PHYSICAL REVIEW D **90,** 125038 (2014)

Seeing asymptotic freedom in an exact correlator of a large-N matrix field theory

Peter Orland*

Baruch College, The City University of New York, 17 Lexington Avenue, New York, New York 10010, USA and The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, New York 10016, USA

(Received 9 October 2014; published 30 December 2014)

Exact expressions for correlation functions are known for the large-N (planar) limit of the $(1+1)$-dimensional $\mathrm{SU}(N)\times\mathrm{SU}(N)$ principal chiral sigma model. These were obtained with the form-factor bootstrap, an entirely nonperturbative method. The large-N solution of this asymptotically free model is far less trivial than that of the $\mathrm{O}(N)$ sigma model (or other isovector models). Here we study the Euclidean two-point correlation function $N^{-1}\langle\mathrm{Tr}\Phi(0)^{\dagger}\Phi(x)\rangle$, where $\Phi(x)\sim Z^{-1/2}U(x)$ is the scaling field and $U(x)\in\mathrm{SU}(N)$ is the bare field. We express the two-point function in terms of the spectrum of the operator $\sqrt{-d^2/du^2}$, where $u\in(-1,1)$. At short distances, this expression perfectly matches the result from the perturbative renormalization group.

DOI: 10.1103/PhysRevD.90.125038 PACS numbers: 03.70.+k, 11.10.-z

Green's functions of quantum chromodynamics (QCD) cannot be calculated at large separations analytically. Currently, only numerical lattice calculations suffice for this purpose. On the other hand, perturbation theory can be used to understand short-distance behavior in any asymptotically free theory, such as QCD. In lower dimensions, there are field-theoretic models with asymptotic freedom, which can be studied mathematically. A nontrivial example is the principal chiral sigma model (PCSM) of a matrix field $U(x)\in\mathrm{SU}(N)$, $N\geq 2$, where x^0 and x^1 are the time and space coordinates, respectively. Here, the large-N limit of the PCSM will be considered. The PCSM is a matrix model, not an isovector model [such as the $\mathrm{O}(N)$ or $\mathbb{C}P(N-1)$ sigma models or the Gross-Neveu model]. The PCSM's large-N limit has not been solved by saddle-point methods. Its Feynman diagrams are truly planar, not linear. Finally, the PCSM has nontrivial field renormalization, even in the large-N limit; this means that its correlation functions are not those of a free field theory, in this limit. In all of these respects, the PCSM resembles QCD substantially better than isovector field theories.

In this paper, an exact expression for a correlation function of the large-N PCSM is studied at short distances, where it is found to obey a power-law decay law. At large distances, this correlation function has exponential decay. Thus, the solution clearly illustrates both ultraviolet freedom and an infrared mass gap. Furthermore, the ultraviolet behavior of this nonperturbatively obtained correlation function has precisely the behavior expected from the perturbative renormalization group. The key to the short-distance behavior is the spectrum of an interesting integro-differential operator on functions of the open interval $(-1,1)$.

The PCSM has the action

$$S=\frac{N}{2g_0^2}\int d^2x\eta^{\mu\nu}\mathrm{Tr}\partial_\mu U(x)^{\dagger}\partial_\nu U(x), \tag{1}$$

where $\mu,\nu=0,1$, $\eta^{00}=1$, $\eta^{11}=-1$, $\eta^{01}=\eta^{10}=0$, where g_0 is the coupling (which is held fixed as $N\to\infty$). This action is invariant under the global transformation $U(x)\to V_L U(x) V_R$, for two constant matrices $V_L, V_R\in\mathrm{SU}(N)$. The renormalized field operator $\Phi(x)$ is an average of $U(x)$ over a region of size b, where $\Lambda^{-1}<b\ll m^{-1}$, where Λ is an ultraviolet cutoff and m is the mass of the fundamental excitation.

For matrix models in more than one dimension, there is no general approach to summing the planar diagrams. The PCSM, however, has the virtue of being integrable. Integrability is not sufficient to determine Green's functions, although the S matrix has been known for three decades [1]. Recently, both integrability and the $1/N$ expansion were combined to find the $N\to\infty$ limit of Green's functions [2,3]. This was done using Smirnov's axioms for form factors [4]. The form-factor bootstrap method has a long history [5]. A detailed comparison of the $1/N$ expansion and form factors of the O(N) sigma model is in Ref. [6].

