Minimal anomaly-free chiral fermion sets and gauge coupling unification

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We look for minimal chiral sets of fermions beyond the standard model that are anomaly free and, simultaneously, vectorlike particles with respect to color $SU(3)$ and electromagnetic $U(1)$. We then study whether the addition of such particles to the standard model particle content allows for the unification of gauge couplings at a high energy scale, above 5.0×10^{15} GeV so as to be safely consistent with proton decay bounds. The possibility to have unification at the string scale is also considered. Inspired in grand unified theories, we also search for minimal chiral fermion sets that belong to $SU(5)$ multiplets, restricted to representations up to dimension 50. It is shown that, in various cases, it is possible to achieve gauge unification provided that some of the extra fermions decouple at relatively high intermediate scales.

I. INTRODUCTION

The standard model (SM) of particle physics is a very successful theory that describes the fundamental known particles and their interactions based on the gauge symmetry $SU(3) \times SU(2) \times U(1)$. So far, the discovered elementary fermions are chiral Weyl particles and thus cannot have any gauge-invariant mass term. Once the SM gauge symmetry is spontaneously broken at the electroweak scale, fermions acquire masses and the remaining symmetry gives rise to interactions whose nature is vectorlike with respect to color $SU(3)$ and electromagnetic $U(1)$.

In the SM, the strong α_s, weak isospin α_w, and hypercharge α_y gauge couplings are not related among themselves by any symmetry principle. It is well known that the gauge couplings $\alpha_{1,2,3} \equiv \kappa_{1,2,3}\alpha_{y,w,s}$, where κ_i are normalization constants, evolve with the energy scale according to the renormalization group equations (RGEs). At one-loop level, one verifies that α_1 and α_2 unify around 10^{13} GeV, while α_2 and α_3 unify at 10^{17} GeV for the canonical normalization $\kappa_1 = 5/3$ and $\kappa_2 = \kappa_3 = 1$, like in $SU(5)$ or $SO(10)$ group. This fact already suggests a possible unification of the three couplings, either by changing the normalization factor κ_1 associated to the hypercharge, or by adding new particle content that properly modifies the running of the couplings. In the former case, complete unification is achieved for $\kappa_1 \approx 13/10$, assuming that the SM is an effective theory valid up to the unification scale around 10^{17} GeV. It is interesting that this scale is close to the scale predicted

in string theories [1]. In the second case, it has been noted that extending the SM with a fourth generation of quarks and leptons gives freedom for the convergence of gauge couplings to a common value at a scale around 3×10^{15} GeV [2].

Adding new chiral fermions to the SM particle content unavoidably raises the question of gauge anomaly cancellation. In general, this is not a trivial issue, since the anomaly-free conditions depend on the transformation properties of the new fermions under the SM gauge group. Furthermore, it is also necessary to guarantee that after electroweak symmetry breaking the theory remains vectorlike with respect to color $SU(3)$ and electromagnetic $U(1)$, in order to conserve parity. Remarkably, these properties are verified between quarks and leptons within each generation of the SM [3–5]. On the other hand, adding only vectorlike fermions to the theory does not bring any anomaly constraint, since their contribution to the gauge anomalies adds to zero. Another motivation for introducing vectorlike particles is that it is possible to construct gauge-invariant mass terms for them, and the masses are not necessarily below the electroweak scale, implying that these particles can decouple from the low-energy spectrum of the theory. These are among the reasons for following this path for gauge coupling unification in several extensions of the SM (see, e.g., Refs. [6–11]).

Concerning the possibility of extending the SM with anomaly-free chiral fermion sets, one can find numerical analyses [12] and theoretical studies [13–17] in the literature. In particular, patterns for adding anomaly-free charge-vectorial chiral sets of fermions that acquire mass through their coupling to the SM Higgs doublet have been investigated in Refs. [13,14]. Furthermore, a general structure of exotic generations and the corresponding analysis of possible quantum numbers were presented in

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Ref. [15], assuming the gauge and Higgs structure of the SM. Finally, a complete description of chiral anomaly-free sets for gauge theories with additional $U(1)$ groups can be found in Ref. [17]; however, the authors restricted their analysis to chiral sets that only include vectorlike fermions with respect to non-Abelian groups. We also remark that in the above works the problem of gauge coupling unification has not been addressed.

The aim of our work is to extend previous studies in order to find minimal chiral sets of fermions beyond the SM that are not only anomaly free but also lead to gauge coupling unification at high energies, inspired by grand unified theories (GUTs) or string theories. The paper is organized as follows. In Sec. II, we study the problem of finding the minimal sets of chiral fermions beyond the SM that lead to anomaly-free solutions. We then proceed in Sec. II A to explore the sets that lead to the unification of the gauge couplings at one-loop level. In our analysis, we restrict to cases in which the unification scale is above 5.0×10^{15} GeV, so as to be safely consistent with proton decay bounds [18]. Furthermore, we allow for the decoupling of the beyond-the-SM fermions at some intermediate energy scale between the electroweak and the unification scales. We also look for solutions in which gauge and gravitational couplings match at a unique (string) scale. We then proceed in Sec. III, inspired by GUTs, to study the possibility of having anomaly-free solutions with chiral fermion multiplets belonging to $SU(5)$ representations up to dimension 50. For the anomaly-free sets obtained, we analyze whether the gauge couplings unify at GUT and/or string energy scales. Finally, our conclusions are presented in Sec. IV.

II. MINIMAL ANOMALY-FREE CHIRAL FERMION SETS

In this work, we search for new chiral sets of particles beyond the SM that are anomaly free and at the same time QED and QCD vectorlike at low energies (i.e., electric and color currents are vectorlike), so that parity symmetry is not spoiled. Without loss of generality, we assume that all chiral fields are left handed. To be free of anomalies, the new chiral fermions must verify the following anomaly conditions with respect to the SM gauge group:

$$[SU(3)\text{-}SU(3)\text{-}SU(3)]\colon \sum_R A_3(R)d_2(R) = 0, \quad (1a)$$

$$[SU(3)\text{-}SU(3)\text{-}U(1)]\colon \sum_R y_R t_3(R)d_2(R) = 0, \quad (1b)$$

$$[SU(2)\text{-}SU(2)\text{-}U(1)]\colon \sum_R y_R t_2(R)d_3(R) = 0, \quad (1c)$$

$$[U(1)\text{-}U(1)\text{-}U(1)]\colon \sum_R y_R^3 d_2(R)d_3(R) = 0, \quad (1d)$$

$$[\text{gravity-gravity-}U(1)]\colon \sum_R y_R d_2(R)d_3(R) = 0, \quad (1e)$$

where $d_i(R)$, $A_i(R)$, and $t_i(R)$ are respectively the dimension, cubic anomaly index, and Dynkin index of the representation R with respect to the subgroup G_i. For a given representation R of a group G, the integer cubic index $A(R)$ is defined through the relation [19]

$$\text{Tr}(\{T_R^a, T_R^b\}T_R^c) = A(R)\text{Tr}(\{T^a, T^b\}T^c), \quad (2)$$

where T_R^a are the generators of G for the representation R, and T^a for the fundamental representation. Clearly, for the conjugate representation $\bar{R}$ one has $A(\bar{R}) = -A(R)$, which in turn implies $A(R) = 0$ for any real representation. The Dynkin index of a representation R can be determined through the Dynkin labels [20,21] as

$$t_m(R) = \frac{d_m(R)}{2m(m^2-1)}\left[\sum_{j=1}^{m-1} j(m-j)(\ell_j^2 + m\ell_j)\right.$$
$$\left. + 2\sum_{j=2}^{m-1}\sum_{k<j}(m-j)k\ell_m\ell_k\right], \quad (3)$$

where the Dynkin label ℓ_j is the number of columns with j boxes in a Young diagram. Obviously, $t_m(\bar{R}) = t_m(R)$. The Dynkin index in Eq. (3) is properly normalized such that it is equal to $1/2$ for the fundamental representation in $SU(N)$ groups. In particular [22],

$$t_1(R) = y_R^2, \qquad t_2(R) = \frac{d_2(R)[d_2^2(R)-1]}{12}, \quad (4)$$

for the subgroups $U(1)_Y$ and $SU(2)$, respectively; y_R is the hypercharge for the representation R. The corresponding Dynkin index $t_3(R)$ and cubic anomaly index $A_3(R)$ for $SU(3)$ are presented in Table I. An important constraint on the Dynkin index $t_2(R)$ comes from Witten's anomaly condition, i.e., $\sum_R t_2(R)$ must be an integer number, in order to avoid the global $SU(2)$ gauge anomaly [23]. Restricting ourselves, for simplicity, to $SU(2)$ representations up to dimension five ($d_2(R) \leq 5$), this condition implies that the number of Weyl fermion doublets is even.

Notice that the set of equations given in Eqs. (1) is invariant under an overall rescaling of the hypercharge y_R of all multiplets; therefore, one has to choose properly the overall normalization of the hypercharge in order to fulfil

TABLE I. The Dynkin and cubic anomaly indices, t_3 and A_3, for the smallest irreducible representations of $SU(3)$.

