PHYSICAL REVIEW D **90,** 125039 (2014)

Relation between Lyapunov exponents and decoherence for real scalar fields in de Sitter spacetime

G. de Souza,[1,†] K. M. Fonseca-Romero,[2,‡] Marcos Sampaio,[3,§] and M. C. Nemes[1,*]

[1]*Departamento de Física, ICEx, Universidade Federal de Minas Gerais, P.O. Box 702, Belo Horizonte, Minas Gerais 30.161-970, Brazil*

[2]*Facultad de Ciencias, Departamento de Física, Universidad Nacional de Colombia Bogotá, Carrera 30 Calle 45-03, C.P. 111321 Bogotá, Colombia*

[3]*Department of Mathematical Sciences, Centre for Particle Theory, Durham University, South Road, Durham DH1 3LE, United Kingdom*

(Received 8 May 2014; published 30 December 2014)

We investigate the relationship between orbital instability and decoherence in de Sitter (dS) spacetime. We consider a simple quadratic toy model proposed by Brandenberger, Laflamme and Mijić of two interacting scalar fields in a dS background. It admits a modewise separation, with each mode consisting of a pair of nonautonomous coupled harmonic oscillators. We show that the (classical) maximal Lyapunov exponent of every mode equals the asymptotic rate of (quantum) von Neumann entropy production of each oscillator, assuming an initial vacuum. We find that for moderately long times after horizon crossing, orbital instability, entropy and single-mode squeezing are larger for increasing coupling strength. If the entropy of an oscillator increases more rapidly than squeezing, for example in the strong-coupling regime for not too high frequencies, the noise of every quadrature of the asymptotic state will be larger than the vacuum noise. The results suggest the possibility that simple, nonlinear interacting physical processes with unstable or chaotic classical counterparts may provide an important contribution to the effectiveness of the classicalization of cosmological scalar fields during a dS stage of spacetime expansion.

DOI: 10.1103/PhysRevD.90.125039 PACS numbers: 03.65.Yz, 04.62.+v

I. INTRODUCTION

The first studies of classicalization of primordial density fluctuations used quantum optics tools in the theory of cosmological perturbations. Such fluctuations were shown to unitarily evolve from an initial vacuum into a highly squeezed vacuum state in a purely de Sitter stage of spacetime expansion. In the time-asymptotic limit, where the squeezing is large, quantum expectation values calculated from the evolved state were found to become indistinguishable from classical averages calculated from a stochastic distribution [1,2]. However, the evolution of isolated fluctuations is isoentropic. In order to evaluate the entropy of primordial fluctuations and describe their classicalization at quantum-state level, one needs to consider environment-induced decoherence, an irreversible process in which the system of interest loses quantum coherences and increases its von Neumann entropy. Because gravity has infinite range and couples to all sources of energy, interactions with some sort of environment are unavoidable. Therefore, environmentally induced decoherence that will certainly play a crucial role in the classicalization of primordial density fluctuations must be taken into account.

The entropy increase in usual models [3–5] occurs because of dynamically generated entanglement correlations between the system and the environment, which is assumed to consist of infinite degrees of freedom. The large environment is unaccessible in its entirety, and tracing it out leads to entropy generation at system level. On the other hand, it has been shown that in Minkowski spacetimes the coupling to a small environment (consisting of one or few degrees of freedom), in the presence of classical dynamical instabilities, can result in much stronger decoherence effects [6–8]. It has been found that for such interacting systems, displaying classical Lyapunov instability, von Neumann entropy generation rates at observed system level either coincide or are favored by the positive maximal Lyapunov exponents. This type of behavior is conjectured [9] to hold up to a certain level of generality, but still not much is known beyond specific examples.

In the present paper, we investigate such an example in the context of the quantum-to-classical transition of a massless scalar field state over de Sitter (dS) spacetime. We will consider for this purpose a simple, solvable model proposed in [10] of two interacting massless scalar fields coupled through a bilinear derivative interaction potential. One of the fields is taken to represent the system of interest, which can be thought as any massless real scalar field

*Deceased.
†Corresponding author.
souza.gustavohenrique@gmail.com
‡kmfonsecar@unal.edu.co
§marcos.sampaio@durham.ac.uk

 © 2014 American Physical Society

producing density fluctuations during a dS stage of inflationary spacetime expansion.[1] The other field represents an unobservable environment. The action for the model is quadratic and in reciprocal space it reduces to a modewise interaction between the system of interest and the unobservable field—that is, to a collection of pairs of interacting harmonic oscillators. Thus, for each mode of the system, the environment consists of a single degree of freedom. When considered over Minkowski spacetime the composite system's classical dynamics for such an action is of course stable, with a phase-space flow consisting of bounded periodic orbits. However, over dS spacetime the system becomes nonautonomous and unstable, with a positive maximal Lyapunov exponent μ that we will calculate to be equal to the background spacetime inflation rate (i.e., exponential expansion rate) given by the Hubble parameter H.

Assuming an initial vacuum state for a composite system mode, we will evaluate the asymptotic growth rate μ_S of its entanglement of formation (EoF)—that is, the von Neumann entropy for an observed system mode. We find that $\mu = \mu_S$, allowing us to calculate a logarithmic divergence for the EoF modulated by the inflation rate and to demonstrate the superhorizon relationship between the entropy generation at system level and the classical exponential orbit separation rate, after entanglement fluctuations cease.

We also examine the relationship between orbital instability and decoherence from the point of view of the nonclassical depth. This is a measure of the system's state decoherence that is also sensitive to its squeezing properties and focuses on the emergence of a phase-space representation for it interpretable as a classical probability distribution, which is a very important aspect of decoherence to take into account in cosmological contexts. We will show how orbital instability influences the nonclassical depth, and thus the effectiveness of the classicalization of the system in this sense. Although the maximal Lyapunov exponent is independent of model details and is given here only by the spacetime inflation rate, the actual instantaneous exponential orbit separation rate for a given mode increases monotonically as a function of the coupling strength. We shall find that it is proportional to the entropy for late times, in such a way that entropy values will get larger when we shift from the weak to the strong coupling extremes. On the other hand, because Gaussianity is preserved here in the course of evolution, the nonclassical depth of a system mode's state after horizon crossing will be given by the asymptotic balance between single-mode squeezing and von Neumann entropy. This indicates that orbital instability will influence the nonclassical depth asymptotics. We will quantify this influence, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes. We will prove that although in the weak-coupling regime every mode will evolve into a highly quadrature squeezed state as expected, in the strong-coupling limit all modes of the observed field evolve into a state with noise larger than the vacuum noise in every phase-space direction (zero nonclassical depth) except for the very high-frequency sector. As we will discuss, these results suggest the possibility that simple, nonlinear interacting physical processes with unstable or chaotic classical dynamical counterparts may provide an important contribution to the effectiveness of the classicalization of cosmological scalar fields during a dS stage of spacetime expansion.

This paper is organized as follows. First, in Sec. II, we introduce the model, present its solution in the Heisenberg representation and calculate its classical maximal Lyapunov exponent. The Heisenberg picture solution makes it very easy to write the time evolution of the composite system Robertson-Schrödinger covariance matrix. We continue by presenting some background material on how to compute the quantities relevant to our analysis of decoherence in terms of the covariance matrix in Sec. III. Our results will be presented in Sec. IV and will be finally discussed in Sec. V.