In this paper, we study an exact nonperturbative expression for the two-point function of the scaling field $\Phi(x)$, found in the second listing of Ref. [2]. The scaling field Φ is normalized by $\langle 0|\Phi(0)_{b_0a_0}|P,\theta,a_1,b_1\rangle=N^{-1/2}\delta_{a_0a_1}\delta_{b_0b_1}$, where the ket on the right is a one particle ($r=1$) state, with rapidity θ. This field is a complex $N\times N$ matrix, which is not directly proportional to the unitary matrix $U(x)$. Nonetheless we write $\Phi(x)\sim Z(g_0,\Lambda)^{-1/2}U(x)$, which means that (the time ordering is optional)

*orland@nbi.dk

$$\frac{1}{N}\langle 0|\mathrm{Tr}\Phi(x)\Phi(0)^{\dagger}|0\rangle = \mathrm{Z}(g_0,\Lambda)^{-1}\frac{1}{N}\langle 0|\mathrm{Tr}U(x)U(0)^{\dagger}|0\rangle. \tag{2}$$

It would be interesting to know the relation of the scaling field used in lattice simulations [7] to that defined above, which is not yet clear to the author. Particle masses are given by the sine formula: $m_r = m\sin(\pi N^{-1}r)/\sin(\pi N^{-1})$, $r = 1,\ldots,N-1$, but in the large-N limit, only the $r=1$, $r=N-1$ states (the elementary particle and antiparticle) survive. The binding energies of the other states vanish. The residues of their poles in S-matrix elements also vanish.

The renormalization factor $\mathrm{Z}(g_0(\Lambda),\Lambda)$ vanishes as $\Lambda\to\infty$ and the coupling $g_0(\Lambda)$ runs so that the mass gap $m(g_0(\Lambda),\Lambda)$ is independent of Λ. For $m|x|\gg 1$, the expression (2) decays exponentially, as expected. We find that for $m|x|\ll 1$, the time-ordered product of two scaling field operators behaves as

$$\frac{1}{N}\langle 0|\mathcal{T}\mathrm{Tr}\Phi(x)\Phi(0)^{\dagger}|0\rangle = C_2(\ln m|x|)^2 + C_1\ln m|x| + C_0 + O(1/\ln m|x|), \tag{3}$$

for some constants C_2, C_1, etc. The leading term is exactly what a perturbative-renormalization-group analysis implies. We consider this to be a striking validation of the form-factor bootstrap.

Let us recall the argument for (3) (see for example, Ref. [8]). For convenience, we perform the Wick rotation $x^0\to \mathrm{i}x^0$, to obtain the regularized Euclidean correlation function $G(|x|,\Lambda) = N^{-1}\langle 0|\mathcal{T}\mathrm{Tr}\Phi(x)\Phi(0)^{\dagger}|0\rangle$. This function and the coupling $g_0(\Lambda)$ satisfy the renormalization group equations

$$\frac{\partial\ln G(R,\Lambda)}{\partial\ln\Lambda} = \gamma(g_0) = \gamma_1 g_0^2 + \cdots,$$
$$\frac{\partial g_0^2(\Lambda)}{\partial\ln\Lambda} = \beta(g_0^2) = -\beta_1 g_0^4 + \cdots, \tag{4}$$

respectively. The coefficients of the anomalous dimension $\gamma(g_0)$ and the beta function $\beta(g_0)$ are $\gamma_1 = (N^2-1)/(2\pi N^2)$ and $\beta_1 = 1/(4\pi)$. For large Λ, $G(R,\Lambda)$ becomes a function of the product of the two variables $G(R\Lambda)$. Integrating (4) yields the leading behavior

$$G(R,\Lambda)\sim C[\ln(R\Lambda)]^{\gamma_1/\beta_1}. \tag{5}$$

As $N\to\infty$, the power γ_1/β_1 approaches 2.

The exact Wightman function (in Minkowski spacetime) of the product of two fields (that is, with no time ordering) is $\mathcal{W}(x) = N^{-1}\langle 0|\mathrm{Tr}\Phi(x)\Phi(0)^{\dagger}|0\rangle$. This function is [2]

$$\mathcal{W}(x) = \int_{-\infty}^{\infty}\frac{d\theta_1}{4\pi}e^{\mathrm{i}p_1\cdot x} + \frac{1}{4\pi}\sum_{l=1}^{\infty}\int_{-\infty}^{\infty}d\theta_1\cdots \times\int_{-\infty}^{\infty}d\theta_{2l+1}e^{\mathrm{i}\sum_{j=1}^{2l+1}p_j\cdot x}\prod_{j=1}^{2l}\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}, \tag{6}$$

where θ_j are rapidities and $p_j = m(\cosh\theta_j, \sinh\theta_j)$ are the corresponding momentum vectors, for $j = 1,\ldots 2l+1$. The right-hand side of (6) is difficult to evaluate. For spacelike separation $x^0 = 0$, it decays exponentially with $|x^1|$. Our purpose here is to evaluate (6) for small timelike separation $x^1 = 0$, $x^0\ll m^{-1}$. For this case, the Wightman function is equal to the time-ordered expectation value on the left-hand side of (3). Equation (6) or an approximation to it has not yet been obtained in any program to solve the large-N PCSM directly from the action (1). Perhaps, one day, this will be done (a recent proposal is in Ref. [9]).

