as well as the position of the bound state supported by it. In correspondence with this, in the case of $0 < C < 1$ not shown here, a localization of the wave function of the bound state is shifted to the $x < 0$ region in comparison with the case $C > 1$.

The upper index in notation for the wave function of the bound state is introduced bearing in mind a generalization for the case of a perturbed Lamé system with various bound states supported both in lower and upper forbidden bands.

Other physical and nonphysical eigenfunctions of $H_{0,1}$ are given by $A_{0,1}\Psi_\pm^\alpha(x)$. They correspond to the same permitted and prohibited values of energy as the eigenstates $\Psi_\pm^\alpha(x)$ of the periodic Lamé Hamiltonian. This shows that the introduced nonperiodic defect is reflectionless; physical Bloch states are transformed into the Bloch states.

Asymptotically, in the limit $x \to -\infty$, the potential has a form of the one-gap periodic Lamé potential, $V_{0,1}(x) \to V_{0,1}^{-\infty}(x) = V_{0,0}(x+\beta_1^-)$. In another limit $x \to +\infty$, we have $V_{0,1}(x) \to V_{0,1}^{+\infty}(x) = V_{0,0}(x-\beta_1^-)$. So, the defect produces a phase shift between the asymptotically periodic one-gap potentials that is equal to $-2\beta_1^-$. This observation follows also directly from (4.1). Asymptotically, we have $\mathcal{F}_+(x;\beta_1^-,C_1) \to C_1F(x;\beta_1^-)$ when $x \to -\infty$, and $\mathcal{F}_+(x;\beta_1^-,C_1) \to C_1^{-1}F(-x;\beta_1^-)$ for $x \to \infty$. Employing the results discussed below (3.5), we can write

$$A_{0,1}A_{0,1}^\dagger \to H_{0,0}(x \pm \beta_1^-) - \varepsilon_1^- \quad \text{for } x \to \mp\infty. \tag{4.7}$$

We get the phase displacement

$$\Delta\phi(\beta_1^-) = -2\beta_1^-, \qquad \varepsilon_1^- = -\mathrm{cd}^2\beta_1^- < 0, \tag{4.8}$$

where we indicate the discrete energy level of the bound state of $H_{0,1}$. The potential $V_{0,1}(x)$ may be treated as a soliton defect in the background of the one-gap periodic Lamé system.

Notice that in the limit $C_1 \to \infty$ (or $C_1 \to 0$) the soliton "goes" to infinity, and in correspondence with Eq. (4.3), $H_{0,1}$ transforms into the shifted Lamé Hamiltonian $H_{0,0}(x+\beta_1^-)$ [or $H_{0,0}(x-\beta_1^-)$].

Before we proceed further, let us show that the infinite-period limit of the obtained system with a periodicity defect corresponds to a reflectionless system of a generic form (2.17) with two bound states of energies $\mathcal{E}_0 = 0$ and $\mathcal{E}_1 = 1 - \kappa_1^2 < 0$. To this aim, we apply the limit $k \to 1$

to the operator (4.2). The nonphysical eigenfunction $\mathcal{F}_+(1)$ of the Lamé system in this limit transforms into the eigenfunction (2.16), whose explicit form is

$$\varphi_1(x;\kappa_1,\tau_1) = \frac{1}{\cosh x} W(\cosh x, \sinh \kappa_1(x+\tau_1)). \tag{4.9}$$

Indeed, in the indicated limit $Z(\beta|1) = \tanh\beta$, and $\mathrm{z}(\beta_1^-)$, defined in Table I, reduces to $\mathrm{z}(\beta_1^-) \to \tanh\beta_1^- + \frac{1}{\sinh\beta_1^- \cosh\beta_1^-} = \mathrm{cotanh}\,\beta_1^- \equiv \kappa_1$, where $1 < \kappa_1 < \infty$ since $\mathbf{K} \to \infty$, and then $\beta_1^- \in (0,\infty)$. We have also $\frac{\Theta(x\pm\beta|1)}{\Theta(x|1)} = \frac{\cosh(x\pm\beta)}{\cosh x}$. Introducing the notation $C_1 \equiv \exp \kappa_1\tau_1$, where τ_1 is an arbitrary real parameter, we find that $\mathcal{F}_+(1)$ transforms into $\frac{1}{\cosh x}(\cosh(x+\beta_1^-)\exp(-\kappa_1(x+\tau_1)) + \cosh(x-\beta_1^-)\exp(\kappa_1(x+\tau_1)))$. This function reduces, up to inessential nonzero multiplicative constant $\sinh\beta_1^-$, to (4.9). Then, in correspondence with the discussion of Sec. II B, the limit of the operator (4.2) is the Darboux generator, which intertwines the reflectionless Pöschl–Teller Hamiltonian (2.7) with the Hamiltonian operator (2.17). Thus, we conclude that the infinite-period limit of (4.3) corresponds to the reflectionless system (2.17).

To introduce several discrete energies into the spectrum of the one-gap Lamé system by making use of its nonphysical states from the lower prohibited band, consider first the case of the two bound states. It is not difficult to show that the Wronskian $W(\mathcal{F}_+(1),\mathcal{F}_-(2)) = \mathcal{F}_+(1)\mathcal{F}'_-(2) - \mathcal{F}'_+(1)\mathcal{F}_-(2)$, where $\mathcal{F}_+(1) = \mathcal{F}_+(x;\beta_1^-,C_1)$, $\mathcal{F}_-(2) = \mathcal{F}_-(x;\beta_2^-,C_2)$, takes strictly negative values, $W(x) < 0$, if $\mathbf{K} > \beta_1^- > \beta_2^- > 0$; see the Appendix. The corresponding energies of the nonphysical eigenstates of $H_{0,0}$ are ordered then as $0 > \mathcal{E}(\beta_1^- + i\mathbf{K}') > \mathcal{E}(\beta_2^- + i\mathbf{K}')$. With such a choice of the states, we can construct the Darboux–Crum transformation producing a nonperiodic deformation of Lamé system, which in addition to the one-gap spectrum of $H_{0,0}(x)$ has two discrete energy values $\varepsilon_j^- = \mathcal{E}(\beta_j^- + i\mathbf{K}')$, $j = 1, 2$,

$$H_{0,2} = -\frac{d^2}{dx^2} + V_{0,2}(x),$$
$$V_{0,2}(x) = V_{0,0}(x) - 2\frac{d^2}{dx^2}(\log W(\mathcal{F}_+(1),\mathcal{F}_-(2))). \tag{4.10}$$

The discrete energy levels ε_1^- and ε_2^- correspond, respectively, to the two bound states

$$\Psi_{0,2}^{;1}(x;\beta_1^-,C_1,\beta_2^-,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\mathcal{F}_-(1))}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))}, \tag{4.11}$$

$$\Psi_{0,2}^{;2}(x;\beta_1^-,C_1,\beta_2,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\mathcal{F}_+(2))}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))}. \tag{4.12}$$

Other physical and nonphysical eigenstates of the system (4.10) are given by

$$\Psi_{0,2;\pm}^{\alpha}(x;\beta_1^-,C_1,\beta_2^-,C_2) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\Psi_\pm^\alpha)}{W(\mathcal{F}_+(1),\mathcal{F}_-(2))} \tag{4.13}$$

and correspond to the Darboux–Crum mapping of the eigenstates (2.2) of the initial Lamé system. The energies of these states are defined by the values of the parameter α exactly in the same way as for the system (2.1). In accordance with (4.1), expressions (4.11) and (4.12) for the bound states correspond to linear combinations of the eigenstates (4.13) with $\alpha = \beta_1^- + i\mathbf{K}'$ and $\alpha = \beta_2^- + i\mathbf{K}'$, respectively.

Let us take now n states

$$\mathcal{F}_{s_j}(j) = \mathcal{F}_{s_j}(x;\beta_j^-,C_j) \quad \text{with} \quad \mathbf{K} > \beta_1^- > \beta_2^- > \ldots > \beta_n^- > 0, \tag{4.14}$$

where s_j corresponds to a linear combination of the form (4.1) with index $+(-)$ for j odd (even). Then, by applying the Darboux–Crum construction on the basis of these eigenstates, we obtain a nonperiodic deformation $H_{0,n}$ of the Lamé system $H_{0,0}$ with n bound states with energies $0 > \varepsilon_1^- > \varepsilon_2^- > \ldots\varepsilon_n^- > -\infty$.

The potential of this system is given by a generalization of Eq. (4.10), in which the Wronskian has to be changed for

$$\mathbb{W}_{0,n}(x) = W(\mathcal{F}_+(1), \quad \mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n)). \tag{4.15}$$

The n bound states of energies ε_j^- are described by the normalizable wave functions

$$\Psi_{0,n}^{;j}(x;\beta_1^-,C_1,\ldots,\beta_n^-,C_n) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n),\mathcal{F}_{-s_j}(j))}{\mathbb{W}_{0,n}}, \quad j = 1,\ldots,n, \tag{4.16}$$

while other corresponding eigenstates of $H_{0,n}$ are given by the generalization of Eq. (4.13),

$$\Psi_{0,n;\pm}^{\alpha}(x;\beta_1^-,C_1,\ldots,\beta_n^-,C_n) = \frac{W(\mathcal{F}_+(1),\mathcal{F}_-(2),\ldots,\mathcal{F}_{s_n}(n),\Psi_\pm^\alpha)}{\mathbb{W}_{0,n}}. \tag{4.17}$$

As in the case (4.10), bound states (4.16) may be obtained from (4.17) by putting there $\alpha = \beta_j^- + i\mathbf{K}'$, $j = 1,\ldots,n$, and changing the wave functions $\Psi_\pm^\alpha$ on the rhs for the corresponding linear combinations of them.

Applying then the limit $x\to-\infty$ to the Wronskian $\mathbb{W}_{0,n}(x)$, we find that it transforms, up to a multiplicative constant, into $\mathbb{W}_{0,n}(x)=W(F(x;\beta_1^-)(x),\ldots,F(x;\beta_n^-))$. Asymptotically, we get a potential $V_{0,n}^{-\infty}(x)=\lim_{x\to-\infty}(-2\frac{d^2}{dx^2}\log\mathbb{W}_{0,n}(x))=V_{0,0}(x+b)$, where $b=\sum_{j=1}^n\beta_j$. Analogously, in another limit $x\to+\infty$, we get the asymptotic form of the potential $V_{0,n}^{+\infty}(x)=V_{0,0}(x-b)$. The phase displacement produced by the n solitons (defects) is

$$\Delta\phi(\beta^-)=-2\sum_{j=1}^n\beta_j^-, \tag{4.18}$$

which generalizes the one-soliton effect (4.8).

The eigenstates of the system $H_{0,n}$ (4.16) and (4.17) can be presented in an alternative form [30],

$$\Psi(x;\beta_1^-,C_1,\ldots,\beta_n^-,C_n)=\mathbb{A}_{0,n}\Psi(x),\qquad \mathbb{A}_{0,n}=A_{0,n}A_{0,n-1}\ldots A_{0,1}, \tag{4.19}$$

where the wave function on the lhs corresponds to (4.16) for the choice $\Psi=\mathcal{F}_{-s_j}(j)$ on the rhs, while it corresponds to the eigenfunctions (4.17) for the choice $\Psi=\Psi_\pm^\alpha$ on the rhs. The operator $\mathbb{A}_{0,n}$ is a differential operator of order n, which is constructed in terms of the recursively defined first-order differential operators (4.2) and

$$A_{0,j}=(\mathbb{A}_{0,j-1}\mathcal{F}_{s_j}(j))\frac{d}{dx}\frac{1}{(\mathbb{A}_{0,j-1}\mathcal{F}_{s_j}(j))}=\frac{d}{dx}+\mathcal{W}_{0,j},\qquad j=2,\ldots, \tag{4.20}$$

where

$$\mathcal{W}_{0,j}=\Omega_{0,j}-\Omega_{0,j-1},\qquad \Omega_{0,j}=-(\log\mathbb{W}_{0,j})_x, \tag{4.21}$$

and $\mathbb{W}_{0,1}\equiv\mathcal{F}_+(1)$. Equations (4.20) and (4.21) can also be used for $j=1$ by putting $\mathbb{W}_{0,0}=1$. Note here that, making use of Eqs. (4.19), it is easy to see that in the case of the two-soliton defect, particularly, the bound states (4.12) and (4.11) are reduced modulo multiplicative constants to the functions $\mathcal{F}_+(1)/\mathbb{W}_{0,2}$ and $\mathcal{F}_-(2)/\mathbb{W}_{0,2}$, respectively. This shows explicitly that the first function describing the discrete ground state is nodeless, while the second wave function corresponding to the first excited bound state has exactly one zero as it should be for the lowest bound states in the spectrum.

Relation (4.19) means that the operator $\mathbb{A}_{0,n}$ maps the eigenstates of the Lamé system (2.1) into the corresponding eigenstates of $H_{0,n}$. Its n-dimensional kernel is spanned by the eigenstates $\mathcal{F}_{s_j}(j)$, $j=1,\ldots,n$. These relations reflect the fact that the Darboux–Crum transformation of order n corresponds to a composition of n subsequent Darboux maps $H_{0,0}\to H_{0,1}\to\ldots\to H_{0,n}$. In accordance with this, the operators $\mathbb{A}_{0,n}$ and $\mathbb{A}_{0,n}^\dagger$ intertwine the Hamiltonian operator $H_{0,n}(x)$ with the Lamé Hamiltonian $H_{0,0}(x)$,

$$\mathbb{A}_{0,n}H_{0,0}=H_{0,n}\mathbb{A}_{0,n},\qquad \mathbb{A}_{0,n}^\dagger H_{0,n}=H_{0,0}\mathbb{A}_{0,n}^\dagger. \tag{4.22}$$

The products of the operator $\mathbb{A}_{0,n}$ and its conjugate are

$$\mathbb{A}_{0,n}\mathbb{A}_{0,n}^\dagger=\prod_{j=1}^n(H_{0,n}-\varepsilon_j^-),\qquad \mathbb{A}_{0,n}^\dagger\mathbb{A}_{0,n}=\prod_{j=1}^n(H_{0,0}-\varepsilon_j^-). \tag{4.23}$$

Alternative representation given by Eqs. (4.19) and (4.20) is valid for arbitrary Darboux–Crum transformations generated on the basis of n eigenstates of a generic Schrödinger Hamiltonian [30]. In the particular case of the one-gap Lamé system $H=H_{0,0}$ and the choice of eigenstates $\psi_j(x)=\Psi_+^{\alpha_j}(x)$, each of which, as we saw in the previous section, generates the translation of the Lamé system for $\alpha_j+i\mathbf{K}'$, we obtain the Darboux–Crum transformation producing the translation of $H_{0,0}(x)$ for $\sum_{j=1}^n\alpha_j+in\mathbf{K}'$. Taking into account that the system (2.1) besides the real period $2\mathbf{K}$ possesses also the imaginary period $2i\mathbf{K}'$, the shift produced by the Darboux–Crum transformation reduces to $\sum_{j=1}^{2r}\alpha_j$ in the case of even $n=2r$ and to $\sum_{j=1}^{2r+1}\alpha_j+i\mathbf{K}'$ when $n=2r+1$ is odd. Making use of this observation, it is obvious that when the total shift produced by the Darboux–Crum transformation reduces to a nontrivial period $2\mathbf{K}n_1+2i\mathbf{K}'n_2$ of the system (2.1) with $n_1^2+n_2^2\geq 2$, the corresponding higher-order generator $\mathbb{A}_n$ gives us the integral (multiplied in a generic case by a polynomial in $H_{0,0}$ [47]) of the one-gap Lamé system. This is the analog of the integral (2.13) of the reflectionless Pöschl–Teller system (2.7), which is the Lax–Novikov integral $\mathcal{P}_{0,0}$ for the system (2.1),

$$i\mathcal{P}_{0,0}=\frac{d^3}{dx^3}+(1+k^2-3k^2\mathrm{sn}^2x)\frac{d}{dx}-3k^2\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x. \tag{4.24}$$

In the limit $k\to 1$, it transforms into (2.13). The kernel of this third-order differential operator is spanned by eigenfunctions $\mathrm{dn}x$, $\mathrm{cn}x$, and $\mathrm{sn}x$, which correspond to the edges of the allowed bands. In correspondence with this, it admits an infinite number of factorizations. Particularly, it can be presented in the form

$$i\mathcal{P}_{0,0}=A_{1/\mathrm{cn}\,x}A_{\mathrm{cn}\,x/\mathrm{dn}\,x}A_{\mathrm{dn}\,x}, \tag{4.25}$$

where $A_{\mathrm{dn}\,x}$ is defined by relation of the form (3.1) with $\psi(x) = \mathrm{dn}\,x$, etc.