II. THE BLM MODEL

Let us begin by introducing the model we will consider, describing its exact Heisenberg picture evolution and calculating its classical maximal Lyapunov exponent. This model was first proposed and used in investigations of decoherence of cosmological perturbations by Brandenberger, Laflamme and Mijić in [10], and for this reason we call it the BLM model. It describes a bipartite system of two coupled massless fields, the system of interest ϕ and the unobservable field ψ, over a curved spacetime with metric $g_{\mu\nu}$. The action of the BLM model reads (natural units will be used throughout the text)

$$S = \int d^4x\sqrt{g}\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi + \partial_\mu\psi\partial^\mu\psi + 2\lambda\partial_\mu\phi\partial^\mu\psi], \qquad (1)$$

where $g = -\det(g_{\mu\nu})$ and λ is the dimensionless coupling parameter normalized such that $\lambda \neq 0$, $|\lambda| < 1$. The cases $\lambda = \pm 1$ are excluded because when $\lambda = \pm 1$, (1) reduces to the action of a single isolated field $\tilde{\phi}_\pm = \phi \pm \psi$.

The weak- and strong-coupling limits are given by $\lambda \to 0$ and $|\lambda| \to 1$ respectively. Over the dS background, the metric is $ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)$, $a(\eta) = -(H\eta)^{-1}$, where η is the conformal time $\eta(t) = \int_\infty^t \frac{ds}{a(s)}$. In this case we have $g = a^4$ and $a(t) = e^{Ht}$, where the Hubble parameter $H \equiv \frac{1}{a}\frac{da}{dt}$ is a constant.

Since the action is quadratic, the system is exactly solvable. In order to write its exact solution in the

[1]For definitiveness, the reader may consider inflaton fluctuations in a first-order approximation, neglecting backreaction effects.

Heisenberg picture, we begin by expanding the fields in terms of their Fourier components, $\phi = \sum_{\vec{k}} \phi_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}}$ and $\psi = \sum_{\vec{k}} \psi_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}}$ in a large box of fixed comoving volume. Let $\Pi_{\phi,\vec{k}}(\eta)$ and $\Pi_{\psi,\vec{k}}(\eta)$ be the momenta conjugate to the Fourier field components. The Hamiltonian of the BLM model then reads $H = \sum_{\vec{k}} H_{\vec{k}}$, where

$$H_{\vec{k}} = \frac{1}{2a^2(1-\lambda^2)} \left(\Pi^2_{\phi,\vec{k}} + \Pi^2_{\psi,\vec{k}} - 2\lambda \Pi_{\phi,\vec{k}} \Pi_{\psi,\vec{k}} \right) + \frac{a^2k^2}{2} (\phi^2_{\vec{k}} + \psi^2_{\vec{k}} + 2\lambda \phi_{\vec{k}} \psi_{\vec{k}}). \quad (2)$$

For simplicity, we have assumed that our comoving length units are such that the box volume in the Fourier expansion reduces to unity.

Let us define new field modes which diagonalize the Hamiltonian $H_{\vec{k}}(\eta)$, using the symplectic transformation (the subindex $\vec{k}$ was dropped to simplify the notation)

$$\begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$
$$\begin{pmatrix} \pi_- \\ \pi_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Pi_\phi \\ \Pi_\psi \end{pmatrix}.$$

In terms of these new fields, the Hamiltonian (2) reads $H_{\vec{k}} = H_+ + H_-$, where

$$H_\pm = \frac{\pi^2_\pm}{2a^2(\eta)(1\pm\lambda)} + \frac{a^2(\eta)k^2(1\pm\lambda)}{2} \phi^2_\pm. \quad (3)$$

We can readily solve the equations of motion for these field modes:

$$\frac{d\hat{\phi}_\pm}{d\eta} = \frac{1}{i\hbar} [\hat{\phi}_\pm, \hat{H}_\pm] = \frac{1}{a^2(1\pm\lambda)} \hat{\pi}_\pm,$$
$$\frac{d\hat{\pi}_\pm}{d\eta} = \frac{1}{i\hbar} [\hat{\pi}_\pm, \hat{H}_\pm] = -a^2(1\pm\lambda)k^2 \hat{\phi}_\pm. \quad (4)$$

The general solution of the resulting second-order equation for these fields,

$$\frac{d}{d\eta}\left(a^2 \frac{du}{d\eta} \right) + a^2k^2 u = 0, \quad (5)$$

is a linear combination of Hankel functions, u and u^*, with

$$u(\eta) = \frac{1}{\sqrt{2}} \frac{H\eta}{k^{1/2}} e^{ik\eta} \left(1 + \frac{1}{k\eta} \right). \quad (6)$$

After some algebra, it can be shown that

$$\begin{pmatrix} \phi_\pm(\eta) \\ \pi_\pm(\eta) \end{pmatrix} = \begin{pmatrix} x(\eta) & \frac{y(\eta)}{1\pm\lambda} \\ (1\pm\lambda) z(\eta) & w(\eta) \end{pmatrix} \begin{pmatrix} \phi_\pm(\eta_0) \\ \pi_\pm(\eta_0) \end{pmatrix} \quad (7)$$

is the solution of the equations of motion (4). The functions $x(\eta)$, $y(\eta)$, $z(\eta)$, and $w(\eta)$ are given by

$$x(\eta) = -i(u^*_\eta v_0 - u_\eta v^*_0) = \frac{k\eta \cos k(\eta-\eta_0) - \sin k(\eta-\eta_0)}{k\eta_0},$$
$$y(\eta) = i(u^*_\eta u_0 - u_\eta u^*_0) = -\frac{H^2}{k^3} (k(\eta-\eta_0) \cos k(\eta-\eta_0) - (1 + k^2\eta_0\eta) \sin k(\eta-\eta_0)),$$
$$z(\eta) = -i(v^*_\eta v_0 - v_\eta v^*_0) = -\frac{k}{H^2\eta_0\eta} \sin k(\eta-\eta_0),$$
$$w(\eta) = i(v^*_\eta u_0 - v_\eta u^*_0) = \frac{k\eta_0 \cos k(\eta-\eta_0) + \sin k(\eta-\eta_0)}{k\eta},$$

where $v(\eta) \equiv a^2 u'_\eta$ and the prime here stands for differentiation with respect to conformal time.

