Skewness dependence of generalized parton distributions, conformal OPE, and the AdS/CFT correspondence

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The traditional idea of the Pomeron/Reggeon description for hadron scattering is now being given theoretical foundation in gravity dual descriptions, where Pomeron corresponds to an exchange of spin-$j \in 2\mathbb{Z}$ states in the graviton trajectory. Deeply virtual Compton scattering (DVCS) is essentially a two-to-two scattering process of a hadron and a photon, and hence one should be able to study nonperturbative aspects (the generalized parton distribution [GPD]) of this process by the Pomeron/Reggeon process in gravity dual. We find, however, that even one of the most developed formulations of gravity dual, Pomeron [Brower-Polchinski-Strassler-Tan (BPST), 2006], is not able to capture skewness dependence of GPD properly. Therefore, we compute Reggeon wave functions on AdS_5 so that the formalism of BPST can be generalized. These wave functions are used to determine the DVCS amplitude, bring it to the form of conformal operator product expansion/collinear factorization, and extract a holographic model of GPD, which naturally fits into the framework known as "dual parametrization," or the "(conformal) collinear factorization approach."

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I. INTRODUCTION

Scattering processes of hadrons involve nonperturbative information of QCD. When it comes to scattering with the center of mass energy higher than the QCD scale, lattice computation will not have enough computation power in a near future, yet perturbative QCD is able to say something only about the hard components involved in the scattering. This is where holographic descriptions of strongly coupled gauge theories may find a role to play. Although we cannot expect gravitational "dual" descriptions to be both calculable and perfectly equivalent to the QCD of the real world at the same time, we still hope to be able to learn nonperturbative aspects of hadrons at the qualitative level, using calculable holographic dual descriptions of nearly conformal strongly coupled gauge theories.

String theory started out as the dual resonance model describing scattering amplitudes of hadrons. One of its major problems as a theory of hadrons was a "prediction" that the amplitude of the elastic scattering of two hadrons falls off exponentially, e^{Bt} in the momentum transfer squared t for some $B > 0$, although the amplitude is known in reality to fall off in a power law in $|-t|$ for hard scattering. The prediction, however, is now understood as that of string theory with a flat background metric; the amplitude of elastic scattering turns into such a power law indeed when the target space of string theory has a warped metric. At the qualitative level, string theory on a warped spacetime—holographic (gravitational dual) descriptions—can be a viable theory of hadron scattering [1–3].

The holographic technique can be used to study not just amplitudes of hadron scattering as a whole, but also to extract the information of partons within hadrons [2]. Parton distribution functions (PDFs) are defined by the inverse Mellin transformation of hadron matrix elements of gauge singlet parton-bilinear operators in QCD, and gravity dual descriptions can be used to determine matrix elements of the gauge singlet operators. The PDF extracted in this way satisfies Dokshitzer-Gribov-Lipatov-Altarelli-Parisi [(DGLAP), q^2-evolution] and Balitsky-Fadin-Kuraev-Lipatov [(BFKL), $\ln(1/x)$-evolution] equations (e.g., [4–7]); just like in perturbative QCD [8], these two evolution equations follow from how the saddle point j^* moves in the complex angular momentum j-plane integral (the inverse Mellin transform). The holographic description for the PDF and the generalized parton distribution (GPD) also shows crossover transition between this DGLAP/BFKL behavior and the Regge behavior [3] (see also [7]). Thus, the parton information studied in this way may be used to understand nonperturbative issues associated with partons in a hadron at qualitative level.

In this article, we study two-body–two-body scattering between a hadron and a photon (that is possibly virtual) in gravitational dual descriptions; $\gamma^*(q_1) + h(p_1) \to \gamma^{(*)}(q_2) + h(p_2)$. A special case of this scattering—the forward scattering with $q_1 = q_2$ and $p_1 = p_2$—has been studied extensively in the literature (e.g., [2,4–7]) for the study of deep inelastic scattering (DIS) and PDF, and some references also cover the case of nonforward elastic scattering

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$[(q_1)^2 = (q_2)^2,\ (q_1 - q_2)^2 \neq 0]$. This article extends the analysis so that all kinds of skewed ($q_1^2 \neq q_2^2$) cases are covered. In hadron physics, therefore, the kinematics needed for deeply virtual Compton scattering, hard exclusive vector meson production and timelike Compton scattering processes [9] is covered in this analysis. With the full skewness dependence included in this analysis, it is also possible to use the result of this study to bridge a gap between data in such scattering processes at nonzero skewness [10] and the transverse profile of partons in a hadron, which is encoded by GPD at zero skewness [11].

From a theoretical perspective, the task of this article is to generalize the formalism of [2,3] (see also [4,6,7]) so that it can be used for two-body-to-two-body scattering that is not necessarily elastic. Pomeron/Reggeon propagators have been treated as if they were for a scalar field in [2,3,7], but they correspond to an exchange of stringy states with nonzero (arbitrarily high) spins; for the study of scattering with nonzero skewness, the polarization of a higher spin state propagator should also be treated with care (see also the approach in [12,13]).

It is a notoriously difficult problem to compute scattering of strings on a curved background geometry. We do not pretend that the generalization of the formalism in this article is something derived from string theory without a flaw. This is rather an attempt at capturing an approximately correct picture of nonperturbative aspects in hadron scattering that string theory would predict in the distant future. We are forced to rely sometimes on physics intuition—and to ignore subtleties or corrections that are not under our control—when we face situations where not enough techniques have been developed in string theory at the moment.

This article is organized as follows. We begin in Sec. II A with a review of parametrization of GPD in terms of conformal OPE (operator product expansion) because the expansion in a series of conformal primary operators becomes the key concept in using AdS/CFT correspondence (cf. [5]). After plainly stating what needs to be done in the gravity dual approach in Sec. II B, we proceed to explain our basic gravity dual setting and an idea of how to construct a scattering amplitude of our interest by using string field theory in Secs. III and IV. Section V shows the results of computing wave functions of spin-j fields on AdS_5, while a more detailed account of the derivation of wave functions is given in Appendix A. Classification of eigenmodes that turn out to be relevant for the "twist-2" operators in later sections is given in Sec. V A, and wave functions are presented as analytic functions of the complex spin (angular momentum) variable j in Sec. V B. These wave functions are organized into irreducible representations of conformal algebra in Sec. V C; the representation for spin-j primary operators contain more eigenmode components than those treated by the Pomeron exchange amplitude in the formalism of [3], indicating that more contributions are needed in the scattering amplitude with nonvanishing skewness than in the formalism of [3]. These wave functions (and propagators) are used in Sec. VI in organizing scattering amplitude on AdS_5. The amplitude obtained in this way can be cast into the form of conformal OPE, from which one can also extract GPD as a function of kinematical variables. We are not committed to a particular form of implementing confining effects in the holographic description, as discussed in Sec. V D. Some qualitative aspects of the GPD profile are examined in Sec. VII. This paper in Phys. Rev. D is based on the preprints [14].

Not surprisingly, holographic models of GPD so obtained provide a special subclass of GPD models that have been called dual parametrization or (conformal) collinear factorization approach in the QCD/hadron community [15–18]. After all, it is the combination of the dual resonance model and the AdS/CFT correspondence that are being used.

We found that interesting preprints [12,13] cover a subject that is closely related to our study in Secs. V and VI and in Appendix A. References [12,13] mainly deal with correlation functions of CFTs as functions of spacetime coordinates, whereas we deal with them in this article as functions of incoming/outgoing momenta, and confinement effects are also implemented, so that we can study hadron scattering processes.

II. OUR APPROACH: CONFORMAL OPE AND GRAVITY DUAL

A. Review: Conformal OPE of DVCS amplitude

1. Notation and conventions

Deeply virtual Compton scattering (DVCS) $\gamma^* + h \to h + \gamma$, hard exclusive vector meson production (VMP) $e + h \to e + h + V$ and timelike Compton scattering (TCS) processes $e + h \to e + h + e^+e^-$ are shown in Figs. 1(a), 1(c), and 1(d), respectively. As a part of all these processes, the photon-hadron two-body-to-two-body scattering amplitude,

$$\mathcal{M}(\gamma^* h \to \gamma^{(*)} h) = (\epsilon_1^\mu) T_{\mu\nu} (\epsilon_2^\nu)^*, \tag{1}$$

is involved.[1] This two-body-to-two-body scattering amplitude with this exclusive choice of the final states (Fig. 2) is truly nonperturbative information, and this is the subject of this article. Because the "final state" photon is required to be on-shell $q_2^2 = 0$ in DVCS and timelike[2] $q_2^2 < 0$ in VMP

[1]There are two contributions from (a) the $\gamma^* + h \to \gamma + h$ virtual Compton scattering and (b) the Bethe-Heitler process in the leptoproduction process of a photon on a target hadron h: $\ell + h \to \ell + \gamma + h$, and they interfere. They can be separated experimentally, however, by exploiting kinematical dependence and polarization [19]. It thus makes sense to focus only on the amplitude (a).

[2]We use the $(-, +++)$ metric throughout this paper.

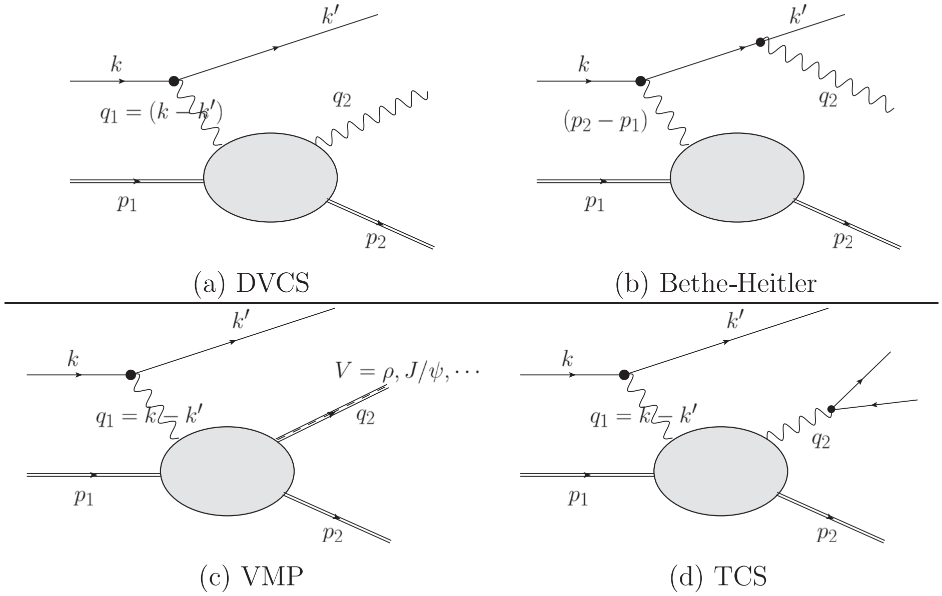


FIG. 1. Panels (a),(b) are diagrams contributing to the leptoproduction process of photon on a hadron, $\ell + h \to \ell + \gamma + h$, (c) is the exclusive vector meson production process, and, finally, (d) is the timelike Compton scattering process.

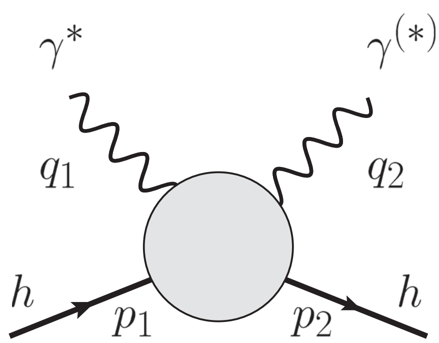


FIG. 2. Photon-hadron two-body-to-two-body scattering amplitude.

and TCS, we are interested in developing a theoretical framework to calculate this nonperturbative amplitude in the case where q_2^2 is different from spacelike $q_1^2 > 0$.