To study the two-point function at short distances, it is convenient to Wick rotate the time variable to Euclidean space as above. Setting $x^1 = 0$ and replacing x^0 by $\mathrm{i}R$, $R>0$, changes the phases in (6) by $\exp\mathrm{i}p_j\cdot x\to\exp - mR\cosh\theta_j$. We define $L = \ln\frac{1}{mR}$. As mR becomes small, $\exp -mR\cosh\theta_j$ becomes approximately the characteristic function of $(-L,L)$, equal to unity for $-L<\theta<L$ and zero everywhere else. This is mathematically similar to the formation of walls in the Feynman-Wilson gas [10]. This trick was used to find the scaling behavior of Ising-model correlation functions [11] from the exact form factors [12]. The short-distance Euclidean two-point function is

$$G(mR) = \frac{L}{2\pi} + \frac{1}{4\pi}\sum_{l=1}^{\infty}\int_{-L}^{L}d\theta_1\cdots \times\int_{-L}^{L}d\theta_{2l+1}\prod_{j=1}^{2l}\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}. \tag{7}$$

Notice that the first term of (6), which is the Wightman function of a free massive field, corresponds to the first term of (7) which is the Euclidean correlation function of a massless field. The expression (7) is the partition function of a polymer in a box of size $2L$. The j^{th} atom in the polymer chain is located at θ_j. There is a long-range potential energy $\ln[(\theta_j-\theta_{j+1})^2+\pi^2]$, between atoms connected on the chain.

It is convenient to rescale the integration variables by $\theta_j = Lu_j$, so that (7) becomes

$$G(mR) = \frac{L}{2\pi} + \frac{L}{4\pi}\sum_{l=1}^{\infty}\int_{-1}^{1}du_1\cdots \times\int_{-1}^{1}du_{2l+1}\prod_{j=1}^{2l}\frac{1}{L[(u_j-u_{j+1})^2+(\pi/L)^2]}. \tag{8}$$

There is a close relation between the terms of (8) and the fractional-power-Laplace operator $\Delta^{1/2} = \sqrt{-d^2/du^2}$. The spectrum of $\Delta^{\alpha/2}$, with real $\alpha \in (0,2)$, is a subject of active mathematical investigation [13,14]. The self-adjoint extension of the operator $\Delta^{1/2}$ on $u \in (-1,1)$ has an infinite set of discrete eigenvalues λ_n, of the eigenfunctions $\varphi_n(u)$, $n = 1,2,\ldots$, with $0 < \lambda_1 < \lambda_2 < \ldots$, with $\varphi_n(\pm 1) = 0$. Another polymer statistical system in which a fractional power of the second derivative plays a role is described in Ref. [15].

Here is a quick introduction to the operator $\Delta^{1/2}$, via the Poisson semigroup. Let us forget the restriction to the open interval and extend the rapidity variables to the real line $(-\infty,\infty)$. Consider the transfer operators $P(a)$, whose matrix elements are defined by $\langle u'|P(a)|u\rangle = a[(u'-u)^2 + a^2]^{-1}\pi^{-1}$, where u' and u are arbitrary real numbers. These operators form the Poisson semigroup [13], with the composition law $P(a)P(b) = P(a+b)$. Specifically, $P(a) = \exp -a\Delta^{1/2}$, where $\Delta^{1/2} = \sqrt{-d^2/du^2}$.