To obtain dynamically generated correlations between the system and auxiliary (unobservable) field parties at a given instant $\eta \geq \eta_0$, starting from a factorized initial condition $\hat{\rho}_T(\eta_0) = \rho_\phi(\eta_0) \otimes \rho_\psi(\eta_0)$, we have to express the nondiagonal fields at time η in terms of these diagonalizing field coordinates at time η_0. The relation, in matrix form, is

$$\begin{pmatrix} \phi^\eta \\ \Pi^\eta_\phi \\ \psi^\eta \\ \Pi^\eta_\psi \end{pmatrix} = \underbrace{\begin{pmatrix} x & \frac{y}{1-\lambda^2} & 0 & \frac{\lambda y}{1-\lambda^2} \\ z & w & -\lambda z & 0 \\ 0 & \frac{\lambda y}{1-\lambda^2} & x & \frac{y}{1-\lambda^2} \\ -\lambda z & 0 & z & w \end{pmatrix}}_{=\mathbb{M}(\eta,\eta_0)} \begin{pmatrix} \phi^0 \\ \Pi^0_\phi \\ \psi^0 \\ \Pi^0_\psi \end{pmatrix}. \quad (8)$$

This equation can also be rewritten in a more compact form as

$$\boldsymbol{X}(\eta) = \mathbb{M}(\eta,\eta_0) \boldsymbol{X}(\eta_0), \quad (9)$$

where we have defined the vector

$$\boldsymbol{X}(\eta) = (\phi(\eta), \Pi_\phi(\eta), \psi(\eta), \Pi_\psi(\eta))^T. \quad (10)$$

The creation and annihilation operators for the reduced system and unobservable field are defined as

$$a_{1\vec{k}}(\eta) = \frac{1}{\sqrt{2}} \left(\phi_{\vec{k}}(\eta) + i\Pi_{\phi\vec{k}}(\eta) \right) = (a^\dagger_{1\vec{k}}(\eta))^\dagger, \quad (11)$$
$$a_{2\vec{k}}(\eta) = \frac{1}{\sqrt{2}} \left(\psi_{\vec{k}}(\eta) + i\Pi_{\psi\vec{k}}(\eta) \right) = (a^\dagger_{2\vec{k}}(\eta))^\dagger, \quad (12)$$

with the usual boson commutation relations $[a_{j\vec{k}}, a^\dagger_{j'\vec{k}'}] = \delta_{j,j'} \delta_{\vec{k},\vec{k}'}$, $j, j' = 1, 2$, being satisfied at any time $\eta \geq \eta_0$.

Notice that (8) is the description of the quantum dynamics for a given mode in the Fock Space $\mathcal{F}_{\vec{k},\eta_0}$ determined by the reference vacua $|0^{\phi}_{\vec{k}}\rangle$ and $|0^{\psi}_{\vec{k}}\rangle$ annihilated respectively by $a_{1\vec{k}}(\eta_0)$ and $a_{2\vec{k}}(\eta_0)$ at time η_0.

Observe that the matrix relation (8) also describes the classical phase-space flow associated to the dynamics of a mode under the BLM model. That is, one just has to consider $(\phi^0\Pi^0_\phi\psi^0\Pi^0_\psi)^T$ as an initial condition in phase-space and $(\phi^\eta\Pi^\eta_\phi\psi^\eta\Pi^\eta_\psi)^T$ as the time-evolved generalized coordinates. The time evolution of the distance between two neighboring points, $d(\eta)=\|\mathbf{X}_1(\eta)-\mathbf{X}_2(\eta)\|=\|\delta\mathbf{X}(\eta)\|=\sqrt{\delta\mathbf{X}^T(\eta)\delta\mathbf{X}(\eta)}$, gives the maximal Lyapunov exponent

$$\mu=\lim_{\eta\to0^-}\lim_{d(\eta_0)\to0}-\frac{H}{2\ln(-H\eta)}\ln\frac{d^2(\eta)}{d^2(\eta_0)}.$$

Here, $\mathbf{X}(\eta)$ is the vector defined in Eq. (10). Taking into account that $\mathbf{r}(\eta)$ evolves according to Eq. (8), we see that the square of the distance varies as $d^2(\eta)=\delta\mathbf{X}^T(\eta_0)\mathbb{N}\delta\mathbf{X}(\eta_0)$. The matrix $\mathbb{N}=\mathbb{M}^T(\eta,\eta_0)\mathbb{M}(\eta,\eta_0)$ is a 4×4 matrix of the form

$$\mathbb{N}=\begin{pmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{B} & \mathbb{A}\end{pmatrix},$$

where $\mathbb{A}$ and $\mathbb{B}$ are 2×2 symmetric matrices. The characteristic polynomial of $\mathbb{N}$ is the product of two quadratic polynomials; hence, the eigenvalues of $\mathbb{N}$ can be explicitly calculated. If we denote by μ_i, $i=\pm$, the roots with the largest real parts, the maximal Lyapunov exponent reads

$$\mu=\lim_{\eta\to0^-}\max_{i=\pm}-\frac{H}{2\ln(-H\eta)}\ln\Re(\mu_i(\eta,\eta_0)),$$

where $\Re(\mu_i(\eta,\eta_0))$ stands for the real part of the eigenvalue μ_i, $i=\pm$. Making an expansion around $\eta=0$, we obtain

$$\mu_\pm=\frac{H^4\eta_0^2(k\eta_0\cos k\eta_0+\sin k\eta_0)^2+k^4(1\pm\lambda)^2\sin^2k\eta_0}{H^4k^2\eta_0^2\eta^2}$$
$$=\frac{C}{\eta^2},\qquad C>0.$$

Finally, we find that the maximal Lyapunov exponent coincides with the Hubble parameter H giving the background spacetime inflation rate (i.e., exponential expansion rate),

$$\mu=\lim_{\eta\to0^-}-\frac{H(\ln C-2\ln(-\eta))}{2\ln(-H\eta)}=H. \tag{13}$$

III. QUANTIFYING DECOHERENCE

For the sake of completeness and to establish a notation for the sequence, we continue by briefly describing the quantities we will use to measure the decoherence process of the observed system. As mentioned in the introduction, we restrict our attention in this work to evaluate the time evolution of these quantities for an initial vacuum. It is a modewise factorized state of the form $\rho(\eta_0)=\prod_{\vec{k}}\rho_{\phi\vec{k}}(\eta_0)\otimes\rho_{\psi\vec{k}}(\eta_0)$, where the labels ϕ and ψ refer to the corresponding subsystem and where $\rho_{\phi\vec{k}}(\eta_0)=|0^\phi_{\vec{k}}\rangle\langle0^\phi_{\vec{k}}|$, $\rho_{\psi\vec{k}}(\eta_0)=|0^\psi_{\vec{k}}\rangle\langle0^\psi_{\vec{k}}|$. The initial state we refer to when we speak of a global vacuum initial condition for a given mode is $|0^\phi_{\vec{k}}\rangle\langle0^\phi_{\vec{k}}|\otimes|0^\psi_{\vec{k}}\rangle\langle0^\psi_{\vec{k}}|$. In the BLM model different modes do not interact and we can focus here on a fixed $\vec{k}$. So we omit from now on reference to mode labels whenever possible.

It is in general a very difficult task to calculate the evolution of this type of quantity for an interacting bipartite system. In the present case, however, our quadratic Hamiltonian will preserve the Gaussian character of an initial global vacuum in the course of evolution: the full composite system quantum state will be a generic two-mode squeezed vacuum at every instant. And for Gaussian states, these information-theoretic quantities can be written directly in terms of the Robertson-Schrödinger covariance matrix (CM), whose evolution can be easily found in terms of the Heisenberg picture solution to dynamics.