$SU(3)$-irrep	3	6	8	10	15	15′
t_3	$\frac{1}{2}$	$\frac{5}{2}$	3	$\frac{15}{2}$	10	$\frac{35}{2}$
A_3	1	7	0	27	14	77

the requirement of having vectorlike particles after the electroweak symmetry breaking. Obviously, having only one extra chiral fermion would imply that it must belong to an adjoint representation with vanishing hypercharge. Furthermore, it is straightforward to demonstrate that, in the presence of only two extra multiplets, the conditions given in Eqs. (1) and the requirement of having vectorlike particles after the electroweak symmetry breaking imply that the multiplets must necessarily form a vectorlike set; otherwise, they must be chiral fermions belonging to adjoint representations, both with zero hypercharge. Indeed, let us rewrite Eqs. (1) for the case of two arbitrary multiplets,

$$A_3(R_1)d_2(R_1) + A_3(R_2)d_2(R_2) = 0, \qquad (5a)$$

$$y_{R_1}t_3(R_1)d_2(R_1) + y_{R_2}t_3(R_2)d_2(R_2) = 0, \qquad (5b)$$

$$y_{R_1}t_2(R_1)d_3(R_1) + y_{R_2}t_2(R_2)d_3(R_2) = 0, \qquad (5c)$$

$$y_{R_1}^3 d_2(R_1)d_3(R_1) + y_{R_2}^3 d_2(R_2)d_3(R_2) = 0, \qquad (5d)$$

$$y_{R_1}d_2(R_1)d_3(R_1) + y_{R_2}d_2(R_2)d_3(R_2) = 0. \qquad (5e)$$

From Eqs. (5d)–(5e), we immediately conclude that $y_{R_1}^2 = y_{R_2}^2$. Therefore, there are just two possibilities. If $y_{R_1} = y_{R_2} = y_R$ then Eq. (5e) becomes

$$y_R(d_2(R_1)d_3(R_1) + d_2(R_2)d_3(R_2)) = 0. \qquad (6)$$

Since $d_i(R) > 0$, the unique solution is $y_R = 0$. This condition leads to two types of solutions, depending on how the remaining Eq. (5a) is satisfied. One possibility is that anomaly cancellation occurs between both multiplets, for instance, as in the multiplet combination $(\mathbf{6}, \mathbf{1})_0 \oplus (\bar{\mathbf{3}}, \mathbf{7})_0$.[1] The other case is when Eq. (5a) is verified independently for each multiplet [i.e., $A_3(R_1) = A_3(R_2) = 0$], which is similar to the case of adding just one chiral fermion.

Instead, if $y_{R_1} = -y_{R_2} \neq 0$ then Eq. (5e) becomes

$$d_2(R_1)d_3(R_1) = d_2(R_2)d_3(R_2), \qquad (7)$$

which when substituted in Eq. (5c) implies

$$t_2(R_1)d_2(R_2) = t_2(R_2)d_2(R_1). \qquad (8)$$

Using Eq. (4), we then conclude that $d_2(R_1) = d_2(R_2)$. In other words, the two multiplets must have the same SU(2) representation. One can now rewrite the remaining anomaly relations simply as

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[1]Hereafter, chiral multiplets are represented as $(d_3(R), d_2(R))_{y_R}$, following the standard notation.

$$\frac{d_3(R_1)}{d_3(R_2)} = \frac{t_3(R_1)}{t_3(R_2)} = -\frac{A_3(R_1)}{A_3(R_2)} = 1. \qquad (9)$$

If we now require that particles are vectorlike with respect to color then, by definition, the SU(3) representations of R_1 and R_2 must be conjugates of each other, unless they are adjoint. Combining these results, we obtain the vectorlike set of two multiplets,

$$(\mathbf{d}, \mathbf{d}')_y \oplus (\bar{\mathbf{d}}, \mathbf{d}')_{-y}. \qquad (10)$$

The question naturally arises whether it is possible to find anomaly-free sets with only three chiral multiplets. Considering SU(3) representations with dimensions $\mathbf{d} \equiv d_3(R) \leq 10$ and SU(2) representations with $d_2(R) \leq 5$, and restricting our search to hypercharges given by rational numbers, we obtain four different sets of solutions, which are presented in Table II. The parameter z in Table II reflects the fact that the hypercharge for each chiral particle can be rescaled by an overall amount. Note that all multiplets in each minimal set have the same SU(3) dimension $\mathbf{d}$. This is so because we restrict our analysis to $d_3(R) \leq 10$. If one relaxes this constraint, solutions with different SU(3) dimensions are found; for instance, the set $(\mathbf{15}, \mathbf{1})_{z/6} \oplus (\bar{\mathbf{6}}, \mathbf{2})_{-z/3} \oplus (\mathbf{1}, \mathbf{3})_{z/2}$ is anomaly free. Furthermore, for the sets P1 and P4, Witten's anomaly condition forces $\mathbf{d}$ to be an even number.

If one requires that the three chiral multiplets in each set give rise to vectorlike particles (with respect to the electric charge) after the gauge symmetry breaking, the following condition must also be verified:

$$\sum_{p=1}^{3} \sum_{j_p} [j_p + y_p(z)]^m = 0, \qquad (11)$$

for any odd positive integer m. In the above equation, $j_p = -s_p, -s_p + 1, \ldots, s_p - 1, s_p$ with $2s_p + 1 = d_2(R_p)$. This condition would then fix the value of z. For $m = 1$ or 3, Eq. (11) is automatically satisfied for any value of z by virtue of the anomaly cancellation conditions given in Eqs. (1). The value of z is therefore determined by taking $m = 5$. This leads to $|z| = 0, 1$, or 3, regardless of the

TABLE II. Minimal anomaly-free chiral sets for an arbitrary SU(3) dimension $\mathbf{d} \leq 10$ and SU(2) dimension $d_2(R) \leq 5$. For the sets P1 and P4, Witten's anomaly condition forces $\mathbf{d}$ to be an even number.

Set	Particle content				
P1	$(\mathbf{d}, \mathbf{1})_{5z/6}$	$\oplus$	$(\mathbf{d}, \mathbf{2})_{-2z/3}$	$\oplus$	$(\bar{\mathbf{d}}, \mathbf{3})_{z/6}$
P2	$(\mathbf{d}, \mathbf{1})_{7z/6}$	$\oplus$	$(\mathbf{d}, \mathbf{3})_{-5z/6}$	$\oplus$	$(\bar{\mathbf{d}}, \mathbf{4})_{z/3}$
P3	$(\mathbf{d}, \mathbf{1})_{3z/2}$	$\oplus$	$(\mathbf{d}, \mathbf{4})_{-z}$	$\oplus$	$(\bar{\mathbf{d}}, \mathbf{5})_{z/2}$
P4	$(\mathbf{d}, \mathbf{2})_{4z/3}$	$\oplus$	$(\mathbf{d}, \mathbf{3})_{-7z/6}$	$\oplus$	$(\bar{\mathbf{d}}, \mathbf{5})_{z/6}$

chosen set. However, we must also ensure that the solutions are simultaneously vectorlike with respect to $SU(3)$. Obviously, when $\mathbf{d} = 1$ or 8, this is already guaranteed. On the other hand, for $\mathbf{d} = 3, 6$, or 10, one can easily show that this requirement imposes $|z| = 1$.

A. Gauge coupling unification

Let us now study the possibility of achieving gauge coupling unification with the anomaly-free sets previously found. We shall work in the framework of the SM, extended by new anomaly-free chiral sets of fermions. In our approach, we do not modify the Higgs and gauge sectors of the SM. More precisely, we assume that if an additional Higgs or gauge particle content beyond the SM is present, it does not affect the evolution of the running gauge couplings. Clearly, the new fermions would have to acquire masses by some mechanism that relies on an extended theory. A simple possibility is to extend the scalar sector and construct renormalizable Yukawa terms for the new chiral fields. Let us consider, for instance, the set P1 of Table II, with $\psi_1 \sim (\mathbf{8}, \mathbf{1})_{5/2}$, $\psi_2 \sim (\mathbf{8}, \mathbf{2})_{-2}$, $\psi_3 \sim (\mathbf{8}, \mathbf{3})_{1/2}$. We then introduce the scalar fields $H \sim (\mathbf{1}, \mathbf{2})_{-1/2}$, $\Phi \sim (\mathbf{1}, \mathbf{4})_{3/2}$, $\Delta \sim (\mathbf{1}, \mathbf{3})_{-1}$ and write the Yukawa interaction terms $\epsilon_{ab}\psi_1^a\psi_2^b H^b$, $\epsilon_{ad}\epsilon_{be}\epsilon_{cf}\psi_2^a\psi_3^{bc}\Phi^{def}$, and $\epsilon_{ac}\epsilon_{be}\epsilon_{df}\psi_3^{ab}\psi_3^{cd}\Delta^{ef}$. The new fermions become massive once the neutral components of the scalar fields acquire vacuum expectation values, i.e., $\langle H^1 \rangle \neq 0$, $\langle \Phi^{222} \rangle \neq 0$, and $\langle \Delta^{11} \rangle \neq 0$. The limitation of this mechanism is that the masses of the new physical fermions are tightly constrained by electroweak precision data. Indeed, since the new scalar fields should carry nontrivial charges under the SM group, they also contribute to the electroweak gauge boson masses. As it turns out, the inclusion of such scalars leads to gauge coupling unification only if some of their masses are much higher than the electroweak scale. Moreover, for the cases where unification is achieved, the masses of the new fermions cannot be all lowered to the electroweak scale. Thus, this mass mechanism is disfavored in our minimal framework.