Explicitly, the square root of the Laplacian on a function $f(u)$, vanishing for $u \notin (-1,1)$, is [13]

$$\Delta^{1/2} f(u) = \frac{1}{\pi}\int_{-1}^{1} du' \mathrm{PV} \frac{f(u') - f(u)}{(u'-u)^2}, \qquad (9)$$

where PV denotes the principal value. This operator has an infinite set of discrete eigenvalues λ_n, of the eigenfunctions $\varphi_n(u)$, $\Delta^{1/2}\varphi_n = \lambda_n \varphi_n$, $n = 1,2,\ldots$, with $0 < \lambda_1 < \lambda_2 < \cdots$, with $\varphi_n(\pm 1) = 0$. Now for $u, u' \in (-1,1)$, we define the operator $H(L)$ by

$$\frac{1}{L[(u-u')^2 + (\pi/L)^2]} = \langle u'| e^{-\frac{\pi}{L}H(L)} |u\rangle. \qquad (10)$$

Then (9), (10) and a straightforward calculation show that $H(L)$ is an approximation to $\Delta^{1/2}$, i.e., $H(L) = \Delta^{1/2} + O(1/L)$, with spectrum

$$H(L)\varphi_n(u,L) = \lambda_n(L)\varphi_n(u,L), \qquad \int_{-1}^{1} du |\phi_n(u,L)|^2 = 1,$$
$$\lambda_n(L) = \lambda_n + O(1/L), \qquad \varphi_n(u,L) = \varphi_n(u) + O(1/L). \qquad (11)$$

Summing over l in Eq. (8) yields, from (11),

$$G(mR) = \frac{L}{4\pi}\int_{-1}^{1} du' \int_{-1}^{1} du \langle u'| \frac{1}{1 - e^{-2\pi H(L)/L}} |u\rangle$$
$$= \frac{L}{4\pi}\sum_{n=1}^{\infty} \left| \int_{-1}^{1} du \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-2\pi\lambda_n/L + O(1/L^2)}}. \qquad (12)$$

Parenthetically, we note that $\int_{-1}^{1} du \phi_n(u,L) = 0$ for even n. We split (12) into two sums:

$$G(mR) = \frac{L}{4\pi}\sum_{\lambda_n \le L/2\pi} \left| \int_{-1}^{1} du \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-2\pi\lambda_n/L + O(1/L^2)}}$$
$$+ \frac{L}{4\pi}\sum_{\lambda_n > L/2\pi} \left| \int_{-1}^{1} du \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-2\pi\lambda_n/L + O(1/L^2)}}. \qquad (13)$$

The second term in (13) cannot diverge as $L \to \infty$, hence gives no contribution to either C_1 or C_2. For

$$\frac{L}{4\pi}\sum_{\lambda_n > L/2\pi} \left| \int_{-1}^{1} du\, \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-2\pi\lambda_n/L + O(1/L^2)}}$$
$$\lesssim \frac{L}{4\pi}\sum_{\lambda_n > L/2\pi} \left| \int_{-1}^{1} du\, \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-1}},$$

and the sum over n on the right-hand side is roughly

$$\sum_{\lambda_n > L/2\pi} \left| \int_{-1}^{1} du\, \varphi_n(u,L)\right|^2 \sim \frac{1}{L}.$$

The first term in (13) may be expanded in powers of $1/L$ to yield

$$\frac{L}{4\pi}\sum_{\lambda_n \le L/2\pi} \left| \int_{-1}^{1} du\, \varphi_n(u,L)\right|^2 \frac{1}{1 - e^{-2\pi\lambda_n/L + O(1/L^2)}}$$
$$= \frac{L}{4\pi}\sum_{\lambda_n \le L/2\pi} \left| \int_{-1}^{1} du\, \varphi_n(u)\right|^2 \frac{L}{2\pi\lambda_n} + O(L).$$

Extending the sum over n from zero to infinity gives the leading coefficient in (3):

$$C_2 = \frac{1}{8\pi^2}\sum_{n=1}^{\infty} \left| \int_{-1}^{1} du\, \varphi_n(u)\right|^2 \lambda_n^{-1}. \qquad (14)$$

An upper bound on the leading coefficient C_2 is obtained by replacing λ_n in (14) by λ_1, and using completeness: $\sum_n |\int_{-1}^{1} \phi_n(u)|^2 = 2$. Thus $C_2 < \frac{1}{4\pi^2\lambda_1} = 0.0219$, from the best known value of $\lambda_1 = 1.1577$, found in the second and third listings of Ref. [14]. It is interesting that without much detailed knowledge of the properties of $H(L)$ or of the square root of the Laplacian, we have established the ultraviolet behavior (3) of the two-point correlation function. An evaluation of C_1 would require a better understanding of the spectrum of $H(L)$.

To conclude, we believe the correlators of $SU(\infty) \times SU(\infty)$ PCSM are now understood almost as well as those of the Ising model [12]. The exact $N \to \infty$ correlation function argued for in Ref. [2] displays massive behavior at large distances. We have found precisely the short-distance behavior predicted with the perturbative beta function and anomalous dimension. This strengthens our confidence in the form factors [2,3], which led to this result.

