Monopoles, Dirac operator, and index theory for fuzzy $SU(3)/(U(1)\times U(1))$

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The intersection of the ten-dimensional fuzzy conifold Y_F^{10} with $S_F^5\times S_F^5$ is the compact eight-dimensional fuzzy space X_F^8. We show that X_F^8 is (the analogue of) a principal $U(1)\times U(1)$ bundle over fuzzy $SU(3)/(U(1)\times U(1))(\equiv\mathcal{M}_F^6)$. We construct $\mathcal{M}_F^6$ using the Gell-Mann matrices by adapting Schwinger's construction. The space $\mathcal{M}_F^6$ is of relevance in higher dimensional quantum Hall effect and matrix models of D-branes. Further we show that the sections of the monopole bundle can be expressed in the basis of $SU(3)$ eigenvectors. We construct the Dirac operator on $\mathcal{M}_F^6$ from the Ginsparg-Wilson algebra on this space. Finally, we show that the index of the Dirac operator correctly reproduces the known results in the continuum.

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I. INTRODUCTION

Fuzzy spaces emerge naturally in the discussion of various theories like quantum Hall effect (QHE) and matrix models of D-branes in the presence of certain background fields. In the context of QHE in two dimensions, the Hilbert space of the lowest Landau level corresponds to symmetric representations of $SU(2)$. The observables for the lowest Landau level then correspond to observables of S_F^2 (for example, see [1–3]).

Higher dimensional ($d>2$) generalizations of QHE are interesting for various reasons. For example, they extend the notion of incompressibility to higher dimensions. In QHE in $d>2$, the Landau problem is replaced by a particle moving on a compact coset space in the presence of background gauge fields (say, monopoles). One such coset space is $SU(3)/(U(1)\times U(1))$. The intersection of the ten-dimensional conifold $Y^{10}(\equiv\{z_\alpha, w_\beta\colon\ z_\alpha w_\alpha=0, \alpha,\beta=1,2,3\})$ with $S^5\times S^5$ is a $U(1)\times U(1)$ monopole bundle on this coset space. For a particle moving on this coset space in the presence of background $U(1)$ monopoles, the Hilbert space of the lowest Landau level has an exact correspondence with the representations of $SU(3)$ (for instance, see [4]). Consequently, the observables for the lowest Landau level are observables of fuzzy $SU(3)/(U(1)\times U(1))(\equiv\mathcal{M}_F^6)$. Thus in the presence of background fields, the natural description of the space becomes fuzzy and the emergent compact fuzzy space like $\mathcal{M}_F^6$ and the monopoles on it are relevant in the understanding of such QHE.

$\mathcal{M}_F^6$ is described by matrix algebras on the carrier space of the representations of $SU(3)$. This space appears in the study of matrix models describing branes. The low energy effective action of a N (coincident) D-brane system is that of a $U(N)$ Yang-Mills theory. It is well known that the corresponding transverse geometry is inherently noncommutative [5–9]. Owing to the non-Abelian nature of such theories, the N Dp-brane system couples to Ramond-Ramond (RR) field strengths of degree $\geq p+4$ [10,11]. In particular, when the RR background is chosen to be proportional to the $SU(3)$ structure constants, an eight-matrix model has the action

$$S=T_0\mathrm{Tr}\left[\frac{1}{2}\dot{\phi}_i{}^2+\frac{1}{4}[\phi_i,\phi_j]^2-\frac{i}{3}\kappa f_{ijk}\phi_i[\phi_j,\phi_k]\right]. \quad (1)$$

Here, $i=1,2,\ldots,8$, κ is a coupling constant, ϕ_i's are $N\times N$ matrices and f_{ijk}'s are the $SU(3)$ structure constants. This describes N coincident Dp-branes (with $p\leq 1$). One of the ground state configurations of the above action is $\mathcal{M}_F^6$ [12].

$SU(3)/(U(1)\times U(1))$ is of particular relevance in string theory. For instance, a seven-dimensional space with G_2 holonomy can have a conical singularity on this space [13]. Understanding how this holonomy appears in the fuzzy case is an interesting question by itself. We leave it for a separate investigation in the future. Here we focus primarily on the construction of $\mathcal{M}_F^6$.

To discuss higher dimensional QHE, we should introduce fermions (electrons) on $\mathcal{M}_F^6$. To this end, we construct the Dirac operator on $\mathcal{M}_F^6$. The Dirac operator is also necessary to discuss supersymmetry on this fuzzy space, which is of interest to many. In a separate context, since this is a finite dimensional model, it is important to study the fermion-doubling problem with the Dirac operator. The construction of the Dirac operator on a fuzzy coset space like $\mathcal{M}_F^6$ is nontrivial and intrinsically interesting. A beautiful (expected) relation exists between the index of the Dirac operator and the topological objects on $\mathcal{M}_F^6$, which we make explicit.