If we insist on the unification of the gauge couplings at a high energy scale, we are led to consider an alternative mechanism that allows for a successful unification, without imposing tight constraints on the masses of the new fermionic degrees of freedom. In such a case, these extra particles could decouple from the theory at intermediate energy scales. An attractive possibility is to enlarge the fermionic sector by considering mirror fermions [24], as in chiral gauge theories based on noncommutative geometry [25]. The advantage of this approach is that it can provide a dynamical mechanism in which fermions belonging to the physical subspace acquire masses at intermediate scales while their mirror partner states get masses higher than the unification scale [26,27]. Obviously, a detailed analysis would be required to fully implement this paradigm in our framework. In what follows, we shall assume that whatever

mechanism is chosen, it allows for the decoupling of some chiral fermions at intermediate mass scales, much higher than the electroweak scale.

The evolution of gauge couplings $\alpha_i (i = 1, 2, 3)$ is governed by the RGEs. Assuming the presence of N chiral fermions with intermediate mass scales $M_I, I = 1, ..., N$, and $M_Z \leq M_I \leq \Lambda$, the one-loop solution at the unification scale Λ can be written as

$$\alpha_i^{-1}(M_Z) = \alpha_U^{-1} + \frac{1}{2\pi}B_i \ln\left(\frac{\Lambda}{M_Z}\right), \quad (12)$$

with

$$\alpha_U \equiv \kappa_1\alpha_y(\Lambda) = \kappa_2\alpha_w(\Lambda) = \kappa_3\alpha_s(\Lambda), \quad (13)$$

and $\alpha_U \lesssim 1$ to ensure that the perturbative regime holds [28]. In Eq. (12),

$$B_i = \frac{1}{\kappa_i}\left(b_i + \sum_{I=1}^{N} b_i^I r_I\right), \quad (14)$$

where the "running weight" r_I is defined for each intermediate energy scale M_I as

$$r_I = \frac{\ln(\Lambda/M_I)}{\ln(\Lambda/M_Z)}, \quad (15)$$

and takes values in the interval $0 \leq r_I \leq 1$. The one-loop beta coefficients b_i account for the SM contribution, while b_i^I are the contributions of the intermediate particles above the threshold M_I. For a given particle in a representation R, the one-loop beta coefficients b_i^R are computed using the formula

$$b_i^R = s(R)t_i(R)\prod_{j\neq i}d_j(R), \quad (16)$$

where $s(R) = 1/6$ for a real scalar, $1/3$ for a complex scalar, $-11/3$ for a gauge boson, $2/3$ for a chiral fermion, and $4/3$ for a vectorlike fermion. When applied to the SM particle content, this gives $b_1 = 41/6$, $b_2 = -19/6$, and $b_3 = -7$.

In order to verify whether the minimal anomaly-free sets in Table II lead to the unification of the gauge couplings, we need to solve Eqs. (13). Each set of solutions in Table II is composed of $N = 3$ chiral multiplets. Thus, there are two equations to determine four unknowns: the intermediate mass scales of the three extra particles, given by the running weights $r_{1,2,3}$, and the unification scale Λ.

We choose to vary the running weights r_2 and r_3 in the allowed range [0, 1], and then determine the scale of the first particle in the set,

$$r_1 = \frac{BB'_{12} - B'_{23}}{\Delta^1_{23} - B\Delta^1_{12}}, \tag{17}$$

where

$$B'_{ij} = B'_i - B'_j, \qquad B'_i = \frac{1}{\kappa_i}\left(b_i + \sum_{I=2}^{N} b_i^I r_I\right), \tag{18}$$

and

$$\Delta^I_{ij} = \frac{b_i^I}{\kappa_i} - \frac{b_j^I}{\kappa_j}. \tag{19}$$

The unification scale Λ is obtained through the relation

$$\ln\left(\frac{\Lambda}{M_Z}\right) = \frac{\tilde{B}}{B_1 - B_2}. \tag{20}$$

The constants B and $\tilde{B}$, defined in terms of parameters at M_Z scale, are given by [29,30]

$$B \equiv \frac{\sin^2\theta_w - \frac{\kappa_2\alpha}{\kappa_3\alpha_s}}{\frac{\kappa_2}{\kappa_1} - (1 + \frac{\kappa_2}{\kappa_1})\sin^2\theta_w},$$

$$\tilde{B} \equiv \frac{2\pi}{\alpha}\left[\frac{1}{\kappa_1} - \left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)\sin^2\theta_w\right]. \tag{21}$$

Using the experimental data at the electroweak scale, chosen here at $M_Z = 91.1876$ GeV [31],

$$\alpha^{-1} = 127.944 \pm 0.014,$$

$$\alpha_s = 0.1185 \pm 0.0006,$$

$$\sin^2\theta_w = 0.23126 \pm 0.00005, \tag{22}$$

and considering $\kappa_1 = \kappa_2 = \kappa_3 = 1$ we obtain

$$B = 0.308 \pm 0.001, \qquad \tilde{B} = 431.4 \pm 0.1. \tag{23}$$

If one uses instead the $\mathsf{SU}(5)$ normalization, then

$$B = 0.718 \pm 0.003, \qquad \tilde{B} = 185.0 \pm 0.2. \tag{24}$$

In Figs. 1–4, we present the allowed ranges of the intermediate scales M_I and the unification scale Λ as functions of the $\mathsf{SU}(3)$ dimension **d** for the 16 solutions found within the chiral sets of Table II. In all cases, we require $\Lambda \geq 5 \times 10^{15}$ GeV, so that we are safely consistent with proton decay bounds. First, we note that no solution with $z = 0$, i.e., with vanishing hypercharge, and **d** $= 1$ or 8, leads to successful unification. For set P1, gauge coupling unification is attained only for the hypercharge normalization $z = 3$ and for **d** $= 8$. The required intermediate scales turn out to be above 10^{13} GeV. In the case of set P2, when $z = 1$, solutions were found only with
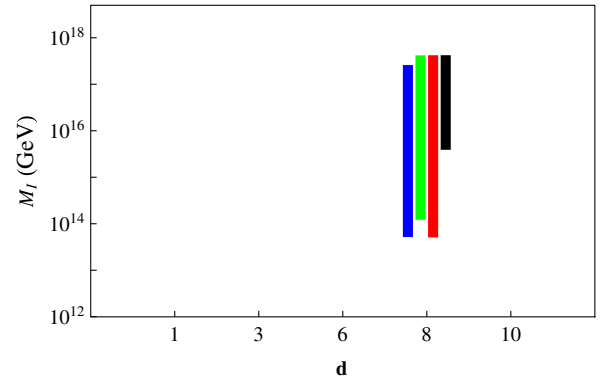

FIG. 1 (color online). The allowed range of the intermediate scales M_I and the unification scale Λ as functions of the $\mathsf{SU}(3)$ dimension **d** for the solution P1 of Table II with $z = 3$. For each value of **d**, the colored bars correspond (from left to right) to the energy scales M_1 $((\mathbf{d}, \mathbf{1})_{5z/6})$, M_2 $((\mathbf{d}, \mathbf{2})_{-2z/3})$, M_3 $((\bar{\mathbf{d}}, \mathbf{3})_{z/6})$, and Λ, respectively.

d $= 3, 6$, or 8. In this case, the mass of the $\mathsf{SU}(2)$ singlet, M_1, can take values at the TeV scale or even lower, while the other two masses are above 10^9 GeV. When $z = 3$, solutions were found for the allowed values of **d** $= 1$ or 8.