The sense of the factorization (4.25) is the following. The first factor on the right, $A_{\mathrm{dn}\,x}$, in accordance with its definition, annihilates dn$x$, the lower edge state of the valence band that is proportional to the limit of $F(x;\beta^-)$ for $\beta^- = \mathrm{K}$. Acting on the wave function sn$x$, which corresponds to the lower edge of the conduction band, the operator $A_{\mathrm{dn}\,x}$ translates it, as well as all other eigenstates of the Lamé system, for the half-period $\mathbf{K}$, $\mathrm{sn}(x+\mathbf{K}) = \mathrm{cn}\,x/\mathrm{dn}\,x$, and then this sn function with a shifted argument is annihilated by the operator $A_{\mathrm{cn}\,x/\mathrm{dn}\,x}$. Acting on the wave function $\mathrm{cn}\,x$, which describes the upper edge state of the valence band, the $A_{\mathrm{dn}\,x}$ transforms it into $\mathrm{cn}(x+\mathbf{K})$, while the subsequent action of the $A_{\mathrm{cn}\,x/\mathrm{dn}\,x}$ transforms this into $\mathrm{cn}(x+\mathbf{K}+i\mathbf{K}') = -ik'/k\mathrm{cn}\,x$, which is annihilated finally by the first-order operator $A_{1/\mathrm{cn}\,x}$. In a similar way, one can construct five other factorizations of $\mathcal{P}_{0,0}$ having a simple interpretation in terms of the Darboux transformations (translations) generated by the edge states. Relation (4.27) corresponds here to the Darboux–Crum transformation that generates the total shift for the nontrivial period $2\mathbf{K}n_1 + 2i\mathbf{K}'n_2$ with $n_1 = n_2 = 1$ in correspondence with the discussion presented above.

The Lamé system's integral $\mathcal{P}_{0,0}$ satisfies the Burchnall–Chaundy relation

$$\mathcal{P}_{0,0}^2 = H_{0,0}(H_{0,0} - k'^2)(H_{0,0} - 1), \tag{4.26}$$

which lies in the basis of the hidden bosonized nonlinear supersymmetry of the one-gap Lamé system [18]. The zeros of the third-order polynomials in $H_{0,0}$ correspond to the energies of the edges of the allowed bands of (2.1). In the limit $k \to 1$, (4.26) transforms into relation (2.14), in which the double factor H_1^2 originates from the first two factors in (4.26) and roots in the shrinking of the valence band.

By analogy with the Lax–Novikov integral (2.13) for the reflectionless Pöschl–Teller system with one bound state, we can find the analogous integral for the $H_{0,n}$ system,

$$\mathcal{P}_{0,n} = \mathbb{A}_{0,n}\mathcal{P}_{0,0}\mathbb{A}_{0,n}^{\dagger}, \qquad [\mathcal{P}_{0,n}, H_{0,n}] = 0, \tag{4.27}$$

which is the differential operator of the order $2n+3$. In correspondence with (4.26) and (4.23), it satisfies the Burchnal–Chaundy relation

$$\mathcal{P}_{0,n}^2 = H_{0,n}(H_{0,n} - k'^2)(H_{0,n} - 1)\prod_{j=1}^{n}(H_{0,n} - \varepsilon_j^-)^2. \tag{4.28}$$

The systems $H_{0,0}$ and $H_{0,n}$ can be intertwined not only by the operators $\mathbb{A}_{0,n}$ and $\mathbb{A}_{0,n}^{\dagger}$ but also by the operators

$$\mathbb{B}_{0,n} = \mathbb{A}_{0,n}\mathcal{P}_{0,n} \quad \text{and} \quad \mathbb{B}_{0,n}^{\dagger}. \tag{4.29}$$

B. Intermediate forbidden band

Let us consider the intermediate prohibited band (gap) and the linear combinations of eigenstates (2.2) in it,

$$\Phi_+(1) \equiv \Phi_+(x;\beta_1^+, C_1) = C_1\Psi_+^{\beta_1^+}(x) + \frac{1}{C_1}\Psi_-^{\beta_1^+}(x), \tag{4.30}$$

$$\Phi_-(2) \equiv \Phi_-(x;\beta_2^+, C_2) = C_2\Psi_+^{\beta_2^+}(x) - \frac{1}{C_2}\Psi_-^{\beta_2^+}(x), \tag{4.31}$$

where $0 < \beta_l^+ < \mathbf{K}$ and C_l, $l = 1, 2$ are arbitrary real constants restricted by the condition $C_l > 0$. Taking into account relation (2.4), the linear combinations used here differ effectively in sign in comparison to those employed in (4.1). This is related to the fact that the eigenvalue $\mathcal{E}(\beta^- + i\mathbf{K}')$ is an increasing function of the real parameter β^- in the lower prohibited band, while $d\mathcal{E}(\beta^+)/d\beta^+ < 0$ in the intermediate, upper forbidden band. Both these functions have an infinite number of zeros on the real line. The choice of any of these two functions as the function ψ in operator (3.1) produces by means of the first-order Darboux transformation a singular partner for the system $H_{0,0}(x)$.

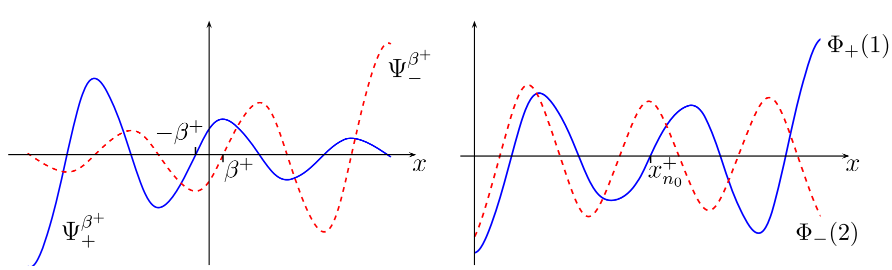

FIG. 6 (color online). Zeros of $\Psi_\pm^{\beta^+}(x)$ are in the equidistant points $2n\mathbf{K} \mp \beta^+$, and the amplitudes of these two functions increase exponentially in opposite directions. The amplitudes of the oscillating states $\Phi_\pm$ increase exponentially in both directions. The graphic on the right corresponds to the case $\beta_1^+ < \beta_2^+$.

Our next goal is to show how, by appropriate use of the second-order Darboux–Crum transformation applied to $H_{0,0}$, one can generate a regular system with two bound states in the gap.

Zeros of the nonphysical eigenfunctions $\Psi_+^{\beta^+}(x)$ are $-\beta^+ + 2n\mathbf{K}$, while the infinite set of zeros of the eigenstates $\Psi_-^{\beta^+}(x)$ is $\beta^+ + 2n\mathbf{K}$, $n \in \mathbb{Z}$. On the open intervals $(-\beta^+, \beta^+) + 2n\mathbf{K}$, functions $\Psi_+^{\beta^+}(x)$ and $\Psi_-^{\beta^+}(x)$ take nonzero values of the opposite sign, whereas on the open intervals $(\beta^+, 2\mathbf{K} - \beta^+) + 2n\mathbf{K}$, they take values of the same sign. Therefore, zeros of the linear combination (4.30) of $\Psi_+^{\beta^+}(x)$ and $\Psi_-^{\beta^+}(x)$ with $\beta_1^+ = \beta^+$ are inside the first of the indicated set of the open intervals, and zeros of (4.31) with $\beta_2^+ = \beta^+$ are inside the second set of the intervals. Since Φ_+ and Φ_- are linearly independent eigenstates of the same eigenvalue $\mathcal{E}(\beta^+)$, in correspondence with the oscillation theorem, each of the indicated open intervals contains exactly one zero of the respective function.

We want to generate a nontrivial nonsingular Darboux–Crum transformation based on the pair of the eigenfunctions (4.30) and (4.31). For this, the Wronskian of these functions should take nonzero nonconstant values. The choice

$$0 < \beta_1^+ < \beta_2^+ < \mathbf{K} \Leftrightarrow \mathcal{E}(\beta_1^+) > \mathcal{E}(\beta_2^+) \tag{4.32}$$

guarantees then that the intervals containing zeros of the functions (4.30) and (4.31) do not intersect, and between each two neighbor zeros x_n^+ and x_{n+1}^+ of the $\Phi_+(x; \beta_1^+, C_1)$, there will appear exactly one zero x_n^- of the $\Phi_-(x; \beta_2^+, C_2)$,

$$x_n^+ \in \mathcal{I}_n^+(1), \qquad x_n^- \in \mathcal{I}_n^-(2), \qquad \mathcal{I}_n^+(1) \cap \mathcal{I}_{n'}^-(2) = \emptyset, \tag{4.33}$$

where

$$\mathcal{I}_n^+(1) = (-\beta_1^+, \beta_1^+) + 2n\mathbf{K},$$
$$\mathcal{I}_n^-(2) = (\beta_2^+, 2\mathbf{K} - \beta_2^+) + 2n\mathbf{K}. \tag{4.34}$$

The amplitudes of the oscillating functions $\Psi_+^{\beta^+}(x)$ and $\Psi_-^{\beta^+}(x)$ increase exponentially for $x \to -\infty$ and $x \to +\infty$, respectively. As a consequence, in the limit $x \to +\infty$, the zeros x_n^+ tend to the right edges of the intervals $\mathcal{I}_n^+(1)$, while x_n^- tend to the left edges of the intervals $\mathcal{I}_n^-(2)$. In another limit $x \to -\infty$, the corresponding zeros tend to the opposite edges of the indicated intervals.

The Wronskian of the eigenfunctions (4.30) and (4.31) obeys the relation

$$\frac{d}{dx} W(y_1, y_2) = (\mathcal{E}(\beta_1^+) - \mathcal{E}(\beta_2^+)) y_1(x) y_2(x), \tag{4.35}$$

where $y_1 = \Phi_+(1)$ $y_2(x) = \Phi_-(2)$. From (4.35), it follows that zeros $x_n^\pm$ correspond exactly to the local extrema of the Wronskian. Let us choose a zero $x_{n_0}^+$ of y_1, $y_1(x_{n_0}^+) = 0$, such that $y_1'(x_{n_0}^+) > 0$. Then, in principle, we have two possibilities: either (i) $y_2(x_{n_0}^+) > 0$ or (ii) $y_2(x_{n_0}^+) < 0$. In case i, we find that $W(x_n^\pm) < 0$, while in case ii, we would have $W(x_n^\pm) > 0$ for any $n \in \mathbb{Z}$. Differentiation of (4.35) in x shows that in case i the zeros x_n^- and x_n^+ correspond to the local maxima and minima of the Wronskian, respectively. In case ii, the role of these zeros as local maxima and minima would be interchanged. Then, in case i, we conclude that the Wronskian takes strictly negative values for all x, while in case ii, it would be a strictly positive function. Though in both cases we would have a nodeless Wronskian, let us show that case i, illustrated on Fig. 6, is realized here. In the limits $x \to \pm\infty$, in correspondence with definition (4.30), (4.31), we have

$$\lim_{x \to +\infty} W(\Phi_+(1), \Phi_-(2)) = -\frac{1}{C_1 C_2} W(\Psi_-^{\beta_1^+}(x), \Psi_-^{\beta_2^+}(x)), \tag{4.36}$$

$$\lim_{x \to -\infty} W(\Phi_+(1), \Phi_-(2)) = C_1 C_2 W(\Psi_+^{\beta_1^+}(x), \Psi_+^{\beta_2^+}(x)). \tag{4.37}$$

Using these relations and the above-described behavior of the zeros of the functions $\Phi_+(1)$ and $\Phi_-(2)$ in the limit $x \to +\infty$, the corresponding local extrema values of W are given by

$$\lim_{x_n^\pm \to +\infty} W(x_n^\pm) = -\frac{1}{C_1 C_2} \frac{\mathrm{H}'(0)\mathrm{H}(\beta_2^+ - \beta_1^+)}{\Theta^2(\beta_j)} \times \exp((\beta_j^+ + 2n\mathbf{K})(\mathrm{Z}(\beta_1^+) + \mathrm{Z}(\beta_2^+))), \quad n \gg 1, \tag{4.38}$$

where $j = 1, 2$ and $\beta_1^+(\beta_2^+)$ corresponds here to $x_n^+(x_n^-)$. For the limits $x_n^\pm \to -\infty$, we have a similar expression with a unique change of the coefficient $1/(C_1 C_2)$ for $C_1 C_2$. Taking into account that $\mathrm{H}'(0) = \sqrt{2kk'\mathbf{K}/\pi} > 0$, and that $\mathrm{H}(\beta_2^+ - \beta_1^+) > 0$ because $0 < \beta_2^+ - \beta_1^+ < \mathbf{K}$, we conclude finally that $\mathbb{W}_{2,0}(x) = W(\Phi_+(1), \Phi_-(2))$ takes strictly negative values on all the real line. Additionally, we conclude that $-\mathbb{W}_{2,0}(x)$ blows up exponentially in both limits $x \to \pm\infty$.

Similarly to (4.10), we construct now the Hamiltonian

$$H_{2,0} = -\frac{d^2}{dx^2} + V_{2,0}(x),$$
$$V_{2,0}(x) = V_{0,0}(x) - 2\frac{d^2}{dx^2} \log W(\Phi_+(1), \Phi_-(2)). \tag{4.39}$$

This quantum system has the same spectrum as the Lamé system except two additional discrete energy levels $\varepsilon_l^+ \equiv \mathcal{E}(\beta_l^+)$, $l = 1, 2$. These are described by the wave functions given by relations of the form (4.11), (4.12) with $\mathcal{F}_\pm(j)$ there changed for corresponding functions $\Phi_\pm(l)$. With some algebraic manipulations, the wave eigenfunctions can be presented in the form

$$\Psi_{2,0}^{1;}(x) = \text{const}\frac{\Phi_-(2)}{\mathbb{W}_{2,0}}, \qquad H_{2,0}\Psi_{2,0}^{1;}(x) = \varepsilon_1^+\Psi_{2,0}^{1;}(x), \tag{4.40}$$

$$\Psi_{2,0}^{2;}(x) = \text{const}\frac{\Phi_+(1)}{\mathbb{W}_{2,0}}, \qquad H_{2,0}\Psi_{2,0}^{2;}(x) = \varepsilon_2^+\Psi_{2,0}^{2;}(x). \tag{4.41}$$

The amplitude of these oscillating functions tends exponentially to zero in both limits $x \to \pm\infty$, which confirms their bound state nature; see Fig. 7. The relations (4.36) and (4.37) tell us that the Darboux–Crum transformation generated on the basis of the states appearing there on the right-hand sides produces a potential translated in $(\beta_1^+ + i\mathbf{K}') + (\beta_2^+ + i\mathbf{K}')$. Using this fact and taking into account the imaginary period $2i\mathbf{K}'$ of $V_{0,0}(x)$, we find that

$$V_{2,0}^{-\infty}(x) = \lim_{x\to-\infty} V_{2,0}(x) = V_{0,0}(x + \beta_1^+ + \beta_2^+),$$

and, analogously,

$$V_{2,0}^{+\infty}(x) = \lim_{x\to+\infty} V_{2,0}(x) = V_{0,0}(x - \beta_1^+ - \beta_2^+).$$

Therefore, similarly to the case of soliton defects corresponding to the bound states in the lower forbidden band, the two-soliton defect associated with the presence of the two bound states in the intermediate (upper) prohibited band produces the phase shift described by Eq. (4.18) with $n = 2$ and β_j^- there changed for β_l^+, where the parameters β_1^+ and β_2^+ obey the condition (4.32). The bound states here are described by infinitely oscillating wave functions, which have an infinite number of zeros and exponentially decreasing amplitudes. This situation contrasts with the bound states introduced into the lower forbidden band, where the wave functions are also exponentially decreasing but have a finite number of zeros, similarly to the nature of ordinary bound states.