Just like in the QCD/hadron literature, we use the following notation for Lorentz invariant kinematical variables:

$$p^\mu = \frac{1}{2}(p_1^\mu + p_2^\mu), \qquad q^\mu = \frac{1}{2}(q_1^\mu + q_2^\mu),$$
$$\Delta^\mu = p_2^\mu - p_1^\mu = q_1^\mu - q_2^\mu, \tag{2}$$

$$x = \frac{-q^2}{2p \cdot q}, \qquad \eta = \frac{-\Delta \cdot q}{2p \cdot q},$$
$$s = W^2 = -(p+q)^2, \qquad t = -\Delta^2. \tag{3}$$

η is called skewness; in the scattering process of our interest, $q_1^2 = q^2 + \Delta^2/4 + q \cdot \Delta$ and $q_2^2 = q^2 + \Delta^2/4 - q \cdot \Delta$ are not the same; hence, the skewness does not vanish. We will focus on high-energy scattering; for a typical energy scale of hadron masses/confinement scale Λ, we assume that

$$\Lambda^2 \ll q_1^2, W^2, \quad \text{while } |t| \lesssim \mathcal{O}(\Lambda). \tag{4}$$

The photon-hadron scattering amplitude (Fig. 2) in the real-world QCD (where all charged partons are fermions), the Compton tensor is given by the hadron matrix element with the insertion of two QED currents J^μ,

$$T^{\mu\nu} = i \int d^4x e^{-iq\cdot x} \langle h(p_2)| T\{J^\nu(x/2) J^\mu(-x/2)\} |h(p_1)\rangle. \tag{5}$$

For simplicity, we assume that the target hadron is a scalar and further pay attention only to the structure function V_1 appearing in the gauge-invariant decomposition[3] of the Compton tensor:

$$T^{\mu\nu} = V_1 P[q_1]^{\mu\rho} P[q_2]_\rho^\nu + V_2 (p \cdot P[q_1])^\mu (p \cdot P[q_2])^\nu$$
$$+ V_3 (q_2 \cdot P[q_1])^\mu (q_1 \cdot P[q_2])^\nu$$
$$+ V_4 (p \cdot P[q_1])^\mu (q_1 \cdot P[q_2])^\nu$$
$$+ V_5 (q_2 \cdot P[q_1])^\mu (p \cdot P[q_2])^\nu + A \epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma}. \tag{7}$$

[3] Here, we introduced a convenient notation:

$$P[q]_{\mu\nu} = \left[\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right]. \tag{6}$$

These structure functions, $V_{1,2,3,4,5}(x,\eta,t,q^2)$, should be expressed in terms of the kinematical variables x,η, and t, and one of the primary purposes of this article is to study how the structure functions depend on the skewness η.

2. Light-cone operator product expansion

The light-cone OPE can be applied to the product of currents $T\{J^\nu J^\mu\}$ before evaluating it as a hadron matrix element. Let the expansion be

$$i\int d^4x e^{-iq\cdot x}T\{J^\nu(x/2)J^\mu(-x/2)\} = \sum_I C^{\mu\nu}_{I\rho_1\ldots\rho_{j_I}}(q)\mathcal{O}_I^{\rho_1\ldots\rho_{j_I}}(0;q^2) \tag{8}$$

for some basis of local operators $\mathcal{O}_I^{\rho_1\cdots\rho_{j_I}}$ renormalized at $\mu^2=q^2$. $C^{\mu\nu}_{I\rho_1\cdots\rho_{j_I}}$'s are the corresponding Wilson coefficients renormalized at $\mu^2=q^2$. If we were to evaluate these local operators on the right-hand side with the same state for both bra and ket, $\langle h(p_2)|$ and $|h(p_1)\rangle$, with $p_2^\mu=p_1^\mu$, then the Compton tensor and its structure functions do not receive nonzero contributions from local operators that are given by the total derivative of some other local operators. In the case of our interest, however, such operators do contribute.

Let us take a series of operators in QCD that are called twist-2 operators in the weak coupling limit. The twist-2 operators in the flavor-nonsinglet sector are labeled by two integers, j,l,

$$\mathcal{O}^\alpha_{j,l} \coloneqq [(-i)^{j+l-1}\partial^{\mu_{j+1}}\cdots\partial^{\mu_{j+l}}\overline{\Psi}_a\gamma^{\mu_1}(\overleftrightarrow{D})^{\mu_2}\cdots(\overleftrightarrow{D})^{\mu_j}\lambda^\alpha_{ab}\Psi_b]_{\mathrm{t.s.t.l.}}(0;q^2), \tag{9}$$

with an $N_F\times N_F$ flavor matrix $(\lambda^\alpha)_{ab}$. Similarly, in the flavor-singlet sector, there are two series of twist-2 operators with the label j,l, given by quark bilinear and gluon bilinear. Here, these operators are made totally symmetric and traceless (t.s.t.l) in the $j+l$ Lorentz indices so that they transform in irreducible representations of the Lorentz group SO(3, 1), $\overleftrightarrow{D}\coloneqq\vec{D}-\overleftarrow{D}$.

Suppose that the hadron matrix element of the operator $\mathcal{O}^\alpha_{j,l}$ is given by

$$\langle h(p_2)|\mathcal{O}^\alpha_{j,l}|h(p_1)\rangle = \sum_{k=0}^{j}[\Delta^{\mu_1}\cdots\Delta^{\mu_{k+l}}p^{\mu_{k+l+1}}\cdots p^{\mu_{j+l}}]_{\mathrm{t.s.t.l.}}A_{j,k}(t;q^2)(-2)^{j-k}; \tag{10}$$

the reduced matrix element $A^\alpha_{j,k}(t)$ is nonperturbative information and cannot be determined by perturbative QCD. If we pay attention only to Wilson coefficients $C^{\mu\nu}_{j,l,\alpha;\mu_1\cdots\mu_{j+l}}$ that are proportional to $\eta^{\mu\nu}$ and are to write them as[4]

$$\eta^{\mu\nu}C^\alpha_{j,l}\frac{q_{\rho_1}\cdots q_{\rho_{j+l}}}{(q^2)^{j+l}}, \tag{11}$$

then the twist-2 flavor-nonsinglet contribution to the structure function V_1 becomes

$$V_1 \simeq \sum_{j,l}C^\alpha_{j,l}\frac{1}{x^{j+l}}\sum_{k=1}^{j}A^\alpha_{j,k}(t;q^2)\eta^{k+l} \eqqcolon \sum_j C^\alpha_j(\vartheta)\frac{1}{x^j}A^\alpha_j(\eta,t;q^2), \tag{12}$$

where $\vartheta\coloneqq(\eta/x)$, $C^\alpha_j(\vartheta)\coloneqq\sum_{l=0}^\infty C^\alpha_{j,l}\vartheta^l$, and $A^\alpha_j(\eta,t)\coloneqq\sum_{k=0}^j\eta^kA^\alpha_{j,k}(t)$. If the structure function V_1 receives contributions only from even $j\in\mathbb{Z}$, then this j summation is rewritten as

$$V_1(x,\eta,t;q^2)\simeq-\int\frac{dj}{4i}\frac{1+e^{-\pi ij}}{\sin(\pi j)}C^\alpha_j(\vartheta)\frac{1}{x^j}A^\alpha_j(\eta,t;q^2) \tag{13}$$

in the form of inverse Mellin transformation; here, $C^\alpha_j(\vartheta;q^2)$ and $A^\alpha_j(\eta,t;q^2)$ are now meant to be holomorphic functions on j (possibly with some poles and cuts) that coincide with the original ones at $j\in2\mathbb{Z}$. Precisely the same story also holds true for the flavor-singlet sector.

Because the structure function is given by the inverse Mellin transform of a product of three factors, namely, (a) the signature factor $\mp[1\pm e^{-\pi ij}]/\sin(\pi j)$, (b) the Wilson coefficients C^α_j, and (c) the hadron matrix elements A^α_j, it can be regarded as a convolution of inverse Mellin transforms of those three factors. The inverse Mellin transform of the signature factor becomes

$$\int\frac{dj}{2\pi i}\frac{1}{x^j}\frac{\pi}{2}\frac{\mp[1\pm e^{-\pi ij}]}{\sin(\pi j)}=\frac{-1}{2}\left[\frac{1}{1-x+i\epsilon}\pm\frac{1}{1+x}\right], \tag{14}$$

which corresponds to propagation of the parton in a perturbative calculation [20], and the inverse Mellin transform of the matrix element is called the generalized parton distribution:

$$H^\alpha(x,\eta,t;\mu^2=q^2)=\int\frac{dj}{2\pi i}\frac{1}{x^j}A^\alpha_j(\eta,t;\mu^2=q^2). \tag{15}$$

GPD $H^\alpha(x,\eta,t;\mu^2)$ of a hadron h is nonperturbative information, just like the ordinary PDF, which is obtained

[4]In the leading order of QCD perturbation, $C^\alpha_{j,0}=-[1+(-1)^j]$ for $j=2,4,\ldots$ and $(\lambda^\alpha)_{ab}=[\mathrm{diag}(4/9,1/9,1/9)]_{\mathrm{t.l.}}$.

by simply setting $\eta = 0$ and $t = 0$. For a phenomenological fit of the experimental data of DVCS and VMP, some function form of the GPD needs to be assumed because of the convolution involved in the scattering amplitude [10]. Setting up a model (and assuming a function form) for the nonperturbative information in terms of $A_j(\eta, t; q^2)$ rather than the GPD itself $[H(x, \eta, t; q^2)]$ is called dual parametrization [15–18], and some phenomenological Ansätze have been proposed. In this article, we aim at deriving a qualitative form of $A_j(\eta, t)$ by using a gravitational dual (that is analytic in j), instead of assuming the form of $A_j(\eta, t)$ by hand.

3. Renormalization and OPE in dilatation eigenbasis

Remembering that the distinction between the $\gamma^* + h \to \gamma + h$ scattering amplitude and GPD originates from the factorization into the Wilson coefficients and local operators (and their matrix elements), one will notice that the GPD defined in this way should depend on the choice of the basis of local operators. Although the choice of operators $\mathcal{O}^\alpha_{j,l}$, with $j \geq 1$ and $l \geq 0$, in (9) appears to be the most natural (and intuitive) one for the twist-2 operators in the flavor-nonsinglet sector, there is nothing wrong with taking different linear combinations of these operators as a basis when the corresponding Wilson coefficients also become linear combinations of what they are for $\mathcal{O}^\alpha_{j,l}$. Given the fact that the operators $\mathcal{O}^\alpha_{j,l}$ mix with one another under renormalization, it should not be compulsory for us to stick to the basis $\mathcal{O}^\alpha_{j,l}$.