FIG. 2 (color online). The allowed range of the intermediate scales M_I and the unification scale Λ as functions of the $\mathsf{SU}(3)$ dimension **d** for the solution P2 of Table II with $z = 1$ (upper panel) and $z = 3$ (lower panel). For each value of **d**, the colored bars correspond (from left to right) to the energy scales M_1 $((\mathbf{d}, \mathbf{1})_{7z/6})$, M_2 $((\mathbf{d}, \mathbf{3})_{-5z/6})$, M_3 $((\bar{\mathbf{d}}, \mathbf{4})_{z/3})$, and Λ, respectively.
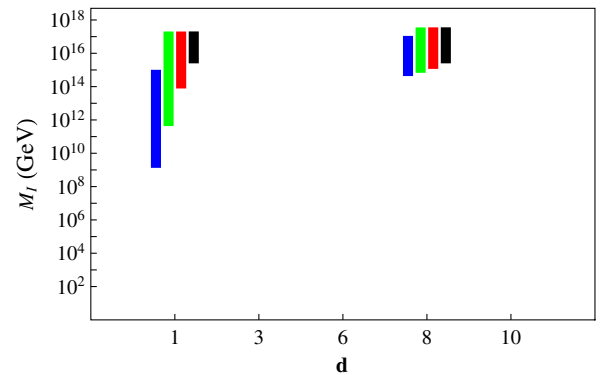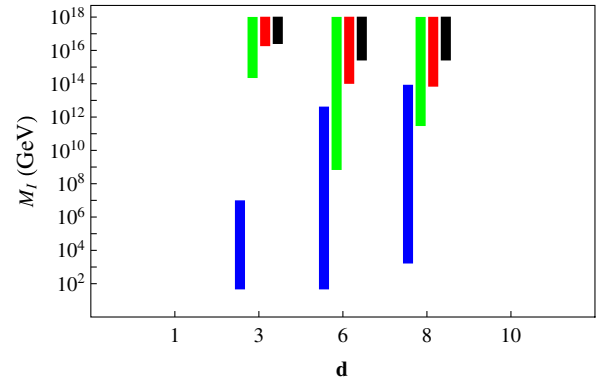
FIG. 3 (color online). The allowed range of the intermediate scales M_I and the unification scale Λ as functions of the $\mathsf{SU}(3)$ dimension **d** for the solution P3 of Table II with $z = 1$ (upper panel) and $z = 3$ (lower panel). For each value of **d**, the colored bars correspond (from left to right) to the energy scales M_1 $((\mathbf{d}, \mathbf{1})_{3z/2})$, M_2 $((\mathbf{d}, \mathbf{4})_{-z})$, M_3 $((\bar{\mathbf{d}}, \mathbf{5})_{z/2})$, and Λ, respectively.

We note that for **d** = 8 unification is achieved for quite high intermediate scales, between 10^{14} GeV and the unification scale. The mass and unification behavior in the set P3 is quite similar to the set P2. In this case, when $z = 1$, solutions were found also for **d** = 10. Finally, the set P4 unifies for $z = 1$ with any $\mathsf{SU}(3)$ even dimension, while for $z = 3$ solutions were found with **d** = 8. The intermediate scales in this set have values above 10^{10} and 10^{15} GeV, for $z = 1$ and $z = 3$, respectively.

Let us now analyze whether the above solutions also lead to the unification of gauge and gravitational couplings at the string scale. In this case, in addition to Eqs. (12), the unification scale Λ must also satisfy the constraint

$$\alpha_U = \alpha_{\text{string}} = \frac{1}{4\pi}\left(\frac{\Lambda}{\Lambda_S}\right)^2, \qquad (25)$$

where $\Lambda_S = 5.27 \times 10^{17}$ GeV is the string scale that takes into account one-loop string effects in the weak coupling limit [32].

Since we have now three equations for unification, namely Eqs. (13) and (25), we choose to determine the running weights of the first two particles of each set, r_1 and
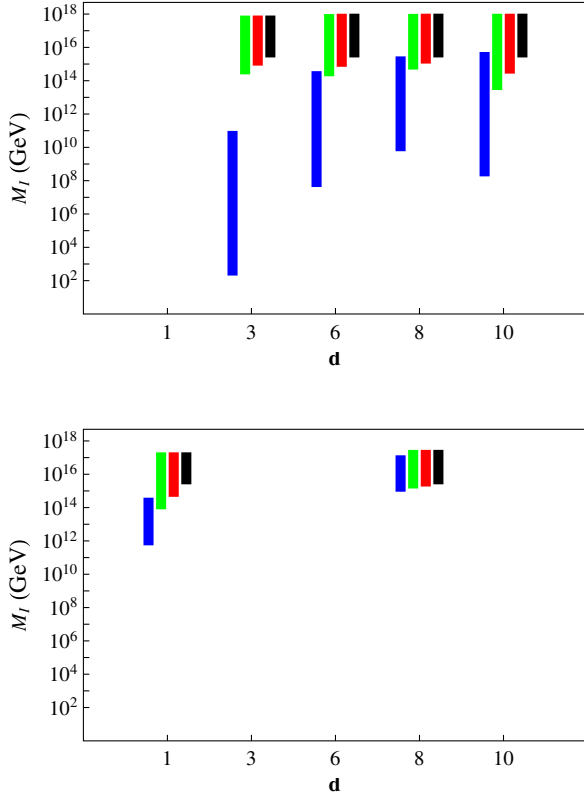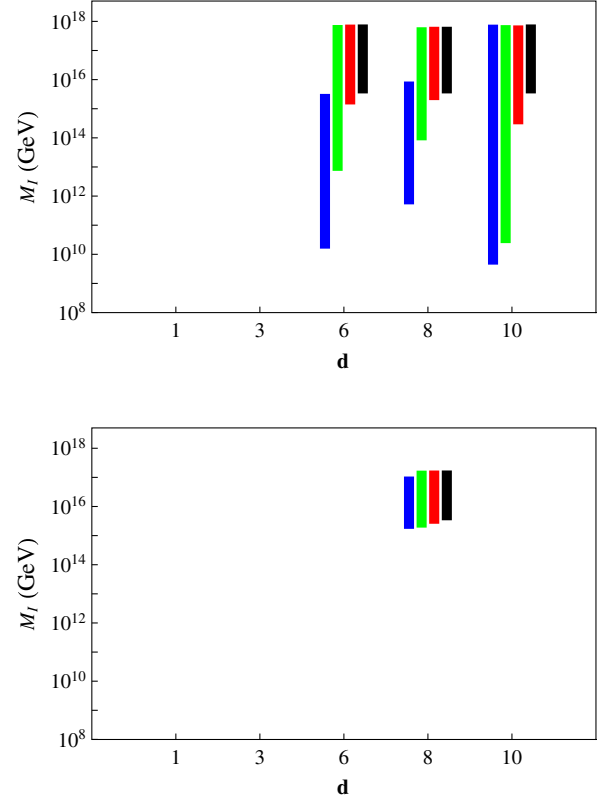
FIG. 4 (color online). The allowed range of the intermediate scales M_I and the unification scale Λ as functions of the $\mathsf{SU}(3)$ dimension **d** for the solution P4 of Table II with $z = 1$ (upper panel) and $z = 3$ (lower panel). For each value of **d**, the colored bars correspond (from left to right) to the energy scales M_1 $((\mathbf{d}, \mathbf{2})_{4z/3})$, M_2 $((\mathbf{d}, \mathbf{3})_{-7z/6})$, M_3 $((\bar{\mathbf{d}}, \mathbf{5})_{z/6})$, and Λ, respectively.

r_2, and the unification scale Λ, while r_3 is free to take values within the allowed range [0, 1]. We find

$$r_1 = \beta_2 - \frac{\tilde{\beta}_2}{\ln\left(\Lambda/M_Z\right)}, \qquad r_2 = -\beta_1 + \frac{\tilde{\beta}_1}{\ln\left(\Lambda/M_Z\right)}. \qquad (26)$$

The coefficients β_i and $\tilde{\beta}_i$ are given by

$$\beta_i = \frac{B_{12}''\Delta_{23}^i - B_{23}''\Delta_{12}^i}{\Delta_{12}^2\Delta_{23}^1 - \Delta_{23}^2\Delta_{12}^1},$$

$$\tilde{\beta}_i = \tilde{B}\,\frac{\Delta_{23}^i - B\Delta_{12}^i}{\Delta_{12}^2\Delta_{23}^1 - \Delta_{23}^2\Delta_{12}^1}, \qquad (27)$$

where

$$B_{ij}'' = B_i'' - B_j'', \qquad B_i'' = \frac{1}{\kappa_i}\left(b_i + \sum_{I=3}^{N} b_i^I r_I\right). \qquad (28)$$

The unification scale Λ is then given by

$$\left(\frac{\Lambda_S}{\Lambda}\right)^2 = -c\,\mathrm{W}_k\left(-\frac{\Lambda_S^2}{cM_Z^2}e^{-\frac{a}{4\pi c}}\right), \qquad (29)$$

where

$$a = \alpha_i^{-1}(M_Z) + \frac{1}{2\pi}\left(\tilde{\beta}_2\frac{b_i^1}{\kappa_i} - \tilde{\beta}_1\frac{b_i^2}{\kappa_i}\right),$$

$$c = \frac{1}{16\pi^2}\left(\beta_2\frac{b_i^1}{\kappa_i} - \beta_1\frac{b_i^2}{\kappa_i} + B_i''\right), \qquad (30)$$

for any choice $i = 1, 2, 3$. The function $W_k(x)$ is the Lambert W function, with $k = 0$ for $c < 0$ and $k = -1$ for $c > 0$. One can verify that the contribution of the running weight r_3 cancels out in the constants a and c. Thus, the unification scale is uniquely determined for any given set of chiral multiplets.