Remember that the CM is the real symmetric matrix Σ given in terms of second-order correlation functions for the fields and their respective momenta as

$$\Sigma_{ij}=\mathrm{tr}\left(\frac{1}{2}\{X_i,X_j\}\rho\right)-\mathrm{tr}(X_i\rho)\mathrm{tr}(X_j\rho), \tag{14}$$

where ρ denotes the full (two-mode) state and the X_i are entries of the four-dimensional vector $\boldsymbol{X}^T=(\phi,\Pi_\phi,\psi,\Pi_\psi)=(X_1,X_2,X_3,X_4)$.[2] Its evolution can be calculated from our Heisenberg picture solution of the model, which gives us the dynamics of the second-order correlation functions in (14) for our fields/momenta in a form such that the CM at time η can be written as a linear transformation of Σ at the earlier time η_0:

$$\sigma_{ij}(\eta)=\sum_{mn}f^{mn}_{ij}(\eta,\eta_0)\sigma_{mn}(\eta_0). \tag{16}$$

Expressions for the relevant $f^{mn}_{ij}(\eta,\eta_0)$ are given in the appendix.

Let us explain then how we will evaluate von Neumann entropy generation and nonclassicality in terms of the CM. We first split the CM in blocks as

[2]Notice that the elements of this vector satisfy the canonical commutation relations $[X_i,X_j]=i\tilde{\Lambda}_{ij}$ where

$$\tilde{\Lambda}=\mathrm{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}_1,\begin{pmatrix}0&1\\-1&0\end{pmatrix}_2\right). \tag{15}$$

$$\Sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{17}$$

where A, B and C are 2×2 matrices, and notice that the submatrix A corresponds to the CM of the Gaussian state obtained by tracing out the second (environmental) mode—that is, the reduced system state. This state is a single-mode squeezed thermal state (STS), which we denote from now on by ρ_ϕ. It is possible to show [11,12] that the von Neumann entropy $S(\rho_\phi)$ for such a STS is given in the present notations as the following function of the determinant D_A of A: $S(\rho_\phi) = f(\sqrt{D_A})$, where

$$f(x) = \left(x + \frac{1}{2}\right) \ln \left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) \ln \left(x - \frac{1}{2}\right).$$

Since the full system evolves unitarily and we assume an initial global vacuum, the quantum state ρ will always be pure in the course of dynamics. It follows that the von Neumann entropy of the system or the auxiliary field is a direct measure of the quantum entanglement between the parties. More precisely, it coincides with the so-called EoF, defined by $\mathrm{EoF}(\rho) = S(\mathrm{tr}_\psi \rho)$. It is convenient to notice that the average number of excitations ν for ρ_ϕ is related to $S(\rho_\phi) = \mathrm{EoF}(\rho)$ by the formula $S = f(\nu + \frac{1}{2})$ [13] and is also given in terms of its CM, by

$$\nu = \sqrt{D_A} + \frac{1}{2}. \tag{18}$$

Concerning the nonclassicality degree of the observed system, there are several quantifiers available besides the entropy that focus on different aspects of the decoherence process. We will also evaluate here a quantifier that is sensible to the squeezing properties of the observed system and focuses on the emergence of a phase-space representation for it interpretable as a classical probability distribution, which is a very important aspect to take into account when examining the quantum-to-classical transition in cosmological contexts. This is in agreement, for example, with the approach taken in [5]. More precisely, the idea is to consider that the reduced system is classical when it admits a positive, regular Glauber-Sudarshan P-representation[3] for its state. In this case, it is known [16] that the state's second-order quantum coherence function $g^{(2)}$ will be ≥ 1, which for a single-mode state is a drastic restriction for detectable quantum effects to show up in its excitation statistics.[4] Taking this into account, a proper quantifier of the effectiveness of the decoherence process must measure how distant the system state is from having a positive and regular P-function.

There is a nonclassicality measure which performs exactly this task. It is the nonclassical depth, introduced independently by Lee [18,19] and Lütkenhaus and Barnett [20]. In a nutshell, the rationale behind the nonclassical depth of a state with P-function $P(\alpha^*, \alpha)$ is to look at its Cahill R-function

$$R(\alpha^*, \alpha, \tau) = \frac{1}{\pi\tau} \int d^2u \exp\left(-\frac{1}{\tau}|\alpha - u|^2\right) P(u^*, u) \tag{19a}$$

$$R(\alpha^*, \alpha, 0) = P(\alpha^*, \alpha) \tag{19b}$$

as a convolution transformation with a Gaussian kernel $\frac{1}{\pi\tau}\exp(-\frac{1}{\tau}|\alpha - u|^2)$. This convolution mask is broader for larger τ, in such a way that the resulting smoothing effect on the output function is enhanced as τ increases. The definition of the nonclassical depth goes then as follows. First, if a given value of τ is large enough so that the R-function corresponding to the P-function of a given quantum state becomes acceptable as a classical phase-space distribution—that is, it becomes a positive ordinary function, normalizable—then we say that τ completes the smoothing operation [relative to the convolution transformation (19b)] for the considered state. Next, let Ω denote the set of all τ that will complete the smoothing operation of the state's P-function. The nonclassical depth for this state is defined as the quantity $\tau_m \equiv \inf_{\tau\in\Omega}\tau$.

It is immediate that we will have $\tau_m = 0$ for a state only when its P-function is already acceptable as a classical phase-space distribution. This is the case, for instance, for an arbitrary coherent state. Moreover, it is possible to show that for $\tau = 1$ one always has $R \equiv Q$ [19], where $Q = Q(\alpha)$ is the Husimi Q-function. The Q-function is acceptable as a classical phase-space distribution function for any quantum state. This establishes the upper bound and lower bounds for τ_m: we have $0 \leq \tau_m \leq 1$ for any state.

For a STS, the nonclassical depth can be easily evaluated in terms of the CM. If $\epsilon_<$ denotes the smallest eigenvalue of the CM—called the generalized squeeze variance (GSV) for the corresponding state—then it can be shown [19] that its nonclassical depth is

$$\tau_m = \max\left(\frac{1 - 2\epsilon_<}{2}, 0\right). \tag{20}$$

For the present purposes, this formula reduces the evaluation of the nonclassical depth of the observed system state ρ_ϕ in the course of evolution to keeping track of the GSV.

There is a simple expression for the GSV of a STS such as ρ_ϕ. In the previous notations, it is given by the dispersion [21]

[3]Here, regular means no more singular than a Dirac delta, which describes the P-function for a coherent state [14,15].

[4]To obtain second-order correlation effects one has to consider multimode field states, which would show up in a nontrivial interacting theory. This type of effect was investigated in the context of the statistics of inflaton quanta in [17].

$$\epsilon_< = \left(\nu + \frac{1}{2}\right)e^{-2|Z|}, \tag{21}$$

where $|Z|$ is the single-mode squeezing strength. This formula, albeit simple, will be important in our analysis in the next section of the effectiveness of the decoherence process for ρ_ϕ. It shows that it is the relative ratio between the quantities $\nu + \frac{1}{2}$ and $e^{-2|Z|}$, related respectively to entropy/thermalization and single-mode squeezing, which determines the nonclassical depth. We must be able then to write it in terms of the CM. This is done by noticing that $|Z|$ can be written after some algebra in terms of the CM by

$$|Z| = \frac{1}{2}\log\left(\frac{T_A}{2\sqrt{D_A}} + \sqrt{\frac{T_A^2}{4D_A} - 1}\right), \tag{22}$$

which taking Eq. (18) into account gives

$$\epsilon_< = \frac{2D_A}{T_A + \sqrt{T_A^2 - 4D_A}} = \frac{\sqrt{D_A}}{V_A}, \tag{23}$$

where $T_A = \mathrm{tr}A$, $D_A = \det\ A$ and $2\sqrt{D_A}V_A = T_A + \sqrt{T_A^2 - 4D_A}$. The formula also shows that in our context the limiting values of the nonclassical depth have the following interpretation: a value of τ_m close to the limit $\frac{1}{2}$ means that the STS ρ_ϕ is a highly quadrature squeezed state, while $\tau_m = 0$ means that its noise is larger than the vacuum noise in every phase-space direction and no squeezed generalized quadrature exists.