Our construction of $\mathcal{M}_F^6$ derives from a Schwinger-like construction using the Gell-Mann matrices and six

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independent oscillators. With these six oscillators, the ten-dimensional fuzzy conifold Y_F^{10} can be constructed as in [14,15]. $X_F^8 = Y_F^{10} \cap (S_F^5 \times S_F^5)$ describes a subspace of this six-dimensional oscillator's Hilbert space, which is the carrier space of all the representations of $SU(3)$. In Sec. II, we show that there exists a Hopf-like map $X_F^8 \to \mathcal{M}_F^6$.

X_F^8 is a $U(1) \times U(1)$ principal bundle over $\mathcal{M}_F^6$. The monopoles can be characterized by linear maps from one representation space of $SU(3)$ to another [14–18]. Such sections are rectangular matrices that map a $\mathcal{M}_F^6$ of a given size to another of a different size. In Sec. III, we construct these matrices in the basis of the $SU(3)$ D-matrices.

In Sec. IV, we show that a Ginsparg-Wilson (GW) algebra is associated with $\mathcal{M}_F^6$. The Dirac operator can be constructed using the elements of this GW algebra, which are functions of the generators of $SU(3)$. We compute the index of the Dirac operator using the representations of $SU(3)$ and their quadratic Casimir values as in [19]. The index is equal to $\mathrm{Tr}(F \wedge F \wedge F)$.

II. CONSTRUCTION OF FUZZY $SU(3)/(U(1) \times U(1))$

$\mathbb{C}_F^6$ is described by six independent oscillators $\hat{a}_\alpha, \hat{b}_\alpha$ ($\alpha = 1, 2, 3$),

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \qquad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta},$$
$$[\hat{b}_\alpha, \hat{b}_\beta] = 0, \qquad [\hat{b}_\alpha, \hat{b}_\beta^\dagger] = \delta_{\alpha\beta},$$
$$[\hat{a}_\alpha, \hat{b}_\beta] = 0, \qquad [\hat{a}_\alpha, \hat{b}_\beta^\dagger] = 0. \tag{2}$$

These oscillators act on the Hilbert space $\mathcal{F}$ spanned by the eigenstates of the number operators $\hat{N}_a (\equiv \hat{a}_\alpha^\dagger \hat{a}_\alpha)$ and $\hat{N}_b (\equiv \hat{b}_\alpha^\dagger \hat{b}_\alpha)$,

$$\mathcal{F} \equiv \mathrm{span}\{|n_a^1, n_a^2, n_a^3; n_b^1 n_b^2, n_b^3\rangle : \ n_a^\alpha, n_b^\alpha = 0, 1, 2, \ldots\}. \tag{3}$$

A fuzzy conifold is described by a subalgebra in $\mathbb{C}_F^6$ [14,15]. We define the operator

$$\hat{\mathcal{O}} \equiv \sum_{\alpha=1}^{3} \hat{b}_\alpha \hat{a}_\alpha, \tag{4}$$

which has as its kernel

$$\ker(\hat{\mathcal{O}}) = \mathrm{span}\{|\cdot\rangle \in \mathcal{F} : \ \hat{\mathcal{O}}|\cdot\rangle = 0\} \subset \mathcal{F}. \tag{5}$$

(We use the symbol $|\cdot\rangle$ to denote the state $|n_a^1, n_a^2, n_a^3; n_b^1 n_b^2, n_b^3\rangle$.) The algebra of $(\hat{a}_\alpha, \hat{b}_\beta)$'s restricted to $\ker(\hat{\mathcal{O}})$ describes a ten-dimensional fuzzy conifold Y_F^{10}.

For convenience of normalization, we will also work with the set of operators

$$\hat{\chi}_\alpha \equiv \hat{a}_\alpha \frac{1}{\sqrt{N_a}}, \qquad \hat{\xi}_\alpha \equiv \hat{b}_\alpha \frac{1}{\sqrt{N_b}}, \tag{6}$$

which satisfy

$$\hat{\chi}_\alpha^\dagger \hat{\chi}_\alpha = 1, \qquad \hat{\xi}_\alpha^\dagger \hat{\xi}_\alpha = 1. \tag{7}$$

The $\hat{\chi}_\alpha$'s (or $\hat{\xi}_\alpha$'s) are well defined if we exclude the states for which $\hat{N}_a = 0$ (or $\hat{N}_b = 0$). Then the algebra generated by $\hat{\chi}_\alpha$'s and $\hat{\xi}_\alpha$'s describes $S_F^5 \times S_F^5$.

The operator $\hat{\mathcal{O}}' \equiv \hat{\xi}_\alpha \hat{\chi}_\alpha$ also vanishes in $\ker(\hat{\mathcal{O}})$. Therefore the algebra of the $\hat{\chi}_\alpha$'s and $\hat{\xi}_\alpha$'s restricted to $\ker(\hat{\mathcal{O}})$ describes an eight-dimensional fuzzy space X_F^8. Informally, we can think of X_F^8 as $Y_F^{10} \cap (S_F^5 \times S_F^5)$.