Out of the 16 solutions, only 11 are consistent with string-scale unification. The results are summarized in Table III. In all cases, the unification scale Λ is in the range $[2.7, 3.2] \times 10^{17}$ GeV, which implies $34 \leq \alpha_{\text{string}}^{-1} \leq 48$. The intermediate mass M_3 is very close to the unification scale

for all solutions, and no solutions were found with intermediate scales below the TeV scale. Moreover, solutions with the hypercharge normalization $z = 3$ were obtained only for set P1 with **d** = 8.

We conclude this section by noticing that the minimal anomaly-free sets given in Table II, although leading to unification with only three chiral fermion multiplets, do not exhibit the usual quantum numbers under the SM group, such as those present in SM GUT embeddings into SU(5) or SO(10) groups.

III. SU(5)-INSPIRED ANOMALY-FREE CHIRAL FERMION SETS

In view of our previous results, and inspired by simplicity in SU(5) as a gauge group, we may ask what the minimal chiral sets are, besides the SM combination $\bar{5} \oplus 10$, that fulfil our three requirements, namely, be anomaly free, vectorlike with respect to the color and electric charges, and, finally, lead to gauge unification.

We shall only consider SU(5) representations with dimensions less than or equal to 50. For the SU(5)

TABLE III.　Minimal chiral sets of solutions for which there is string-scale unification. The particle content of each set is defined in Table II. The mass ordering M_1, M_2, and M_3 corresponds to the ordering of the multiplets adopted for each set in Table II.

Set	z	d	Λ [10^{17} GeV]	M_1 [GeV]	M_2 [GeV]	M_3 [GeV]
P1	3	8	2.7	$[2.7, 7.3] \times 10^{16}$	$[1.2, 2.7] \times 10^{17}$	$[2.2, 2.7] \times 10^{17}$
P2	1	3	2.7	$[2.1, 26] \times 10^{4}$	$[4.4, 26] \times 10^{16}$	$[1.3, 2.7] \times 10^{17}$
	1	6	3.0	$[3.7, 45] \times 10^{10}$	$[6.7, 40] \times 10^{14}$	$[1.4, 3.0] \times 10^{17}$
	1	8	2.9	$[9.4, 110] \times 10^{11}$	$[8.7, 52] \times 10^{15}$	$[1.4, 2.9] \times 10^{17}$
P3	1	3	2.7	$[8.6, 12] \times 10^{9}$	$[2.2, 2.7] \times 10^{17}$	$[2.5, 2.7] \times 10^{17}$
	1	6	2.8	$[1.3, 4.4] \times 10^{13}$	$[1.2, 2.8] \times 10^{17}$	$[1.9, 2.8] \times 10^{17}$
	1	8	2.8	$[1.9, 4.1] \times 10^{14}$	$[1.7, 2.8] \times 10^{17}$	$[2.2, 2.8] \times 10^{17}$
	1	10	3.0	$[1.1, 9.2] \times 10^{14}$	$[4.7, 20] \times 10^{16}$	$[1.5, 3.0] \times 10^{17}$
P4	1	6	2.8	$[3.5, 13] \times 10^{14}$	$[9.1, 28] \times 10^{16}$	$[2.4, 2.8] \times 10^{17}$
	1	8	2.8	$[2.2, 3.6] \times 10^{15}$	$[1.8, 2.8] \times 10^{17}$	$[2.6, 2.8] \times 10^{17}$
	1	10	3.2	$[3.1, 320] \times 10^{15}$	$[8.3, 490] \times 10^{14}$	$[1.6, 2.8] \times 10^{17}$

TABLE IV.　Particle multiplets and one-loop beta coefficients for the SU(5) representations with dimensions less than or equal to 50.

Label	Multiplet	SU(5)-rep	(b_1, b_2, b_3)	Label	Multiplet	SU(5)-rep	(b_1, b_2, b_3)
1	$(\mathbf{1}, \mathbf{2})_{1/2}$	5, 45	$(1/3, 1/3, 0)$	12	$(\mathbf{1}, \mathbf{2})_{-3/2}$	40	$(3, 1/3, 0)$
2	$(\mathbf{3}, \mathbf{1})_{-1/3}$	5, 45, 50	$(2/9, 0, 1/3)$	13	$(\mathbf{8}, \mathbf{1})_{1}$	40	$(16/3, 0, 2)$
3	$(\mathbf{1}, \mathbf{1})_{1}$	10	$(2/3, 0, 0)$	14	$(\mathbf{3}, \mathbf{3})_{-1/3}$	45	$(2/3, 4, 1)$
4	$(\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$	10, 40	$(8/9, 0, 1/3)$	15	$(\mathbf{3}, \mathbf{1})_{4/3}$	45	$(32/9, 0, 1/3)$
5	$(\mathbf{3}, \mathbf{2})_{1/6}$	10, 15, 40	$(1/9, 1, 2/3)$	16	$(\bar{\mathbf{3}}, \mathbf{2})_{-7/6}$	45, 50	$(49/9, 1, 2/3)$
6	$(\mathbf{1}, \mathbf{3})_{1}$	15	$(2, 4/3, 0)$	17	$(\bar{\mathbf{6}}, \mathbf{1})_{-1/3}$	45	$(4/9, 0, 5/3)$
7	$(\mathbf{6}, \mathbf{1})_{-2/3}$	15	$(16/9, 0, 5/3)$	18	$(\mathbf{8}, \mathbf{2})_{1/2}$	45, 50	$(8/3, 8/3, 4)$
8	$(\mathbf{1}, \mathbf{4})_{-3/2}$	35	$(6, 10/3, 0)$	19	$(\mathbf{1}, \mathbf{1})_{-2}$	50	$(8/3, 0, 0)$
9	$(\bar{\mathbf{3}}, \mathbf{3})_{-2/3}$	35, 40	$(8/3, 4, 1)$	20	$(\bar{\mathbf{6}}, \mathbf{3})_{-1/3}$	50	$(4/3, 8, 5)$
10	$(\bar{\mathbf{6}}, \mathbf{2})_{1/6}$	35, 40	$(2/9, 2, 10/3)$	21	$(\mathbf{6}, \mathbf{1})_{4/3}$	50	$(64/9, 0, 5/3)$
11	$(\overline{\mathbf{10}}, \mathbf{1})_{1}$	35	$(20/3, 0, 5)$				

multiplets contained in these representations [21], we give in Table IV the corresponding one-loop beta coefficients b_i. The representations are labeled from 1 to 21 according to their quantum numbers.

Applying the anomaly constraints in Eqs. (1) to the 21 particle species of Table IV, we obtain the following system of linear equations for the number of multiplets, $n_I, I = 1, \ldots, 21$, of each particle type:

$$n_2 - n_4 + 2n_5 + 7n_7 - 3n_9 - 14n_{10} - 27n_{11} + 3n_{14} - n_{15} - 2n_{16} - 7n_{17} - 21n_{20} + 7n_{21} = 0,$$

$$n_2 + 2n_4 - n_5 + 10n_7 + 6n_9 - 5n_{10} - 45n_{11} - 18n_{13} + 3n_{14} - 4n_{15} + 7n_{16} + 5n_{17} - 18n_{18} + 15n_{20} - 20n_{21} = 0,$$

$$n_1 + n_5 + 8n_6 - 30n_8 - 16n_9 + 2n_{10} - 3n_{12} - 8n_{14} - 7n_{16} + 8n_{18} - 16n_{20} = 0,$$

$$9n_1 - 4n_2 + 36n_3 - 32n_4 + n_5 + 108n_6 - 64n_7 - 486n_8 - 96n_9 + 2n_{10} + 360n_{11} - 243n_{12}$$
$$+ 288n_{13} - 12n_{14} + 256n_{15} - 343n_{16} - 8n_{17} + 72n_{18} - 288n_{19} - 24n_{20} + 512n_{21} = 0,$$

$$n_1 - n_2 + n_3 - 2n_4 + n_5 + 3n_6 - 4n_7 - 6n_8 - 6n_9 + 2n_{10} + 10n_{11} - 3n_{12} + 8n_{13} - 3n_{14}$$
$$+ 4n_{15} - 7n_{16} - 2n_{17} + 8n_{18} - 2n_{19} - 6n_{20} + 8n_{21} = 0. \tag{31}$$