Under the perturbation of QCD, the flavor-nonsinglet twist-2 operators are renormalized under

$$\mu \frac{\partial}{\partial \mu} [\mathcal{O}_{j-m,m}(0;\mu^2)] = -[\gamma^{(j)}]_{mm'} [\mathcal{O}_{j-m',m'}(0;\mu^2)]; \qquad (16)$$

because operators can mix only with those with the same number of Lorentz indices, the anomalous dimension matrix $[\gamma]$ is block diagonal in the basis of $\mathcal{O}^\alpha_{j,l}$. The $j \times j$ matrix for the operators $\mathcal{O}^\alpha_{j-m,m}$ $(m = 0, \ldots, j-1)$ is denoted by $[\gamma^{(j)}]$. This matrix is upper triangular in this basis, and the diagonal entries are given by the anomalous dimensions of the twist-2 spin-j operators without a total derivative:

$$[\gamma^{(j)}]_{mm} = \gamma(j - m). \qquad (17)$$

Therefore, the eigenvalue of the anomalous dimension matrix is $\{\gamma(j-m)\}_{m=0,\ldots,j-1}$ in this diagonal block, and the corresponding operator $\overline{\mathcal{O}}^\alpha_{j-m-1,m}$ is a linear combination of operators $\mathcal{O}_{j-m',m'}$, with $m' = m, \ldots, j-1$ [21]. The corresponding Wilson coefficient $\overline{C}^\alpha_{j-m-1,m}$ for such an operator is a linear combination of $\mathcal{C}^\alpha_{j-m',m'}$, with $m' = m, \ldots, 0$. In this operator basis, matrix elements and Wilson coefficients renormalize multiplicatively, without mixing.[5]

In this new basis of local operators, the structure function becomes

$$\begin{aligned} V_1 &\simeq \sum_{n,K} \overline{C}^\alpha_{n,K} \frac{1}{x^{n+1+K}} \sum_k \overline{A}^\alpha_{n+1,k}(t;\mu^2) \eta^{K+k} \\ &=: \sum_n \overline{C}^\alpha_n(\vartheta) \frac{1}{x^{n+1}} \overline{A}^\alpha_{n+1}(\eta, t; \mu^2), \end{aligned} \qquad (18)$$

where

$$\overline{C}^\alpha_n(\vartheta) = \sum_{K=0}^{\infty} \overline{C}^\alpha_{n,K} \vartheta^K, \qquad (19)$$

and $\overline{A}^\alpha_{n+1,k}(t;\mu^2)$ is the reduced matrix element of the operator[6] $\overline{\mathcal{O}}^\alpha_{n,0}(0;\mu^2)$. The structure function is therefore written as yet another inverse Mellin transform,

$$V_1 \simeq -\int \frac{dj}{4i} \frac{1 + e^{-\pi i j}}{\sin(\pi j)} \overline{C}^\alpha_{j-1}(\vartheta) \frac{1}{x^j} \overline{A}^\alpha_j(\eta, t; \mu^2). \qquad (20)$$

Yet another GPD can also be defined by using $\overline{A}^\alpha$ instead of $A^\alpha_j(\eta, t; q^2)$:

$$\overline{H}^\alpha(x, \eta, t; \mu^2) = \int \frac{dj}{2\pi i} \frac{1}{x^j} \overline{A}^\alpha_j(\eta, t; \mu^2). \qquad (21)$$

When it comes to the description of the $\gamma^* + h \to \gamma + h$ scattering amplitude as a whole, it does not matter which operator basis is used. In order to talk about the distribution of partons in the transverse directions in a hadron, we only need GPD at $\eta = 0$. The newly defined GPD $\overline{H}$ does just as good a job as the H defined in (15); they are the same at $\eta = 0$.

Even within the dual parametrization approach, it has been advantageous to use this operator basis because it becomes much easier to implement a phenomenological assumption (a function form) of $\overline{A}^\alpha_j(\eta, t; \mu^2)$ that is consistent with renormalization group flow [15].

[5] In reality, the anomalous dimension matrix depends on the coupling constant α_s, and α_s changes over the scale. Thus, the eigenoperator of the renormalization/dilatation also changes over the scale. In scale invariant theories (and in theories with only slow running in α_s), however, this multiplicative renormalization is exact or a good approximation (cf. [22]).

[6] Just like $\mathcal{O}_{j,l} = (-i\partial)^l \mathcal{O}_{j,0}$, there is a relation $\bar{\mathcal{O}}_{n,K} = (-i\partial)^K \bar{\mathcal{O}}_{n,0}$ in the new basis. This is why all the hadron matrix elements of $\bar{\mathcal{O}}_{n,K}$ can be parametrized by $\bar{A}_{n+1,k}$, just like those of $\mathcal{O}_{j,l}$ are by $A_{j,k}$. Here, n corresponds to the conformal spin, which is sometimes denoted by j in the literature. In this article, however, we maintain $j = n + 1$.

4. Conformal OPE

Although the hadron matrix element is essentially nonperturbative and is not calculable within perturbative QCD, more discussion has been made of the Wilson coefficients $\overline{C}^{\alpha}_{n,K}$. They still have to be calculated order by order in perturbation theory, if one is interested strictly in the QCD of the real world. If one is interested in gauge theories that are more or less "similar" to QCD, however, stronger statements can be made for a system with higher symmetry: conformal symmetry. One can think of $\mathcal{N}=4$ super Yang-Mills theory or $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge theory of [23] as an example of theories with exact (super)conformal symmetry. The QED probe in the real-world QCD can be replaced by gauging global symmetries [such as (a part of) SU(4) R symmetry of $\mathcal{N}=4$ super Yang-Mills theory and $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry of [23]]. By applying the conformal symmetry, one can derive stronger statements on the Wilson coefficients of primary operators appearing in the OPE.

Suppose that we are interested in the OPE of two primary operators, A and B, that are both scalar under SO(3,1). If we take the basis of local operators for the expansion to be primary operators $\overline{\mathcal{O}}_n$ (with j_n Lorentz indices and an l_n scaling dimension) and their descendants $\partial^K\overline{\mathcal{O}}_n$ (with j_n+K Lorentz indices), then in the OPE,

$$T\{A(x)B(0)\} = \sum_n \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_n+j_n)} \sum_{K=0}^{\infty} c_{n,K}\frac{x^{\rho_1}\cdots x^{\rho_{j_n+K}}}{(x^2)^{j_n+K}} \times [\partial^K\overline{\mathcal{O}}_n(0)]_{\rho_1\cdots\rho_{j_n+K}}. \tag{22}$$

The conformal symmetry determines all the coefficients of the descendants $c_{n,K}$ $(K\geq 1)$ in terms of that of the primary operator, $c_{n,0}=: c_n$. Ignoring the mixture of nontraceless contributions, one finds that [24]

$$T\{A(x)B(0)\} \simeq \sum_n \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_n+j_n)} x^{\rho_1}\cdots x^{\rho_{j_n}} c_n \times {}_1F_1\left(\frac{l_A-l_B+l_n+j_n}{2}, l_n+j_n; x\cdot\partial\right) \times [\overline{\mathcal{O}}_n(0)]_{\rho_1\cdots\rho_{j_n}}. \tag{23}$$

A question of real interest to us is the OPE of conserved currents J^ν and J^μ. They are not scalars of SO(3,1), but the same logic as in [24] can be used also to show that, in the terms with Wilson coefficients proportional to $\eta^{\mu\nu}$,

$$T\{J^\nu(x)J^\mu(0)\} \simeq \eta^{\mu\nu}\sum_n \left(\frac{1}{x^2}\right)^{3-\frac{\tau_n}{2}} x^{\rho_1}\cdots x^{\rho_{j_n}} c_n \times {}_1F_1\left(\frac{l_n+j_n}{2}, l_n+j_n; x\cdot\partial\right) \times [\overline{\mathcal{O}}_n(0)]_{\rho_1\cdots\rho_{j_n}} + \cdots, \tag{24}$$

where $\tau_n := l_n - j_n$ is the twist, a mixture of the nontraceless (and hence higher twist) contributions is ignored, and terms with Wilson coefficients without $\eta^{\mu\nu}$ are all omitted here. The scaling dimension of conserved currents $l_A=l_B=3$ has been used. The momentum space version of the OPE is [25]

$$i\int d^4x\, e^{-iq_2\cdot x} T\{J^\nu(x)J^\mu(0)\} \simeq \eta^{\mu\nu}\sum_n \frac{(2\pi)^2\Gamma(\frac{l_n+j_n-2}{2})}{4^{2-\frac{\tau_n}{2}}\Gamma(3-\frac{\tau_n}{2})} c_n \frac{(-2i)^{j_n} q_2^{\rho_1}\cdots q_2^{\rho_{j_n}}}{(q_2^2)^{\frac{\tau_n}{2}-1}(q_2^2)^{j_n}} \times {}_2F_1\left(\frac{l_n+j_n}{2}, \frac{l_n+j_n}{2}-1, l_n+j_n; \frac{-2iq_2\cdot\partial}{q_2^2}\right)\overline{\mathcal{O}}_n(0) + \cdots, \tag{25}$$

or, equivalently [18],

$$i\int d^4(x-y) e^{-iq\cdot(x-y)} T\{J^\nu(x)J^\mu(y)\} \simeq \eta^{\mu\nu}\sum_n \frac{(2\pi)^2\Gamma(\frac{l_n+j_n-2}{2})}{4^{2-\frac{\tau_n}{2}}\Gamma(3-\frac{\tau_n}{2})} c_n \frac{(-2i)^{j_n} q^{\rho_1}\cdots q^{\rho_{j_n}}}{(q^2)^{\frac{\tau_n}{2}-1}(q^2)^{j_n}} \times {}_2F_1\left(\frac{l_n+j_n-2}{4}, \frac{l_n+j_n}{4}, \frac{l_n+j_n}{2}; \left(\frac{iq\cdot\partial}{q^2}\right)^2\right)\overline{\mathcal{O}}_n\left(\frac{x+y}{2}\right) + \cdots. \tag{26}$$

Either in the form of (25) or (26), the primary operators $\overline{\mathcal{O}}_n$ and the corresponding coefficients c_n are renormalized multiplicatively.

B. AdS/CFT approach

In AdS/CFT correspondence, type IIB string theory on $\mathrm{AdS}_5\times W$ with a five-dimensional Einstein manifold W corresponds to a gauge theory on $\mathbb{R}^{3,1}$ with an exact conformal symmetry; theories with an exact conformal symmetry, however, are qualitatively different from the QCD in the real world. But the type IIB string on a geometry that is close to $\mathrm{AdS}_5\times W$, except with the confining end in the infrared, may be used to extract a qualitative lesson on strongly coupled gauge theories with confinement, which are not qualitatively different from the QCD.

In a dual pair of a conformal field theory (CFT) and a string theory on a background $\mathrm{AdS}_5 \times W$, primary operators of the CFT are in one-to-one correspondence with string states on AdS_5, and their correlation functions can be calculated by using the wave functions of the string states on AdS_5. When the background geometry is changed from $\mathrm{AdS}_5 \times W$ to some warped geometry that is nearly AdS_5 with an end in the infrared, then the wave functions might be used to calculate matrix elements of the corresponding "primary" operators in an almost conformal theory. The correspondence between the operators and string states can be made precise, because they are both classified in terms of the representation of the conformal algebra, which is shared by both of the dual theories.

In order to determine GPD $\overline{H}$ in gravitational dual descriptions, it is therefore sufficient to determine wave functions of string states corresponding to the primary operators of interest. Although there is plenty of literature discussing the correspondence between the (superconformal) primary operators and string states at the supergravity level, it is known that the flavor-singlet twist-2 operators (labeled by the spin j) correspond to the stringy excitations with arbitrary high spin j that are in the same trajectory as the graviton [3,26]. Our task is, therefore, to determine the wave functions of such string states. Needless to say, one has to fix all of the gauge degrees of freedom associated with string component fields (not just the general coordinate invariance associated with the graviton) before working out the mode decomposition. Furthermore, wave functions need to be grouped together properly so that they form an irreducible representation of the conformal group in order to establish correspondence with a primary operator of the gauge theory side, which also forms an irreducible representation of the conformal group, along with its descendants.

It will be clear by the end of this article that all such technical work is necessary and essential for the purpose of extracting skewness dependence of GPD.

There are two different (but equivalent) ways to study the DVCS $\gamma^* + h \to \gamma^{(*)} + h$ amplitude and GPD in gravitational dual descriptions. One is to determine the hadron matrix elements of spin-j primary operators by using appropriate wave functions; GPD $\overline{H}$ is obtained by the inverse Mellin transform of the matrix elements. Using the Wilson coefficients that are governed by the conformal symmetry [see (26)], the DVCS amplitude will also be obtained. Conversely, the other way is to calculate disc/sphere amplitude directly, with the vertex operators given (approximately) by using the wave functions associated with the target hadron (see Secs. III and IV). We will identify the structure of conformal OPE in the expression for the $\gamma^* + h \to \gamma^{(*)} + h$ scattering amplitude in gravity dual [see (160), (163), (178)], with the Wilson coefficient for the twist-2 operators precisely as predicted by conformal symmetry (26). That also makes it possible to read out hadron matrix elements, and to extract the GPD. In these approaches, one can hope to work also for higher twist contributions, in principle, but we are not ambitious enough to do that in this article. In this article, we will proceed with the latter approach.