The system (4.39) is also characterized by the Lax–Novikov integral, which in the present case is the differential operator of order 7,

$$\mathcal{P}_{2,0} = \mathbb{A}_{2,0}\mathcal{P}_{0,0}\mathbb{A}_{2,0}^\dagger, \qquad [\mathcal{P}_{2,0}, H_{2,0}] = 0. \tag{4.42}$$

The second-order operators $\mathbb{A}_{2,0}$ and $\mathbb{A}_{2,0}^\dagger$ intertwining the Lamé system $H_{0,0}$ with $H_{2,0}$ have the form (4.20) and (4.19) with the functions $\mathcal{F}_+(1)$ and $\mathcal{F}_-(2)$ changed here, respectively, for $\Phi_+(1)$ and $\Phi_-(2)$. They satisfy relations of the form (4.23) with $n = 2$, where $H_{0,n}$ has to be changed for $H_{2,0}$, and constants ε_j^- have to be changed for corresponding energy values ε_l^+, $l = 1, 2$, of the nonphysical eigenstates from the intermediate prohibited band we used in the construction.

Analogously to the discussion presented in the previous subsection, it is not difficult to show that the infinite-period limit applied to the system (4.39) corresponds to the reflectionless system given by potential (2.18).

The described procedure of the introduction of the periodicity defects with eigenvalues within the intermediate prohibited band can be generalized for the case of an arbitrary even number of the solitons. This can be done in a systematic way by choosing linear combinations of the wave functions of the form (4.30) and (4.31) with alternating lower indices + and −, cf. (4.15), with the restriction on the parameters β^+, which generalizes that from (4.32),

$$0 < \beta_1^+ < \beta_2^+ < \dots \beta_{2\ell}^+ < \mathbf{K} \Leftrightarrow \mathcal{E}(\beta_1^+) > \mathcal{E}(\beta_2^+)\dots > \mathcal{E}(\beta_{2\ell}^+). \tag{4.43}$$

In the basis of such a construction, there is the property $|\mathbb{W}_{2\ell,0}(x)| > 0$ guaranteed by the choice (4.43), where $\mathbb{W}_{2\ell,0}(x)$ is the Wronskian of the corresponding 2ℓ nonphysical eingestates of the Lamé system,

$$\mathbb{W}_{2\ell,0}(x) = W(\Phi_+(1), \Phi_-(2), \dots, \Phi_+(2\ell-1), \Phi_-(2\ell)). \tag{4.44}$$

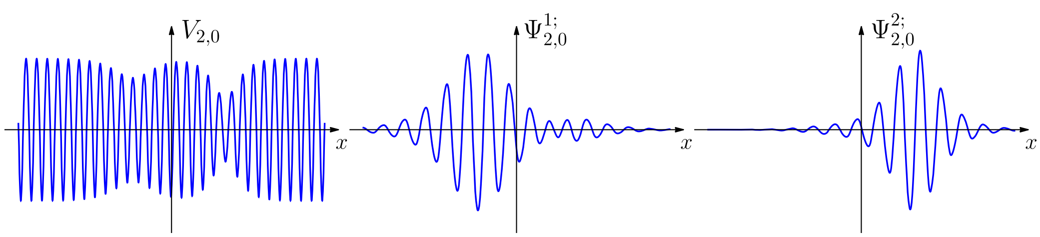

FIG. 7 (color online). Each of the two pulse-type bound states of the system $H_{2,0}$ is localized in one of the two periodicity defects of the potential $V_{2,0}$, which are showing up as compression modulations. The states also reveal a small tunnelling (asymmetry) in the direction of the other deformation.

The proof of this property is given in the Appendix.[5] As a generalization of (4.39) and (4.42), the Hamiltonian and Lax–Novikov integral are given here by the relations

$$H_{2\ell,0} = H_{0,0} - 2\frac{d^2}{dx^2}\log \mathbb{W}_{2\ell,0}, \tag{4.45}$$

$$\mathcal{P}_{2\ell,0} = \mathbb{A}_{2\ell,0}\mathcal{P}_{0,0}\mathbb{A}^{\dagger}_{2\ell,0}, \qquad [\mathcal{P}_{2\ell,0}, H_{2\ell,0}] = 0. \tag{4.46}$$

They satisfy the Burchnall–Chaundy relation of the form

$$\mathcal{P}^2_{2\ell,0} = H_{2\ell,0}(H_{2\ell,0} - k'^2)(H_{2\ell,0} - 1)\prod_{l=1}^{2\ell}(H_{2\ell,0} - \varepsilon_l^+)^2. \tag{4.47}$$

Here, $\varepsilon_l^+ = \mathcal{E}(\beta_l^+)$ are the eigenvalues of the bound states

$$\Psi^{l;}_{2\ell,0}(x;\beta_1^+, C_1, \ldots, \beta^+_{2\ell}, C_{2\ell}) = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \Phi_{(-1)^l}(l))}{\mathbb{W}_{2\ell,0}}, \quad l = 1, \ldots, 2\ell. \tag{4.48}$$

Other physical and nonphysical eigenstates of $H_{2\ell,0}$ of eigenvalues $\mathcal{E}(\alpha)$ are given by

$$\Psi^{\alpha}_{2\ell,0;\pm}(x;\beta_1^+, C_1, \ldots, \beta^+_{2\ell}, C_{2\ell}) = \frac{W(\Phi_+(1), \Phi_-(2), \ldots, \Phi_-(2\ell), \Psi^{\alpha}_{\pm})}{\mathbb{W}_{2\ell,0}}. \tag{4.49}$$

From this picture with even number $2\ell \geq 2$ of bound states in the intermediate forbidden band, one can obtain systems that contain odd number $2\ell - 1$ of discrete energy levels in the same prohibited band of the initial one-gap Lamé system. This can be achieved by sending any one of the 2ℓ solitons to infinity.

Let us see how this procedure works in the case of the system (4.39). For the sake of definiteness, we send the first soliton, associated with the higher discrete energy level $\mathcal{E}(\beta_1^+)$, to infinity. Another case corresponding to the limit associated with the soliton related to the lower discrete energy level can be realized in a similar way. To send the indicated soliton to infinity, we take a limit $C_1 \to \infty$. In analogous way, one can also consider the limit $C_1 \to 0$.

In the limit $C_1 \to \infty$, the potential $V_{2,0}(x)$ given by Eq. (4.39) transforms into

$$\lim_{C_1\to\infty} V_{2,0}(x) \equiv \breve{V}_{1,0}(x;\beta_1^+) = V_{0,0}(x) - 2\frac{d^2}{dx^2}\log W(\Psi_+^{\beta_1^+}, \Phi_-(2)). \tag{4.50}$$

The Hamiltonian $\breve{H}_{1,0}(x;\beta_1^+) = -\frac{d^2}{dx^2} + \breve{V}_{1,0}(x;\beta_1^+)$ possesses single bound state of energy ε_2^+, which can be obtained as a limit of the bound eigenstate $\Psi^{2;}_{2,0}(x)$ of $H_{2,0}$,

$$\lim_{C_1\to\infty}\Psi^{2;}_{2,0}(x) = \breve{\Psi}^{1;}_{1,0}(x); \tag{4.51}$$

see Fig. 8. In correspondence with the results of Sec. III, the Darboux transformation based on the single eigenfunction $\Psi_+^{\beta_1^+}(x)$ produces the Treibich–Verdier potential, $V_{0,0}(x) - 2\frac{d^2}{dx^2}\log\Psi_+^{\beta_1^+} = V_{0,0}(x + \beta_1^+ + i\mathbf{K}')$, and we can present (4.50) in the equivalent form

$$\breve{V}_{1,0}(x) = V_{0,0}(x + \beta_1^+ + i\mathbf{K}') - 2\frac{d^2}{dx^2}\left(\log\frac{W(\Psi_+^{\beta_1^+}, \Phi_-(2))}{\Psi_+^{\beta_1^+}}\right). \tag{4.52}$$

Function $W(\Psi_+^{\beta_1^+}, \Phi_-(2))/\Psi_+^{\beta_1^+}$ appearing in the argument of the logarithm is an eigenfunction of the system $H_{0,0}(x + \beta_1^+ + i\mathbf{K}')$. The Bloch-like eigenstates of this Hamiltonian operator can be obtained from the corresponding eigenstates of the Lamé system $H_{0,0}(x)$, $\Psi^{\alpha}_{\pm}(x + \beta_1^+ + i\mathbf{K}') = N_{\pm}(\alpha)\breve{\Psi}^{\alpha}_{\pm}(x + \beta_1^+)$, where

$$\breve{\Psi}^{\alpha}_{\pm}(x) = \frac{\Theta(x \pm \alpha)}{\mathrm{H}(x)}e^{\mp xZ(\alpha)} \tag{4.53}$$

and $N_{\pm}(\alpha) = \exp\left(\mp i(\frac{\alpha\pi}{2\mathbf{K}} + \mathbf{K}'Z(\alpha)\right)$. Therefore, we have

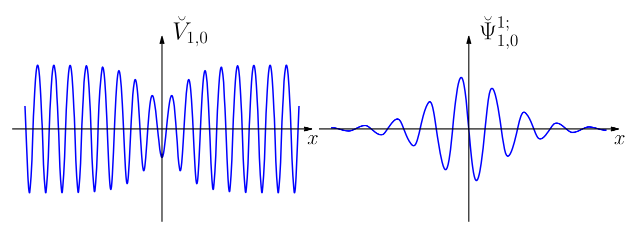

FIG. 8 (color online). Sending one soliton to infinity results in a potential supporting one bound state less. System $\breve{H}_{1,0}$ is related with the Lamé system $H_{0,0}$ by the Darboux–Crum transformation of the second order, while it is related with the singular Treibich–Verdier system by the first-order Darboux transformation. The symmetric state (presented by odd function here) is centered in the soliton deformation of the potential, and the tunnelling related to the soliton sent to infinity disappears.

[5] Like in the procedure shortly discussed in Sec. II B corresponding to the reflectionless Pöschl–Teller system, the defects also can be introduced in such a way that their associated energies will appear between the already placed discrete energy levels, but the final picture will be described equivalently by the Darboux–Crum transformation based on the Wronskian (4.44).

$$\frac{W(\Psi_+^{\beta_1^+},\Psi_\pm^{\beta_2^+})}{\Psi_+^{\beta_1^+}} = C_\pm \breve{\Psi}_\pm^{\beta_2^+}(x+\beta_1^+). \tag{4.54}$$

Putting in both sides of the last relation $x = -\beta_1^+$ (or $x = \mp\beta_2^+$ to escape simple poles at both sides), we define the real nonzero constants $C_\pm$ in (4.54),

$$C_\pm = \mp \frac{\mathrm{H}(\beta_2^+ \mp \beta_1^+)\mathrm{H}'(0)}{\Theta(\beta_1^+)\Theta(\beta_2^+)} \exp\left(\pm\beta_1^+ \mathrm{Z}(\beta_2^+)\right). \tag{4.55}$$

Making a shift $x \to x - \beta_1^+$ in (4.50), all this gives us

$$V_{1,0}(x) \equiv \breve{V}_{1,0}(x-\beta_1^+) = 1 + k'^2 - 2\frac{\mathbf{E}}{\mathbf{K}} - 2\frac{d^2}{dx^2}\log\chi_{1,0}^{\beta_2^+}, \tag{4.56}$$

$$\chi_{1,0}^{\beta_2^+}(x) = \breve{C}_2\Theta(x+\beta_2^+)\exp(-x\mathrm{Z}(\beta_2^+)) + \frac{1}{\breve{C}_2}\Theta(x-\beta_2^+)\exp(x\mathrm{Z}(\beta_2^+)). \tag{4.57}$$

Here, a real constant $\breve{C}_2$ is given in terms of C_2 by $\breve{C}_2 = C_2\sqrt{\frac{\mathrm{H}(\beta_2^+-\beta_1^+)}{\mathrm{H}(\beta_2^++\beta_1^+)}}\exp(\beta_1^+\mathrm{Z}(\beta_2^+)) > 0$, and we have taken into account the relation $\frac{d^2}{dx^2}\log\mathrm{H}(x) = \mathrm{dn}^2(x+i\mathbf{K}') - \frac{\mathbf{E}}{\mathbf{K}}$. In the limit $C_1 \to \infty$, the Wronskian in the denominator of the eigenstate (4.40) of energy $\mathcal{E}(\beta_1^+)$ of the system $H_{2,0}$ blows up exponentially, and this state disappears. On the other hand, the state (4.41) transforms into the bound state of energy $\mathcal{E}(\beta_2^+)$ of the system $H_{1,0}(x) = -\frac{d^2}{dx^2} + V_{1,0}(x)$,

$$\Psi_{2,0}^{2;}(x-\beta_1^+) \to \breve{\Psi}_{1,0}^{1;}(x-\beta_1^+) = \mathrm{const}\frac{\mathrm{H}(x)}{\chi_{1,0}^{\beta_2^+}(x)}. \tag{4.58}$$

The presence of this bound state in the spectrum of $H_{1,0}(x)$ is the unique difference in comparison with the spectrum of the one-gap Lamé system $H_{0,0}(x)$. The system $H_{1,0}(x)$ is related with $H_{0,0}(x)$, however, by the second-order Darboux–Crum transformation of the form (4.50) with x changed there for $x - \beta_1^+$. On the other hand, the system $H_{1,0}(x)$ can be related with the singular Treibich–Verdier system described by the potential $V_{0,0}(x+i\mathbf{K}')$, by the first-order Darboux transformation based on the function $\breve{\Psi}_\pm^\alpha(x-\beta_1^+)$ given by Eq. (4.53), which is the eigenfunction of the singular PT-invariant Hamiltonian operator $H_{0,0}(x+i\mathbf{K}')$. This picture is analogous to that for the Pöschl–Teller system when we want to introduce there the bound state between the already existing bound state and the continuous part of the spectrum; see Sec. II B.

In correspondence with the described picture, the system $H_{1,0}(x)$ is characterized by the irreducible Lax–Novikov integral

$$\mathcal{P}_{1,0}(x) = A_\psi \mathcal{P}_{0,0}(x+i\mathbf{K}')A_\psi^\dagger, \qquad \psi = \breve{\Psi}_+^{\beta_1^+}(x-\beta_1^+), \tag{4.59}$$

which is the differential operator of order 5, where $\mathcal{P}_{0,0}(x)$ is the Lax–Novikov integral (4.24) of the Lamé system $H_{0,0}(x)$. In (4.59), one can take, equivalently, $\psi = \Psi^{\beta_1^+}(x-\beta_1^+ + i\mathbf{K}')$.

Notice a remarkable similarity of the potential $V_{1,0}$ given by Eqs. (4.56) and (4.57) with the potential $V_{0,1}$ defined by Eqs. (4.4) and (4.5). The important difference of both potentials is, however, that $\mathrm{Z}(\beta_2^+)$ presents in the structure of $V_{1,0}$, while in the structure of the potential $V_{0,1}$, there appears $\mathrm{z}(\beta_1^-)$ defined in Table I. Unlike the nodeless bound state (4.6) of the system $V_{0,1}$, the bound state (4.58) of the system $V_{1,0}$ has an infinite number of zeros at $x_n = 2n\mathbf{K}$, and its amplitude, like that of the wave function (4.6), decreases exponentially as x goes to $\pm\infty$.

When $x \to \pm\infty$, Hamiltonian $H_{1,0}(x)$ asymptotically transforms into $H_{0,0}(x \mp \beta_2^+) - \mathcal{E}(\beta_2^+)$, and we get the phase displacement $\Delta\phi(\beta_2^+) = -2\beta_2^+$ generated by the one-soliton potential defect, which supports one bound state within the upper prohibited band of the original one-gap Lamé system.

Let us notice that one can also introduce an odd number of bound states into the gap by taking, instead of (4.32), the set of parameters $0 = \beta_1^+ < \beta_2^+ < \ldots\beta_{2\ell}^+ < \mathbf{K}$, or $0 < \beta_1^+ < \beta_2^+ < \ldots\beta_{2\ell}^+ = \mathbf{K}$. This assumes the change of the state $\Phi_+(1)$ in Wronskian (4.44) for $\mathrm{sn}\,x$ in the first case, or $\Phi_-(2\ell)$ for $\mathrm{cn}\,x$ in the second case. Such alternatives, however, do not give anything new. They are reproduced just by taking, respectively, limits $\beta_1^+ \to 0$ or $\beta_{2\ell}^+ \to \mathbf{K}$ in the general picture presented in this subsection.