Besides the anomaly constraints, we must also require that the new low-energy fermion states form vectorlike sets with respect to the color $SU(3)$ and electromagnetic $U(1)$. This requirement leads to the additional constraints

$$\mathbf{1}_1\colon n_1 + n_3 + n_6 - n_8 - n_{12} = 0,$$

$$\mathbf{3}_{-1/3}\colon n_2 + n_5 - n_9 + n_{14} = 0,$$

$$\mathbf{\bar{3}}_{-2/3}\colon n_4 - n_5 + n_9 - n_{14} + n_{16} = 0,$$

$$\mathbf{1}_2\colon n_6 - n_8 - n_{12} - n_{19} = 0,$$

$$\mathbf{6}_{-2/3}\colon n_7 - n_{10} - n_{20} = 0,$$

$$\mathbf{1}_{-3}\colon n_8 = 0,$$

$$\mathbf{\bar{3}}_{-5/3}\colon n_9 + n_{16} = 0,$$

$$\mathbf{\bar{6}}_{-1/3}\colon n_{10} + n_{17} + n_{20} = 0,$$

$$\mathbf{\overline{10}}_1\colon n_{11} = 0,$$

$$\mathbf{8}_1\colon n_{13} + n_{18} = 0,$$

$$\mathbf{3}_{-4/3}\colon n_{14} - n_{15} = 0,$$

$$\mathbf{\bar{6}}_{-4/3}\colon n_{20} - n_{21} = 0, \tag{32}$$

where $\mathbf{d}_Q$ denotes states with $SU(3)$ dimension $\mathbf{d}$ and electric charge Q. Substituting Eqs. (32) into Eqs. (31), we verify that the first, second, and last equations in (31) are automatically satisfied, while the remaining two equations can be rewritten as

$$2n_1 + 9n_2 + 3n_3 + 17n_4 - 9n_5 - 5n_6 + 16n_7$$
$$- 18n_{10} + 8n_{13} = 0,$$

$$54n_1 + 243n_2 + 81n_3 + 459n_4 - 243n_5 - 135n_6$$
$$+ 432n_7 - 486n_{10} + 216n_{13} = 0. \tag{33}$$

In Table V, we present the minimal sets of chiral multiplets, with a maximum of $n_s = 6$ number of species

and up to ten multiplets per set, that are anomaly free and, at the same time, lead to vectorlike states at low energies. It is worth noticing that the sets S2 and S7 correspond to one and two additional SM generations, respectively. We shall search among the sets in Table V for solutions that lead to a successful gauge coupling unification at GUT and string scales.

Any self-contained unification scenario must include scalars in order to obtain the proper symmetry breaking. In our minimal $SU(5)$-inspired setup, we shall assume that the breaking of $SU(5)$ into the SM group occurs at the GUT (string) scale and it is achieved through the usual **24** adjoint scalar representation. The breaking of the SM gauge group is then realized via the usual vacuum expectation value of the Higgs field in the **5** representation. Thus, the scalar content is $\Sigma_3 \sim (\mathbf{1}, \mathbf{3})_0$, $\Sigma_8 \sim (\mathbf{8}, \mathbf{1})_0$, $(X, Y)^{\intercal} \sim (\mathbf{3}, \mathbf{2})_{-5/6}$, $H^c \sim (\mathbf{3}, \mathbf{1})_{-1/3}$, and the Higgs doublet $H \sim (\mathbf{1}, \mathbf{2})_{1/2}$. Since the scalars X, Y, and H^c can dangerously mediate proton decay, in what follows we assume that their masses are of the order of the unification scale. Therefore, from the RGE viewpoint, the only relevant scalars are Σ_3, Σ_8, and H. While the mass of the Higgs doublet is required to be at the electroweak scale, the mass scales of Σ_3 and Σ_8 are allowed to vary from M_Z up to the unification scale. Without any further assumption, the latter are expected to be close to each other, i.e., $M_{\Sigma_3} \approx M_{\Sigma_8}$.

We proceed as in the previous section and make use of Eqs. (17)–(20). We randomly vary the running weights r_2 to r_{21} in their allowed range [0, 1], together with the running weights of the scalars Σ_3 and Σ_8. We then determine r_1 using Eq. (17) and calculate the unification scale Λ through Eq. (20).

In Table VI, we list the nine sets of solutions that lead to unification of the gauge couplings for $\Lambda \geq 5 \times 10^{15}$ GeV. As mentioned before, this lower bound is invoked so that we are safely consistent with proton decay bounds. In fact, we have also found unification of the gauge couplings for

TABLE V. Anomaly-free chiral sets obtained from $\mathsf{SU}(5)$ representations up to dimension 50. The multiplet notation follows the convention $(\mathsf{SU}(3), \mathsf{SU}(2))_Y$, and n_s corresponds to the number of different species required in each set. For each solution it is also indicated whether it leads to GUT (G) and/or string (S) scale unification. The sets S9, S19, and S20, marked with *, unify at a scale $\Lambda \lesssim 10^{16}$ GeV.

Set	n_s	Particle content	G	S
S1	4	$3(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{-3/2} \oplus (\mathbf{1},\mathbf{1})_2$	–	–
S2	5	$(\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	–	–
S3	5	$(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_1 \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\mathbf{8},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{1})_{-2}$	–	–
S4	5	$2(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\mathbf{1},\mathbf{1})_{-1} \oplus (\bar{\mathbf{6}},\mathbf{1})_{2/3} \oplus (\mathbf{6},\mathbf{2})_{-1/6} \oplus (\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	–	–
S5	5	$(\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{3})_1 \oplus 3(\mathbf{1},\mathbf{2})_{-3/2} \oplus 2(\mathbf{1},\mathbf{1})_2$	–	–
S6	5	$2(\mathbf{1},\mathbf{2})_{1/2} \oplus 3(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1} \oplus 2(\mathbf{1},\mathbf{2})_{3/2} \oplus (\mathbf{1},\mathbf{1})_{-2}$	–	–
S7	5	$2(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\mathbf{3},\mathbf{1})_{-1/3} \oplus 2(\mathbf{1},\mathbf{1})_{-1} \oplus 2(\mathbf{3},\mathbf{1})_{2/3} \oplus 2(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	✓	–
S8	5	$2(\mathbf{1},\mathbf{1})_{-1} \oplus 2(\mathbf{1},\mathbf{3})_1 \oplus 2(\mathbf{8},\mathbf{1})_1 \oplus 2(\mathbf{8},\mathbf{2})_{-1/2} \oplus 2(\mathbf{1},\mathbf{1})_{-2}$	✓	–
S9	6	$(\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\bar{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{3})_{2/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\bar{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\bar{\mathbf{3}},\mathbf{2})_{-7/6}$	*	–
S10	6	$(\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{8},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\bar{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{8},\mathbf{2})_{1/2}$	✓	✓
S11	6	$(\mathbf{3},\mathbf{1})_{2/3} \oplus (\bar{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{3})_{2/3} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\bar{\mathbf{3}},\mathbf{2})_{-7/6} \oplus (\mathbf{8},\mathbf{2})_{-1/2}$	✓	✓
S12	6	$(\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-2/3} \oplus (\bar{\mathbf{6}},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{2})_{-3/2} \oplus (\mathbf{6},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{1})_2$	–	–
S13	6	$(\bar{\mathbf{6}},\mathbf{1})_{2/3} \oplus 2(\mathbf{8},\mathbf{1})_1 \oplus (\bar{\mathbf{6}},\mathbf{1})_{-1/3} \oplus 2(\mathbf{8},\mathbf{2})_{-1/2} \oplus (\mathbf{6},\mathbf{3})_{1/3} \oplus (\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	✓	✓
S14	6	$2(\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus 2(\mathbf{3},\mathbf{3})_{2/3} \oplus (\mathbf{6},\mathbf{2})_{-1/6} \oplus 2(\bar{\mathbf{3}},\mathbf{2})_{-7/6} \oplus (\bar{\mathbf{6}},\mathbf{3})_{-1/3} \oplus (\mathbf{6},\mathbf{1})_{4/3}$	✓	✓
S15	6	$2(\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus 2(\bar{\mathbf{3}},\mathbf{1})_{-2/3} \oplus 2(\mathbf{3},\mathbf{2})_{1/6} \oplus (\bar{\mathbf{6}},\mathbf{1})_{2/3} \oplus (\mathbf{6},\mathbf{2})_{-1/6} \oplus (\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	✓	✓
S16	6	$2(\bar{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\bar{\mathbf{6}},\mathbf{2})_{1/6} \oplus 2(\mathbf{3},\mathbf{3})_{-1/3} \oplus 2(\bar{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{6},\mathbf{3})_{1/3} \oplus (\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	✓	✓
S17	6	$(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus 2(\bar{\mathbf{3}},\mathbf{1})_{-2/3} \oplus 2(\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{2})_{-3/2} \oplus (\mathbf{1},\mathbf{1})_2$	✓	–
S18	6	$(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\mathbf{1},\mathbf{3})_1 \oplus 3(\mathbf{1},\mathbf{2})_{-3/2} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\bar{\mathbf{8}},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{1})_2$	–	–
S19	6	$3(\mathbf{1},\mathbf{2})_{1/2} \oplus 3(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_1 \oplus (\mathbf{1},\mathbf{2})_{-3/2} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\bar{\mathbf{8}},\mathbf{2})_{-1/2}$	*	–
S20	6	$2(\mathbf{1},\mathbf{2})_{1/2} \oplus 2(\mathbf{1},\mathbf{1})_{-1} \oplus 2(\mathbf{1},\mathbf{3})_{-1} \oplus 2(\mathbf{1},\mathbf{2})_{3/2} \oplus (\bar{\mathbf{8}},\mathbf{1})_{-1} \oplus (\mathbf{8},\mathbf{2})_{1/2}$	*	–

the sets S9, S19, and S20, but the unification scale Λ turns out to be very constrained for those sets ($\Lambda \lesssim 10^{16}$ GeV). The values of α_U^{-1}, the maximum value of the unification scale $\Lambda_{\max}$, as well as the minimum and maximum values of the intermediate mass scales obtained for each set are also given in Table VI. In all cases, the mass scale $M_{\Sigma_3} \simeq M_{\Sigma_8}$ can take any value from M_Z up to the unification scale.