III. GRAVITY DUAL SETTINGS

A number of warped solutions to the type IIB string theory have been constructed, and they are believed to be dual to some strongly coupled gauge theories. When the geometry is close to $\mathrm{AdS}_5 \times W$ with some five-dimensional Einstein manifold W, with weak running of the anti–de Sitter (AdS) radius along the holographic radius, the corresponding gauge theory will also have approximate conformal symmetry, and the gauge coupling constant runs slowly. If the $\mathrm{AdS}_5 \times W$ geometry has a smooth end at the infrared as in [27], then the dual gauge theory will end up with confinement. Gravitational backgrounds in the type IIB string theory with the properties we stated above all provide a decent framework for studying qualitative aspects of nonperturbative information associated with gluons/Yang-Mills theory on $3+1$ dimensions.

In studying the $h + \gamma^* \to h + \gamma$ scattering process in a gravitational dual, we need a global symmetry to be gauged weakly, just like QED for QCD. In type IIB D-brane constructions of gauge theories that have a gravity dual, U(1) subgroups of an R symmetry or a flavor symmetry on D7-branes can be used as the models of the electromagnetic U(1) symmetry. Therefore, we have in mind gravity dual models on a background that is approximately $\mathrm{AdS}_5 \times W$ with a nontrivial isometry group on W, or with a D7-brane configuration on it, as in [2].

Our interest, however, is not so much in writing down an exact mathematical expression based on a particular gravity dual model that is equivalent to a particular strongly coupled gauge theory, but more in extracting qualitative information of partons in hadrons of confining gauge theories in general. It is, therefore, more suitable for this purpose to use a simplified setup that carries common (and essential) features of the type IIB models that we described above. Throughout this article, we assume a pure $\mathrm{AdS}_5 \times W$ metric background,

$$ds^2 = G_{MN} dx^M dx^N = g_{mn} dx^m dx^n + R^2 (g_W)_{ab} d\theta^a d\theta^b, \tag{27}$$

$$g_{mn} dx^m dx^n = e^{2A(z)} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \quad e^{2A(z)} = \frac{R^2}{z^2}; \tag{28}$$

that is, we ignore the running effect, and we do not specify the five-dimensional manifold W. The dilaton vacuum expectation value is simply assumed to be constant, $e^\phi = g_s$. A confining effect—the infrared end of this geometry—can be introduced, for example, by sharply cutting off the AdS_5 space at $z = \Lambda^{-1}$ (hard wall models), or by similar alternatives (soft wall models). We are not committed to a particular implementation of the infrared cutoff in this article (see the discussion in Sec. V D), except in a couple of places where we write down some concrete expressions for illustrative purposes (Secs. VII A and VII D). The energy scale Λ associated with (any form of implementation of) the infrared cutoff corresponds to the confining energy scale in the dual gauge theories. When we consider (a simplified version of the) models with D7-branes for flavor, we assume that the D7-brane world volume wraps on a three-cycle on W and extends all the way down to the infrared end of the holographic radius z; i.e., all of $0 \le z \le \Lambda^{-1}$. This corresponds to assuming massless quarks. In this article, we will not pay attention to physics where spontaneous chiral symmetry breaking is essential.

As we stated earlier, we would like to work out the $h + \gamma^* \to h + \gamma^{(*)}$ scattering amplitude by using the gravity dual models. This is done by summing up sphere/disc amplitudes, along with those with higher genus world sheets. We will restrict our attention to kinematical regions where saturation is not important (i.e., large q^2 and/or not too small x, and large N_c). That allows us to focus only on sphere/disc amplitudes, with the insertion of four vertex operators corresponding to the incoming and outgoing hadron h and (possibly virtual) photon γ.

As a string-based model of the target hadron h [that is SO(3,1) scalar], we have in mind either a scalar "glueball"[7] that has nontrivial R-charge, or a scalar meson made of matter fields. The former corresponds to a vertex operator [in the $(-1,-1)$ picture]

$$V(p) = :e^{ip_\mu \cdot X^{\hat{\mu}}} \psi^m \tilde{\psi}^n g_{mn} \Phi(Z; m_n) Y(\Theta):, \qquad (29)$$

where $Y(\Theta)$ is a "spherical harmonics" on W, and the latter to

$$V(p) = :e^{ip_\mu \cdot X^{\hat{\mu}}} \psi \Phi(Z; m_n):, \qquad (30)$$

where ψ corresponds to the D7-brane fluctuations in its transverse directions. $\Phi(Z)$ is the wave function on AdS_5, with the argument promoted to the field on the world sheet [3]. The vertex operators above are approximate expressions in the $(\alpha'/R^2) \sim 1/\sqrt{\lambda}$ expansion (e.g., [28]) in a theory formulated with a nonlinear σ model given by (27). If we are to employ the hard wall implementation of the infrared boundary, with the AdS_5 metric in the bulk without modification, then the wave function $\Phi(Z; m_n)$ is of the form

$$\sqrt{t_h} \Phi(z; m_n) = 2\Lambda z^2 \frac{J_{\ell_\phi - 2}(j_{\ell_\phi - 2, n} \Lambda z)}{|J'_{\ell_\phi - 2}(j_{\ell_\phi - 2, n})|}. \qquad (31)$$

[7]By glueball, we only mean a bound state of fields in super Yang-Mills theory.

This wave function is that of the nth lightest hadron corresponding to some scalar operator with conformal dimension ℓ_ϕ; the hadron mass $m_n = j_{\ell_\phi - 2, n} \Lambda$ is given by the nth zero of the Bessel function $J_{\ell_\phi - 2}$. We will comment on the normalization factor $\sqrt{t_h}$ in later sections, though it disappears from the expression for physical observables.

The "photon" current in the correlation function/matrix element $T^{\nu\mu}$ in the gauge theory description corresponds to the insertion of vertex operators associated with non-normalizable wave functions, rather than with the normalizable wave functions (31) for the target hadron state. If we are to employ an R-symmetry current as the string-based model of the QED current, then the corresponding closed string vertex operator is

$$V(q) = :e^{iq_\mu \cdot X^{\hat{\mu}}} v_a(\Theta) A_m(Z; q)(\psi^a \tilde{\psi}^m + \psi^m \tilde{\psi}^a):, \qquad (32)$$

with some Killing vector $v_a \partial/\partial\theta^a$ on W. The vertex operator in the case of a D7-brane U(1) current is

$$V(q) = :e^{iq_\mu \cdot X^{\hat{\mu}}} A_m(Z; q) \psi^m:. \qquad (33)$$

The wave function $A_m(Z; q)$ on AdS_5 is of the form

$$A_\mu(z; q) = \left[\delta^{\hat{\kappa}}_\mu - \frac{q_\mu q^{\hat{\kappa}}}{q^2}\right] \epsilon_\kappa(q)(qz) K_1(qz) + q_\mu \frac{q^{\hat{\kappa}} \epsilon_\kappa(q)}{2q^2} (qz)^2 K_2(qz), \qquad (34)$$

$$A_z(z; q) = -i\partial_z \frac{q^{\hat{\kappa}} \epsilon_\kappa(q)}{2q^2} (qz)^2 K_2(qz). \qquad (35)$$

Here, q stands for $\sqrt{q^2}$, although it sometimes imply four-momentum q^μ, depending on the local context in this article. The rationale for our choice of the terms proportional to $(q \cdot \epsilon)$ will be explained later on in Appendix A.4, but those terms should not be relevant in the final result because of the gauge invariance of $T^{\nu\mu}$. When the infrared boundary is implemented by the hard wall, $K_1(qz)$ should be replaced by $K_1(qz) + [K_0(q/\Lambda)/I_0(q/\Lambda)] I_1(qz)$, and $K_2(qz)$ by an arbitrary linear combination of $K_2(qz)$ and $I_2(qz)$.

It is not as easy to calculate the sphere/disc amplitudes in practice, however. It has been considered that the parton contributions to $\gamma^* + h \to \gamma^{(*)} + h$ scattering are given by an amplitude with states in the leading trajectory with arbitrary high spin being exchanged [3]. These fields are not scalar on AdS_5 but come with multiple degrees of freedom associated with polarizations. Such polarization of higher spin fields propagating on AdS_5 needs to be treated properly—including such issues as covariant derivatives and kinetic mixing among different polarizations (diagonalization of the Virasoro generator L_0)—in gravity dual descriptions in order to be able to discuss the skewness dependence of the GPD/DVCS amplitude. The direct impact of the curved background geometry can be implemented through the nonlinear σ model on the world sheet, but one has to define the vertex operators as a composite operator properly in such an interacting theory. The Ramond-Ramond background is an essential ingredient in making the warped background metric stable, yet a nonzero Ramond-Ramond background cannot be implemented in the Neveu-Schwarz-Ramond (NSR) formalism.

Instead of using a world-sheet calculation in the NSR formalism when implementing the effect of a curved background (27), we use string field theory action on flat space in this article and make it covariant. Because the gravity dual setup of our interest is in type IIB string theory, we are thus supposed to use superstring field theory for closed string and open string modes. In order to avoid technical complications associated with the interacting superstring field theories, however, we employ a sort of toy-model approach by using the cubic string field theory for bosonic string theory.

In our toy-mode approach, we deal with the cubic string field theory on AdS_5 ($\times$ some internal compact manifold), and ignore instability of the background geometry. The probe photon in this toy-model gravity dual setup will be the massless vector state of bosonic string theory with the wave function (34), (35). The target hadron can be any scalar states, (say, the tachyon) with the wave function (31). We are to construct a toy-model amplitude of the $h + \gamma^* \to h + \gamma^{(*)}$ scattering by using the two-to-two scattering of the massless photon and some scalar in bosonic string theory on the AdS_5 background. In short, this is to maintain the spirit of the setup in [2,3] and use the bosonic cubic string field theory to compute and obtain something concrete, from which qualitative lessons are to be extracted for the setup of our interest.

One of the costs of this approach (without the technical complexity of interacting superstring field theory) is that we have to restrict our attention to the Reggeon exchange (flavor-nonsinglet) amplitude because the cubic bosonic string field theory deals with open strings, not the closed (i.e., flavor neutral) string. The amplitude constructed in this way is certainly not faithful to the equations of type IIB string theory, either. Since our motivation is not in constructing yet another exact solution to superstring theory, however, we still expect that this (flavor-nonsinglet) toy-mode amplitude in bosonic string still maintains some fragrance of hadron scattering amplitude to be calculated in superstring theory. This discussion continues in Sec. VII C.

IV. CUBIC STRING FIELD THEORY

Section IV A summarizes the technical details of cubic string field theory that we will need in later sections. We then proceed in Sec. IV B to explain an idea of how to reproduce disc amplitude only from string field theory t-channel amplitude, using photon-tachyon scattering on a flat spacetime background as an example. This idea of constructing amplitude is generalized in Sec. VI for scattering on a warped spacetime, and we will see that this construction of the amplitude allows us to cast the amplitude almost immediately into the form of conformal OPE (25), (26).

A. Action of the cubic SFT on a flat spacetime

The action of the cubic string field theory (SFT) is given by [29]

$$S = -\frac{1}{2\alpha'}\int \left(\Phi * Q_B\Phi + \frac{2}{3}g_o\Phi * \Phi * \Phi\right), \quad (36)$$

$$= -\frac{1}{2\alpha'}\left(\Phi \cdot Q_B\Phi + \frac{2g_o}{3}\Phi \cdot \Phi * \Phi\right), \quad (37)$$

where g_o is a coupling constant of mass dimension $(1 - D/2)$, where $D = 26$ is the spacetime dimensions of bosonic string theory.[8] Q_B is the Becchi-Rouet-Stora-Tyutin (BRST) operator, and $*$ and $\cdot$ are the star product and inner product of the string fields, respectively; all of the technical details we need for this article are summarized below in this section, but more information is found, e.g., in [29,30]. The string field Φ is, as a ket state, expanded in terms of the Fock states as in

$$\Phi = |\Phi\rangle = \phi(x)|\downarrow\rangle + (A_M(x)\alpha^M_{-1} + C(x)b_{-1} + \overline{C}(x)c_{-1})|\downarrow\rangle + \left(f_{MN}(x)\frac{1}{\sqrt{2}}\alpha^M_{-1}\alpha^N_{-1} + ig_M(x)\frac{1}{\sqrt{2}}\alpha^M_{-2} + h(x)b_{-1}c_{-1} + \cdots\right)|\downarrow\rangle, \quad (38)$$

with component fields $\phi, A_M, C, \overline{C}, f_{MN}, g_M, h, \ldots$; we have already chosen the Feynman-Siegel gauge here. We will eventually be interested only in the states with a vanishing ghost number, $N_{\rm gh} = 0$, because states with a nonzero ghost number do not appear in the t-channel/s-channel exchange for the disc amplitude.