A. $X_F^8 \to \mathcal{M}_F^6 \equiv$ fuzzy $SU(3)/(U(1) \times U(1))$

Using the matrices $T_i = \frac{1}{2}\lambda_i$ ($\lambda_i =$ Gell-Mann matrices, $i = 1, 2, \ldots, 8$) satisfying

$$[T_i, T_j] = i f_{ijk} T_k, \qquad \{T_i, T_j\} = \frac{1}{3}\delta_{ij} + d_{ijk} T_k, \tag{8}$$

we can define a Schwinger-like construction (similar to [20]),

$$\hat{y}_i = \hat{a}_\alpha^\dagger (T_i)^{\alpha\beta} \hat{a}_\beta - \hat{b}_\alpha (T_i)^{\alpha\beta} \hat{b}_\beta^\dagger, \tag{9}$$

$$\hat{s}_i = \hat{\chi}_\alpha^\dagger (T_i)^{\alpha\beta} \hat{\chi}_\beta - \hat{\xi}_\alpha (T_i)^{\alpha\beta} \hat{\xi}_\beta^\dagger. \tag{10}$$

The $\hat{y}_i$'s obey

$$\hat{y}_i^\dagger = \hat{y}_i, \qquad [\hat{y}_i, \hat{y}_j] = i f_{ijk} \hat{y}_k, \tag{11}$$

and the Casimirs are $\hat{C}_2 \equiv \hat{y}_i \hat{y}_i$ and $\hat{C}_3 \equiv d_{ijk} \hat{y}_i \hat{y}_j \hat{y}_k$,

$$\hat{C}_2 = \frac{1}{3}[\hat{N}_a^2 + \hat{N}_b^2 + \hat{N}_a \hat{N}_b + 3(\hat{N}_a + \hat{N}_b)] - \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}}, \tag{12}$$

$$\hat{C}_3 = \frac{1}{18}(\hat{N}_a - \hat{N}_b)(2\hat{N}_a + \hat{N}_b + 3)(\hat{N}_a + 2\hat{N}_b + 3) + \frac{\hat{N}_a - \hat{N}_b}{2} \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}}. \tag{13}$$

The Hilbert space $\mathcal{F}$ can be split into the subspaces $\mathcal{F}_{n_a, n_b}$,

$$\mathcal{F}_{n_a, n_b} \equiv \mathrm{span}\left\{|\cdot\rangle : \ \sum_\alpha n_a^\alpha = n_a, \sum_\alpha n_b^\alpha = n_b\right\}. \tag{14}$$

$\mathcal{F} = \oplus \mathcal{F}_{n_a, n_b}$ where the direct sum $\oplus$ is over n_a and n_b. In $\tilde{\mathcal{F}}_{n_a, n_b} \equiv \mathcal{F}_{n_a, n_b} \cap \ker(\hat{\mathcal{O}})$, the Casimirs take fixed values,

$$\hat{C}_2|_{\tilde{\mathcal{F}}_{n_a, n_b}} = c_2 \mathbb{I}, \qquad \hat{C}_3|_{\tilde{\mathcal{F}}_{n_a, n_b}} = c_3 \mathbb{I}, \tag{15}$$

with

$$c_2 = \frac{1}{3}[n_b^2 + n_a^2 + n_b n_a + 3(n_a + n_b)] = \text{fixed},$$
$$c_3 = \frac{1}{18}(n_a - n_b)(2n_a + n_b + 3)(n_a + 2n_b + 3) = \text{fixed}. \tag{16}$$

Also in $\tilde{\mathcal{F}}_{n_a,n_b}$,

$$\hat{s}_i\hat{s}_i = \text{fixed}, \qquad d_{ijk}\hat{s}_i\hat{s}_j\hat{s}_k = \text{fixed}. \tag{17}$$

Therefore, the algebra of $\hat{s}_i$'s restricted to $\tilde{\mathcal{F}}_{n_a,n_b}$ describes a six-dimensional fuzzy space $\mathcal{M}_F^6$.

Each $\tilde{\mathcal{F}}_{n_a,n_b}$ is a carrier space of a finite dimensional irrep of $SU(3)$. This representation is characterized by a pair of positive integers $(p,q) = (n_a, n_b)$[1] and is of dimension

$$\dim_{SU(3)} = \frac{1}{2}(n_b + n_a + 2)(n_b + 1)(n_a + 1). \tag{18}$$

The $\hat{y}_i$'s (and $\hat{s}_i$'s) are square matrices in $\tilde{\mathcal{F}}_{n_a,n_b}$. Thus $\mathcal{M}_F^6$ is the fuzzy version of $SU(3)/(U(1)\times U(1))$ and (9) is a noncommutative $U(1)\times U(1)$ fibration.