C. Bound states in both forbidden bands

One can introduce periodicity defects into the Lamé system by constructing the potentials that support bound states in both lower and upper forbidden bands. Similarly to the already discussed cases, the construction is based on the property that the Wronskian

$$\mathbb{W}_{2\ell,n}(x) = W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell), \mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n)) \tag{4.60}$$

is a nodeless smooth function on all the real line; see the Appendix. In this way, the most general family of the one-gap Hamiltonians with $2\ell + n$ defects (solitons) introduced into the periodic background of Lamé potential $V_{0,0}(x)$ is defined by

$$H_{2\ell,n} = H_{0,0} - 2\frac{d^2}{dx^2}\log\mathbb{W}_{2\ell,n}(x). \tag{4.61}$$

The defects correspond to 2ℓ bound states in the spectral gap and n bound states in the lower prohibited band.

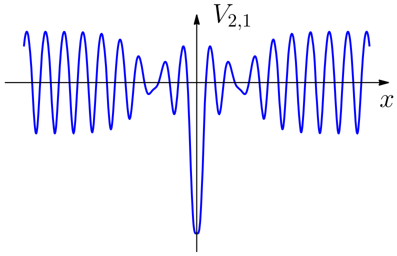

FIG. 9 (color online). Potential supporting two bound states in the gap and one bound state in the lower forbidden band. The defects in the form of the two compression modulations and a potential soliton well can be displaced arbitrarily in the periodic background.

In Fig. 9 is shown the form of the potential for the simplest case $\ell = n = 1$.

Each member of the family of Hamiltonians (4.61) possesses a nontrivial integral

$$\mathcal{P}_{2\ell,n} = \mathbb{A}_{2\ell,n}\mathcal{P}_{0,0}\mathbb{A}^{\dagger}_{2\ell,n}, \qquad [\mathcal{P}_{2\ell,n}, H_{2\ell,n}] = 0, \tag{4.62}$$

satisfying the relation

$$\mathcal{P}^2_{2\ell,n} = H_{2\ell,n}(H_{2\ell,n} - k'^2)(H_{2\ell,n} - 1) \times \prod_{l=1}^{2\ell}(H_{2l,n} - \varepsilon^+_l)^2 \prod_{j=1}^{n}(H_{2\ell,n} - \varepsilon^-_j)^2. \tag{4.63}$$

Here, $\mathbb{A}_{2\ell,n}$ is the differential operator of order $2\ell + n$, which is defined by $\mathbb{A}_{2\ell,n} = A_{2\ell,n}\ldots A_{2\ell,1}\mathbb{A}_{2\ell,0}$, where

$$A_{2\ell,j} = \frac{\mathbb{W}_{2\ell,j}}{\mathbb{W}_{2\ell,j-1}}\frac{d}{dx}\frac{\mathbb{W}_{2\ell,j-1}}{\mathbb{W}_{2\ell,j}}, \qquad j = 1,\ldots,n. \tag{4.64}$$

The first-order differential operator $A_{2\ell,n}$ and its conjugate generate the intertwining relations

$$A_{2\ell,n}H_{2\ell,n-1} = H_{2\ell,n}A_{2\ell,n}, \qquad A^{\dagger}_{2\ell,n}H_{2\ell,n} = H_{2\ell,n-1}A^{\dagger}_{2\ell,n} \tag{4.65}$$

and factorize the neighbor Hamiltonians $H_{2\ell,n}$ and $H_{2\ell,n-1}$ in the form

$$A_{2\ell,n}A^{\dagger}_{2\ell,n} = H_{2\ell,n} - \varepsilon^-_n, \qquad A^{\dagger}_{2\ell,n}A_{2\ell,n} = H_{2\ell,n-1} - \varepsilon^-_n. \tag{4.66}$$

The 2ℓ bound states of $H_{2\ell,n}$ of energies ε^+_l, $l = 1,\ldots,2\ell$, within the gap are given by

$$\Psi^{l;}_{2\ell,n} = \frac{W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n),\Phi_{(-1)^l}(l))}{\mathbb{W}_{2\ell,n}}, \tag{4.67}$$

while the n bound states of energies ε^-_j, $j = 1,\ldots,n$, in the lower prohibited band have the form

$$\Psi^{;j}_{2\ell,n} = \frac{W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n),\mathcal{F}_{-s_j}(j))}{\mathbb{W}_{2\ell,n}}. \tag{4.68}$$

Here, we do not indicate explicitly the parameters that define the functions $\Psi^{l;}_{2\ell,n}$ and $\Psi^{;j}_{2\ell,n}$ being in general of the form $\Psi(x;\beta^+_1,C^+_1,\ldots,\beta^+_{2\ell},C^+_{2\ell},\beta^-_1,C^-_1,\ldots\beta^-_n,C^-_n)$. Other, physical as well as nonphysical, eigenstates of $H_{2\ell,n}$ of eigenvalues $\mathcal{E}(\alpha)$ are given by

$$\Psi^{\alpha}_{2\ell,n;\pm} = \frac{W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n),\Psi^{\alpha}_{\pm})}{\mathbb{W}_{2\ell,n}}. \tag{4.69}$$

It is always possible to eliminate any of the bound states from the spectrum taking the limit $C^{\pm}_r \to 0$, or $C^{\pm}_r \to \infty$ for the corresponding parameter. In the case we take such a limit for the parameter C^+_l of the state $\Phi_{(-1)^{l+1}}(l)$, we obtain $H_{2\ell,n}(x) \to \breve{H}_{2\ell-1,n}(x;\beta^+_l)$, where $\breve{H}_{2\ell-1,n}(x;\beta^+_l)$ is the Hamiltonian of the system with $2\ell - 1$ bound states in the gap. Similarly to the case discussed in the previous subsection, the $\breve{H}_{2\ell-1,n}(x;\beta^+_l)$ can also be obtained by the Darboux–Crum transformation of order $2\ell - 1 + n$ applied to the singular Treibich–Verdier system. The Lax–Novikov integral $\breve{\mathcal{P}}_{2\ell-1,n}(x;\beta^+_l)$ of $\breve{H}_{2\ell-1,n}(x;\beta^+_l)$ appears from (4.62) via the indicated limit through the reduction, $\mathcal{P}_{2\ell,n}(x) \to (\breve{H}_{2\ell-1,n}(x;\beta^+_l) - \varepsilon^+_l)\breve{\mathcal{P}}_{2\ell-1,n}(x;\beta^+_l)$. On the other hand, if we take one of the two specified limits for the parameter C_j^-, we obtain the Hamiltonian $\tilde{H}_{2\ell,n-1}(x;\beta_j^-)$, which corresponds to the system $H_{2\ell,n-1}(x)$ of the form (4.61) with the displaced argument, $x \to x + \beta_j^-$. The initial parameters β_i^- with $i = j+1, \ldots, n$ transform into the parameters β_i^-, $i = j, \ldots, n-1$, of the resulting system, and the same happens with the corresponding parameters C_i^-. Moreover, all parameters $C^\pm$ undergo rescaling, $C_l^+ \to c_l^+(\beta_l^+, \beta_j^-)C_l^+$, $l = 1, \ldots, 2\ell$, $C_i^- \to c_i^-(\beta_i^-, \beta_j^-)C_i^-$, $i = 1, \ldots, n-1$, where $c_l^+ > 0$ and $c_i^- > 0$ are some functions of the indicated arguments, whose explicit form we do not write down in detail here.

Notice that in the most general case of one-gap quantum system $H_{2\ell-m,n} = -\frac{d^2}{dx^2} + V_{2\ell-m,n}(x)$ supporting $2\ell - m + n \geq 1$ bound states, relation (4.18) is generalized for

$$\Delta\phi = -2\sum_{j=1}^{n}\beta_j^- - 2\sum_{l=1}^{2\ell-m}\beta_l^+, \tag{4.70}$$

where $n \geq 0$, $2\ell - m \geq 0$, $m = 0, 1$, and the omission of the corresponding sum is assumed when $n = 0$ or $\ell = 0$. This is the net phase displacement between $x = +\infty$ and $x = -\infty$ periodic asymptotics of the potential $V_{2\ell-m,n}(x)$, which is the one-gap Lamé potential $V_{0,0}(x)$ perturbed by $n \geq 0$ soliton defects of the potential well type and $2\ell - m \geq 0$, periodicity defects of the compression modulations nature.

In conclusion of this section, let us note that the notion of Hill's discriminant (Lyapunov function) is defined for a Schrödinger equation with periodic potential, and reflects coherently the properties of the eigenstates under the shift of the quantum system for its period [23,41]. The Darboux–Crum transformations that do not violate the periodicity of the potential produce isospectral systems and do not change the corresponding discriminants [42,43]. The systems we constructed here are *almost isospectral* to the one-gap Lamé system. Their potentials are not periodic functions, and so Hill's discriminant cannot be defined for them in a usual way. It can be considered only in the regions $x \to -\infty$ and $x \to +\infty$, where the periodicity (with a relative phase displacement defect) is restored asymptotically. At the same time, it is necessary to bare in mind that the Lyapunov function reflects the stability properties of the points in the spectrum: for periodic quantum systems, two linearly independent Bloch–Floquet states correspond to all the points inside the allowed bands, while the edge points are treated as nonstable because there one of the two solutions is unbounded [41]. Since the periodicity defects we constructed introduce into the spectrum of the Lamé system only the discrete energy values corresponding to nondegenerate bound states, one can say that they do not change the properties of stability of the spectrum of the initial system.

V. EXOTIC SUPERSYMMETRY

According to the analysis presented above, any pair of the Hamiltonians $H_{2\ell_1-m_1,n_1}$ and $H_{2\ell_2-m_2,n_2}$, where $m_{1,2} = 0, 1$, can be related by means of the two pairs of intertwining operators. One pair of mutually conjugate operators intertwines the Hamiltonians directly. Another pair has higher differential order and does the same job via a virtual periodic one-gap system. The operators of the second pair involve in their structure the Lax–Novikov integral of the Lamé system $H_{0,0}$, or of its analog corresponding to the singular on the real line Treibich–Verdier one-gap system. Each of the subsystems in the pair $(H_{2\ell_1-m_1,n_1}, H_{2\ell_2-m_2,n_2})$ is also characterized by its proper Lax–Novikov integral. As a result, if we consider the extended system given by the matrix 2×2 Schrödinger operator composed from the pair of the indicated Hamiltonians, it will be described not just by the $\mathcal{N} = 2$ linear or nonlinear supersymmetry as it would be expected for the ordinary pair of Darboux(–Crum) related quantum mechanical systems. Instead, as in the case of nonperiodic reflectionless systems, it will be characterized by an exotic nonlinear $\mathcal{N} = 4$ supersymmetric structure that involves the two nontrivial bosonic generators composed from the Lax–Novikov integrals of the subsystems.

From the perspective of physical applications, the most interesting case corresponds to the pairs of the Schrödinger Hamiltonians, which can be related by the mutually conjugate first-order Darboux intertwiners alongside with the pair of higher-order intertwiners. It is this case that we consider in this section in detail.

We start from the general discussion of the picture corresponding to a basic case, from which other cases can be obtained via certain limiting procedures. Then, we illustrate this by considering the simplest examples, which reveal all the peculiarities of the exotic supersymmetric structure.

A. Exotic supersymmetry with the first-order supercharges: Generic picture

The first-order differential operators $A_{2\ell,n}$ and $A_{2\ell,n}^\dagger$ intertwine the Hamiltonians $H_{2\ell,n-1}$ and $H_{2\ell,n}$,

$$A_{2\ell,n}H_{2\ell,n-1} = H_{2\ell,n}A_{2\ell,n}, \qquad H_{2\ell,n-1}A_{2\ell,n}^\dagger = A_{2\ell,n}^\dagger H_{2\ell,n}, \tag{5.1}$$

and factorize them,

$$A_{2\ell,n}^\dagger A_{2\ell,n} = H_{2\ell,n-1} - \varepsilon_n^-, \qquad A_{2\ell,n}A_{2\ell,n}^\dagger = H_{2\ell,n} - \varepsilon_n^-, \tag{5.2}$$

where $\varepsilon_n^- = \mathcal{E}(\beta_n^- + i\mathbf{K}')$. These relations allow us to consider the extended system described by the Hamiltonian

$$\mathcal{H}_{2\ell,n} = \begin{pmatrix} H_{2\ell,n-1} & 0 \\ 0 & H_{2\ell,n} \end{pmatrix} \quad (5.3)$$

and by the pair of matrix operators

$$S^1_{2\ell,n} = \begin{pmatrix} 0 & A^\dagger_{2\ell,n} \\ A_{2\ell,n} & 0 \end{pmatrix}, \qquad S^2_{2\ell,n} = i\sigma_3 S^1_{2\ell,n}. \quad (5.4)$$

Taking the trivial integral $\Gamma = \sigma_3$ as a $\mathbb{Z}_2$-grading operator, we identify $\mathcal{H}_{2\ell,n}$ as the bosonic operator, $[\Gamma, \mathcal{H}_{2\ell,n}] = 0$, and $S^a_{2\ell,n}$, $a = 1, 2$, as the fermionic ones, $\{\Gamma, S^a_{2\ell,n}\} = 0$. They generate a superalgebra of $\mathcal{N} = 2$ supersymmetric quantum mechanics,

$$[\mathcal{H}_{2\ell,n}, S^a_{2\ell,n}] = 0, \qquad \{S^a_{2\ell,n}, S^b_{2\ell,n}\} = 2\delta^{ab}(\mathcal{H}_{2\ell,n} - \varepsilon^-_n). \quad (5.5)$$

By the redifinition of the Hamiltonian via an additive shift, $\mathcal{H}_{2\ell,n} - \varepsilon^-_n \to \mathcal{H}_{2\ell,n}$, one can transform (5.5) into the standard form of $\mathcal{N} = 2$ superalgebra describing the system with the zero energy of the nondegenerate ground state appearing in the spectrum of the "lower" subsystem of the extended matrix system. Since the subsystems $H_{2\ell,n-1}$ and $H_{2\ell,n}$ possess the nontrivial Lax–Novikov integrals being differential operators of orders $4\ell + 2n + 1$ and $4\ell + 2n + 3$, the extended system (5.3) possesses also two nontrivial bosonic integrals that we define in the form

$$P^1_{2\ell,n} = \begin{pmatrix} (H_{2\ell,n-1} - \varepsilon^-_n)\mathcal{P}_{2\ell,n-1} & 0 \\ 0 & \mathcal{P}_{2\ell,n} \end{pmatrix},$$
$$P^2_{2\ell,n} = \sigma_3 P^1_{2\ell,n}. \quad (5.6)$$

We introduced here the additional factor in the upper component whereby the upper and lower components of these integrals are operators of the same differential order. The commutation relations

$$[\mathcal{H}_{2\ell,n}, P^a_{2\ell,n}] = 0, \qquad [P^a_{2\ell,n}, P^b_{2\ell,n}] = 0,$$
$$[P^1_{2\ell,n}, S^a_{2\ell,n}] = 0 \quad (5.7)$$

extend the superalgebraic relations (5.5) and show that the integral $P^1_{2\ell,n}$ is the bosonic central charge. On the other hand, the nontrivial commutator $[P^2_{2\ell,n}, S^a_{2\ell,n}]$ generates the second pair of the fermionic supercharges $Q^a_{2\ell,n}$, which are the matrix differential operators of the order $2(2\ell + n + 1)$. As we shall see, the anticommutator of $Q^a_{2\ell,n}$ with $Q^b_{2\ell,n}$ produces a polynomial in matrix Hamiltonian $\mathcal{H}_{2\ell,n}$, while the anticommutator of $Q^a_{2\ell,n}$ with $S^b_{2\ell,n}$ generates the central charge $P^1_{2\ell,n}$. The second bosonic integral $P^2_{2\ell,n}$ generates finally a kind of a rotation between the supercharges $S^a_{2\ell,n}$ and $Q^a_{2\ell,n}$.