The Hilbert space of one string state is spanned by the Fock states given (in this gauge) by

$$\prod_{a=1}^{h_a}\alpha^{M_a}_{-n_a}\prod_{b=1}^{h_b}b_{-l_b}\prod_{c=1}^{h_c}c_{-m_c}|\downarrow\rangle, \quad (39)$$

with $1 \le n_1 \le n_2 \le \cdots \le n_{h_a}$, $1 \le l_1 < l_2 < \cdots < l_{h_b}$, and $1 \le m_1 < m_2 < \cdots < m_{h_c}$. Let us use $Y \coloneqq \{\{n_a\}\text{'s}, \{l_b\}\text{'s}, \{m_c\}'s\}$ as the label distinguishing

[8] The sign of the interaction term is just a matter of convention, because field redefinition for all of the component fields $\Phi \to -\Phi$ is always possible. Under this redefinition, however, the covariant derivative can be either $\partial_m - i\rho(A_\mu)$ or $\partial_m + i\rho(A_m)$. The sign convention above is for $\partial_m - i\rho(A_m)$, following the convention of Sec. 6.5 of Polchinski's textbook.

different Fock states of string on a flat spacetime. The mass of these Fock states is determined by

$$\alpha' k^2 + (N^{(Y)} - 1) = 0, \quad N^{(Y)} = \sum_{a=1}^{h_a} n_a + \sum_{b=1}^{h_b} l_b + \sum_{c=1}^{h_c} m_c. \tag{40}$$

A component field corresponding to a Fock state may be further decomposed into a multiple irreducible representation of the Lorentz group, but at least the rank-h_a totally symmetric traceless tensor representation is always contained. The Fock states of particular interest to us are the ones in the leading trajectory: $Y = \{1^N, 0, 0\}$, so that all the n_a's are 1, $h_b = h_c = 0$, and $N^{(Y)} = h_a$. The totally symmetric traceless tensor component field of these states is denoted by $(N!)^{-1/2} A^{(Y)}_{M_1 \cdots M_{h_a}}$.

The kinetic term—the first term of (36), (37)—is written down in terms of the component fields as follows:

$$-\frac{1}{2\alpha'} \Phi \cdot Q_B \Phi = \frac{1}{2} \int d^{26}x \, \mathrm{tr}\left[\phi(x)\left(\partial^2 + \frac{1}{\alpha'}\right)\phi(x) + A_M(x)\partial^2 A^M(x) + f_{MN}(x)\left(\partial^2 - \frac{1}{\alpha'}\right) f^{MN}(x) + g_M(x)\left(\partial^2 - \frac{1}{\alpha'}\right) g^M(x) - h(x)\left(\partial^2 - \frac{1}{\alpha'}\right) h(x) + \cdots\right]. \tag{41}$$

The totally symmetric tensor component field of the Fock states in the leading trajectory $Y = \{1^N, 0, 0\}$ has a kinetic term

$$\frac{1}{2} \int d^{26}x \, \mathrm{tr}\left[A^{M_1 \ldots M_j}\left(\partial^2 - \frac{N-1}{\alpha'}\right) A_{M_1 \ldots M_j}\right]. \tag{42}$$

The cubic string field theory action in the Feynman-Siegel gauge has two nice properties: First, the kinetic terms of those Fock states do not mix in the flat spacetime background, and second, the second derivative operators are simply given by the d'Alembertian operator, without complicated restrictions or mixing among various polarizations in the component fields.

The second term of the action (36), (37) gives rise to interactions involving three component fields. Interactions involving Fock states with small excitation level N are [30]

$$\begin{aligned} -\frac{1}{2\alpha'}\frac{2g_o}{3}\Phi \cdot \Phi * \Phi &= -\int d^{26}x \frac{g_o \lambda_{\mathrm{sft}}}{3\alpha'} \hat{E}(\mathrm{tr}[\phi^3(x)] \\ &+ \sqrt{\frac{8\alpha'}{3}} \mathrm{tr}\left[(-iA_M)(\phi \overleftrightarrow{\partial}^M \phi)\right] \\ &- \frac{8\alpha'}{9\sqrt{2}} \mathrm{tr}[f_{MN}(\phi \overleftrightarrow{\partial}^M \overleftrightarrow{\partial}^N \phi)] - \frac{5}{9\sqrt{2}} \mathrm{tr}[f^M_M \phi^2] \\ &+ \frac{2\sqrt{\alpha'}}{3} \mathrm{tr}[(\partial_M g^M)\phi^2] - \frac{11}{9} \mathrm{tr}[h\phi^2]) + \cdots, \end{aligned} \tag{43}$$

where $\lambda_{\mathrm{sft}} = 3^{9/2}/2^6$ [31], $\overleftrightarrow{\partial}^M = (\overleftarrow{\partial^M} - \overrightarrow{\partial^M})$, and

$$\hat{E} = \exp\left[-2\alpha' \ln\left(\frac{2}{3^{3/4}}\right)(\partial^2_{(1)} + \partial^2_{(2)} + \partial^2_{(3)})\right]. \tag{44}$$

The $\partial^2_{(1,2,3)}$ designates the taking of derivatives of the first, second, and third fields.[9]

Interactions involving totally symmetric leading trajectory states are also of interest to us. The tachyon-tachyon–$Y = \{1^N, 0, 0\}$ cubic coupling with N derivatives is given by

$$-\frac{g_o \lambda_{\mathrm{sft}}}{\alpha'} \int d^{26}x \, \hat{E} \, \mathrm{tr}\left[A^{(Y)}_{M_1 \cdots M_N}(\phi(-i\overleftrightarrow{\partial}^{M_1}) \cdots (-i\overleftrightarrow{\partial}^{M_N})\phi)\right]\left(\frac{8\alpha'}{27}\right)^{\frac{N}{2}} \frac{1}{\sqrt{N!}} \tag{45}$$

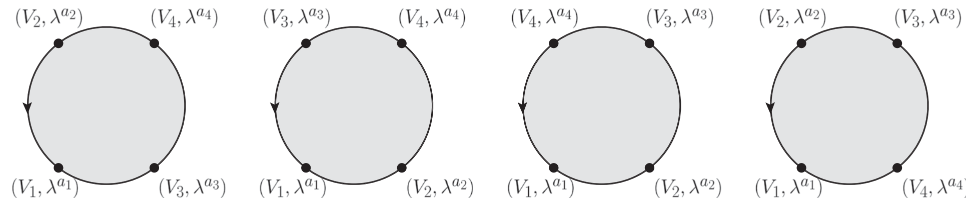
in the interaction part of the action. The photon (level-1 state)-photon–$Y = \{1^N, 0, 0\}$ coupling in the cubic string field theory includes

$$-\frac{g_o \lambda_{\mathrm{sft}}}{\alpha'} \int d^{26}x \, \hat{E} \, \mathrm{tr}\left[A^{(N)}_{M_1 \cdots M_N}(A_L(-i\overleftrightarrow{\partial}^{M_1}) \cdots (-i\overleftrightarrow{\partial}^{M_N})A_K)\left(\frac{8\alpha'}{27}\right)^{\frac{N}{2}} \frac{\eta^{KL}\frac{16}{27}}{\sqrt{N!}} + \cdots\right], \tag{46}$$

where we kept only the terms that have N derivatives and are proportional to η^{KL}, as they are necessary in deriving (61).

[9] Concretely,

$$\hat{E}A(x)B(x)C(x) = \left[\left(\frac{27}{16}\right)^{\frac{\alpha'}{2}\partial^2} A(x)\right]\left[\left(\frac{27}{16}\right)^{\frac{\alpha'}{2}\partial^2} B(x)\right]\left[\left(\frac{27}{16}\right)^{\frac{\alpha'}{2}\partial^2} C(x)\right].$$



(a) 1342 (b) 1243 (c) 1234 (d) 1432

FIG. 3. Disc amplitudes with two photon vertex operators (V_1 and V_2) and two tachyon vertex operators (V_3 and V_4) inserted. Kinematical amplitudes given by the disc amplitudes above are multiplied by the Chan-Paton factors $\mathrm{tr}[\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}]$ in (a), $\mathrm{tr}[\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}]$ in (b), $\mathrm{tr}[\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}]$ in (c) and $\mathrm{tr}[\lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}]$ in (d), respectively. The two disc amplitudes (a),(b) become $\mathcal{M}_{\mathrm{Ven}}(s,t)$, while (c),(d) become $\mathcal{M}_{\mathrm{Ven}}(u,t)$.

B. Cubic SFT scattering amplitude and *t*-channel expansion

Before proceeding to study the $h+\gamma^* \to h+\gamma^{(*)}$ scattering amplitude by using the cubic string field theory on the warped spacetime background, let us remind ourselves how to obtain t-channel operator product expansion from the amplitude calculation based on string field theory, by using tachyon-photon scattering on the flat spacetime as an example.

Let us consider the disc amplitude of tachyon-photon scattering. The vertex operators labeled by $i=1,2$, $V_i = :\epsilon^i_M \partial X^M e^{ik_i\cdot X}:$ are for photon incoming ($i=1$) and outgoing ($i=2$) states, which come with Chan-Paton matrices λ^{a_i}. Tachyon incoming ($i=3$) and outgoing ($i=4$) states correspond to vertex operators $V_i = :e^{ik_i\cdot X}:$ with Chan-Paton matrices λ^{a_i}. The photon-tachyon scattering amplitude $A+\phi \to A+\phi$ in bosonic open string theory (Veneziano amplitude) is given by[10]

$$\mathcal{M}_{\mathrm{Ven}}(s,t) = -\left(\frac{g_o^2}{\alpha'}\right)\frac{\Gamma(-\alpha' t-1)\Gamma(-\alpha' s-1)}{\Gamma(-\alpha'(s+t)-1)}\epsilon_M(k_2)\epsilon_N(k_1)\left\{\left[\eta^{MN} - \frac{k_1^M k_2^N}{k_1\cdot k_2}\right](\alpha' s+1) + 2\alpha'\left(\left[p^M - k_1^M\frac{k_2\cdot p}{k_1\cdot k_2}\right] - \frac{k_2^M}{2}\right)\left(\left[p^N - k_2^N\frac{k_1\cdot p}{k_2\cdot k_1}\right] - \frac{k_1^N}{2}\right)(\alpha' t+1)\right\}, \tag{47}$$

which is to be multiplied by the Chan-Paton factor $\mathrm{Tr}[\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}\lambda^{a_1} + \lambda^{a_4}\lambda^{a_2}\lambda^{a_1}\lambda^{a_3}]$ [see Figs. 3(a) and 3(b)]. If the Chan-Paton matrices of a pair of incoming and outgoing vertex operators, λ^{a_1} and λ^{a_2}, commute with each other,[11] then the Chan-Paton factors from Figs. 3(c) and 3(d) are the same, and the total kinematical part of the amplitude for this Chan-Paton factor becomes $\mathcal{M}_{\mathrm{Ven}}(s,t) + \mathcal{M}_{\mathrm{Ven}}(u,t)$.