Note that in the above construction neither n_a nor n_b can be chosen to be zero. In case the construction is done with only one set of oscillators (either $\hat{a}_\alpha$'s or $\hat{b}_\alpha$'s), one would get fuzzy $\mathbb{C}P^2$, as in [11,21]. Nevertheless, for $n_a \gg n_b$ (or $n_b \gg n_a$), $\mathcal{M}_F^6$ looks like fuzzy $\mathbb{C}P^2$ in some sense [11].

III. NONCOMMUTATIVE LINE BUNDLE

Let $\mathcal{H}_{n_a,n_b\to l_a,l_b}$ be the space of linear operators Φ, which map $\tilde{\mathcal{F}}_{n_a,n_b}$ to $\tilde{\mathcal{F}}_{l_a,l_b}$,

$$\Phi\colon\ \tilde{\mathcal{F}}_{n_a,n_b} \to \tilde{\mathcal{F}}_{l_a,l_b}, \qquad \Phi \in \mathcal{H}_{n_a,n_b\to l_a,l_b}. \tag{19}$$

In general, these Φ's are rectangular matrices.

$\mathcal{H}_{n_a,n_b\to n_a,n_b}$ is a noncommutative algebra which maps $\tilde{\mathcal{F}}_{n_a,n_b} \to \tilde{\mathcal{F}}_{n_a,n_b}$ and any $\Phi \in \mathcal{H}_{n_a,n_b\to n_a,n_b}$ is a square matrix. In this algebra, the rotations are generated by the adjoint action of $\hat{y}_i^{(n_a,n_b)}$ ($\hat{y}_i^{(n_a,n_b)}$ is the restriction of $\hat{y}_i$ in $\tilde{\mathcal{F}}_{n_a,n_b}$),

$$Ad(\hat{y}_i^{(n_a,n_b)})\Phi \equiv \hat{\mathcal{L}}_i\Phi \equiv [\hat{y}_i^{(n_a,n_b)}, \Phi]. \tag{20}$$

$\hat{\mathcal{L}}_i$'s generate a $SU(3)$,

$$[\hat{\mathcal{L}}_i, \hat{\mathcal{L}}_j] = i f_{ijk}\hat{\mathcal{L}}_k. \tag{21}$$

When $n_a \neq l_a, n_b \neq l_b$, the spaces $\mathcal{H}_{n_a,n_b\to l_a,l_b}$ are noncommutative bimodules and any $\Phi \in \mathcal{H}_{n_a,n_b\to l_a,l_b}$ is a rectangular matrix. For the bimodules, the generators of the $SU(3)$ in (21) act by a left and a right multiplication,

$$\hat{\mathcal{L}}_i\Phi = \hat{y}_i^{(l_a,l_b)}\Phi - \Phi\hat{y}_i^{(n_a,n_b)}. \tag{22}$$

This $SU(3)$ action is reducible, and we will give its explicit decomposition shortly. Any element $\Phi \in \mathcal{H}_{n_a,n_b\to l_a,l_b}$ can be expanded in the basis of the eigenvectors of $\hat{\mathcal{L}}_3$, $\hat{\mathcal{L}}_8$, $\hat{\mathcal{L}}_i\hat{\mathcal{L}}_i$ and $d_{ijk}\hat{\mathcal{L}}_i\hat{\mathcal{L}}_j\hat{\mathcal{L}}_k$.

To construct the basis vectors, let us start as follows. The operator

$$\hat{f} = (\hat{a}_3^\dagger)^{\tilde{l}_a}(\hat{a}_2)^{\tilde{n}_a}(\hat{b}_2^\dagger)^{\tilde{l}_b}(\hat{b}_3)^{\tilde{n}_b} \tag{23}$$

is an element of $\mathcal{H}_{n_a,n_b\to l_a,l_b}$ if $(\tilde{l}_a, \tilde{n}_a, \tilde{l}_b, \tilde{n}_b)$ are positive integers satisfying

$$\kappa_a \equiv l_a - n_a = \tilde{l}_a - \tilde{n}_a, \quad \kappa_b \equiv l_b - n_b = \tilde{l}_b - \tilde{n}_b. \tag{24}$$

It is easy to see that

$$\begin{aligned}
\hat{U}_+\hat{f} &\equiv (\hat{\mathcal{L}}_1 + i\hat{\mathcal{L}}_2)\hat{f} = 0,\\
\hat{V}_+\hat{f} &\equiv (\hat{\mathcal{L}}_4 - i\hat{\mathcal{L}}_5)\hat{f} = 0,\\
\hat{W}_+\hat{f} &\equiv (\hat{\mathcal{L}}_6 - i\hat{\mathcal{L}}_7)\hat{f} = 0,\\
\hat{\mathcal{L}}_3\hat{f} &= \frac{1}{2}(\tilde{n}_a + \tilde{l}_b)\hat{f},\\
\hat{\mathcal{L}}_8\hat{f} &= -\frac{1}{2\sqrt{3}}(2\tilde{l}_a + 2\tilde{n}_b + \tilde{n}_a + \tilde{l}_b)\hat{f}.
\end{aligned} \tag{25}$$