Taking in (5.3) the limit $C^+_l \to \infty$ or $C^+_l \to 0$ with l chosen from the set $1, \ldots, 2\ell$, we obtain another extended system:

$$\breve{\mathcal{H}}_{2\ell-1,n} = \begin{pmatrix} \breve{H}_{2\ell-1,n-1} & 0 \\ 0 & \breve{H}_{2\ell-1,n} \end{pmatrix}. \quad (5.8)$$

As we saw, the application of the limits $C^+_l \to \infty$ or $C^+_l \to 0$ to the corresponding Lax–Novikov integrals of the subsystems produces the reducible operators. The irreducible nonsingular Lax–Novikov integrals of $\breve{H}_{2\ell-1,n-1}$ and $\breve{H}_{2\ell-1,n}$ have orders $4\ell + 2n - 1$ and $4\ell + 2n + 1$ and include in their structure the Lax–Novikov integral of the singular Treibich–Verdier one-gap system. The bosonic integrals $\breve{P}^a_{2\ell-1,n}$ of the extended matrix system (5.8) are constructed from $\breve{\mathcal{P}}_{2\ell-1,n-1}$ and $\breve{\mathcal{P}}_{2\ell-1,n}$ like in (5.6). Again, $\breve{P}^1_{2\ell-1,n}$ will play the role of the central charge of the nonlinear superalgebra, while the commutator $[\breve{P}^2_{2\ell-1,n}, \breve{S}^a_{2\ell-1,n}]$ will generate the second pair of the supercharges $\breve{Q}^a_{2\ell-1,n}$. The exotic superalgebra of the system (5.8) will have as a result a form similar to that for the system (5.3).

Let us change index n for $n + 1$ in (5.3) and take one of the two limits

$$\lim_{C^-_{n+1} \to 0,\infty} H_{2\ell,n+1}(x) = \tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1}), \quad (5.9)$$

where the upper and lower signs on the rhs correspond, respectively, to the 0 and ∞ cases. In such a limit, we get the extended system described by the Hamiltonian,

$$\tilde{\mathcal{H}}_{2\ell,n} = \begin{pmatrix} H_{2\ell,n} & 0 \\ 0 & \tilde{H}_{2\ell,n} \end{pmatrix}, \quad (5.10)$$

where $\tilde{H}_{2\ell,n}$ corresponds to one of the indicated limits, $\tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1})$. Here, we have used the definition of the functions (4.1) and have taken into account that for the function (2.6) the identity $F(-x; \beta^-) = F(x; -\beta^-)$ is valid. The initial subsystems $H_{2\ell,n}$ and $H_{2\ell,n+1}$ in (5.3) with n changed for $n + 1$ are related by the first-order intertwining operators $A_{2\ell,n+1}$ and $A^\dagger_{2\ell,n+1}$. Then, the pair of $H_{2\ell,n}(x)$ and $\tilde{H}_{2\ell,n}(x; \mp\beta^-_{n+1})$ in (5.10) is related by the first-order intertwining operators

$$X_{2\ell,n}(x; \mp\beta^-_{n+1}) \equiv \lim_{C^-_{n+1} \to 0,\infty} A_{2\ell,n+1}$$
$$= \frac{\hat{\mathbb{W}}_{2\ell,n}(F(x; \mp\beta^-_{n+1}))}{\mathbb{W}_{2\ell,n}} \frac{d}{dx} \frac{\mathbb{W}_{2\ell,n}}{\hat{\mathbb{W}}_{2\ell,n}(F(x; \mp\beta^-_{n+1}))} \quad (5.11)$$

and $X^\dagger_{2\ell,n}(x; \mp\beta^-_{n+1})$, where $\hat{\mathbb{W}}_{2\ell,n}(f(x)) \equiv W(\Phi_+(1), \ldots, \Phi_-(2\ell), \mathcal{F}_+(1) \ldots \mathcal{F}_{s_n}(n), f(x))$. The subsystems in

(5.10) are completely isospectral, and the exotic supersymmetry in this case has a structure similar to that of the system (5.3). However, unlike (5.3), the system (5.10) is characterized by the spontaneously broken exotic supersymmetry, and this fact, as we shall see, is properly reflected by the "fine structure" of the nonlinear superalgebra.

Another interesting case that could be mentioned corresponds to the limit

$$\lim_{\beta^-_{n+1}\to\beta^-_n} H_{2\ell,n+1} = \lim_{\beta^-_{n+1}\to\beta^-_n} \tilde{H}_{2\ell,n}(x;\mp\beta^-_{n+1}) = H_{2\ell,n-1}. \quad (5.12)$$

However, if we apply such a limit to the system (5.3) with index n changed for $n+1$, we obtain just a system of the form $\mathcal{H}_{2\ell,n}$ but with the permuted upper and lower corresponding Hamiltonians.

B. Unbroken exotic supersymmetry

Consider now the simplest case of the extended systems (5.3) with $\ell = 0$, $n = 1$. Besides the first-order operators $A_{0,1}$ and $A^\dagger_{0,1}$, the pair of Hamiltonians $H_{0,0}$ and $H_{0,1}$ is intertwined by the differential operators of order 4, $B_{0,1} = A_{0,1}\mathcal{P}_{0,0}(x)$ and $B^\dagger_{0,1}$. The systems $H_{0,0}$ and $H_{0,1}$ are also characterized by the Lax–Novikov integrals $\mathcal{P}_{0,0}(x)$ and $\mathcal{P}_{0,1}(x) = A_{0,1}\mathcal{P}_{0,0}(x)A^\dagger_{0,1}$. Besides the integrals of the form (5.4) and (5.6), the extended matrix system is characterized also by the pair of the supercharges

$$Q^1_{0,1} = \begin{pmatrix} 0 & B^\dagger_{0,1} \\ B_{0,1} & 0 \end{pmatrix}, \qquad Q^2_{0,1} = i\sigma_3 Q^1_{0,1}. \quad (5.13)$$

The fermionic integrals $S^a_{0,1}$ and $Q^a_{0,1}$ and the bosonic integrals $P^a_{0,1}$ together with the Hamiltonian $\mathcal{H}_{0,1}$ generate the nonlinear superalgebra

$$\{S^a, S^b\} = 2\delta^{ab}(\mathcal{H} - \varepsilon^-_1),$$
$$\{Q^a, Q^b\} = 2\delta^{ab}(\mathcal{H} - \varepsilon^-_1)C_3(\mathcal{H}), \quad (5.14)$$

$$\{S^a, Q^b\} = 2\delta^{ab}P^1, \quad (5.15)$$

$$[P^2, S^a] = -2i\epsilon^{ab}(\mathcal{H} - \varepsilon^-_1)Q^b,$$
$$[P^2, Q^a] = -2i\epsilon^{ab}(\mathcal{H} - \varepsilon^-_1)C_3(\mathcal{H})S^b, \quad (5.16)$$

$$[P^1, Q^a] = 0, \qquad [P^1, S^a] = 0, \quad (5.17)$$

where $C_3(\mathcal{H}) = \mathcal{H}(\mathcal{H} - k'^2)(\mathcal{H} - 1)$, ϵ^{ab} is the antisymmetric tensor, $\epsilon^{12} = 1$, and for the sake of simplicity, we omit the lower indices. The unique nondegenerate state with energy $\mathcal{E} = \varepsilon^-_1$ appearing in the spectrum of subsystem $H_{0,1}$ is annihilated by the shifted Hamiltonian $\mathcal{H} - \varepsilon^-_1$ and by all the integrals S^a, Q^a, and P^a. This means that the exotic supersymmetry of the extended Schrödinger system is unbroken. The doubly degenerate energy values corresponding to the edges of the allowed bands of the subsystems are the zeros of the third-order polynomial appearing in the superalgebra structure: $C_3(\mathcal{E}) = 0$ for $\mathcal{E} = 0, k'^2, 1$. This reflects the property that the corresponding edge states of the subsystems are detected by the fourth-order supercharges Q^a as well as by the bosonic integrals P^a; all these operators annihilate them. One can also show that the physical eigenstates $\Psi^\alpha_\pm$ and $A_{0,1}\Psi^\alpha_\pm$ of the upper and lower subsystems inside their valence and conduction bands possessing the quasimomentum of the opposite sign (they correspond to the different lower indices of the Bloch states) are distinguished by the bosonic integrals P^a.

The second relation $[P^1, S^a] = 0$ from (5.17) can be rewritten as a nonlinear differential equation for the superpotential $\mathcal{W}_{0,1}(x)$ shown in Fig. 10, see Eq. (4.21). This corresponds here to the first equation of the stationary mKdV hierarchy, which can be associated with the extended system with one nonperiodic soliton defect introduced into the one-gap Lamé system. At the same time, the equation $[\mathcal{H}, P^1] = 0$ can be presented in the form of the nonlinear differential equations of the third order for the potentials $V_\pm(x) \equiv \mathcal{W}^2_{0,1} \pm \mathcal{W}'_{0,1} + \varepsilon^-_1$. These equations correspond to the first equation of the stationary KdV hierarchy, which can be associated with the one-gap Lamé system itself and with its deformation $V_-(x)$ produced by the one-soliton defect introduced into the periodic background of the one-gap Lamé system.

The generic case of the extended systems (5.3) and (5.8) is described by the exotic nonlinear superalgebras of the

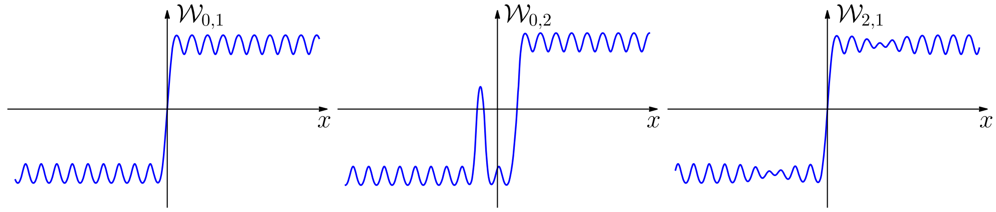

FIG. 10 (color online). Topological superpotentials in the form of the kink that incorporate one bound state into the spectrum. On the left is shown the superpotential that relates the systems $H_{0,0}$ with $H_{0,1}$. The superpotential in the center corresponds to the pair of the systems $H_{0,1}$ and $H_{0,2}$, while that on the right corresponds to the pair of $H_{2,0}$ and $H_{2,1}$.

same form. The unique difference is that the third-order polynomial $C_3(\mathcal{H})$ appearing here will be changed for the structure polynomials of the form (4.63), which are associated with the square of the corresponding Lax–Novikov integrals.

C. Spontaneously broken exotic supersymmetry

The case of the spontaneously broken exotic supersymmetry realized in the one-gap systems with the nonperiodicity defects can be illustrated by the extended system with the mutually displaced one-gap Lamé systems $H_{0,0}(x)$ and $\tilde{H}_{0,0}(x;\beta^-) = H_{0,0}(x+\beta^-)$. Though such systems are periodic, all the principle features of the structure of the exotic supersymmetry we observe in this case appear also in the extended systems composed from the completely isospectral systems with soliton defects.

The isospectral Hamiltonians $H_{0,0}(x)$ and $H_{0,0}(x+\beta^-)$ are connected by the first-order differential operator

$$X_{0,0}(x;\beta^-) = F(x;\beta^-)\frac{d}{dx}\frac{1}{F(x;\beta^-)} = \frac{d}{dx} + \Delta_{0,0}(x;\beta^-) \tag{5.18}$$

and by its Hermitian conjugate operator, where

$$\Delta_{0,0}(x,\beta^-) = \mathrm{Z}(x) - \mathrm{Z}(x+\beta^-) + \mathrm{z}(\beta^-) \tag{5.19}$$

is the superpotential shown in Fig. 11. To simplify notations, in what follows in this subsection, we omit lower indices in Hamiltonians, intertwining operators, and corresponding Lax–Novikov integrals and put $\beta^- = \beta$. Recall that $0 < \beta < \mathbf{K}$.

The operator (5.18) and its conjugate factorize the Hamiltonians,

$$X^\dagger(x;\beta)X(x;\beta) = H(x) - \varepsilon(\beta),$$
$$X(x;\beta)X^\dagger(x;\beta) = H(x+\beta) - \varepsilon(\beta), \tag{5.20}$$

and intertwine them,

$$X(x;\beta)H(x) = H(x+\beta)X(x;\beta),$$
$$X^\dagger(x;\beta)H(x+\beta) = H(x)X^\dagger(x;\beta), \tag{5.21}$$

where $\varepsilon(\beta) = \mathcal{E}(\beta + i\mathbf{K}') = -\mathrm{cs}^2\beta$. These first-order intertwining operators are related by $X^\dagger(x;\beta) = -X(x+\beta;-\beta)$, that follows from the identity $1/F(x;\beta) = F(x+\beta;-\beta)\exp(-\beta\mathrm{z}(\beta))$, and corresponds according to (5.11) to the limit $C^- \to \infty$ of the first-order operator $A_{0,1}$. In this limit, the topologically nontrivial superpotential $\mathcal{W}_{0,1}$ transforms into the topologically trivial superpotential $\Delta_{0,0}$, see Figs. 10 and 11. One can construct the second intertwiner being the differential operator of the order 2 by taking a composition of the two first-order intertwiners (5.18),

$$\mathcal{G}(x;\beta',\beta) = X(x+\beta';\beta-\beta')X(x;\beta'), \tag{5.22}$$

$\mathcal{G}(x;\beta',\beta)H(x) = H(x+\beta)\mathcal{G}(x;\beta,\beta')$, where we assume that $\beta' \neq \beta$. The first factor on the rhs in (5.22) intertwines the $H(x)$ with the Hamiltonian of the virtual system $H(x+\beta')$, and then this is intertwined by the second factor with $H(x+\beta)$. Notice also that $\mathcal{G}^\dagger(x;\beta',\beta) = \mathcal{G}(x+\beta;\beta'-\beta,-\beta)$.

One could think here that in this way intertwining operators of the higher order $n > 2$ can be constructed, but this is impossible because of the identity [22,47]

$$\mathcal{G}(x;\beta',\beta) = \mathcal{G}(x;\beta'',\beta) + G(\beta,\beta',\beta'')X(x;\beta), \tag{5.23}$$

from where it follows that the third-order differential operator

$$X(x+\beta';\beta-\beta')\mathcal{G}(x;\beta'',\beta')$$
$$= -(H(x+\beta) - \varepsilon(\beta'-\beta))X(x;\beta)$$
$$+ G(\beta',\beta'',\beta)\mathcal{G}(x;\beta',\beta), \tag{5.24}$$

which intertwines $H(x)$ and $H(x+\beta)$, reduces effectively to the first- and second-order intertwining operators $X(x;\beta)$ and $\mathcal{G}(x;\beta',\beta)$. Here, we used the notations $G(\beta,\beta',\beta'') \equiv g(\beta,-\beta') - g(\beta,-\beta'')$,

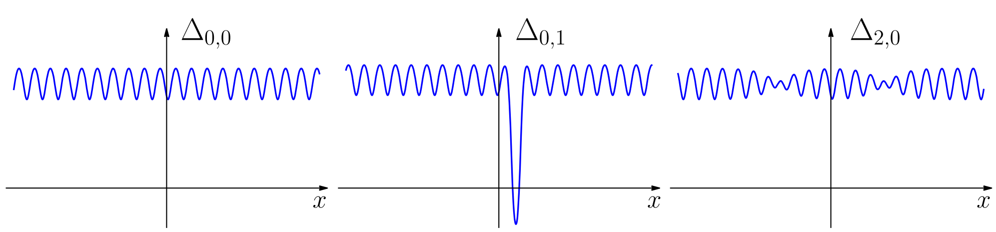

FIG. 11 (color online). Nontopological superpotentials that allow us to displace the periodic potential network of the Lamé system as well as the nonperiodic defects in it. The nontrivial displacements of the defects correspond to a nonlinear interaction between the soliton defects themselves and to their interaction with the periodic background (see Fig. 4). According to Fig. 10, these superpotentials are obtained by sending the kink, and the associated ground state of the Hamiltonian (5.3), to minus infinity, which generates the supersymmetry breaking. The shown superpotentials relate the following isospectral pairs: $H_{0,0}$ and $\tilde{H}_{0,0}$ (on the left), $H_{0,1}$ and $\tilde{H}_{0,1}$ (in the center), and $H_{2,0}$ and $\tilde{H}_{2,0}$ (on the right).

$$g(\beta,\beta')\equiv \mathrm{ns}\,\beta\,\mathrm{ns}\,\beta'\mathrm{ns}(\beta+\beta')(1-\mathrm{cn}\,\beta\,\mathrm{cn}\,\beta'\mathrm{cn}(\beta+\beta')). \tag{5.25}$$

The relation (5.23) reflects effectively a kind of the "gauge" nature of the parameter β', which appears in the structure of $\mathcal{G}(x;\beta',\beta)$ and is associated with a virtual system $H(x+\beta')$. On the other hand, from the same relation and definition (5.22), one finds that the second-order operator

$$Y(x;\beta)=X(x+\beta';\beta-\beta')X(x;\beta')-g(\beta,-\beta')X(x;\beta) \tag{5.26}$$

is invariant under the change $\beta'\to\beta''$. Thus, being a certain linear combination of (5.18) and (5.22), $Y(x;\beta)$ is the "gauge-invariant" second-order intertwining operator, $Y(x;\beta)H(x)=H(x+\beta)Y(x;\beta)$, which does not depend on the value of the virtual parameter in spite of its appearance on the rhs in (5.26). The conjugate operator acts in the opposite direction, and similarly to the first order intertwining operator, we have $Y^\dagger(x;\beta)=Y(x+\beta;-\beta)$.