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. Axel Cortés Cubero, whose suggestions led to the expression (8), and Dr. Timothy Budd for discussions about the Poisson semigroup and the fractional-power Laplacian. This work was supported in part by a grant from the PSC-CUNY.

[1] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**, 121 (1983); P. B. Wiegmann, Phys. Lett. **141B**, 217 (1984); E. Abdalla, M. C. B. Abdalla, and M. Lima-Santos, Phys. Lett. **140B**, 71 (1984); P. B. Wiegmann, Phys. Lett. **142B**, 173 (1984); L. D. Faddeev and N. Yu. Reshetikhin, Ann. Phys. (N.Y.) **167**, 227 (1986). Bethe's *Ansatz* for an alternative large-*N* limit, with vanishing mass gap, is in V. A. Fateev, V. A. Kazakov, and P. B. Wiegmann, Nucl. Phys. **B424**, 505 (1994).

[2] P. Orland, Phys. Rev. D **84**, 105005 (2011); **86**, 045023 (2012).

[3] A. Cortés Cubero, Phys. Rev. D **86**, 025025 (2012); A. Cortés Cubero and P. Orland, Phys. Rev. D **88**, 025044 (2013). There is an error in some of the higher-point form factors in these papers, which has been corrected in A. Cortés Cubero, Phys. Rev. D **90**, 065002 (2014).

[4] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, Adv. Series in Math. Phys. No. 14 (World Scientific, Singapore, 1992).

[5] S. N. Vergeles and V. M. Gryanik, Sov. J. Nucl. Phys. **23**, 704 (1976); P. H. Wiesz, Phys. Lett. **67B**, 179 (1977); M. Karowski and P. Wiesz, Nucl. Phys. **B139**, 455 (1978).

[6] H. M. Babujian, A. Foerster, and M. Karowski, J. High Energy Phys. 11 (2013) 089.

[7] R. Narayanan, H. Neuberger, and E. Vicari, J. High Energy Phys. 04 (2008) 094.

[8] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Pulishers, Chur, 1987), Sec. 2.1. There is an error in Eq. (8.36) of Polyakov's book; The correct result may be found in P. Rossi and E. Vicari, Phys. Rev. D **49**, 6072 (1994); Eq. (168), and in P. Rossi, M. Campostrini, and E. Vicari, Phys. Rep. **302**, 143 (1998), Eqs. (7.5) and (7.6).

[9] A. Cherman, D. Dorigoni, G. V. Dunne, and M. Unsal, Phys. Rev. Lett. **112**, 021601 (2014).

[10] R. P. Feynman, Phys. Rev. Lett. **23**, 1415 (1969); K. G. Wilson, in *Proceedings of the Fourteenth Scottish Universities Summer School in Physics, 1973*, edited by R. L. Crawford and R. Jennings (Academic Press, New York, 1974).

[11] J. L. Cardy and G. Mussardo, Nucl. Phys. **B340**, 387 (1990); V. P. Yurov and Al. B. Zamolodchikov, Int. J. Mod. Phys. A **06**, 3419 (1991).

[12] B. Berg, M. Karowski, and P. Weisz, Phys. Rev. D **19**, 2477 (1979); T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. B **13**, 316 (1976); M. Sato, T. Miwa, and M. Jimbo, Proc. Jpn. Acad. A **53**, 6 (1977); Publ. RIMS, Kyoto Univ. **16**, 531 (1980); B. Schroer and T. T. Truong, Nucl. Phys. **B144**, 80 (1978).

[13] J. Elliott, Trans. Am. Math. Soc. **76**, 300 (1954); J. Elliott and W. Feller, Trans. Am. Math. Soc. **82**, 392 (1956); R. Banuelosa and T. Kulczycki, J. Funct. Anal. **211**, 355 (2004); **234**, 199 (2006); A. Zoia, A. Rossi, and M. Kardar, Phys. Rev. E **76**, 021116 (2007); E. Katzav and M. Adda-Bedia, J. Phys. A **41**, 022002 (2008).

[14] T. Kulczycki, M. Kwaśnicki, J. Małecki, and A. Stós, Proc. London Math. Soc. **101**, 589 (2010); M. Kwaśnicki, J. Funct. Anal. **262**, 2379 (2012); E. Katzav and M. Adda-Bedia, Eur. Phys. Lett. **83**, 30006 (2008).

[15] R. D. Pisarski, Phys. Rev. D **34**, 670(R) (1986).