So $\hat{f}$ is the highest weight vector of the $SU(3)$ representation characterized by (p,q), with

$$p = \tilde{l}_a + \tilde{l}_b + \tilde{n}_a + \tilde{n}_b, \quad q = \tilde{l}_a + \tilde{n}_b, \quad p \geq q \geq 0. \tag{26}$$

The quadratic and the cubic Casimirs for this representation take values

$$\mathcal{C}_2 = \frac{1}{3}(p^2 + q^2 - pq + 3p), \tag{27}$$

$$\mathcal{C}_3 = \frac{1}{18}(p - 2q)(2p - q + 3)(q + p + 3), \tag{28}$$

while the dimension is

$$d = \frac{1}{2}(p - q + 1)(p + 2)(q + 1). \tag{29}$$

The lower weight vectors belonging to the same irrep (p,q) are generated by the action of the lowering operators $\hat{U}_-(\equiv \hat{\mathcal{L}}_1 - i\hat{\mathcal{L}}_2)$, $\hat{V}_-(\equiv \hat{\mathcal{L}}_4 + i\hat{\mathcal{L}}_4)$ and $\hat{W}_-(\equiv \hat{\mathcal{L}}_6 + i\hat{\mathcal{L}}_7)$ on $\hat{f}$.

A generic vector belonging to the (p,q) irrep is labeled by m_3 and m_8—the $\hat{\mathcal{L}}_3$ and $\hat{\mathcal{L}}_8$ values, respectively,

[1]The Casimirs of a (p,q) representation of $SU(3)$ are

$$\hat{C}_2 = \frac{1}{3}[p^2 + q^2 + pq + 3(p+q)]\mathbb{I},$$

$$\hat{C}_3 = \frac{1}{18}(p-q)(2p+q+3)(p+2q+3)\mathbb{I}.$$

TABLE I. Allowed values of (p,q) for $\kappa_a, \kappa_b \geq 0$.

$\tilde{n}_a$	$\tilde{l}_a$	$\tilde{n}_b$	$\tilde{l}_b$	p	q
0	κ_a	0	κ_b	$\kappa_a + \kappa_b$	κ_a
1	$\kappa_a + 1$	0	κ_b	$\kappa_a + \kappa_b + 2$	$\kappa_a + 1$
0	κ_a	1	$\kappa_b + 1$	$\kappa_a + \kappa_b + 2$	$\kappa_a + 1$
...	...	...	...	...	...
n_a	l_a	n_b	l_b	$J_a + J_b$	$l_a + n_b$

$$\hat{\mathcal{L}}_3 \Psi_{p,q}^{m_3,m_8} = m_3 \Psi_{p,q}^{m_3,m_8},$$
$$\hat{\mathcal{L}}_8 \Psi_{p,q}^{m_3,m_8} = m_8 \Psi_{p,q}^{m_3,m_8},$$
$$\hat{\mathcal{L}}_i \hat{\mathcal{L}}_i \Psi_{p,q}^{m_3,m_8} = \mathcal{C}_2 \Psi_{p,q}^{m_3,m_8},$$
$$d_{ijk} \hat{\mathcal{L}}_i \hat{\mathcal{L}}_j \hat{\mathcal{L}}_k \Psi_{p,q}^{m_3,m_8} = \mathcal{C}_3 \Psi_{p,q}^{m_3,m_8}. \quad (30)$$

In the following, we specify the allowed values of (p,q) (i.e. which irreps occur). In (24), κ_a and κ_b can be both positive and negative. The ranges of the pairs $(\tilde{l}_a, \tilde{n}_a)$ and $(\tilde{l}_b, \tilde{n}_b)$ are different for each choice of the sign of κ_a and κ_b. Consequently, the irreps appearing in such maps also differ. We find the irreps for each case.

Case 1: $l_a \geq n_a$ and $l_b \geq n_b$

In this case, both $\kappa_a, \kappa_b \geq 0$. The ranges of $\tilde{n}_a$ and $\tilde{n}_b$ are

$$0 \leq \tilde{n}_a \leq n_a, \qquad 0 \leq \tilde{n}_b \leq n_b. \quad (31)$$

Therefore the allowed values of p and q are shown in Table I, where $J_a \equiv l_a + n_a$ and $J_b \equiv l_b + n_b$.

Case 2: $l_a \leq n_a$ and $l_b \geq n_b$

Here, $\kappa_a \leq 0$ and $\kappa_b \geq 0$ and

$$0 \leq \tilde{l}_a \leq l_a, \qquad 0 \leq \tilde{n}_b \leq n_b. \quad (32)$$

Hence p and q take the values shown in Table II.

Case 3: $l_a \geq n_a$ and $l_b \leq n_b$

TABLE II. Allowed values of (p,q) for $\kappa_a \leq 0, \kappa_b \geq 0$.