For further details on the aspects of Gaussian information theory discussed in this section, we refer the reader to [21–25].

IV. RESULTS

We are now in position to present our results. We begin by the study of the entropy production at system level, followed by the analysis of the nonclassical depth/GSV.

A. Von Neumann entropy generation

As we discussed in the previous section, for a global vacuum initial condition the von Neumann entropy $\mathcal{S}(\rho_\phi)$ of the reduced system state (or, equivalently, the EoF of the full system state) accounts for both its degree of mixing and the amount of bipartite entanglement. We have seen that it is given by the determinant D_A of the CM A for ρ_ϕ. The calculation of $A(\eta)$ for an initial vacuum is relatively easy because, at time η_0, the only nonzero elements of the two-mode CM are those of the diagonal, all equal to $1/2$. The determinant of the CM at time η is given by

$$D_{\mathrm{vacua}}(\eta) = \frac{1 + d_2(\eta)\lambda^2 + d_4(\eta)\lambda^4 + x^2(\eta)z^2(\eta)\lambda^6}{4(1-\lambda^2)^2}, \tag{24}$$

where $d_2(\eta) = x^2z^2 - 2x^2w^2 + 2xyzw + 2y^2z^2 + y^2w^2$ and $d_4(\eta) = x^2w^2 + y^2z^2 - 2x^2z^2$. It is worth noticing that the entropy of entanglement increases monotonically with D_A in the interval $D_A \in [1/4, \infty)$. By performing an asymptotic expansion for $\eta \to 0^-$, we find $D_A \approx A(\eta_0; \lambda; k; H)\eta^{-2}$, where $A(\eta_0; \lambda; k; H) > 0$ is a function only of η_0, λ, k and H.

We define the asymptotic entropy generation rate as

$$\mu_S = \lim_{\eta\to 0^-} -\frac{H}{\ln(-H\eta)}S(\eta).$$

Taking into account the previous discussion we find $\mu_S = H$. For large values of D_A, $S(\rho_\phi)$ the approximations

$$S(\rho_\phi) \approx 1 + \ln\left(\sqrt{D_A} + \frac{1}{2}\right) \approx \frac{1}{2}\ln D_A$$

are good and we can write

$$\mu_S = \lim_{\eta\to 0^-} -\frac{H}{2\ln(-H\eta)}\ln D_A(\eta).$$

Taking into account the asymptotic expansion for D_A this gives us $\mu_S = H = \mu$, as claimed. Thus, the maximal Lyapunov exponent and the asymptotic entropy generation rate at reduced system level are equal.

From this equality, we see that for late times in the superhorizon regime the von Neumann entropy of the reduced system state relates to the maximal Lyapunov exponent as

$$S(\rho_\phi; \eta) \approx -\frac{\mu}{H}\ln(-H\eta). \tag{25}$$

As the μ is equal to H this establishes a logarithmic divergence for the late time entropy modulated by the Hubble parameter, $S(\rho_\phi; \eta) \approx -\ln(-H\eta)$. An exception for this behavior will be found only if $k = 0$, because $A(\eta_0; \lambda; k; H)$ vanishes in this case and we get

$$D_A(k\to 0^+) = \frac{\lambda^2(H^4(\eta^3 - \eta_0^3)^2 - 18) + 9\lambda^4 + 9}{36(\lambda^2-1)^2}. \tag{26}$$

But of course, this limit is not physical.

We thus see that although the general dynamical behavior of the EoF is modulated by the absolute value of the wave vector for earlier times, it diverges logarithmically for late times ($\eta \to 0^-$) for any $k \neq 0$, in a way that is completely frequency independent (also of any other model detail) and is modulated only by the spacetime inflation rate. Its early time behavior is markedly different for small and large values of k. In the former case, the EoF remains small and corresponds to the determinant D_A given by (26); in the latter case, entanglement oscillates with an amplitude which grows with k. Entanglement oscillations, which are

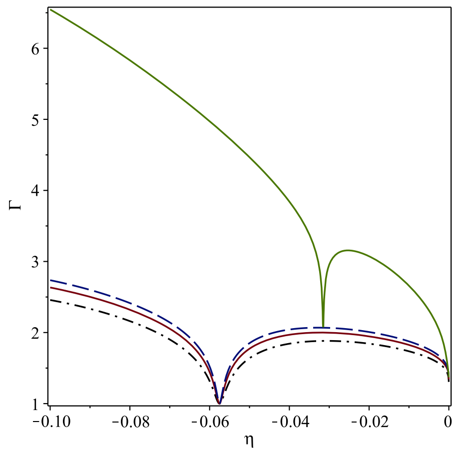

FIG. 1 (color online). Instantaneous orbit separation rate $\Gamma(\eta)$ for an initial vacuum. We scale conformal time into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$. Here, $k = 30$. The dot-dashed black line corresponds to $\lambda = 0.1$, the solid red line to $\lambda = 0.3$, the dashed blue line to $\lambda = 0.5$, and the solid green line to $\lambda = 0.999$.

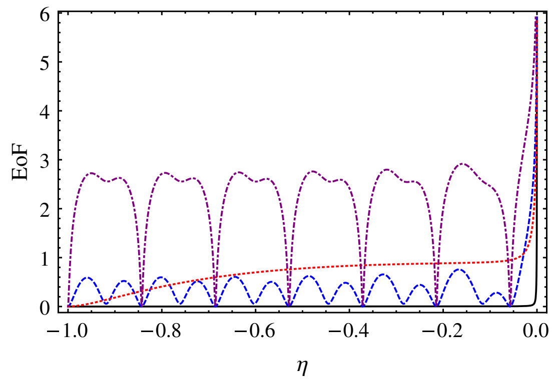

FIG. 2 (color online). EoF as a function of the η for an initial vacuum. We scale conformal time into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$. Entanglement oscillations are seen for initial states in the subhorizon regime ($k = 20$, the dot-dashed purple line corresponds to $\lambda = \frac{9}{10}$, the dashed blue line to $\lambda = \frac{1}{10}$), but not for initial states in the superhorizon regime ($k = \frac{1}{10}$, the solid black line corresponds to $\lambda = \frac{1}{10}$, the dotted red line to $\lambda = \frac{9}{10}$). The entanglement of formation diverges asymptotically in all cases.

characteristic of the subhorizon regime $k\eta > 1$, cease in the superhorizon regime $k\eta \ll 1$, when the EoF diverges asymptotically.