Let us stay focused on $\mathcal{M}_{\mathrm{Ven}}(s,t)$ alone for now. The amplitude proportional to η^{MN} can be expanded, as is well known, as a sum only of t-channel poles[12]:

$$\frac{g_o^2}{\alpha'}\frac{\Gamma(-\alpha' t-1)\Gamma(-\alpha' s)}{\Gamma(-\alpha'(s+t)-1)} = \frac{g_o^2}{\alpha'}\int_0^1 dx\, x^{-\alpha' t-2}(1-x)^{-\alpha' s-1}, \tag{48}$$

$$= \frac{g_o^2}{\alpha'}\sum_{N=0}^{\infty}\frac{-1}{\alpha' t-(N-1)}\frac{(\alpha' s+1)\cdots(\alpha' s+N)}{N!}. \tag{49}$$

The Veneziano amplitude (47) can also be obtained in cubic string field theory [32]. In the cubic SFT, the scattering amplitude consists of two pieces, a collection of t-channel exchange diagrams and s-channel diagrams (Fig. 4):

$$\mathcal{M}_{\mathrm{Ven}}(s,t) = \sum_Y \mathcal{M}_Y^{(t)}(s,t) + \sum_Y \mathcal{M}_Y^{(s)}(s,t). \tag{50}$$

Infinitely many one string states (39) with zero ghost number ($h_b = h_c$)—labeled by Y—can be exchanged in the t channel or the s channel, and the corresponding contributions are in the form of

$$\mathcal{M}_Y^{(t)} = \frac{f_Y^{(t)}(s,t)}{-\alpha' t-1+N^{(Y)}}, \qquad \mathcal{M}_Y^{(s)} = \frac{f_Y^{(s)}(t,s)}{-\alpha' s-1+N^{(Y)}}, \tag{51}$$

[10]Here, $p := (k_3-k_4)/2$, the averaged momentum of the tachyon before and after the scattering, just like in (2).

[11]Just like in the case where both λ^{a_1} and λ^{a_2} are an $N_F\times N_F$ matrix diag$(2/3,-1/3,-1/3)$.

[12]It is also possible to expand this as a sum of s-channel poles only; that is the celebrated s-t duality of the Veneziano amplitude.

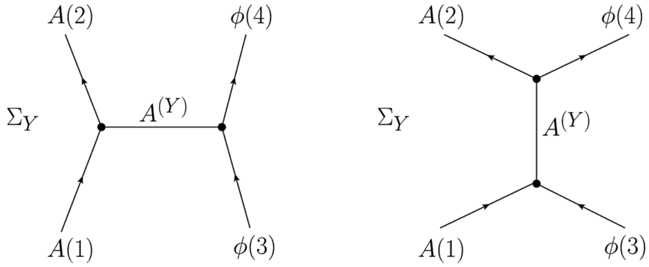


FIG. 4. Two types of diagrams contribute to the photon-tachyon scattering amplitude $\mathcal{M}_{\rm Ven}(s,t)$ in cubic string field theory: the t-channel exchange of one string states labeled by Y (left panel) and the s-channel exchange (right panel).

where $f_Y^{(t)}$ and $f_Y^{(s)}$ are regular functions at finite s and t; $N^{(Y)}$ is the excitation level (40) of a component field $A^{(Y)}$.

Because both the world-sheet calculation (47), (49) and the cubic SFT calculation (50), (51) are the same thing, $\mathcal{M}_{\rm Ven}(s,t)$ in both approaches should be exactly the same functions of (s,t). Therefore, for an arbitrary given value of s, the residue of all the poles in the complex t plane should be the same. We also know that the Veneziano amplitude can be expanded purely in the infinite sum of t-channel poles with t-independent residues. This means that the full Veneziano amplitude (47) can be reproduced just from the t-channel cubic SFT amplitude[13] $\sum_Y \mathcal{M}_Y^{(t)}(s,t)$ through the following procedure:

$$\sum_Y \frac{f_Y^{(t)}(s,t)}{-\alpha' t - 1 + N^{(Y)}} \to \sum_Y \frac{f_Y^{(t)}(s,(N^{(Y)}-1)/\alpha')}{-\alpha' t - 1 + N^{(Y)}} = \mathcal{M}_{\rm Ven}(s,t). \tag{52}$$

To see that this prescription really works, let us take a look at the amplitudes of t-channel exchange of one string states with small excitation level $N^{(Y)} = 0, 1, 2$. Focusing on the amplitude of $A + \phi \to A + \phi$ proportional to η^{MN}, we find that the tachyon exchange in the t channel [Fig. 5(a)] gives rise to the amplitude [33]

$$\mathcal{M}_\phi^{(t)}(s,t) = \left(\frac{g_o \lambda_{\rm sft}}{\alpha'}\right)^2 \left(\frac{2}{3^{3/4}}\right)^{-2\alpha' t - 2\alpha' t + 4} \frac{-1}{t + 1/\alpha'} = \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t + 1} \frac{-1}{\alpha' t + 1}, \tag{53}$$

which is obtained simply by using the ϕ-ϕ-ϕ vertex rule (43) and the A-A-ϕ vertex rule (45). The prescription (52) turns this amplitude into

[13] The t-channel and s-channel amplitudes of the cubic SFT, $\sum_Y \mathcal{M}_Y^{(t)}$ and $\sum_Y \mathcal{M}_Y^{(s)}$, correspond to the integration over $[0, 1/2]$ and $[1/2, 1]$, respectively, in (48) [32]. Thus, $\sum_Y \mathcal{M}_Y^{(s)}$ does not contain a pole in t.

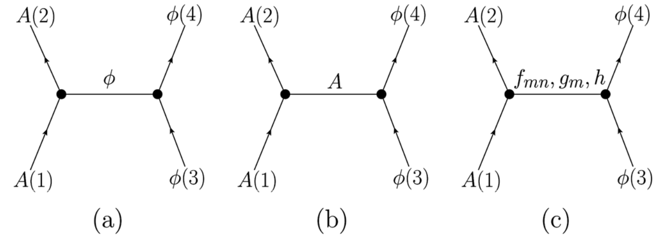


FIG. 5. t-channel exchange diagrams for $A + \phi \to A + \phi$ scattering in cubic string field theory. The tachyon ($N = 0$), photon ($N = 1$), and level-2 states are exchanged in diagrams (a), (b), and (c), respectively.

$$\to \mathcal{M}_\phi(s,t) = \frac{g_o^2}{\alpha'} \frac{-1}{\alpha' t + 1}, \tag{54}$$

which reproduces the $N = 0$ term of (49).

The t-channel exchange of level $N^{(Y)} = 1$ excited states can also be calculated in the cubic string field theory [Fig. 5(b)]. The amplitude proportional to η^{MN} is

$$\mathcal{M}_A^{(t)}(s,t) = \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t} \frac{-1}{\alpha' t} \left[\frac{\alpha'(s-u)}{2}\right], \tag{55}$$

where $(s-u) = (k_{(1)} - k_{(2)}) \cdot (k_{(4)} - k_{(3)})$. Using the relation $\alpha'(s+t+u) = -2$ in the tachyon-photon scattering to eliminate u in favor of s and t, and following the prescription (52)—which is to exploit $\alpha' t = 0$ in the numerator—this amplitude is replaced by [33]

$$\longrightarrow \mathcal{M}_A(s,t) = \frac{g_o^2}{\alpha'} \frac{-(\alpha' s + 1)}{\alpha' t}. \tag{56}$$

Once again, this reproduces the level $N = 1$ contribution to the Veneziano amplitude (49).

A similar calculation for level-2 state exchange can be carried out [Fig. 5(c)]. Using the vertex rule in (43) for the level-2–ϕ-ϕ couplings, and also the interactions among level-2–A-A couplings in the literature, the cubic SFT t-channel amplitude is given by [33]

$$\mathcal{M}_f^{(t)}(s,t) = \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t - 1} \frac{-1}{\alpha' t - 1} \times \left[\frac{(\alpha'(s-u))^2}{8} - \frac{5(\alpha' t + 2)}{16 \cdot 2} + \frac{490}{16^2 \cdot 2}\right], \tag{57}$$

$$\mathcal{M}_g^{(t)}(s,t) = \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t - 1} \frac{-1}{\alpha' t - 1} \left[-\frac{36\alpha' t}{16^2}\right], \tag{58}$$

$$\mathcal{M}_h^{(t)}(s,t) = \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t - 1} \frac{-1}{\alpha' t - 1} \left[-\frac{11^2}{16^2}\right]. \tag{59}$$

After using $\alpha' u = -\alpha'(s+t) - 2$ to eliminate u in favor of s and t, and further following the prescription (52) ($\alpha' t \to 1$

in the numerator), one will see that the level $N^{(Y)} = 2$ amplitude turns into

$$\to (\mathcal{M}_f + \mathcal{M}_g + \mathcal{M}_h)(s,t) = \frac{g_o^2}{\alpha'} \frac{-1}{\alpha' t - 1} \left[\frac{(\alpha' s)^2 + 3(\alpha' s) + 2}{2} \right]. \quad (60)$$

Once again, this is precisely the same as the $N = 2$ contribution to the Veneziano amplitude (49).

Contributions from the t-channel exchange of states in the leading trajectory can also be examined systematically. Using the vertex rule (45), (46) involving the states in the leading trajectory ($Y = \{1^N, 0, 0\}$), one finds that the amplitude proportional to η^{MN} is

$$\mathcal{M}^{(t)}_{\{1^N,0,0\}} \simeq \frac{g_o^2}{\alpha'} \left(\frac{27}{16}\right)^{\alpha' t - (N-1)} \frac{-1}{\alpha' t - (N-1)} \frac{(\alpha'(s-u)/2)^N}{N!}, \quad (61)$$

where we maintained only the terms with the highest power of either s or u. After using the kinematical relation $\alpha'(s+t+u) + 2 = 0$ to eliminate u in favor of s and t, and following the prescription (52) [$\alpha' t \to (N-1)$ in the numerator], we obtain the large-(α's) leading power contribution to the Nth term of (49) with the correct coefficient.

We have, therefore, seen that the prescription (52) allows us to use the t-channel exchange amplitude in the cubic string field theory to construct the full disc scattering amplitude. In Sec. VI, this prescription is extended for the disc scattering amplitudes on a spacetime with a curved background metric, which is the situation of real interest in the context of hadron scattering.

V. MODE DECOMPOSITION ON AdS_5

Let us now proceed to work out mode decomposition of the totally symmetric (traceless) component field on the warped spacetime. The correspondence between the primary operators of the conformal field theory on the (UV) boundary and wave functions on AdS_5 is made clear in this section. The Pomeron/Reggeon wave functions are obtained as a *holomorphic* function of the spin variable j since we need to do so for the further inverse Mellin transformation. The wave functions will then also be used to construct the scattering amplitude of $h + \gamma^* \to h + \gamma^{(*)}$ and GPD in Secs. VI and VII.

Let the bilinear (free) part of the (bulk) action of a rank-j tensor field on AdS_5 be[14]

$$S_{\text{eff kin}} = -\frac{1}{2}\frac{t_{Ay}}{R^3} \int d^4x \int dz \sqrt{-g(z)} g^{m_1 n_1} \cdots g^{m_j n_j} \times \left[g^{m_0 n_0} (\nabla_{m_0} A^{(y)}_{m_1 \cdots m_j})(\nabla_{n_0} A^{(y)}_{n_1 \cdots n_j}) + \left(\frac{c_y}{R^2} + \frac{N^{(y)}_{\text{eff}}}{\alpha'}\right) A^{(y)}_{m_1 \cdots m_j} A^{(y)}_{n_1 \cdots n_j} \right], \quad (62)$$

where we assume that kinetic mixing between different fields is either absent or sufficiently small. Here, the dimensionless parameter $N^{(y)}_{\text{eff}}$ is $N^{(Y)} - 1$ for an $N^{(Y)} \in \mathbf{Z}_{\geq 0}$ for bosonic open string ($j \leq N^{(Y)}$), which would be $4(N^{(Y)} - 1)$ for an $N^{(Y)} \in \mathbf{Z}_{\geq 1}$ for closed string ($j \leq 2N^{(Y)}$). This field is regarded as a reduction of some field with spherical harmonics on the internal manifold,[15] and hence $j \leq h_a$, in general. Another dimensionless coefficient c_y may contain a contribution from the "mass" associated with the spherical harmonics over the internal manifold and may also include the ambiguity (which is presumably of order unity) associated with making d'Alembertian of the flat metric background covariant.[16] The combination $(c_y/R^2 + N^{(y)}_{\text{eff}}/\alpha')$ is denoted by M^2_{eff}.