$\tilde{l}_a$	$\tilde{n}_a$	$\tilde{n}_b$	$\tilde{l}_b$	p	q
0	$-\kappa_a$	0	κ_b	$-\kappa_a + \kappa_b$	0
1	$-\kappa_a + 1$	0	κ_b	$-\kappa_a + \kappa_b + 2$	1
0	$-\kappa_a$	1	$\kappa_b + 1$	$-\kappa_a + \kappa_b + 2$	1
...	...	...	...	...	...
l_a	n_a	n_b	l_b	$J_a + J_b$	$l_a + n_b$

TABLE III. Allowed values of (p,q) for $\kappa_a \geq 0, \kappa_b \leq 0$.

$\tilde{n}_a$	$\tilde{l}_a$	$\tilde{l}_b$	$\tilde{n}_b$	p	q
0	κ_a	0	$-\kappa_b$	$\kappa_a - \kappa_b$	$\kappa_a - \kappa_b$
1	$\kappa_a + 1$	0	$-\kappa_b$	$\kappa_a - \kappa_b + 2$	$\kappa_a - \kappa_b + 1$
0	κ_a	1	$-\kappa_b + 1$	$\kappa_a - \kappa_b + 2$	$\kappa_a - \kappa_b + 1$
...	...	...	...	...	...
n_a	l_a	l_b	n_b	$J_a + J_b$	$l_a + n_b$

TABLE IV. Allowed values of (p,q) for $\kappa_a, \kappa_b \leq 0$.

$\tilde{l}_a$	$\tilde{n}_a$	$\tilde{l}_b$	$\tilde{n}_b$	p	q
0	$-\kappa_a$	0	$-\kappa_b$	$-(\kappa_a + \kappa_b)$	$-\kappa_b$
1	$-\kappa_a + 1$	0	$-\kappa_b$	$-(\kappa_a + \kappa_b) + 2$	$-\kappa_b + 1$
0	$-\kappa_a$	1	$-\kappa_b + 1$	$-(\kappa_a + \kappa_b) + 2$	$-\kappa_b + 1$
...	...	...	...	...	...
l_a	n_a	l_b	n_b	$J_a + J_b$	$l_a + n_b$

When $\kappa_a \geq 0$ and $\kappa_b \leq 0$, the ranges of $\tilde{n}_a$ and $\tilde{l}_b$ are

$$0 \leq \tilde{n}_a \leq n_a, \qquad 0 \leq \tilde{l}_b \leq l_b. \quad (33)$$

Then the allowed values of p and q are as shown in Table III.

Case 4: $l_a \leq n_a$ and $l_b \leq n_b$

In this case, $\kappa_a \leq 0$ and $\kappa_b \leq 0$ and

$$0 \leq \tilde{l}_a \leq l_a, \qquad 0 \leq \tilde{l}_b \leq l_b. \quad (34)$$

Thus the irreps have (p,q) values as shown in Table IV.

Any arbitrary element $\Phi \in \mathcal{H}_{n_a,n_b \to l_a,l_b}$ can be expressed in terms of the $SU(3)$ harmonics as

$$\Phi = \sum_{m_3,m_8,p,q} c_{p,q}^{m_3,m_8} \Psi_{p,q}^{m_3,m_8}, \qquad c_{p,q}^{m_3,m_8} \in \mathbb{C}. \quad (35)$$

These Φ's are identified as the noncommutative analogue of the sections of the associated line bundle.

A. Topological charge

The sections of the associated line bundle carry two topological charges, corresponding to each $U(1)$ fiber. In $\mathcal{H}_{n_a,n_b \to l_a,l_b}$, we can define two topological charge operators,

$$\hat{K}_a \equiv \frac{1}{2}[\hat{N}_a,], \qquad \hat{K}_b \equiv \frac{1}{2}[\hat{N}_b,]. \quad (36)$$

It is easy to see that Φ in (35) has topological charges (κ_a, κ_b) given by

$$\left.\begin{aligned} \hat{K}_a \Phi &= \tfrac{\tilde{l}_a - \tilde{n}_a}{2} \Phi = \tfrac{\kappa_a}{2} \Phi, \\ \hat{K}_b \Phi &= \tfrac{\tilde{l}_b - \tilde{n}_b}{2} \Phi = \tfrac{\kappa_b}{2} \Phi, \end{aligned}\right\} \quad \kappa_a, \kappa_b \in \mathbb{Z}. \quad (37)$$

Therefore Φ is a section of a complex line bundle with topological charges (κ_a, κ_b).

IV. THE DIRAC OPERATOR

$SU(3)/(U(1) \times U(1))$ is a six-dimensional space embedded in $\mathbb{R}^8$. This space is curved and nonsymmetric. Also, as $U(1) \times U(1) \subset \mathrm{Spin}(6) \cong SU(4)$, this space admits a spin structure. In the commutative case, the Dirac operator contains three terms: the kinetic term, the spin-connection term and the background monopole term (if any) [19].

There are many possible ways to obtain the Dirac operator on this space (for example, [22]). In the fuzzy case, we do so by constructing a Ginsparg-Wilson algebra on $\mathcal{M}_F^6$.