One can represent $Y(x;\beta)$ in the explicitly β'-independent form in terms of the superpotential (5.19) and parameter β. However, we do not need here such an expression and will use the representation (5.26).

From the properties of $X(x;\beta)$ and $Y(x;\beta)$, it follows that the third-order operators $X^\dagger(x;\beta)Y(x;\beta)$ and $Y^\dagger(x;\beta)X(x;\beta)$ reduce, up to the additive constants, to the third-order Lax–Novikov integral $\mathcal{P}(x)=\mathcal{P}_{0,0}(x)$ given by Eq. (4.24) and to $\mathcal{P}(x+\beta)$, respectively. Namely, we have

$$X^\dagger(x;\beta)Y(x;\beta)=-i\mathcal{P}(x)-\mathcal{N}_0(\beta),$$
$$X(x;\beta)Y^\dagger(x;\beta)=i\mathcal{P}(x+\beta)-\mathcal{N}_0(\beta) \tag{5.27}$$

and the pair of identity relations, which can be obtained from (5.27) by the Hermitian conjugation. The β-dependent constant $\mathcal{N}_0(\beta)$ is given by[6]

$$\mathcal{N}_0(\beta)=\mathrm{dn}\,\beta\,\mathrm{cn}\,\beta\,\mathrm{ns}^3\beta=\frac{1}{2}\frac{d}{d\beta}\varepsilon(\beta). \tag{5.28}$$

Similarly to (5.21), the second-order intertwining operators generate the second-order polynomial in the isospectral Hamiltonians,

$$Y^\dagger(x;\beta)Y(x;\beta)=\mathcal{N}_2(H(x),\beta),$$
$$Y(x;\beta)Y^\dagger(x;\beta)=\mathcal{N}_2(H(x+\beta),\beta), \tag{5.29}$$

where

$$\mathcal{N}_2(H(x),\beta)=H^2(x)+c_1(\beta)H(x)+c_2(\beta), \tag{5.30}$$

$$c_1(\beta)=-k'^2-\mathrm{ns}^2\beta=\varepsilon(\beta)-1-k'^2,$$
$$c_2(\beta)=\mathrm{dn}^2\beta\mathrm{ns}^4\beta=(\varepsilon(\beta)-1)(\varepsilon(\beta)-k'^2). \tag{5.31}$$

Finally, for the products of the intertwining operators with the Lax–Novikov integral, we obtain

$$-iX(x;\beta)\mathcal{P}(x)=\mathcal{N}_1(H(x+\beta),\beta)Y(x;\beta)+\mathcal{N}_0(\beta)X(x;\beta) \tag{5.32}$$

$$i\mathcal{P}(x)X^\dagger(x;\beta)=\mathcal{N}_1(H(x),\beta)Y^\dagger(x;\beta)+\mathcal{N}_0(\beta)X^\dagger(x;\beta), \tag{5.33}$$

$$iY(x;\beta)\mathcal{P}(x)=\mathcal{N}_2(H(x+\beta),\beta)X(x;\beta)+\mathcal{N}_0(\beta)Y(x;\beta), \tag{5.34}$$

$$-i\mathcal{P}(x)Y^\dagger(x;\beta)=\mathcal{N}_2(H(x),\beta)X^\dagger(x;\beta)+\mathcal{N}_0(\beta)Y^\dagger(x;\beta) \tag{5.35}$$

and four other relations given by the Hermitian conjugation. Here, we introduce the notation

$$\mathcal{N}_1(H(x),\beta)=H(x)-\varepsilon(\beta). \tag{5.36}$$

The operators $X(x;\beta)$ and $Y(x;\beta)$ and their conjugate ones intertwine the Lax–Novikov integrals $\mathcal{P}(x)$ and $\mathcal{P}(x+\beta)$ exactly in the same way as they do this with the corresponding Hamiltonians.

Now, we are in a position to identify the superalgebra of the extended Schrödinger system $\tilde{\mathcal{H}}=\mathrm{diag}(H(x),H(x+\beta))$, which corresponds to (5.10) with $\ell=n=0$ and lower component $\tilde{H}_{0,0}(x;\beta_1^-)$. This extended system is characterized by the two pairs of the fermion integrals $\tilde{S}^a(x;\beta)$ and $\tilde{Q}^a(x;\beta',\beta)$, constructed from the first-, $X^\dagger(x;\beta)$, $X(x;\beta)$, and second-order, $Y^\dagger(x;\beta)$, $Y(x;\beta)$, intertwining operators in the form similar to that in (5.4), and by the two boson integrals $\tilde{P}^1=\mathrm{diag}(\mathcal{P}(x),\mathcal{P}(x+\beta))$ and $\tilde{P}^2=\sigma_3\tilde{P}^1$. These 2×2 matrix operators generate the exotic nonlinear $\mathcal{N}=4$ superalgebra,

$$\{\tilde{S}^a,\tilde{S}^b\}=2\delta^{ab}\mathcal{N}_1(\tilde{\mathcal{H}},\beta),$$
$$\{\tilde{Q}^a,\tilde{Q}^b\}=2\delta^{ab}\mathcal{N}_2(\tilde{\mathcal{H}},\beta), \tag{5.37}$$

$$\{\tilde{S}^a,\tilde{Q}^b\}=-2\epsilon^{ab}\tilde{P}^1-2\delta^{ab}\mathcal{N}_0(\beta), \tag{5.38}$$

[6]Notice here that, for the limit case $\beta=\mathbf{K}$, $\mathcal{N}_0(\mathbf{K})=0$. Then, for the choice $\beta'=\mathbf{K}+i\mathbf{K}'$, the coefficient g in (5.26) turns into zero, and the Hermitian conjugate form of the first relation in (5.27) corresponds to factorization (4.27). Another choice, for instance, $\beta'=i\mathbf{K}'$, gives a factorization $i\mathcal{P}_{0,0}(x)=A_{1/\mathrm{sn}\,x}A_{\mathrm{sn}\,x/\mathrm{dn}\,x}A_{\mathrm{dn}\,x}$.

$$[\tilde{P}^2,\tilde{S}^a]=-2i\mathcal{N}_1(\tilde{\mathcal{H}},\beta)\tilde{Q}^a-2i\mathcal{N}_0(\beta)\tilde{S}^a,$$
$$[\tilde{P}^2,\tilde{Q}^a]=2i\mathcal{N}_2(\tilde{\mathcal{H}},\beta)\tilde{S}^a+2i\mathcal{N}_0(\beta)\tilde{Q}^a, \quad (5.39)$$

$$[\tilde{P}^1,\tilde{Q}^a]=0, \qquad [\tilde{P}^1,\tilde{S}^a]=0, \quad (5.40)$$

where $\mathcal{N}_1(\tilde{\mathcal{H}},\beta)$ and $\mathcal{N}_2(\tilde{\mathcal{H}},\beta)$ are defined as above with the operator argument $H(x)$ changed for $\tilde{\mathcal{H}}$. The matrix Hamiltonian operator $\tilde{\mathcal{H}}$ plays here, as well as in the superalgebra we considered in the previous subsection, the role of the central element. Note that the constants appearing in the structure of $\mathcal{N}_1(\tilde{\mathcal{H}},\beta)$ and $\mathcal{N}_2(\tilde{\mathcal{H}},\beta)$ correspond to the energies of the doubly degenerate states of the system at the edges of the allowed bands: $\mathcal{E}=0,k'^2,1$.

The sub-superalgebra generated by the supercharges $\tilde{S}^a$ and by the Hamiltonian $\tilde{\mathcal{H}}$ with $0<\beta<\mathbf{K}$ corresponds to the case of the spontaneously broken linear (Lie) $\mathcal{N}=2$ supersymmetry. The first-order supercharges do not annihilate the two ground states $\Psi^t_+\equiv(\mathrm{dn}x,0)$ and $\Psi^t_-\equiv(0,\mathrm{dn}(x+\beta))$ being eigenstates of zero energy of the extended system. This is obvious from the first relation from (5.37) and Eq. (5.36). The quantity $-\varepsilon(\beta)=\mathrm{cs}^2\beta>0$ defines here the scale of supersymmetry breaking. The second relation from (5.37) and Eqs. (5.30) and (5.31) show that the second-order supercharges $\tilde{Q}^a$ also do not annihilate these states. These edge states, however, as well as the edge states of energies k'^2 and 1, which correspond to the two other doubly degenerate energy levels of $\tilde{\mathcal{H}}$, are zero modes of the bosonic generators $\tilde{P}^a$.

The limit case $\beta=\mathbf{K}$ corresponding to $\varepsilon=0$ is special here. At $\beta=\mathbf{K}$, the coefficient $\mathcal{N}_0$ turns into zero, and the indicated two ground states are zero modes of the first-order supercharges. The structure of the nonlinear superalgebra (5.37)–(5.40) essentially simplifies because of the disappearance of the three terms in Eqs. (5.38) and (5.39). In this case, the second-order supercharges $\tilde{Q}^a$ annihilate the doubly degenerate states at the edges of the valence and conduction bands of energies k'^2 and 1. Since the second-order supercharges $\tilde{Q}^a$ do not annihilate the degenerate pair of the ground states in this case either, the extended system $\tilde{\mathcal{H}}$ with $\beta=\mathbf{K}$ is characterized by the partially broken exotic nonlinear $\mathcal{N}=4$ supersymmetry.

Notice that, though at $\beta=\mathbf{K}$ the sub-supersymmetry $\mathcal{N}=2$ generated by $\tilde{\mathcal{H}}$ and $\tilde{S}^a$ is unbroken, the subsystems $H(x)$ and $H(x+\beta)$ are completely isospectral, and the superextended system is characterized by the zero Witten index [48]. This is a characteristic peculiarity of the quantum supersymmetric systems composed from the periodic completely isospectral pairs, which was noted for the first time by Braden and Macfarlane [3] for the particular case of the pair of one-gap periodic Lamé systems shifted mutually for the half-period $\beta=\mathbf{K}$ and later was discussed in a more broad context of "self-isospectrality" by Dunne and Feinberg [11]. In the framework of the nonlinear "tri-supersymmetric" structure, it was analyzed then in Refs. [19,42].

In the context of the breaking of the exotic supersymmetry, it is worth noticing that, generally speaking, the second-order supercharges are not defined uniquely here. Instead of $\tilde{Q}^a$, one can take linear combinations of $\tilde{Q}^a$ and $\tilde{S}^a$, for instance, $\hat{Q}^a=\tilde{Q}^a+\gamma\tilde{S}^a$, where γ is a real constant. The particular choice $\gamma=\mathrm{dn}\beta/\mathrm{sn}\beta\,\mathrm{cn}\beta$ gives then the supercharges $\hat{Q}^a$, which satisfy the anticommutation relations $\{\hat{Q}^a,\hat{Q}^b\}=2\delta^{ab}\tilde{\mathcal{H}}(\tilde{\mathcal{H}}+\varrho(\beta))$, where $\varrho(\beta)=k'^2\mathrm{sc}^2\beta$. Hence, for $\beta\neq\mathbf{K}$, the supercharges $\hat{Q}^a$ annihilate the ground states of zero energy of the system $\tilde{\mathcal{H}}$ (while other states from their kernels correspond to nonphysical eigenstates of $\tilde{\mathcal{H}}$). In this case, the exotic supersymmetry generated by $\tilde{S}^a$, $\hat{Q}^a$, $\tilde{P}^a$, and $\tilde{\mathcal{H}}$ should be interpreted as partially broken. However, the second-order supercharges $\hat{Q}^a$, unlike $\tilde{Q}^a$, are not defined for the limit case $\beta=\mathbf{K}$. The supercharges $\hat{Q}^a$ with the indicated choice of the parameter γ correspond to the second-order intertwining generators (5.22) with $\beta'=\mathbf{K}$.

As in the case of the unbroken exotic supersymmetry we considered in the previous subsection, the Lax–Novikov matrix integral $\tilde{P}^1$ plays here the role of the bosonic central charge, and the second relation in (5.40) corresponds to the stationary equation of the mKdV hierarchy for the topologically trivial superpotential $\Delta_{0,0}(x,\beta)$. The relation $[\tilde{\mathcal{H}},\tilde{P}^1]=0$ corresponds to the pair of stationary equations of the KdV hierarchy for the functions $V_\pm(x)=\Delta_{0,0}(x,\beta)^2\pm\Delta'_{0,0}(x,\beta)+\varepsilon(\beta)$, which represent the potentials of the corresponding mutually shifted Schrödinger systems.

The superalgebra (5.37)–(5.40) in comparison with that of the unbroken exotic supersymmetry case (5.14)–(5.17) contains the terms with the coefficient $\mathcal{N}_0(\beta)$ in (5.38) and (5.39), which are absent in (5.15) and (5.16). There are also other obvious differences in these two forms of superalgebras, which reflect properly the unbroken and spontaneously broken character of the exotic supersymmetries and different topological nature of the corresponding superpotentials. At the formal level, some of these differences are associated with a nontrivial limit procedure applied to the fourth-order intertwining operators $B_{0,1}=A_{0,1}\mathcal{P}_{0,1}(x)$ and $B^\dagger_{0,1}$, in terms of which the fourth-order supercharges Q^a were constructed in the previous subsection. In correspondence with the limit (5.11), we have $B_{0,1}\to X(x;\beta)\mathcal{P}(x)$, $\mathcal{P}(x)=\mathcal{P}_{0,0}(x)$. But according to the relation (5.32), the fourth-order intertwining operator we obtain in the limit is reducible, and, finally, instead of the fourth-order intertwining operators, here we have the second-order operators $Y(x;\beta)$ and $Y^\dagger(x;\beta)$, which intertwine the completely isospectral pair of the Schrödinger systems $H(x)=H_{0,0}(x)$ and $H(x+\beta)=H_{0,0}(x+\beta)$.

VI. DISCUSSION AND OUTLOOK

To conclude, we summarize shortly the results and point out further possible research directions.

We showed how, by applying the Darboux–Crum transformations to the quantum one-gap Lamé system, an arbitrary countable number of bound states can be introduced into the forbidden bands of its spectrum. These states are trapped by localized perturbations of the periodic potential background of the initial system. The nature of the perturbations depends on whether they support discrete energy levels in the lower forbidden band, or in the finite gap separating the allowed valence and conduction bands. In the first case, the perturbations have a nature of the smooth soliton potential *wells* superimposed on the background of the Lamé system, while the discrete energy levels in the gap are supported by compression *modulations* of the periodic background. Though both types of perturbations have a soliton nature, to distinguish, we identify them here as the W-type and M-type defects, respectively. The nature of the bound states is essentially different in these two cases. The $n \geq 1$ bound states trapped by the W-type defects are described by the wave functions with finite number $0 \leq j \leq n-1$ of nodes on the real line. In contrast, the bound states supported by the M-type defects have an infinite number of nodes and represent oscillating trapped pulses.

The obtained nonperiodic systems are reflectionless; their physical states inside the valence and conduction bands are described by the Darboux–Crum transformed Bloch states of the Lamé system, just like the scattering states of quantum systems with multisoliton potentials are given by a Darboux–Crum transformation of free particle plane waves. Similarly to the multisoliton reflectionless potentials, which exponentially tend to a constant value corresponding to the free particle case, here the asymptotics of the perturbed potentials corresponds to the periodic one-gap Lamé potential. We show that the net phase displacement (defect) between $x = +\infty$ and $x = -\infty$ periodic asymptotics of the potential are given by a simple sum of the same parameters that determine, via the elliptic dn^2 parametrization, the discrete energy levels.