As can be seen from Table VI, it is possible to achieve unification with some of the intermediate scales taking values as low as the electroweak scale. As an illustration, we present in Fig. 5 the running of the gauge couplings at one-loop level for the set S7, which corresponds to the addition of two complete SM generations. In this example, the unification of the gauge couplings occurs at the scale $\Lambda = 1.3 \times 10^{16}$ GeV with $\alpha_U^{-1} \simeq 35$. The intermediate mass scales are $M_2 = M_5 = 560$ GeV, $M_1 = 6.9 \times 10^{10}$ GeV, and $M_3 = M_4 = 1.0 \times 10^{16}$ GeV. Note that the mass labeling M_i, with $i = 1, \ldots, 21$, follows the multiplet notation given in Table IV. We assume $M_{\Sigma_3} = M_{\Sigma_8} = \Lambda$.

Let us now consider the string scenario. In order to determine if the gauge and gravitational couplings unify in this case, we compute, for each set of solutions, the running weights of the first two particles, r_1 and r_2, using Eq. (26),

allowing the running weights of the remaining particles to randomly vary in the range [0, 1]. The unification scale Λ is determined using Eq. (29). We have found only six sets compatible with string-scale unification, which are presented in Table VII. One can see that, in order to have unification, the presence of sextets and/or octets of $\mathsf{SU}(3)$ is required in all cases.

We conclude this section by commenting on the possibility of constructing anomaly-free solutions with complete $\mathsf{SU}(5)$ representations. Looking at Table V, one can easily verify that none of the sets corresponds to a complete representation of this gauge group. In order to obtain anomaly cancellation within $\mathsf{SU}(5)$, we must consider the $\mathsf{SU}(5)$ anomaly condition

$$[\mathsf{SU}(5)\text{-}\mathsf{SU}(5)\text{-}\mathsf{SU}(5)]: \quad \sum_R A_5(R) = 0, \qquad (34)$$

where the anomaly cubic index $A_5(R)$ is given in Table VIII. For our choice of $\mathsf{SU}(5)$ representations, with dimensions less than or equal to 50, the above equation implies the following relation among the number of $\mathsf{SU}(5)$ multiplets:

TABLE VI.   Chiral sets for which gauge coupling unification is achieved among the $SU(5)$-inspired anomaly-free solutions given in Table V. The minimal scale for unification is set at 5.0×10^{15} GeV. The values of α_U^{-1}, maximum value of the unification scale Λ_{max}, as well as the minimum and maximum values of the intermediate mass scales are also given.

Set	α_U^{-1}	Λ_{max}[GeV]	Intermediate mass scales [GeV]					
			Representation	Minimum	Maximum	Representation	Minimum	Maximum
S7	[30.5, 37.6]	1.0×10^{17}	$(\mathbf{1},\mathbf{2})_{1/2}$	M_Z	7.6×10^{16}	$(\mathbf{3},\mathbf{1})_{-1/3}$	M_Z	1.3×10^{16}
			$(\mathbf{1},\mathbf{1})_{-1}$	8.0×10^3	7.8×10^{16}	$(\mathbf{3},\mathbf{1})_{2/3}$	2.6×10^4	9.4×10^{16}
			$(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	M_Z	5.8×10^7			
S8	[1.4, 18.8]	5.0×10^{17}	$(\mathbf{1},\mathbf{1})_{-1}$	2.2×10^3	2.8×10^{17}	$(\mathbf{1},\mathbf{3})_1$	M_Z	4.0×10^5
			$(\mathbf{8},\mathbf{1})_1$	8.9×10^{14}	4.7×10^{17}	$(\mathbf{8},\mathbf{2})_{-1/2}$	M_Z	6.2×10^7
			$(\mathbf{1},\mathbf{1})_{-2}$	9.4×10^{12}	3.4×10^{17}			
S10	[8.0, 35.0]	5.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	M_Z	4.2×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$	M_Z	4.9×10^{17}
			$(\mathbf{8},\mathbf{1})_{-1}$	6.7×10^3	5.0×10^{17}	$(\mathbf{3},\mathbf{3})_{-1/3}$	M_Z	1.1×10^{12}
			$(\bar{\mathbf{3}},\mathbf{1})_{4/3}$	M_Z	5.0×10^{17}	$(\mathbf{8},\mathbf{2})_{1/2}$	M_Z	4.7×10^{13}
S11	[4.0, 34.5]	5.3×10^{17}	$(\mathbf{3},\mathbf{1})_{2/3}$	M_Z	4.9×10^{17}	$(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	M_Z	2.7×10^{17}
			$(\mathbf{3},\mathbf{3})_{2/3}$	M_Z	9.7×10^{14}	$(\mathbf{8},\mathbf{1})_1$	8.4×10^3	4.6×10^{17}
			$(\bar{\mathbf{3}},\mathbf{2})_{-7/6}$	4.0×10^5	5.1×10^{17}	$(\mathbf{8},\mathbf{2})_{-1/2}$	M_Z	4.3×10^{13}
S13	[1.0, 35.5]	5.3×10^{17}	$(\bar{\mathbf{6}},\mathbf{1})_{2/3}$	7.4×10^4	5.1×10^{17}	$(\mathbf{8},\mathbf{1})_1$	1.9×10^7	5.0×10^{17}
			$(\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	5.5×10^7	5.0×10^{17}	$(\mathbf{8},\mathbf{2})_{-1/2}$	1.4×10^{13}	5.2×10^{17}
			$(\mathbf{6},\mathbf{3})_{1/3}$	M_Z	7.1×10^{13}	$(\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	M_Z	5.0×10^{17}
S14	[1.0, 35.9]	5.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	M_Z	4.7×10^{17}	$(\mathbf{3},\mathbf{3})_{2/3}$	6.4×10^7	4.9×10^{17}
			$(\mathbf{6},\mathbf{2})_{-1/6}$	M_Z	2.8×10^{16}	$(\bar{\mathbf{3}},\mathbf{2})_{-7/6}$	2.9×10^3	5.0×10^{17}
			$(\bar{\mathbf{6}},\mathbf{3})_{-1/3}$	2.5×10^3	4.4×10^{17}	$(\mathbf{6},\mathbf{1})_{4/3}$	M_Z	4.9×10^{17}
S15	[32.3, 37.3]	5.3×10^{17}	$(\mathbf{3},\mathbf{1})_{1/3}$	9.0×10^6	4.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$	1.5×10^{11}	4.3×10^{17}
			$(\mathbf{3},\mathbf{2})_{1/6}$	M_Z	9.8×10^5	$(\bar{\mathbf{6}},\mathbf{1})_{2/3}$	2.1×10^{13}	5.0×10^{17}
			$(\mathbf{6},\mathbf{2})_{-1/6}$	8.1×10^{10}	8.5×10^{16}	$(\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	5.4×10^{12}	4.8×10^{17}
S16	[1.0, 37.1]	5.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	M_Z	5.1×10^{17}	$(\bar{\mathbf{6}},\mathbf{2})_{1/6}$	M_Z	6.0×10^{15}
			$(\mathbf{3},\mathbf{3})_{-1/3}$	9.5×10^7	5.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{4/3}$	M_Z	5.0×10^{17}
			$(\mathbf{6},\mathbf{3})_{1/3}$	880	4.9×10^{17}	$(\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	M_Z	5.0×10^{17}
S17	[31.9, 37.1]	6.0×10^{16}	$(\mathbf{1},\mathbf{2})_{1/2}$	M_Z	2.0×10^{16}	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	M_Z	8.3×10^{15}
			$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$	2.9×10^6	3.3×10^{16}	$(\mathbf{3},\mathbf{2})_{1/6}$	M_Z	2.0×10^7
			$(\mathbf{1},\mathbf{2})_{-3/2}$	6.5×10^{10}	5.0×10^{16}	$(\mathbf{1},\mathbf{1})_2$	6.7×10^{10}	5.9×10^{16}

$$n_5 + n_{10} + 9n_{15} - 44n_{35} - 16n_{40} - 6n_{45} - 15n_{50} = 0. \tag{35}$$

Moreover, imposing the low-energy vectorlike conditions for the particle content of the above multiplets, we obtain the relations

$$n_5 + n_{10} - n_{40} = 0,$$
$$n_{15} - n_{40} - n_{50} = 0,$$
$$n_{15} + n_{45} = 0,$$
$$n_{35} = 0. \tag{36}$$

One can easily verify that once Eqs. (36) are imposed, Eq. (34) is automatically satisfied.