The result in Eq. (25) also shows that the late time entropy relates to the actual instantaneous exponential orbit separation rate $\Gamma(\eta)$ by

$$S(\rho_\phi; \eta) \approx \frac{1}{2}\Gamma(\eta), \tag{27}$$

with $\Gamma(\eta) = \ln \Re(\max_{i=\pm}\mu_i(\eta, \eta_0))$ in the notation of Sec. II. This, on the other hand, displays dependence on the model details, and for a given mode it is larger for stronger couplings. In fact, a numerical study, summarized in Fig. 1, shows that it increases monotonically as a function of $\lambda \in (0, 1)$. The dependence in k only changes its early time oscillatory behavior. As a consequence of this relation, we see that the superhorizon regime values of the von Neumann entropy of the system mode will then be favored by the stronger orbital instability when we shift from the weak- to the strong-coupling extremes. A numerical study of the EoF showed that this is indeed the case: the EoF of course vanishes in the limit of no interaction, $\lambda = 0$, and the numerical investigation summarized in Fig. 2 demonstrated that it also increases with an increasing coupling constant in the interval $\lambda \in (0, 1)$.

B. Effectiveness of decoherence

Let us examine the decoherence process from the point of view of the nonclassical depth. In the previous section, we have seen that the GSV $\epsilon_<$ determines the nonclassical depth and is given by the ratio of $\sqrt{D_A}$ to $e^{2|Z|}$, which measures thermalization (entropy) and squeezing of the system. As $\epsilon_<$ ranges from 0 to $\frac{1}{2}$ the nonclassical depth varies from its maximum Gaussian state value of $\frac{1}{2}$ to its minimum, 0; for $\epsilon_<$ larger than $\frac{1}{2}$ the nonclassical depth is zero. We have also seen in Eq. (27) that the values of the entropy (and thus of $\sqrt{D_A}$) increase in the superhorizon regime when the orbital instability measured by $\Gamma(\eta)$ gains in importance, as we shift from the weak- to the strong-coupling limit. This indicates that orbital instability will influence the nonclassical depth asymptotics. We will quantify this influence, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes.

From Eq. (23), we have to analyze the late time behavior of $\sqrt{D_A}$ and V_A. Their asymptotic expansions can be calculated to be of the form $\sqrt{D_A}(\eta \to 0) = -\frac{1}{\eta}P + O(1)$ and $V_A(\eta \to 0) = -\frac{1}{\eta}Q + O(1)$, where the coefficients P and Q are functions of λ, k, H. We see then that the balance between these quantities is going to be determined by P, Q.

The detailed form of these coefficients is very cumbersome and we will omit the details. For simplicity, we will assume henceforth that conformal time has been scaled into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$, corresponding to the standard cosmic time instant $t = 0$. The resulting asymptotic expansion for $\epsilon_< = \frac{\sqrt{D_A(\eta)}}{V_A(\eta)}$ [Eq. (23)] is

$$\epsilon_<(\eta \to 0) = \frac{\lambda^2}{2k^6(1-\lambda^2)^2}\frac{G_1 + G_2}{G}, \tag{28}$$

where

$$G_1 = G \times (k \cos k - \sin k)^2, \tag{29}$$

$$G_2 = \frac{1}{2}k^4\sin^2(k)[(1-\lambda^2)^2k^4\sin^2(k) + (1+\lambda^2)(k\cos k - \sin k)^2], \quad (30)$$

and

$$G = \frac{1}{2}[(1+\lambda^2)k^4\sin^2(k) + (k\cos k - \sin k)^2]. \quad (31)$$

With this formula available, let us begin by examining the conditions for asymptotic quadrature squeezing. From Eq. (28), the condition $\epsilon_<(\eta \to 0) < \frac{1}{2}$ reduces a constraint on λ and k,

$$\frac{\lambda^2}{(1-\lambda^2)^2} < k^6\frac{G}{G_1+G_2}. \quad (32)$$

In the weak-coupling regime $\lambda \to 0$, the right-hand side of Eq. (32) has a positive finite limit which never exceeds $\frac{9}{10}$. Since the left-hand side tends to zero as $\lambda \to 0$, it follows that (32) will be satisfied for every k as long as λ is small enough. Furthermore, it is not difficult to see from the expressions for G, G_1 and G_2 that the right-hand side of (32) diverges with increasing k, so that taking initial length scales on the particle horizon, $k = 1$, or deeper, $k > 1$, will lead to a larger asymptotic quadrature squeezing. Thus, the typical output reduced system state in the weak-coupling limit is a highly quadrature-squeezed state. This is consistent with what is expected, for example, for isolated massless inflaton fluctuations [2].

On the other hand, the condition for absence of quadrature squeezing (zero nonclassical depth) is $\epsilon_< \geq \frac{1}{2}$, which leads to

$$\frac{\lambda^2}{(1-\lambda^2)^2} \geq k^6\frac{G}{G_1+G_2}. \quad (33)$$

Since the left-hand side of (33) diverges as $\lambda \to 1$, this inequality can hold in the strong-coupling limit as long as we place a restriction in k. In fact, the limit of $k^6G/(G_1+G_2)$ as $\lambda \to 1$ is a positive function of k which diverges when k is equal to one of the roots of the equation $k\cos(k) = \sin(k)$ and when $k \to \infty$. Nonetheless, experimenting numerically with the coupling parameter shows that taking λ close enough to 1 guarantees (33) to hold up to considerably large values of k. Thus, the asymptotic reduced system state $\rho_{\phi,\infty}$ here is qualitatively very different from the weak-coupling case. As we already observed, taking the $|\lambda| \to 1$ limit increases the instantaneous exponential orbit separation rate of the model, reflecting in larger entropy values. In fact, remember that we showed that although the EoF presents a logarithmic divergence for nonzero coupling, its growth increases as λ varies from $\lambda \approx 0$ to $\lambda \approx 1$. The numerical analysis, summarized in Fig. 3, shows that for $|\lambda| \to 1$ this orbit separation rate will become strong enough to render the asymptotic state $\rho_{\phi,\infty}$ quadrature-squeezing free except for very small subhorizon length scales. In conclusion, we see that in the strong-coupling cases when $\tau_m(\rho_{\phi,\infty}) = 0$, decoherence is sufficiently effective to make the noise for the output state exceed the vacuum noise in all phase-space directions. This should be compared with the results described in [5].