The procedure for introducing the W- and the M-type periodicity defects has some important differences. In the first case, the order n of the Darboux–Crum transformation corresponds exactly to the number of the introduced bound states. In the second case, the same is true when the number of discrete energy values is even. The odd number of the discrete energy levels in the gap is obtained by sending one of the already introduced 2ℓ M-type defects to infinity. The resulting potential with $2\ell - 1$ M defects is related to the initial Lamé system by 2ℓ-th-order Darboux–Crum transformation. At the same time, it can be related by the Darboux–Crum transformation of order $2\ell - 1$ with a singular one-gap Treibich–Verdier system obtained by a displacement of the regular Lamé system for one of its two complex half-periods. The indicated complex displacement can itself be generated by the first-order Darboux transformation. This explains the existence of two alternative Darboux–Crum transformations whose orders differ by 1.

The procedure described in this article allows us to construct the irreducible Lax–Novikov integrals of motion for the perturbed systems $H_{2\ell-m,n}$ via the Darboux–Crum dressing of the Lax–Novikov integral of the initial periodic Lamé system $H_{0,0}$. This is similar, again, to the situation with the transparent quantum systems described by multisoliton potentials, for which the Lax–Novikov integrals are the Darboux–Crum dressed form of the momentum operator of the free particle. The Lax–Novikov integrals here are differential operators of order $2(n + 2\ell - m) + 3$ for the system with $n \geq 0$ W-type and $2\ell - m \geq 0$, $m = 0, 1$, M-type defects. The condition of conservation of these integrals generates a nonlinear differential equation of order $2(n + 2\ell - m) + 3$ for the potential $V_{2\ell-m,n}(x)$. This ordinary nonlinear differential equation of odd order in the highest derivative belongs to the stationary KdV hierarchy.

For an extended system composed from an arbitrary pair of the Hamiltonians $H_{2\ell_1-m_1,n_1}$ and $H_{2\ell_2-m_2,n_2}$, which possess $n_i \geq 0$, $i = 1, 2$, discrete energy levels in the lower forbidden band, and $2\ell_i - m_i \geq 0$, $m_i = 0, 1$, bound states in the gap, the presence of the Lax–Novikov integrals has an essential consequence. The whole system is now described not just by an $\mathcal{N} = 2$ linear or nonlinear supersymmetry as would be expected in the case of a Darboux–Crum related pair of ordinary, nontransparent, or not periodic finite-gap, quantum Hamiltonians. Instead, such a system is characterized by an exotic nonlinear $\mathcal{N} = 4$ supersymmetry that, besides two pairs of the fermion supercharges of odd and even differential orders, involves two bosonic generators composed from the Lax–Novikov integrals of the subsystems. We investigated in more detail the most interesting, from the point of view of physical applications, case in which two of the four fermionic supercharges are matrix differential operators of order 1. In this case, one of the matrix Lax–Novikov bosonic integrals plays a role of central charge of a nonlinear superalgebra, and its commutativity with first-order supercharges generates a higher-order differential equation for the superpotential that belongs to the stationary mKdV hierarchy. The second bosonic integral generates rotations between the pair of first-order supercharges and the pair of higher-order supercharges.

When the spectra of Schrödinger superpartners are different only in the lowest discrete energy level present in one of the two subsystems, which corresponds to the almost isospectral case, the superpotential has a topologically nontrivial modulated crystalline kink-type nature. This case is described by an unbroken exotic nonlinear $\mathcal{N} = 4$ supersymmetry, in which the ground state is annihilated by all four supercharges and two bosonic integrals. On the other hand, in the completely isospectral case, the pair of Schrödinger Hamiltonians is characterized by a superpotential of a topologically trivial, modulated kink-antikink-type nature. Such pairs can be obtained from the pairs of the almost isospectral case just by sending the W-type defect associated with the lowest-energy discrete value to infinity. The completely isospectral pairs are described by a spontaneously broken exotic nonlinear $\mathcal{N}=4$ supersymmetry. Unlike the unbroken supersymmetry case, in such systems, the two states corresponding to the lowest doubly degenerate energy value are annihilated (in a generic case) only by the bosonic Lax–Novikov integrals.

When one of the two first-order supercharges is reinterpreted as the matrix Hamiltonian operator, we arrive at the Bogoliubov–de Gennes system, in which the superpotential will play the role of a scalar Dirac potential. The results presented here allow us then, particularly, to obtain new types of self-consistent condensates and associate with them new solutions for the Gross–Neveu model, which correspond to the kink- and kink-antikink-type configurations in the crystalline background. We are going to consider this problem elsewhere.

It is worth noticing that Dirac Hamiltonians with scalar potential appear, in different physical context, in the description of the low-energy charge carriers in graphene and related carbon nanostructures. This fact opens potential applications of the results in physics of condensed matter systems, following the ideas of Refs. [49–51].

The discussed constructions can be generalized to the case of the PT-symmetric one-gap potentials. To achieve this, it is sufficient to apply the complex shift considered in Sec. III to the described Hermitian systems with periodicity defects. Such systems have an immediate application in the context of the PT-symmetric quantum mechanics and optics.

An interesting development of the presented results is to "reconstruct" the time dependence for defects in a periodic background of the one-gap Lamé system in correspondence with dynamics illustrated, as an example, by Fig. 4. This would provide us a new class of solutions for the KdV and mKdV equations. At the same time, it is natural to consider the generalization of the construction to the case of quantum n-gap systems with $n>1$. One can also wonder if, somehow, both W-type and M-type defects are the result of "shrinking" bands from a more generic finite-gap Hamiltonian, under some special limit.

Finally, it would also be very interesting to look for the $(1+1)$-dimensional field theories, in which nontrivial solutions are controlled by a stability operator of the Schrödinger type [52] with the potentials of the nature considered here.

ACKNOWLEDGMENTS

The work of M. P. has been partially supported by Chilean FONDECYT Grant No. 1130017. M. P. thanks Salamanca University and Nuclear Physics Institute of the Academy of Sciences of the Czech Republic (NPI ASCR), where a part of this work was done, for hospitality. A. A. acknowledges the Chilean CONICYT scholarship 21120826 and financial support of Dirección de Postgrado and Vicerrectoria Académica of the Universidad de Santiago de Chile. He thanks Salamanca University for the kind hospitality. F. C. wishes to thank the warm hospitality of NPI ASCR. F. C. is partially supported through FONDECYT Grant No. 11121651, CONICYT Grant No. 79112034, and by the Alexander von Humboldt Foundation. Centro de Estudios Científicos is funded by the Chilean Government through the Centers of Excellence Base Financing Program of CONICYT. V. J. was supported by Project No. RVO61389005 of the NPI ASCR.

APPENDIX: NON-SINGULARITY OF POTENTIALS

We show here that the family of Hamiltonians

$$H_{2\ell,n}=H_{0,0}-2\frac{d^2}{dx^2}(\log W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))) \tag{A1}$$

is given in terms of the *nonsingular* potentials, which correspond to the soliton defects introduced into the periodic background of the one-gap Lamé system. To achieve this, we demonstrate successively that the Wronskians appearing in the structure of $H_{0,n}$, $H_{2\ell,0}$, and, finally, $H_{2\ell,n}$ are nodeless on the real line. The notations we employ are explained in the main text.

1. Lower prohibited band

To show that the potential of $H_{0,n}$ is regular, i.e., has no zeros on the real line, we will demonstrate that

$$(-1)^{\frac{n(n+1)}{2}}W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_{n+1}}(n+1))>0. \tag{A2}$$

First, we define the two sets of functions,

$$f_n(x)\equiv(-1)^n\frac{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_{n+1}}(n+1))}{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))} \tag{A3}$$

and

$$g_n(x)\equiv(-1)^n\frac{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n),\mathcal{F}_{s_{n+2}}(n+2))}{W(\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))}, \tag{A4}$$

which are nonphysical eigenstates of $H_{0,n}$ with eigenvalues ε^-_{n+1} and ε^-_{n+2}, respectively. We will check below that $f_n(x)>0$, while $g_n(x)$ has only one zero.

In correspondence with the definition $\mathbb{W}_{0,0}=1$ introduced in Eq. (4.21), for $n=0$, we have $f_0=\mathcal{F}_+(1)>0$, and $g_0=\mathcal{F}_-(2)$. The second function (plotted for a particular case with $C_2 = 1$ in Fig. 3) has one zero, which we denote by x_0. Thus, we have $g_0(x) > 0$ for $x < x_0$ and $g_0(x) < 0$ for $x > x_0$.

For the case $n = 1$, we also define the functions

$$f(x) = W(\mathcal{F}_+(1), \mathcal{F}_-(2)), \qquad g(x) = W(\mathcal{F}_+(1), \mathcal{F}_+(3)), \tag{A5}$$

which appear in the numerators of (A3) and (A4). Taking into account that $\mathcal{F}$ are solutions of the stationary Schrödinger equation, it is straightforward to check that

$$f'(x) = (\varepsilon_1^- - \varepsilon_2^-)\mathcal{F}_+(1)\mathcal{F}_-(2), \tag{A6}$$

$$g'(x) = (\varepsilon_1^- - \varepsilon_3^-)\mathcal{F}_+(1)\mathcal{F}_+(3). \tag{A7}$$

As $\varepsilon_2^- < \varepsilon_1^- < 0$, we observe that $\mathrm{sign}(f'(x)) = \mathrm{sign}(\mathcal{F}_-(2))$. Then,

$$f(x_0) = \mathcal{F}_+(x_0; \beta_1^-, C_1)\mathcal{F}'_-(x_0; \beta_2^-, C_2) \tag{A8}$$

since $\mathcal{F}_-(x_0; \beta_2^-, C_2) = 0$. From the Schrödinger equation, we have also $\mathcal{F}'_-(x_0; \beta_2^-, C_2) \neq 0$, and from the definition (4.1), it follows that $\mathcal{F}'_-(x_0; \beta_2^-, C_2) < 0$. We have then $f(x_0) < 0$, and hence $\mathrm{sign}(f'(x)) = \mathrm{sign}(\mathcal{F}_-(2))$. Thus, the function $f(x)$ increases monotonically from $f(-\infty) = -\infty$, it takes a maximum negative value $f(x_0) < 0$ at $x = x_0$, and then it decreases again monotonically to $f(\infty) = -\infty$. This means that $f(x) < 0$ and, as a consequence,

$$f_1(x) = -\frac{W(\mathcal{F}_+(1), \mathcal{F}_-(2))}{\mathcal{F}_+(1)} > 0 \tag{A9}$$

for all x.

The derivative $g'(x)$ takes positive values and grows up exponentially for $x \to \pm\infty$. Therefore, $g(x)$ passes through zero only once at some point x_1. The function

$$g_1(x) = -\frac{W(\mathcal{F}_+(1), \mathcal{F}_+(3))}{\mathcal{F}_+(1)} \tag{A10}$$

has then only one zero at this point x_1 and takes positive and negative values for $x < x_1$ and $x > x_1$, respectively. So, we see that the nonphysical eigenstates f_0 and f_1 of $H_{0,0}$ and $H_{0,1}$, respectively, have no zeros, while their eigenfunctions g_0 and g_1 have one zero, where their slope is negative.

We extend now this result by induction for arbitrary n by showing that $f_n(x) > 0$ while g_n has only one zero x_n and that $g_n(x) > 0$ and $g_n(x) < 0$ for $x < x_n$ and $x > x_n$, respectively, and so, $g'_n(x_n) < 0$.

By using the Darboux–Crum construction, we can check that functions $f_n(x)$ and $g_n(x)$ are nonphysical eigenstates of the Schrödinger operator

$$H_{0,n} = H_{0,0} - 2\frac{d^2}{dx^2}\log W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_n}(n)) \tag{A11}$$

with eigenvalues ε_{n+1}^- and ε_{n+2}^-. For $n+1$, we have

$$f_{n+1}(x) = (-1)^{n+1}\frac{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+2}}(n+2))}{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1))} = -\frac{W(f_n, g_n)}{f_n}, \tag{A12}$$

$$W'(f_n, g_n) = (\varepsilon_{n+1}^- - \varepsilon_{n+2}^-)f_n g_n, \tag{A13}$$

from where we obtain that $\mathrm{sign}W'(f_n, g_n) = \mathrm{sign}g_n(x)$. The zero x_n of g_n corresponds therefore to the maximum of $W(f_n, g_n)$,

$$W(f_n, g_n)(x_n) = g'_n(x_n)f_n(x_n) < 0. \tag{A14}$$

Since $\mathrm{sign}W'(f_n, g_n) = \mathrm{sign}g_n(x)$, the function $-W(f_n, g_n)$ decreases for $x < x_n$ and increases for $x > x_n$, and then $-W(f_n, g_n)(x_n) > 0$ for all x. From Eq. (A12), we conclude that $f_{n+1}(x) > 0$ for all x.

Let us change β_{n+1}^- by β_{n+3}^- in the numerator of the function $f_n(x)$ in (A3) and redefine the resulting function as $h_n(x)$. This function takes positive values, $h_n(x) > 0$, and we obtain the following relations:

$$g_{n+1} = (-1)^{n+1}\frac{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1), \mathcal{F}_{s_{n+3}}(n+3))}{W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1))} = -\frac{W(f_n, h_n)}{f_n}, \tag{A15}$$

$$W'(f_n, h_n) = (\varepsilon_{n+1}^- - \varepsilon_{n+3}^-)f_n h_n > 0. \tag{A16}$$

Consequently, $W(f_n, h_n)$ increases exponentially from $-\infty$ to $+\infty$ passing through one zero, which we call x_{n+1}. Since $f_n(x) > 0$ is a regular function, and g_{n+1} has only one zero at x_{n+1}, we find that $g_{n+1}(x) > 0$ for $x < x_{n+1}$ and $g_{n+1}(x) < 0$ for $x > x_{n+1}$.

Finally, from the definition (A3) of $f_n(x)$, we obtain

$$f_n f_{n-1} \ldots f_1 \mathcal{F}_+(1) = ((-1)^{\sum_{i=1}^n i})W(\mathcal{F}_+(1), \ldots, \mathcal{F}_{s_{n+1}}(n+1)), \tag{A17}$$

and since

$$f_n f_{n-1} \ldots, f_1 \mathcal{F}_+(1) > 0, \tag{A18}$$

we demonstrate the necessary relation (A2).

2. Upper prohibited band

To show that $H_{2\ell,0}$ is nonsingular on the whole real line, we show that the Wronskian is a regular nodeless function

$W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell))$, where the functions $\Phi_+(2l-1)$ and $\Phi_-(2l)$, $l=1,2\ldots$ correspond to a generalization of those defined in (4.30) and (4.31) for $\ell=1$.

Before, we showed that $W(\Phi_+(1),\Phi_-(2))<0$ by choosing parameters $0<\beta_1^+<\beta_2^+<\mathbf{K}$. This condition means that $1>\varepsilon_1^+>\varepsilon_2^+>k'^2$ for the eigenvalues of the nonphysical eigenstates $\Phi_+(1)$ and $\Phi_-(2)$ inside the intermediate forbidden band of $H_{0,0}$.

To demonstrate the validity of the formulated statement for the next case $\ell=2$, we define an eigenstate of the one-gap Lamé system with the displaced argument, $x\to x+\beta_3^++i\mathbf{K}'$, in the following form:

$$\check{\Phi}[1](x,\beta_3^+)=\frac{W(\Psi_+^{\beta_3^+}(x),\Phi_+(1))}{\Psi_+^{\beta_3^+}(x)}. \quad (A19)$$

This state has an infinite number of poles at the zeros of $\Psi_+^{\beta_3^+}(x)$. Between each pair of poles, $\check{\Phi}[1](x,\beta_3^+)$ does not change the sign and takes nonzero values. Its sign is inverted in the neighbor regions separated by poles. From the theorem on zeros, the linearly independent state

$$\check{\Phi}[2](x;\beta_3^+)=\frac{W(\Psi_+^{\beta_3^+}(x),\Phi_-(2))}{\Psi_+^{\beta_3^+}(x)} \quad (A20)$$

has also an infinite number of poles, but between each pair of poles, it possesses one zero, which we denote as x_i. The function (A20) preserves the sign when the argument passes through any pole.