The equation of motion (in the bulk part)[17] then becomes

$$g^{m_1 m_2} (\nabla_{m_1} \nabla_{m_2} A^{(y)}_{n_1 \cdots n_j}) - \left(\frac{c_y}{R^2} + \frac{N^{(y)}_{\text{eff}}}{\alpha'}\right) A^{(y)}_{n_1 \cdots n_j} = 0. \quad (64)$$

Solutions to this equation of motion can be obtained from solutions of the following eigenmode equation,[18]

$$\nabla^2 A_{m_1 \cdots m_j} = -\frac{\mathcal{E}}{R^2} A_{m_1 \cdots m_j}, \quad (65)$$

[14] The dimensionless constant t_{Ay} is something like N_c^2 for a mode obtained by a reduction of closed string component fields in higher dimensions. More comments on t_{Ay} for open string states is found in footnote 27.

[15] The internal manifold would be a five-dimensional one, W, for closed string modes in type IIB, and a three-cycle one for open string states on the flavor D7-branes. For sufficiently small x, however, amplitudes of exchanging modes with nontrivial spherical harmonics on these internal manifolds are relatively suppressed, and we are not interested very much.

[16] The ambiguity in c_y/R^2 includes insertion of the curvature tensor,

$$([\nabla_M, \nabla_N])^{Q'}_{Q} = -\Gamma^{Q'}_{QN,M} + \Gamma^{Q'}_{QM,N} + \Gamma^{L}_{QM}\Gamma^{Q'}_{LN} - \Gamma^{L}_{QN}\Gamma^{Q'}_{LM} = \frac{\delta^{Q'}_{M} g_{QN} - \delta^{Q'}_{N} g_{QM}}{R^2}, \quad (63)$$

which vanishes in flat space. Depending on details of how it is inserted, the value of c_y may not be the same for all the individual irreducible components of SO(4,1) in a rank-j tensor field $A_{m_1 \cdots m_j}$.

[17] There is also an IR boundary part of the equation motion. We will come back to this issue in Sec. V D.

[18] The differential operator $\nabla^2 := g^{mn}\nabla_m \nabla_n$ is Hermitian under the measure $d^4x dz \sqrt{-g(z)} g^{m_1 n_1} \cdots g^{m_j n_j}$.

by imposing the on-shell condition

$$\frac{(\mathcal{E}+c_y)}{\sqrt{\lambda}}+N^{(y)}_{\rm eff}=0 \qquad (\text{i.e.,}\quad \mathcal{E}+R^2M^2_{\rm eff}=0). \tag{66}$$

We will work out the eigenmode decomposition for rank-j tensor fields in the following, where we have to work only for a separate j, without referring to the mass parameter.[19]

The eigenmode wave functions are used not just for a construction of solutions to the equation of motions, but also in constructing the Reggeon exchange contributions to the $h+\gamma^*\to h+\gamma^{(*)}$ scattering amplitude. The propagator is proportional to

$$\frac{-i}{\frac{\mathcal{E}+c_y}{\sqrt{\lambda}}+N^{(y)}_{\rm eff}-i\epsilon}\frac{\alpha' R^3}{t_{Ay}}. \tag{67}$$

The mode equation for a rank-j tensor field $A_{m_1\cdots m_j}$ on AdS_5 is further decomposed into those of irreducible representations of SO(4,1). For simplicity of argument, we deal only with the mode equations for the totally symmetric (and traceless) rank-j tensor fields. Namely,

$$A_{m_1\cdots m_j}=A_{m_{\sigma(1)}\cdots m_{\sigma(j)}} \quad \text{for } {}^\forall\sigma\in\mathfrak{S}_j. \tag{68}$$

We call them spin-j fields.

The eigenmode equation (65) for a totally symmetric spin-j field can be decomposed into $j+1$ pieces, labeled by $k=0,\ldots,j$:

$$\begin{aligned}((R^2\Delta_j)&-[(2k+1)j-2k^2+3k])A_{z^k\mu_1\cdots\mu_{j-k}}\\ &+2zk\partial^{\hat\rho}A_{z^{k-1}\rho\mu_1\cdots\mu_{j-k}}+k(k-1)A^{\hat\rho}_{z^{k-2}\rho\mu_1\cdots\mu_{j-k}}\\ &-2z(D[A_{z^{k+1}\cdots}])_{\mu_1\cdots\mu_{j-k}}+(E[A_{z^{k+2}\cdots}])_{\mu_1\cdots\mu_{j-k}}\\ =&-\mathcal{E}A_{z^k\mu_1\cdots\mu_{j-k}}.\end{aligned} \tag{69}$$

Here,

$$A_{z^k\mu_1\cdots\mu_{j-k}}:=A_{\underbrace{z\cdots z}_{k}\mu_1\cdots\mu_{j-k}} \tag{70}$$

and can be regarded as a rank-$(j-k)$ totally symmetric tensor of the SO(3,1) Lorentz group. The SO(3,1) indices with ˆ in the superscript, such as $\hat\rho$ in $\partial^{\hat\rho}$, are raised by the four-dimensional (4D) Minkowski metric $\eta^{\rho\sigma}$ from a subscript σ, not by the five-dimensional (5D) warped metric g^{mn}. $D[a]$ and $E[a]$ are operations creating totally symmetric rank-$(r+1)$ and rank-$(r+2)$ tensors of SO(3,1), respectively, from a totally symmetric rank-r tensor of SO (3,1), a:

$$(D[a])_{\mu_1\cdots\mu_{r+1}}:=\sum_{i=1}^{r+1}\partial_{\mu_i}a_{\mu_1\cdots\check{\mu_i}\cdots\mu_{r+1}}, \tag{71}$$

$$(E[a])_{\mu_1\cdots\mu_{r+2}}:=2\sum_{p<q}\eta_{\mu_p\mu_q}a_{\mu_1\cdots\check{\mu_p}\cdots\check{\mu_q}\cdots\mu_{r+2}}. \tag{72}$$

The differential operator Δ_j in the first term is defined, as in [3], by

$$\begin{aligned}R^2\Delta_j&:=R^2z^{-j}\left[\left(\frac{z}{R}\right)^5\partial_z\left[\left(\frac{R}{z}\right)^3\partial_z\right]\right]z^j+R^2\left(\frac{z}{R}\right)^2\partial^2,\\ &=z^2\partial_z^2+(2j-3)z\partial_z+j(j-4)+z^2\partial^2.\end{aligned} \tag{73}$$

The eigenmode equation (65), (69) is a generalization of the "Schrödinger equation" of [3] determining the Pomeron wave function. As we will see, the single-component Pomeron wave function discussed in [3], etc. corresponds to (93)—that of the $(n,l,m)=(0,0,0)$ eigenmode in our language, and the Schrödinger equation to (90), (A10); there are other eigenmodes, whose wave functions are to be determined in the following.

In the following sections, V A and V B, we simply state the results of the eigenmode decomposition of (65), (69) for spin-j fields. A more detailed account is given in Appendix A.

A. Eigenvalues and eigenmodes for $\Delta^\mu=0$

Because of the $(3+1)$-dimensional translational symmetry in ∇^2, solutions to the eigenmode equations can be classified by the eigenvalues of the generators of translation, $(-i\partial_\mu)$. Until the end of Sec. V B, we will focus on eigenmodes in the form of

$$A_{m_1\cdots m_j}(x,z)=e^{i\Delta\cdot x}A_{m_1\cdots m_j}(z;\Delta) \tag{74}$$

and study the eigenmode equation (65) separately for different eigenvalues Δ^μ.

The eigenmode equation for $\Delta^\mu=0$ and that for $\Delta^\mu\neq 0$ are qualitatively different and need separate study. The eigenmodes for $\Delta^\mu\neq 0$ will be presented in Sec. V B (and Appendix A.2); we begin in Sec. V A (and Appendix A.1) with the eigenmode equation for $\Delta^\mu=0$, which is also regarded as an approximation of the eigenmode equation for $\Delta^\mu\neq 0$ in the asymptotic UV boundary region (at the least, $\Delta z\ll 1$ and z may be as small as R).

For now, we relax the traceless condition on the spin-j field $A_{m_1\cdots m_j}$ $(m_i=0,1,\ldots,3,z)$, and we just assume that the rank-j tensor field $A_{m_1\cdots m_j}$ is totally symmetric.[20]

[19]There are many states with the same value of j, but with different c_y and $N^{(y)}_{\rm eff}$.

[20]This only makes the following presentation more far-reaching; in the end, it is quite easy to identify which eigenmodes fall into the traceless part within $A_{m_1\cdots m_j}$. See (82)–(84) at the end of Sec. V A.

Consider the following decomposition of the space of z-dependent field configuration $A_{m_1\cdots m_j}(z;\Delta=0)$:

$$A_{z^k\mu_1\cdots\mu_{j-k}}(z;\Delta^\mu=0)=\sum_{N=0}^{[(j-k)/2]}(E^N[a^{(k,N)}])_{\mu_1\cdots\mu_{j-k}};\qquad(75)$$

here, $(a^{(k,N)}(z;\Delta^\mu=0))_{\mu_1\cdots\mu_{j-k-2N}}$ is a rank-$(j-k-2N)$ totally symmetric tensor of SO(3,1), and it satisfies the 4D-traceless condition

$$\eta^{\hat\mu_1\hat\mu_2}a^{(k,N)}_{\mu_1\cdots\mu_{j-k-2N}}=0.\qquad(76)$$

Thus, the field configuration can be described by $a^{(k,N)}$'s with $0\le k\le j$, $0\le N\le[(j-k)/2]$. These components form groups labeled by $n=0,\ldots,j$, where the nth group consists of $a^{(k,N)}$'s, with $k+2N=n$; they are all rank-$(j-n)$ totally symmetric tensors of SO(3,1); let us call the subspace spanned by the components in this nth group the nth subspace. The eigenmode equation for $\Delta^\mu=0$ becomes block diagonal under the decomposition into these subspaces labeled by $n=0,\ldots,j$ [see (A3) in the appendix]. Therefore, the eigenmode equation for $\Delta^\mu=0$ can be studied separately for the individual diagonal blocks.

The nth diagonal block contains $[n/2]+1$ components, and hence there are $[n/2]+1$ eigenmodes. Let $\mathcal{E}_{n,l}$ $(l=0,\ldots,[n/2])$ be the eigenvalues in the nth diagonal block. The corresponding eigenmode wave function is of the form

$$(a^{(k,N)}(z;\Delta^\mu=0))_{\mu_1\cdots\mu_{j-n}}=c_{k,l,n}(\epsilon^{(n,l)})_{\mu_1\cdots\mu_{j-n}}z^{2-j-i\nu},\qquad(77)$$

where $\epsilon^{(n,l)}$ is a z-independent, k-independent rank-$(j-n)$ tensor of SO(3,1) ($c_{k,l,n}\in\mathbb{R}$). In the eigenmode equation for $\Delta^\mu=0$, the eigenmode wave functions are all in a simple power of z, and the power is parametrized by $i\nu$ ($\nu\in\mathbb{R}$). The eigenvalues $\mathcal{E}_{n,l}$ are functions of ν; once the mass-shell condition (66) is imposed, the eigenmodes turn into solutions of the equation of motion and $i\nu$ is determined by the mass parameter.

The eigenmodes with smaller (n,l) are as follows:

$$\mathcal{E}_{0,0}=(j+4+\nu^2),\quad a^{(0,0)}(z)_{\mu_1\cdots\mu_j}=\epsilon^{(0,0)}_{\mu_1\cdots\mu_j}z^{2-j-i\nu},\qquad(78)$$

$$\mathcal{E}_{1,0}=(3j+5+\nu^2),\qquad a^{(1,0)}(z)_{\mu_1\cdots\mu_{j-1}}=\epsilon^{(1,0)}_{\mu_1\cdots\mu_{j-1}}z^{2-j-i\nu},\qquad(79)$$

$$\mathcal{E}_{2,0}=(5j+4+\nu^2),$$
$$\begin{pmatrix}a^{(0,1)}(z)_{\mu_1\cdots\mu_{j-2}}\\ a^{(2,0)}(z)_{\mu_1\cdots\mu_{j-2}}\end{pmatrix}=\begin{pmatrix}1\\ -4j\end{pmatrix}\epsilon^{(2,0)}_{\mu_1\cdots\mu_{j-2}}z^{2-j-i\nu},\qquad(80)$$

$$\mathcal{E}_{2,1}=(j+2+\nu^2),$$
$$\begin{pmatrix}a^{(0,1)}(z)_{\mu_1\cdots\mu_{j-2}}\\ a^{(2,0)}(z)_{\mu_1\cdots\mu_{j-2}}\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}\epsilon^{(2,1)}_{\mu_1\cdots\mu_{j-2}}z^{2-j-i\nu}.\qquad(81)$$

Empirically, the j dependence of the eigenvalues in the nth diagonal block appears to be $\mathcal{E}_{n,l}=((2n+1-4l)j+\nu^2+O(1))$ $(l=0,\ldots,[n/2])$ [see (A12)–(A27) in the Appendix for more samples of the eigenvalues], and we promote this j dependence as a rule of the labeling of the eigenmodes with l.