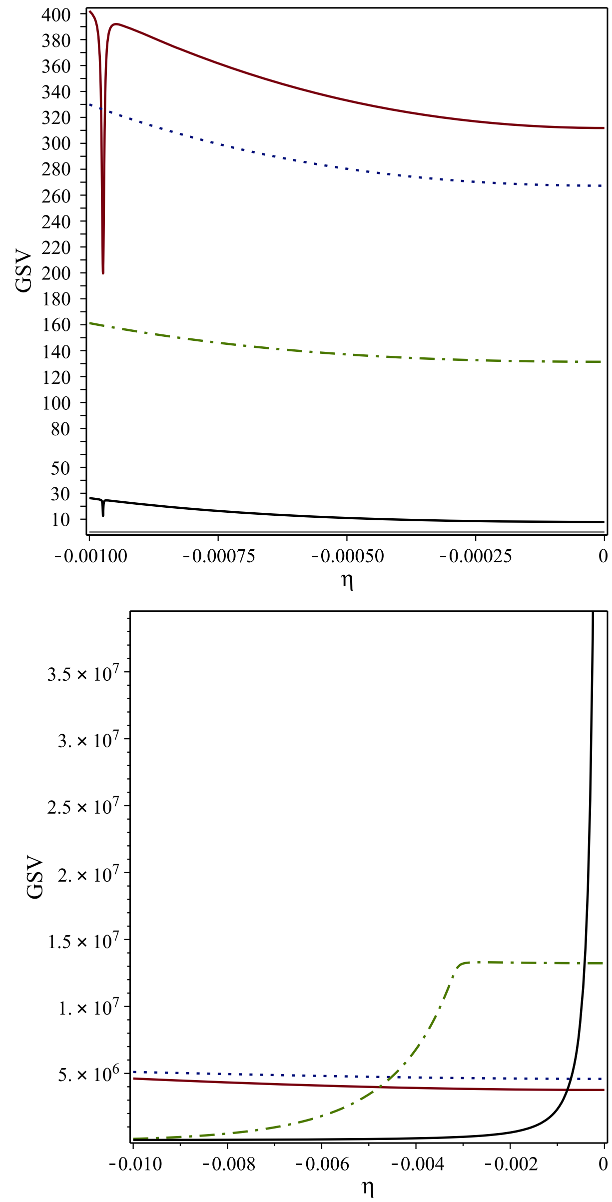

FIG. 3 (color online). GSV asymptotics. The coupling is $\lambda = 0.999999$. (a) Top: the solid black line corresponds to $k = 1000$, with associated initial length scale being 0.1% of the Hubble radius and $\epsilon_<(\eta \to 0) > 10$. The solid red line corresponds to $k = 500$, the dotted blue line to $k = 350$, and the dot-dashed green line to $k = 350$. (b) Bottom: for initial length scales of order $\approx 1\% H$, $\epsilon_<(\eta \to 0)$ is huge, $\approx 10^6$. The dot-dashed green line corresponds to $k = 35$, the dotted blue line to $k = 40$ and the solid red line to $k = 50$. The solid black line is practically the Hubble radius, $k = 1.5$; in this case $\epsilon_<(\eta \to 0) \approx 10^{11}$.

V. CONCLUSIONS

We examined in this work the relation between Lyapunov exponents and decoherence in de Sitter spacetime with spatially flat simultaneity hypersurfaces. We considered a quadratic example model, the BLM model, which reduces modewise to a model of nonautonomous coupled harmonic oscillators. Assuming an initial vacuum, we demonstrated that there is a relationship here between classical orbital instability as measured by the maximal Lyapunov exponent μ and the von Neumann entropy generation rate for the reduced subsystem, μ_S. We found that these are equal, leading to a relation between the entropy and the exponential orbit separation in the late times superhorizon regime of the form

$$S(\rho_\phi;\eta) \approx -\frac{\mu}{H}\ln(-H\eta).$$

Thus, the von Neumann entropy presents in this regime a logarithmic divergence modulated by the background spacetime inflation rate given by the Hubble parameter H and proportional to the maximal Lyapunov exponent. In the present case, the above relation reduces to a logarithmic divergence of $S(\rho_\phi)$ depending only on H. But if such a relationship between entropy and μ holds in greater generality, then other simple interacting processes for the system presenting nonlinearities or other more complicated unstable classical counterparts can lead to greater entropy generation rates. We believe that the example here is then instructive in the sense that it gives an idea of how we can expect the linear relationship between entropy and the maximal Lyapunov exponent of Zurek-Paz type in Minkowski spacetime [9] to be altered in de Sitter spacetime.

The results above consider the exponential orbit separation rate only in the limit and are seen to be independent of the model parameters. But when we consider the actual instantaneous orbit separation rate, we see that it is favored for any given mode by stronger couplings. Thus, we can actually illustrate the effect of stronger orbital instability in the decoherence process of the system even within the present model. We have seen that the late times von Neumann entropy is proportional to the late times instantaneous orbit separation rate. So, is the corresponding entropy generation enough to result in classicality? In this direction, we have also analyzed the superhorizon behavior of the nonclassical depth, which measures the emergence of a phase-space representation of the system oscillator quantum state corresponding to a stochastic distribution. In the present case, it depends upon a competition between the effect of single-mode squeezing and thermalization, as in Eq. (21). This indicates that an influence on the nonclassical depth asymptotics will be present when the orbital instability measured by $\Gamma(\eta)$ gains in importance, as we shift from the weak- to the strong-coupling limit. We verified this in quantitative terms, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes. We showed that in the strong-coupling limit all modes of the observed field evolve into a state with noise larger than the vacuum noise in every phase-space direction (zero nonclassical depth) except for the very high-frequency sector, corresponding to very large k.

This analysis offers then more supporting evidence that increasing the classical orbit instability will increase the effectiveness of classicalization. It suggests that if cosmological perturbations participate in simple but realistic physical processes during a dS stage of expansion, the nonlinear interactions involved could lead to a very significant contribution to their quantum-to-classical transition. These nonlinearities can lead to very complicated dynamics, and increase the rate in which the system explores its phase space. From what we have learned, this can have a sensitive impact on thermalization at observed system level and make quantum correlation effects very difficult to show up on the statistics of the classicalized output state (this has been the subject of several studies in the Minkowski case; see [7]). It is remarkable from an information-theoretic point of view that this can already be seen for a system of coupled harmonic oscillators over expanding spacetimes.

This type of contribution to entanglement entropy generation for cosmological fields has also been discussed, in the different context of isolated self-interacting scalar field perturbations, in [26]. The issue of the quantum-to-classical transition of cosmological perturbations is indeed a very subtle one. If quantum-mechanical features in the correlation structure of fields are to survive a period of inflationary spacetime expansion and help us understand through the cosmic microwave background sky the early history of the Universe, it is determinant that we understand in a clear way the underlying decoherence mechanisms.

ACKNOWLEDGMENTS

G. de Souza acknowledges financial support from Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). Marcos Sampaio thanks CNPq for a research grant and Durham University for the hospitality.

APPENDIX: COVARIANCE MATRIX DYNAMICS

We collect here the expressions of the functions $f^{mn}_{\vec{k}ij}(\eta,\eta_0)$ in (16). They have been obtained by using the Heisenberg picture solution for the BLM model expressed in Eq. (8), Sec. II. The expressions are (we drop the index $\vec{k}$ and the time argument)

$$f_{11}^{11} = x^2, \quad f_{11}^{12} = 2\frac{xy}{1-\lambda^2}, \quad f_{11}^{14} = 2\frac{\lambda xy}{1-\lambda^2}$$

$$f_{11}^{22} = \left(\frac{y}{1-\lambda^2}\right)^2, \qquad f_{11}^{24} = 2\frac{\lambda y^2}{(1-\lambda^2)^2}, \qquad f_{11}^{44} = \left(\frac{\lambda y}{1-\lambda^2}\right)^2$$

$$f_{12}^{11} = xz, \qquad f_{12}^{22} = \frac{yw}{1-\lambda^2}, \qquad f_{12}^{34} = -\frac{\lambda^2 yz}{1-\lambda^2}$$

$$f_{12}^{12} = xw + \frac{yz}{1-\lambda^2}, \qquad f_{12}^{13} = -\lambda xz, \qquad f_{12}^{23} = -\frac{\lambda yz}{1-\lambda^2} \qquad f_{12}^{14} = \frac{\lambda yz}{1-\lambda^2}, \qquad f_{12}^{24} = \frac{\lambda yw}{1-\lambda^2}$$

$$f_{22}^{11} = z^2, \qquad f_{22}^{22} = w^2, \qquad f_{22}^{33} = \lambda^2 z^2 \qquad f_{22}^{12} = 2zw \qquad f_{22}^{13} = -2\lambda z^2 \qquad f_{22}^{23} = -2\lambda zw$$