Now, it is necessary to show that $W(\check{\Phi}[1],\check{\Phi}[2])$ does not have zeros. For this, we redefine the function $\check{\Phi}[2]$ up to a sign in such a way that its derivative in some x_{i_0} will be positive. In the same way, we also redefine, up to a global sign, the function $\check{\Phi}[1](x)$ to have $\check{\Phi}[1](x_{i_0})<0$. Thus, we obtain that

$$W(\check{\Phi}[1],\check{\Phi}[2])(x_i)=\check{\Phi}[1](x_i)\check{\Phi}'[2](x_i)<0, \quad (A21)$$

while

$$W'(\check{\Phi}[1],\check{\Phi}[2])=(\varepsilon_1^+-\varepsilon_2^+)\check{\Phi}[1]\check{\Phi}[2]. \quad (A22)$$

The function $W(\check{\Phi}[1],\check{\Phi}[2])$ has a local extremum at each x_i, and its derivative is positive for $x<x_i$ until a pole and is negative for $x>x_i$ until the next pole since x_i is a local maximum of $W(\check{\Phi}[1],\check{\Phi}[2])(x)$. From here, we conclude that $W(\check{\Phi}[1],\check{\Phi}[2])(x)$ does not have zeros and hence is of one sign.

Because of the identity

$$W(\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(x))=\Psi_+^{\beta_3^+}(x)W(\check{\Phi}[1],\check{\Phi}[2])(x;\beta_3^+), \quad (A23)$$

the Wronskian $W(\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(\pm x))$ has exactly the same zeros as $\Psi_\pm^{\beta_3^+}(x)$. Note that we have $W(\Phi_+(-x;\beta_1^+,1/C_1),\ \Phi_-(-x;\beta_2^+,1/C_2),\Psi_+^{\beta_3^+}(-x))=-W(-\Phi_+(1),\Phi_-(2),\Psi_+^{\beta_3^+}(-x))=-W(\Phi_+(1),\Phi_-(2),-\Psi_+^{\beta_3^+}(-x))$. Using the Wronskian properties, it is easy to see that $W(a,b)(x)=-W(a,b)(-x)$ and $W(a,b,c)(x)=-W(a,b,c)(-x)$, but $W(a,b,c,d)(x)=W(a,b,c,d)(-x)$. Taking in account the above relations, we can write

$$\mathrm{sign}W(\check{\Phi}[1],\check{\Phi}[2])(x;\beta_3^+)=\mathrm{sign}W(\check{\Phi}[1],\check{\Phi}[2])(-x;\beta_3^+). \quad (A24)$$

Thus, the zeros of the nonphysical states of $H_{2,0}$,

$$\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3))}{W(\Phi_+(1),\Phi_-(2))}\quad\text{and}\quad\frac{W(\Phi_+(1),\Phi_-(2),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))}, \quad (A25)$$

are within the intervals $\mathcal{I}_n^+(\beta_3^+)$ and $\mathcal{I}_n^-(\beta_4^+)$, respectively, see Eq. (4.34), where $\mathcal{I}_n^+(3)\bigcap\mathcal{I}_n^-(4)=\emptyset$. As a consequence of the theorem on zeros, their zeros are alternated.

Next, we can check that under the condition $0<\beta_1^+<\beta_2^+<\beta_3^+<\beta_4^+<\mathbf{K}$, the Wronskian

$$W\left(\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3))}{W(\Phi_+(1),\Phi_-(2))},\frac{W(\Phi_+(1),\Phi_-(2),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))}\right)=\frac{W(\Phi_+(1),\Phi_-(2),\Phi_+(3),\Phi_-(4))}{W(\Phi_+(1),\Phi_-(2))} \quad (A26)$$

does not have zeros nor the function $W(\Phi_+(1),\Phi_-(2),\Phi_+(3),\Phi_-(4))$.

This result can be generalized for the case of the Wronskian of 2ℓ states, $W(\Phi_+(1),\Psi_-(2),\ldots,\Phi_+(2\ell-1),\Phi_-(2\ell))$, under the condition $0<\beta_1^+<\beta_2^+<\ldots<\beta_{2\ell}^+<\mathbf{K}$.

Using the identity

$$\begin{aligned}&W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Psi_+^{\beta_{2\ell+1}^+}(x))\\&=W(\Phi_+(1),\ldots,\Phi_-(2\ell-2),\Psi_+^{\beta_{2\ell+1}^+}(x))\\&\quad\times W(\check{\Phi}[1,\ldots,2\ell-1],\check{\Phi}[1,\ldots,2\ell-2,2\ell]),\end{aligned} \quad (A27)$$

we have

$$\begin{aligned}&\Psi_+^{\beta_{2\ell+1}^+}(x)W(\check{\Phi}[1],\check{\Phi}[2])\times W(\check{\Phi}[1,2,3],\check{\Phi}[1,2,4])\times\ldots\\&\quad\times W(\check{\Phi}[1,\ldots,2\ell-2,2\ell-1],\check{\Phi}[1,\ldots,2\ell-2,2\ell])\\&=W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Psi_+^{\beta_{2\ell+1}^+}(x)),\end{aligned} \quad (A28)$$

where

$$\check{\Phi}[1,\ldots,l,l+r](x,\beta^+) = \frac{W(\Psi_+^{\beta^+}(x),\Phi_+(1),\ldots,\Phi_-(2l),\Phi_{s_{2l+r}}(2l+r))}{W(\Psi_+^{\beta^+}(x),\Phi_+(1),\ldots,\Phi_-(2l))} \quad \text{(A29)}$$

and $r=1,2,\ l=0,1,\ldots$. Having in mind all previous demonstrations, it is clear that

$$|W(\check{\Phi}[1,\ldots,2l-2,2l-1],\check{\Phi}[1,\ldots,2l-2,2l])|>0, \quad \text{(A30)}$$

and the functions

$$\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_+(2\ell+1))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \quad \text{and} \quad \frac{W(\Phi_+(1),\ldots,.\Phi_-(2\ell),\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \quad \text{(A31)}$$

have alternating zeros in the intervals $\mathcal{I}_n^+(\beta_{2\ell+1}^+)$ and $\mathcal{I}_n^-(\beta_{2\ell+2}^+)$, respectively. Then,

$$W\left(\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_+(2\ell+1))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))},\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}\right)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell+2))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \quad \text{(A32)}$$

is regular and has no zeros, which means that $W(\Phi_+(1),\ldots,\Phi_-(2\ell+2))$ is nonsinguar and nodeless if and only if $W(\Phi_+(1),\ldots,\Phi_-(2\ell))$ is regular and has no zeros.

Besides, if the potentials of the systems $H_{2\ell,0}$ are nonsingular for all real x, by taking limits $C_l\to\infty$ or $C_l\to 0$, the regularity is preserved, and we get a regular Hamiltonians $H_{2\ell-1,0}$ with $2\ell-1$ states in the gap of the Lamé system.

3. Mixed case

Finally, using the all previous demonstrations, we show that the most general Hamiltonian

$$H_{2\ell,n}=H_{0,0}-2\frac{d^2}{dx^2}(\log W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n))) \quad \text{(A33)}$$

has also a nonsingular potential. To this aim, we define

$$F_{2\ell}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),F(x;\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}, \quad \text{(A34)}$$

which is a nonphysical eigenstate of $H_{2\ell,0}$ with eigenvalue $\mathcal{E}(\beta^-+i\mathbf{K}')$. Using the Wronskian identity

$$W(\tilde{\Phi}_1,\ldots,\tilde{\Phi}_l)=W(W(F,\Phi_1)/F,\ldots,W(F,\Phi_l)/F) = W(F,\Phi_1,\ldots,\Phi_l)/F, \quad \text{(A35)}$$

where $\tilde{\Phi}=W(F,\Phi)/F$, we obtain

$$F_{2\ell}(x;\beta^-)=\frac{W(\tilde{\Phi}_+(1),\ldots,\tilde{\Phi}_-(2\ell))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))}F(x;\beta^-) = G_{2\ell}(x;\beta^-)F(x;\beta^-). \quad \text{(A36)}$$

$\tilde{\Phi}_i$ is the eigenstate of the displaced Lamé system $H_{0,0}(x+\beta^-)$, with the properties similar to those as Φ_i. We have shown that $W(\Phi_+(1),\ldots,\Phi_-(2\ell))$ is nodeless and takes finite values of a definite sign. This implies that $W(\tilde{\Phi}_+(1),\ldots,\tilde{\Phi}_-(2\ell))$ share the same properties. Hence, function $G_{2\ell}(x;\beta^-)$ also possesses the same indicated properties. Taking into account the properties of the functions inside the Wronskian under the reflection $x\to -x$, it is not difficult to show that $\text{sign}G_{2\ell}(x;\beta^-)=\text{sign}G_{2\ell}(x;-\beta^-)$. Having the identity $F(-x;\beta^-)=F(x;-\beta)$, we find that

$$\mathcal{F}_{2\ell,+}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\mathcal{F}_+(\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2l))} \quad \text{(A37)}$$

$$= CG_{2\ell}(x;\beta^-)F(x;\beta^-) + \frac{1}{C}G_{2\ell}(x;-\beta^-)F(-x;\beta^-). \quad \text{(A38)}$$

Since $G_{2\ell}(x;\pm\beta^-)$ take values of the same sign and increase exponentially, the function $\mathcal{F}_{2\ell,+}$ has no zeros. Then,

$$\mathcal{F}_{2\ell,-}(x;\beta^-)=\frac{W(\Phi_+(1),\ldots,\Phi_-(2\ell),\mathcal{F}_-(\beta^-))}{W(\Phi_+(1),\ldots,\Phi_-(2\ell))} \quad \text{(A39)}$$

$$= CG_{2\ell}(x;\beta^-)F(x;\beta^-) - \frac{1}{C}G_{2\ell}(x;-\beta^-)F(-x;\beta^-) \quad \text{(A40)}$$

has only one zero. Here, the functions $\mathcal{F}_{2\ell,\pm}$ are linearly independent eigenstates of the operator $H_{2\ell,0}$ with eigenvalues $\mathcal{E}(\beta^-+i\mathbf{K}')$, which are analogous to the eigenfunctions $\mathcal{F}_\pm$ of the Lamé system $H_{0,0}$; see (4.1). Using the arguments presented in Appendix A 1, one can show that

$$W(\mathcal{F}_{2\ell,+}(1),\ldots,\mathcal{F}_{2\ell,s_{n+1}}(n+1)) \quad \text{(A41)}$$

has no zeros. From the Crum theorem,

$$H_{2\ell,n} = H_{2\ell,0} - 2\frac{d^2}{dx^2}\log W(\mathcal{F}_{2\ell,+}(1),\ldots,\mathcal{F}_{2\ell,s_n}(n))$$
$$= H_{0,0} - 2\frac{d^2}{dx^2}\log \mathbb{W}_{2\ell,n}, \qquad (A42)$$

and it follows that

$$W(\Phi_+(1),\Phi_-(2),\ldots,\Phi_-(2\ell),\mathcal{F}_+(1),\ldots,\mathcal{F}_{s_n}(n)) \qquad (A43)$$

is a smooth and nodeless function.

---

[1] B. Sutherland, Phys. Rev. A **8**, 2514 (1973).
[2] Y. Alhassid, F. Gursey, and F. Iachello, Phys. Rev. Lett. **50**, 873 (1983).
[3] H. W. Braden and A. J. Macfarlane, J. Phys. A **18**, 3151 (1985).
[4] J. Mateos Guilarte, J. Phys. A **23**, 3141 (1990).
[5] A. Saxena and A. R. Bishop, Phys. Rev. A **44**, R2251 (1991).
[6] J. Q. Liang, H. J. W. Muller-Kirsten, and D. H. Tchrakian, Phys. Lett. B **282**, 105 (1992).
[7] Y. Brihaye, S. Giller, P. Kosinski, and J. Kunz, Phys. Lett. B **293**, 383 (1992).
[8] P. M. Sutcliffe, J. Phys. A **29**, 5187 (1996).
[9] D. Boyanovsky, H. J. de Vega, R. Holman, and J. F. J. Salgado, Phys. Rev. D **54**, 7570 (1996).
[10] P. B. Greene, L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. D **56**, 6175 (1997).
[11] G. V. Dunne and J. Feinberg, Phys. Rev. D **57**, 1271 (1998).
[12] G. V. Dunne and K. Rao, J. High Energy Phys. 01 (2000) 019.
[13] F. Finkel, A. Gonzalez-Lopez, A. L. Maroto, and M. A. Rodriguez, Phys. Rev. D **62**, 103515 (2000).
[14] R. S. Maier and D. L. Stein, Phys. Rev. Lett. **87**, 270601 (2001).
[15] G. V. Dunne and M. Shifman, Ann. Phys. (Amsterdam) **299**, 143 (2002).
[16] M. Thies, J. Phys. A **39**, 12707 (2006).
[17] N. Dorey and B. Vicedo, J. High Energy Phys. 07 (2006) 014.
[18] F. Correa, L. M. Nieto, and M. S. Plyushchay, Phys. Lett. B **644**, 94 (2007).
[19] F. Correa, V. Jakubsky, L. M. Nieto, and M. S. Plyushchay, Phys. Rev. Lett. **101**, 030403 (2008).
[20] M. V. Ioffe, J. Mateos Guilarte, and P. A. Valinevich, Nucl. Phys. **B790**, 414 (2008).
[21] F. Correa, G. V. Dunne, and M. S. Plyushchay, Ann. Phys. (Amsterdam) **324**, 2522 (2009).
[22] M. S. Plyushchay, A. Arancibia, and L.-M. Nieto, Phys. Rev. D **83**, 065025 (2011).
[23] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons* (Plenum, New York, 1984).
[24] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin, 1994).
[25] P. Drazin and R. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, England, 1996).
[26] I. Kay and H. E. Moses, J. Appl. Phys. **27**, 1503 (1956).
[27] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).
[28] M. S. Plyushchay and L. M. Nieto, Phys. Rev. D **82**, 065022 (2010).
[29] A. Arancibia, J. M. Guilarte, and M. S. Plyushchay, Phys. Rev. D **87**, 045009 (2013).
[30] A. Arancibia, J. M. Guilarte, and M. S. Plyushchay, Phys. Rev. D **88**, 085034 (2013).
[31] A. Arancibia and M. S. Plyushchay, Phys. Rev. D **90**, 025008 (2014).
[32] F. Correa and V. Jakubsky, Phys. Rev. D**90**, 125003 (2014).
[33] F. Gesztesy and R. Svirsky, Memoir. Am. Math. Soc. **118**, 1 (1996).
[34] D. J. Fernandez C., B. Mielnik, O. Rosas-Ortiz, and B. F. Samsonov, J. Phys. A **35**, 4279 (2002); D. J. Fernandez and A. Ganguly, Phys. Lett. A **338**, 203 (2005); Phys. At. Nucl. **73**, 288 (2010).
[35] E. Corrigan, Theor. Math. Phys. **171**, 655 (2012); E. Corrigan and C. Zambon, Nonlinearity **19**, 1447 (2006).
[36] R. Konik and A. LeClair, Nucl. Phys. **B538**, 587 (1999).
[37] J. Avan and A. Doikou, J. High Energy Phys. 11 (2012) 008.
[38] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1980).
[39] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[40] A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).
[41] W. Magnus and S. Winkler, *Hill's Equation* (Interscience, New York, 1966).
[42] F. Correa, V. Jakubsky, and M. S. Plyushchay, J. Phys. A **41**, 485303 (2008).
[43] K. V. Khmelnytskaya and H. C. Rosu, Ann. Phys. (Amsterdam) **325**, 2512 (2010); K. V. Khmelnytskaya, H. C. Rosu, and A. González, Ann. Phys. (Amsterdam) **325**, 596 (2010).
[44] F. Correa and M. S. Plyushchay, J. Phys. A **40**, 14403 (2007).
[45] J. L. Burchnall and T. W. Chaundy, Proc. London Math. Soc. Ser 2, **21**, 420 (1923); Proc. R. Soc. A **118**, 557 (1928).
[46] A. Treibich and J.-L. Verdier, C. R. Acad. Sci. Paris **311**, 51 (1990); Duke Math. J. **68**, 217 (1992).
[47] A. Arancibia and M. S. Plyushchay, Phys. Rev. D **85**, 045018 (2012).
[48] E. Witten, Nucl. Phys. **B188**, 513 (1981); **B202**, 253 (1982).
[49] V. Jakubsky, L. M. Nieto, and M. S. Plyushchay, Phys. Rev. D **83**, 047702 (2011).
[50] V. Jakubsky and M. S. Plyushchay, Phys. Rev. D **85**, 045035 (2012).
[51] F. Correa and V. Jakubsky, Phys. Rev. D **87**, 085019 (2013).
[52] A. Alonso-Izquierdo and J. Mateos Guilarte, Ann. Phys. (Amsterdam) **327**, 2251 (2012).