We look at the zero charge sector first. On $\mathcal{H}_{n_a,n_b\to n_a,n_b}$, $\hat{y}_i$ has left and right actions

$$\hat{y}_i^L f \equiv \hat{y}_i f, \qquad \hat{y}_i^R f \equiv f\hat{y}_i. \tag{38}$$

These satisfy (11)–(13),

$$\begin{aligned} [\hat{y}_i^L, \hat{y}_j^L] &= i f_{ijk}\hat{y}_k^L, \\ [\hat{y}_i^R, \hat{y}_j^R] &= -i f_{ijk}\hat{y}_k^R, \\ [\hat{y}_i^L, \hat{y}_j^R] &= 0, \\ \hat{y}_i\hat{y}_i \equiv \hat{y}_i^L\hat{y}_i^L &= \hat{y}_i^R\hat{y}_i^R = c_2\mathbb{I}, \\ d_{ijk}\hat{y}_i^L\hat{y}_j^L\hat{y}_k^L &= d_{ijk}\hat{y}_i^R\hat{y}_j^R\hat{y}_k^R = c_3\mathbb{I}. \end{aligned} \tag{39}$$

The γ matrices on $\mathbb{R}^8$ are 16×16 matrices which generate the Clifford algebra

$$\{\gamma_i, \gamma_i\} = 2\delta_{ij}, \qquad \gamma_i^\dagger = \gamma_i, \qquad i = 1, 2, \ldots, 8. \tag{40}$$

Using γ_i's we can construct

$$t_i \equiv \frac{1}{4i} f_{ijk}\gamma_j\gamma_k, \qquad [t_i, \gamma_j] = i f_{ijk}\gamma_k, \tag{41}$$

which generate a $SU(3)$,

$$[t_i, t_j] = i f_{ijk} t_k. \tag{42}$$

We can define

$$\Gamma \equiv \frac{1}{a}\gamma_i\left(\hat{y}_i^L + \frac{1}{3}t_i\right), \qquad \tilde{\Gamma} \equiv -\frac{1}{a}\gamma_i\left(\hat{y}_i^R - \frac{1}{3}t_i\right), \tag{43}$$

where the normalization a is given by

$$a^2\mathbb{I} = \hat{y}_i\hat{y}_i + \frac{1}{3}t_i t_i, \qquad t_i t_i = 3\mathbb{I}, \qquad \hat{y}_i\hat{y}_i = c_2\mathbb{I}. \tag{44}$$

Γ and $\tilde{\Gamma}$ generate a Ginsparg-Wilson algebra,

$$\mathcal{A}_{\mathrm{GW}} = \{\Gamma, \tilde{\Gamma}\colon\ \Gamma^2 = \mathbb{I} = \tilde{\Gamma}^2, \Gamma^\dagger = \Gamma, \tilde{\Gamma}^\dagger = \tilde{\Gamma}\}. \tag{45}$$

From this algebra, one can construct a Dirac operator $\mathcal{D}$

$$\mathcal{D} = a(\Gamma + \tilde{\Gamma}) = \gamma_i\hat{\mathcal{L}}_i + \frac{2}{3}\gamma_i t_i, \qquad \hat{\mathcal{L}}_i = \hat{y}_i^L + \hat{y}_i^R, \tag{46}$$

and a chirality operator Γ_{ch}

$$\Gamma_{\mathrm{ch}} = a(\Gamma - \tilde{\Gamma}) = \gamma_i(\hat{y}_i^L + \hat{y}_i^R). \tag{47}$$

It is easy to check that

$$\{\mathcal{D}, \Gamma_{ch}\} = 0, \tag{48}$$

$$\nu \equiv \mathrm{index}_{\mathcal{D}} = \mathrm{Tr}(\Gamma_{ch}) \quad \text{(index theorem)}. \tag{49}$$

It is interesting to note that in (46), the second term ($\sim\gamma_i t_i$) is the spin-connection term in [19].

The Dirac operator is of the form

$$\mathcal{D} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}. \tag{50}$$

When the action of $\mathcal{D}$ is restricted to the algebra $(\mathcal{H}_{n_a,n_b\to n_a,n_b} \times Mat_{16})$, it is the Dirac operator on $\mathcal{M}_F^6$. In this case, there is no monopole background, and hence the gauge field contribution to the Dirac operator is zero.

On the bimodule $\mathcal{H}_{n_a,n_b\to l_a,l_b}$ with $n_a \neq l_a, n_b \neq l_b$ or either, there is a background monopole. On this bimodule, $\hat{\mathcal{L}}_i$ is the covariant derivative which includes the monopole contribution. Hence, if we restrict $\mathcal{D}$ to the bimodule $(\mathcal{H}_{n_a,n_b\to l_a,l_b} \times Mat_{16})$, we automatically incorporate the background monopole information. There is no need to add the monopole term in the Dirac operator. Rather, restricting the algebra of $\mathcal{D}$ to the proper subspace accounts for monopoles.