$$f_{13}^{12} = \frac{\lambda xy}{1-\lambda^2}, \qquad f_{13}^{13} = x^2, \qquad f_{13}^{14} = \frac{xy}{1-\lambda^2} \qquad f_{13}^{22} = \frac{\lambda y^2}{(1-\lambda^2)^2}, \qquad f_{13}^{23} = \frac{xy}{1-\lambda^2}$$

$$f_{13}^{24} = \left(\frac{y}{1-\lambda^2}\right)^2 + \left(\frac{\lambda y}{1-\lambda^2}\right)^2 \qquad f_{13}^{34} = \frac{\lambda xy}{1-\lambda^2}, \qquad f_{13}^{44} = \frac{\lambda y^2}{(1-\lambda^2)^2}$$

$$f_{14}^{11} = -\lambda xz, \qquad f_{14}^{13} = xz, \qquad f_{14}^{14} = xw - \frac{\lambda^2 yz}{1-\lambda^2}$$

$$f_{14}^{12} = -\frac{\lambda yz}{1-\lambda^2}, \qquad f_{14}^{23} = \frac{yz}{1-\lambda^2}, \qquad f_{14}^{24} = \frac{yw}{1-\lambda^2} \qquad f_{14}^{34} = \frac{\lambda yz}{1-\lambda^2}, \qquad f_{14}^{44} = \frac{\lambda yw}{1-\lambda^2},$$

$$f_{23}^{12} = \frac{\lambda yz}{1-\lambda^2}, \qquad f_{23}^{13} = xz, \qquad f_{23}^{14} = \frac{yz}{1-\lambda^2} \qquad f_{23}^{22} = \frac{yw}{1-\lambda^2}, \qquad f_{23}^{23} = xw - \frac{\lambda^2 yz}{1-\lambda^2}, \qquad f_{23}^{24} = \frac{yw}{1-\lambda^2}$$

$$f_{23}^{33} = -\lambda xz, \qquad f_{23}^{34} = -\frac{\lambda yz}{1-\lambda^2}, \qquad f_{24}^{11} = -\lambda z^2, \qquad f_{24}^{13} = z^2 + \lambda^2 z^2, \qquad f_{24}^{14} = zw$$

$$f_{24}^{12} = -\lambda zw, \qquad f_{24}^{23} = zw, \qquad f_{24}^{24} = w^2 \qquad f_{24}^{33} = -\lambda z^2, \qquad f_{24}^{34} = -\lambda zw,$$

$$f_{33}^{22} = \left(\frac{\lambda y}{1-\lambda^2}\right)^2, \qquad f_{33}^{23} = 2\frac{\lambda xy}{1-\lambda^2}, \qquad f_{33}^{24} = 2\frac{\lambda y^2}{(1-\lambda^2)^2}$$

$$f_{33}^{33} = x^2, \qquad f_{33}^{34} = \frac{xy}{1-\lambda^2}, \qquad f_{33}^{44} = \left(\frac{y}{1-\lambda^2}\right)^2 \qquad f_{34}^{12} = -\frac{\lambda^2 yz}{1-\lambda^2}, \qquad f_{34}^{23} = \frac{\lambda yz}{1-\lambda^2}, \qquad f_{34}^{24} = \frac{\lambda yw}{1-\lambda^2}$$

$$f_{34}^{13} = -\lambda xz \qquad f_{34}^{33} = xz, \qquad f_{34}^{34} = xw + \frac{yz}{1-\lambda^2} \qquad f_{34}^{14} = -\frac{\lambda yz}{1-\lambda^2}, \qquad f_{34}^{44} = \frac{yw}{1-\lambda^2}$$

$$f_{44}^{11} = \lambda^2 z^2, \qquad f_{44}^{13} = -2\lambda z^2, \qquad f_{44}^{14} = -2\lambda zw \qquad f_{44}^{33} = z^2, \qquad f_{44}^{34} = 2zw, \qquad f_{44}^{44} = w^2.$$

We observe that $f_{21}^{mn} = f_{12}^{mn}$, $f_{43}^{mn} = f_{34}^{mn}$, $f_{31}^{mn} = f_{13}^{mn}$, $f_{32}^{mn} = f_{14}^{mn}$, $f_{41}^{mn} = f_{23}^{mn}$, $f_{42}^{mn} = f_{24}^{mn}$. Otherwise, functions f_{ij}^{mn} not appearing above are null.

---

[1] L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. D **42**, 3413 (1990).
[2] C. Kiefer and D. Polarski, Adv. Sci. Lett. **2**, 164 (2009).
[3] P. Martineau, Classical Quantum Gravity **24**, 5817 (2007).
[4] C. P. Burgess, R. Holman, and D. Hoover, Phys. Rev. D **77**, 063534 (2008).
[5] C. Kiefer, I. Lohmar, D. Polarski, and A. Starobinsky, Classical Quantum Gravity **24**, 1699 (2007).
[6] R. Blume-Kohout and W. H. Zurek, Phys. Rev. A **68**, 032104 (2003).
[7] K. M. Fonseca-Romero, J. E. Parreira, L. A M. Souza, M. C. Nemes, and W. Wreszinski, J. Phys. A **41**, 115303 (2008).
[8] H. R. Jauslin, O. Sapin, S. Guérin, and W. F. Wreszinski, J. Math. Phys. (N.Y.) **45**, 4377 (2004).
[9] W. H. Zurek and J. P. Paz, Phys. Rev. Lett. **72**, 2508 (1994).
[10] R. Brandenberger, R. Laflamme, and M. Mijic, Mod. Phys. Lett. A **05**, 2311 (1990).
[11] G. Agarwal, Phys. Rev. A **3**, 828 (1971).
[12] G. Agarwal and H. Huang, Phys. Rev. A **49**, 52 (1994).
[13] J. Parreira, K. M. Fonseca-Romero, and M. C. Nemes, Phys. Rev. D **83**, 064033 (2011).
[14] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).
[15] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1882 (1969).
[16] L. Davidovitch, Rev. Mod. Phys. **68**, 127 (1996).

[17] M. Giovannini, Phys. Rev. D **83**, 023515 (2011).
[18] C. Lee, Phys. Rev. A **44**, R2775 (1991).
[19] C. Lee, Phys. Rev. A **45**, 6586 (1992).
[20] N. Lütkenhaus and S. M. Barnnett, Phys. Rev. A **51**, 3340 (1995).
[21] A. Serafini, F. Illuminati, and S. De Siena, J. Phys. B: At. Mol. Opt. Phys. **37**, L21 (2004).
[22] B. Yurke and M. Potasek, Phys. Rev. A **36**, 3464 (1987).
[23] D. Buono, G. Nocerino, V. D'Auria, A. Porzio, S. Olivares, and M. G. A. Paris, J. Opt. Soc. Am. B **27**, A110 (2010).
[24] J. Laurat, G. Keller, J. A. Oliveira-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso, and F. Illuminati, J. Opt. Soc. Am. B **7**, S577 (2005).
[25] A. Serafini, F. Illuminati, and S. de Siena, J. Phys. B **37**, L21 (2004).
[26] D. Mazur and J. Heyl, Phys. Rev. D **80**, 023523 (2009).