Besides the well-known SM solution, formed by the combination $\bar{5} \oplus 10$, we have found the following minimal anomaly-free sets of complete representations of $SU(5)$,
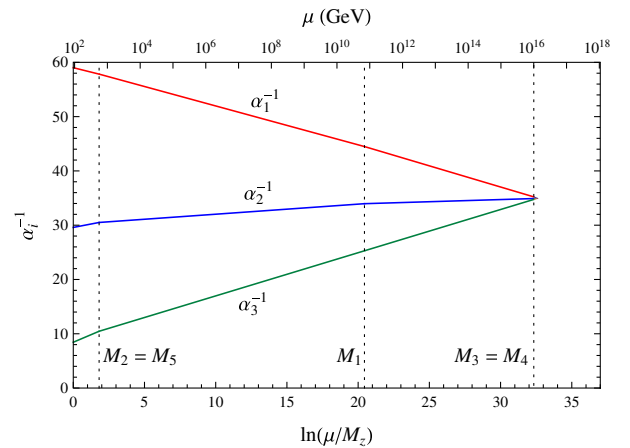

FIG. 5 (color online).   Running of the gauge couplings at one-loop level for the set S7 given in Table VI, which corresponds to the addition of two complete SM generations. A successful unification of the couplings is achieved at the scale $\Lambda = 1.3 \times 10^{16}$ GeV. We assume $M_{\Sigma_3} = M_{\Sigma_8} = \Lambda$.

TABLE VII. As in Table VI, for the sets of solutions that lead to gauge and gravitational coupling unification at the string scale.

Set	$\Lambda[10^{17}$ GeV]	Intermediate mass scales [GeV]					
		Representation	Minimum	Maximum	Representation	Minimum	Maximum
S10	[3.5, 5.3]	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	M_Z	5.0×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$	M_Z	4.6×10^{17}
		$(\mathbf{8},\mathbf{1})_{-1}$	2.9×10^7	4.5×10^{17}	$(\mathbf{3},\mathbf{3})_{-1/3}$	3.5×10^3	2.4×10^{12}
		$(\bar{\mathbf{3}},\mathbf{1})_{4/3}$	9.8×10^7	5.2×10^{17}	$(\mathbf{8},\mathbf{2})_{1/2}$	M_Z	1.1×10^8
S11	[3.6, 5.3]	$(\mathbf{3},\mathbf{1})_{2/3}$	M_Z	4.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	M_Z	8.8×10^{16}
		$(\mathbf{3},\mathbf{3})_{2/3}$	9.6×10^7	3.0×10^{14}	$(\mathbf{8},\mathbf{1})_1$	1.3×10^{11}	4.4×10^{17}
		$(\bar{\mathbf{3}},\mathbf{2})_{-7/6}$	5.7×10^{12}	5.0×10^{17}	$(\mathbf{8},\mathbf{2})_{-1/2}$	M_Z	1.1×10^8
S13	[3.3, 5.3]	$(\bar{\mathbf{6}},\mathbf{1})_{2/3}$	7.9×10^3	5.2×10^{17}	$(\mathbf{8},\mathbf{1})_1$	7.7×10^9	5.2×10^{17}
		$(\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	3.1×10^6	5.2×10^{17}	$(\mathbf{8},\mathbf{2})_{-1/2}$	8.3×10^{12}	5.2×10^{17}
		$(\mathbf{6},\mathbf{3})_{1/3}$	4.8×10^5	1.3×10^{14}	$(\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	3.3×10^6	5.0×10^{17}
S14	[3.2, 5.3]	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	M_Z	5.0×10^{17}	$(\mathbf{3},\mathbf{3})_{2/3}$	7.6×10^{10}	5.2×10^{17}
		$(\mathbf{6},\mathbf{2})_{-1/6}$	M_Z	3.3×10^{15}	$(\bar{\mathbf{3}},\mathbf{2})_{-7/6}$	5.2×10^9	5.1×10^{17}
		$(\bar{\mathbf{6}},\mathbf{3})_{-1/3}$	2.3×10^7	4.3×10^{17}	$(\mathbf{6},\mathbf{1})_{4/3}$	5.8×10^5	5.2×10^{17}
S15	[3.2, 3.3]	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$	1.3×10^7	3.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$	1.7×10^{11}	3.2×10^{17}
		$(\mathbf{3},\mathbf{2})_{1/6}$	M_Z	8.4×10^5	$(\bar{\mathbf{6}},\mathbf{1})_{2/3}$	1.0×10^{14}	3.2×10^{17}
		$(\mathbf{6},\mathbf{2})_{-1/6}$	1.8×10^{11}	5.2×10^{16}	$(\bar{\mathbf{6}},\mathbf{1})_{-1/3}$	7.8×10^{12}	3.2×10^{17}
S16	[3.2, 5.3]	$(\bar{\mathbf{3}},\mathbf{2})_{-1/6}$	M_Z	5.2×10^{17}	$(\bar{\mathbf{6}},\mathbf{2})_{1/6}$	M_Z	3.6×10^{12}
		$(\mathbf{3},\mathbf{3})_{-1/3}$	1.7×10^{11}	5.3×10^{17}	$(\bar{\mathbf{3}},\mathbf{1})_{4/3}$	8.5×10^4	5.2×10^{17}
		$(\mathbf{6},\mathbf{3})_{1/3}$	1.8×10^8	5.2×10^{17}	$(\bar{\mathbf{6}},\mathbf{1})_{-4/3}$	1.1×10^5	5.2×10^{17}

$$\bar{5} \oplus \overline{40} \oplus 50,$$
$$10 \oplus 40 \oplus \overline{50},$$
$$15 \oplus \overline{45} \oplus 50,$$
$$\bar{5} \oplus \overline{15} \oplus \overline{40} \oplus 45,$$
$$10 \oplus 15 \oplus 40 \oplus \overline{45}, \tag{37}$$

which contain at most three or four complete $\mathsf{SU}(5)$ representations. It is interesting to note that the last anomaly-free chiral set contains, in a nontrivial way, one generation of the SM fermions (labeled 1–5 in Table IV).

Concerning gauge coupling unification, it is clear that, unless one allows for mass splittings inside each $\mathsf{SU}(5)$ multiplet, none of the solutions in Eq. (37) would lead to unification. Indeed, the contribution of the new particles [belonging to a complete $\mathsf{SU}(5)$ multiplet] to the beta coefficients at one loop in Eqs. (12)–(15) is the same for each SM subgroup. Therefore, they do not affect the unification condition of Eq. (13). On the other hand, if

one allows for different intermediate scales inside each $\mathsf{SU}(5)$ multiplet, one can then obtain sets of solutions that can account for unification but they are not as minimal as those given in Table VI.

IV. CONCLUSIONS

In this work we have searched for minimal chiral sets of fermions beyond the SM that are anomaly free and, simultaneously, vectorlike particles with respect to color $\mathsf{SU}(3)$ and electromagnetic $\mathsf{U}(1)$. We have studied whether the addition of such particles allows for the unification of gauge couplings at a high energy scale. The possibility to have unification at the string scale has also been considered. We have looked for minimal solutions with chiral fermion sets of arbitrary quantum numbers, which do no fit in standard GUT groups, as well as for those that belong to $\mathsf{SU}(5)$ representations with dimensions less than or equal to 50. In both cases, we have verified that some of the anomaly-free sets can unify at a unification scale above 5×10^{15} GeV and could also lead to the unification of gauge and gravitational couplings at the string scale around 3×10^{17} GeV. Our results are summarized in Figs. 1–4, and Tables III, VI, and VII.

In our framework, we have only considered one-loop corrections to the running of the gauge couplings, allowing the extra fermions to decouple from the theory at arbitrary intermediate scales. A more comprehensive analysis would

TABLE VIII. The cubic index A_5 for irreducible representations of $\mathsf{SU}(5)$ with $d_5(R) \le 50$.

$\mathsf{SU}(5)$-irrep	5	10	15	24	35	40	45	50
A_5	1	1	9	0	-44	-16	-6	-15

require us to include higher loop RGE corrections. Our results show that adding only a minimal chiral content to SM and requiring gauge unification enforces that some of the new particles should decouple from the theory at intermediate scales much larger than the electroweak scale. Since electroweak precision data severely constrain additional scalars charged under the SM group, giving large masses to the new fermions through such scalars seems unrealistic in the present context. Therefore, the problem of generating the required intermediate fermion masses through an alternative mechanism deserves further study. An interesting possibility is to implement a dynamical mechanism by extending the theory with mirror fermions, analogous to that developed in Ref. [27].

Finally, it is worth emphasizing that our search for gauge coupling unification has been focused on nonsupersymmetric scenarios. Supersymmetry commonly arises in the context of string theories; yet it may happen that it is broken at a very high energy scale. In fact, different paths to string-scale unification can be envisaged [1]. Supersymmetry may also be required to maintain the stability of the relevant mass scales, namely, the string and/or gauge unification scale, the intermediate scales for the masses of the new fermions, and the electroweak scale. For instance, invoking low-energy supersymmetry helps in solving the hierarchy problem associated to the presence of quadratic divergences in the Higgs mass. So far, supersymmetry has not been observed at the energy scales that are accessible to present collider experiments. In view of this, nonsupersymmetric extensions of the SM remain plausible alternatives that are worth being investigated.

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