A. Zero modes of the Dirac operator

$SU(3)/(U(1)\times U(1))$ is a nonsymmetric space with positive curvature. On this space, there is an additional connection due to the torsion, which appears in the square of the Dirac operator [19],

$$\begin{aligned} \mathcal{D}^2 = -\nabla^2 &+ \text{curvature} + \text{torsion} \\ &+ \text{possible gauge field contribution}. \end{aligned} \tag{51}$$

The Dirac Laplacian ∇^2 on this coset space is a positive operator. For the Dirac operator to have zero modes, we would need the cancellation of the lowest eigenvalue of the Laplacian with the lowest eigenvalue of the sum of the curvature, torsion and gauge field contributions. These considerations require that the number of zero modes of the Dirac operator on $SU(3)/(U(1)\times U(1))$ is given by the dimension of the $SU(3)$ irrep with the minimum value of the quadratic Casimir $\mathcal{C}_2$ [19]. We adopt the same requirement to compute the number of zero modes in the fuzzy case.

Case 1: $\kappa_a \geq 0$ and $\kappa_b \geq 0$

In this case $\mathcal{C}_2$ takes the minimum value in the representation with $(p, q) = (\kappa_a + \kappa_b, \kappa_a)$,

$$\mathcal{C}_2^{\min} = \frac{1}{3}(\kappa_a^2 + \kappa_b^2 + \kappa_a\kappa_b) + \kappa_a + \kappa_b. \tag{52}$$

The dimension of this representation is the index of $\mathcal{D}$,

$$\nu = d^{\min} = \frac{1}{2}(\kappa_a + \kappa_b + 2)(\kappa_a + 1)(\kappa_b + 1). \tag{53}$$

The discussion for the other cases is similar:

Case 2: $\kappa_a \leq 0$ *and* $\kappa_b \geq 0$

$$(p,q) = (-\kappa_a + \kappa_b, 0),$$
$$\mathcal{C}_2^{\min} = \frac{1}{3}(-\kappa_a + \kappa_b)^2 - \kappa_a + \kappa_b,$$
$$\nu = d^{\min} = \frac{1}{2}(-\kappa_a + \kappa_b + 2)(-\kappa_a + \kappa_b + 1). \qquad (54)$$

Case 3: $\kappa_a \geq 0$ *and* $\kappa_b \leq 0$

$$(p,q) = (\kappa_a - \kappa_b, \kappa_a - \kappa_b),$$
$$\mathcal{C}_2^{\min} = \frac{1}{3}(\kappa_a - \kappa_b)^2 + \kappa_a - \kappa_b,$$
$$\nu = d^{\min} = \frac{1}{2}(\kappa_a - \kappa_b + 2)(\kappa_a - \kappa_b + 1). \qquad (55)$$

Case 4: $\kappa_a \leq 0$ *and* $\kappa_b \leq 0$

$$(p,q) = (-\kappa_a - \kappa_b, -\kappa_b),$$
$$\mathcal{C}_2^{\min} = \frac{1}{3}(\kappa_a^2 + \kappa_b^2 + \kappa_a\kappa_b) - \kappa_a - \kappa_b,$$
$$\nu = d^{\min} = \frac{1}{2}(-\kappa_a - \kappa_b + 2)(-\kappa_a + 1)(-\kappa_b + 1). \qquad (56)$$

When there is no monopole, $\kappa_a = 0 = \kappa_b$ and the index is

$$\nu = 1, \qquad (57)$$

which is consistent with [19,21]. Also, as in the commutative case, the index ν gives $\mathrm{Tr}(F \wedge F \wedge F)$ for the monopole fields.

V. CONCLUSIONS

Our realization of $\mathcal{M}_F^6$ can be used to study large N limits of matrix models of D-branes. Among other things, $\mathcal{M}_F^6$ as the vacua of the matrix model (1) can be in an irreducible or reducible representation. It is easy to generalize the Schwinger construction of $\mathcal{M}_F^6$ by using Brandt-Greenberg oscillators and obtain reducible algebras of $\mathcal{M}_F^6$, as in [18]. We can obtain the quantum states for those reducible $\mathcal{M}_F^6$'s using the prescription of Gelfand-Naimark-Segal. Those quantum states will be inherently mixed and will carry entropy, which is typically large. This information will play a vital role in the understanding of the vacua and tachyon condensations in the matrix model.

Using the Dirac operator and the index theory, one may try to construct the spinor bundle on $\mathcal{M}_F^6$ and thus find the supersymmetric analogue of $X_F^8 \to \mathcal{M}_F^6$. The super conifold in the continuum has various interesting features [23] which should be manifest in the fuzzy version as well. We leave this for a future investigation.

The fermion-doubling problem on this finite dimensional space can also be studied. It has been shown that there is no fermion-doubling on the fuzzy sphere [24,25]. There might be such dramatic consequences on $\mathcal{M}_F^6$ as well.

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