The eigenmode with $l=0$ is found in any one of the diagonal blocks ($n=0,\ldots,j$). Its eigenvalue is

$$\mathcal{E}_{n,0}=(2n+1)j+2n-n^2+4+\nu^2,\qquad(82)$$

and

$$c_{2\bar k,0,2\bar n}=(-)^{\bar k}4^{\bar k}\frac{\bar n!}{(\bar n-\bar k)!}\frac{(j-\bar n+1)!}{(j-\bar n-\bar k+1)!},\quad(n=2\bar n,\bar k=0,\ldots,\bar n),\qquad(83)$$

$$c_{2\bar k+1,0,2\bar n+1}=(-)^{\bar k}4^{\bar k}\frac{\bar n!}{(\bar n-\bar k)!}\frac{(j-\bar n)!}{(j-\bar n-\bar k)!},\quad(n=2\bar n+1,\bar k=0,\ldots,\bar n).\qquad(84)$$

These $(n,l)=(n,0)$ eigenmodes are characterized by the 5D-traceless condition

$$g^{m_1m_2}A_{m_1\cdots m_j}=0.$$

Thus, the eigenmodes within the 5D-traceless (and totally symmetric) component—the spin-j field—for $\Delta^\mu=0$ are labeled simply by $n=0,\ldots,j$.

B. Mode decomposition for nonzero Δ_μ

1. Diagonal block decomposition for the $\Delta^\mu\neq 0$ case

The eigenmode equation (65), (69) is much more complicated in the case of $\Delta^\mu\neq0$ because of the second and fourth terms in (69). The eigenmode equation is still made block diagonal for an appropriate decomposition of the space of field $A_{m_1\cdots m_j}(z;\Delta^\mu)$.

Consider a decomposition

$$A_{z^k\mu_1\cdots\mu_{j-k}}(z;\Delta^\mu)=\sum_{s=0}^{j-k}\sum_{N=0}^{[s/2]}(\tilde E^ND^{s-2N}[a^{(k,s,N)}])_{\mu_1\cdots\mu_{j-k}},\qquad(85)$$

where a new operation $a\mapsto\tilde E[a]$ on a totally symmetric SO(3,1) tensor a,

$$(\tilde{E}[a])_{\mu_1\cdots\mu_{r+2}} := 2\sum_{p<q}\left(\eta_{\mu_p\mu_q} - \frac{\partial_{\mu_p}\partial_{\mu_q}}{\partial^2}\right)a_{\mu_1\cdots\check{\mu_p}\cdots\check{\mu_q}\cdots\mu_{r+2}}, \tag{86}$$

is used. $a^{(k,s,N)}$'s are totally symmetric, 4D-traceless [i.e., (76)] rank-$(j-k-s)$ tensor fields of SO(3,1) that satisfy an additional condition, the 4D-transverse condition:

$$\partial^{\hat{\rho}}(a^{(k,s,N)})_{\rho\mu_2\cdots\mu_{j-k-s}} = i\Delta^{\hat{\rho}}(a^{(k,s,N)})_{\rho\mu_2\cdots\mu_{j-k-s}} = 0. \tag{87}$$

The space of field configuration $A_{m_1\cdots m_j}(z;\Delta^\mu)$ is now decomposed into $a^{(k,s,N)}$'s, with $0\le k\le j$, $0\le s\le j-k$, $0\le N\le [s/2]$; these components form groups labeled by $m=0,\ldots,j$, where the mth group consists of $a^{(k,s,N)}$'s, with $k+s=m$; they are all rank-$(j-m)$, totally symmetric 4D-traceless and 4D-transverse tensors of SO(3,1); let us call the subspace spanned by the components in this mth group the mth subspace. The eigenmode equation for $\Delta^\mu\neq 0$ becomes block diagonal under the decomposition into these subspaces labeled by $m=0,\ldots,j$. The eigenmode equation for the mth sector is given by (A39) in Appendix A.2. The mth subspace should have

$$\sum_{s=0}^{m}([s/2]+1) \tag{88}$$

eigenmodes.

Eigenvalues $\mathcal{E}$ are determined in terms of the characteristic exponent in the expansion of the solution in the power series of z. Let the first term in the expansion be $z^{2-j-i\nu}$; the eigenvalues are functions of ν then. Because the indicial equation at the regular singular point $z\simeq 0$ allows us to determine the eigenvalues in terms of ν, the eigenvalues in the case of $\Delta^\mu\neq 0$ cannot be different from the ones we have already known in the $\Delta^\mu=0$ case. In the mth diagonal block, the eigenvalues consist of $\mathcal{E}_{n,l}$, with $0\le n\le m$, $0\le l\le [n/2]$.

To summarize, the eigenmodes in the totally symmetric rank-j tensor field of SO(4,1) are labeled by (n,l,m) and Δ^μ and ν. Their eigenvalues $\mathcal{E}_{n,l}$ depend only on n and l (with $0\le n\le j$ and $0\le l\le [n/2]$) and ν. Corresponding eigenmodes are denoted by

$$\begin{aligned}A(x,z)^{n,l,m;\Delta,\nu}_{z^k\mu_1\cdots\mu_{j-k}} &= e^{i\Delta\cdot x}A^{n,l,m}_{z^k\mu_1\cdots\mu_{j-k}}(z;\Delta^\mu,\nu)\\ &= e^{i\Delta\cdot x}\sum_{N=0}^{[s/2]}\tilde{E}^N D^{s-2N}[\epsilon^{(n,l,m)}]\frac{b^{(j-m)}_{s,N}}{\Delta^{s-2N}}\Psi^{(j);s,N}_{i\nu;n,l,m}(-\Delta^2,z).\end{aligned} \tag{89}$$

$\epsilon^{(n,l,m)}$ is a (z-independent) totally symmetric 4D-traceless 4D-transverse rank-$(j-m)$ tensor of SO(3,1), and all the s's appearing in the expression above are understood as $s=m-k$. $b^{(r)}_{s,N}$ is a constant whose definition is given in (A38) in the Appendix.

2. Single-component Pomeron wave function

The Pomeron wave function that has been discussed in the literature (e.g., [3]) does not look as awful as (89). To our knowledge, the Pomeron wave function in the literature in the context of hadron high-energy scattering has been a single component one, $\Psi_{i\nu}(t,z)$. How is $A^{n,l,m}_{m_1\cdots m_j}(z;\Delta^\mu,\nu)$ related to $\Psi_{i\nu}(-\Delta^2;z)$?

In the block diagonal decomposition of the eigenmode equation, there is only one subspace where the diagonal block is 1×1. That is the $m=0$ subspace, which consists only of $a^{(0,0,0)}$. The eigenmode equation is

$$\left[\Delta_j - \frac{j}{R^2}\right]a^{(0,0,0)}(z;\Delta^\mu) = -\frac{\mathcal{E}}{R^2}a^{(0,0,0)}(z;\Delta^\mu). \tag{90}$$

This equation, as well as (A10) in the $\Delta^\mu=0$ case, corresponds to the Schrödinger equation in [3] determining the Pomeron wave function. It should be noted, however, that we consider that ∇^2 is the operator relevant to the eigenmode decomposition[21] rather than Δ_j; furthermore, the operator ∇^2 and Δ_j has a simple relation $\nabla^2=\Delta_j - j/R^2$ only on this $m=0$th subspace of a totally symmetric rank-j tensor field of SO(4,1).

The eigenvalue is

$$\mathcal{E}_{0,0} = (j+4+\nu^2), \tag{91}$$

when we define the first term in the power series expansion of z to be $z^{2-j-i\nu}$. The eigenmode wave function is

$$a^{(0,0,0)}(z;\Delta^\mu)_{\mu_1\cdots\mu_j} = \epsilon^{(0,0,0)}_{\mu_1\cdots\mu_j}\Psi^{(j)}_{i\nu}(-\Delta^2,z), \tag{92}$$

$$\Psi^{(j)}_{i\nu}(-\Delta^2,z) := \frac{2}{\pi}\sqrt{\frac{\nu\sinh(\pi\nu)}{2R}}e^{(j-2)A}K_{i\nu}(\Delta z), \tag{93}$$

where $e^{2A(z)}=(R/z)^2$ is the warp factor introduced in (28). The normalization factor is determined [3][22] so that it satisfies the normalization condition[23]

[21]Thus, the propagator (67) uses the eigenvalue of ∇^2, rather than that of Δ_j. The eigenvalue $\mathcal{E}$ of ∇^2 in the $m=0$th subspace is $(j+4+\nu^2)$ as in (91), instead of $(4+\nu^2)$. Reference [3] uses a mode $h_{mn}\propto z^{-2}(\eta_{\mu\nu},\delta_{zz})$ of the spin-2 field to fix the details of (65), (66) and (90). This $h_{mn}\propto z^{-2}(\eta_{\mu\nu},\delta_{zz})$ mode, however, corresponds to the $(n,l)=(2,1)$ mode of the spin-$j=2$ field in (A17), rather than the 5D-traceless 5D-transverse mode $(n,l)=(0,0)$. The eigenvalue $\mathcal{E}_{2,1}=(2+j+\nu^2)$ with $j=2$ becomes $(4+\nu^2)$, though.

[22]The Pomeron wave function in [7] was of the form (124), which becomes (93) in the limit of $\Lambda\to 0$, while we keep z and Δ^μ fixed.

[23]The normalization condition is generalized to (99) later on.

$$\int d^4x \int dz\sqrt{-g(z)}e^{-2jA}[e^{i\Delta\cdot x}\Psi^{(j)}_{i\nu}(-\Delta^2,z)] \times [\Psi^{(j)}_{i\nu'}(-\Delta'^2,z)e^{-i\Delta'\cdot x}] = (2\pi)^4\delta^4(\Delta-\Delta')\delta(\nu-\nu'). \tag{94}$$

The single component Pomeron/Reggeon wave function $\Psi^{(j)}_{i\nu}(-\Delta^2,z)$ is now understood as $\Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2,z)$.

3. 5D-traceless 5D-transverse Modes

The eigenmode equation (65) for a totally symmetric rank-j tensor field of SO(4,1) should be closed within its 5D-traceless component. The subspace of 5D-traceless component is characterized by the 5D-traceless condition

$$g^{m_1m_2}A_{m_1\cdots m_j}(z;\Delta^\mu) = 0. \tag{95}$$

The fact that the Hermitian operator ∇^2 maps this subspace to itself implies that the eigenmode equation of ∇^2 is block diagonal, when the space of (not-necessarily-5D-traceless) $A_{m_1\cdots m_j}$ is decomposed into the sum of the 5D-traceless subspace and its orthogonal complement. The collection of the eigenmodes with $l=0$ corresponds to the subspace of 5D-traceless field configuration.

Similarly, one can think of a subspace of field configuration satisfying both the 5D-traceless condition (95) and the 5D-transverse condition

$$g^{nm_1}\nabla_n A_{m_1m_2\cdots m_j} = 0. \tag{96}$$

Obviously this is a subspace of the subspace of the 5D-traceless modes we discussed above. Since the Hermitian operator ∇^2 on AdS_5 maps this new subspace also to itself, the eigenmode equation of ∇^2 should also become block diagonal when the subspace of 5D-traceless modes is decomposed into this new subspace and its orthogonal complement.

As we will see in Appendix A.3, there is only one such mode satisfying this set of conditions (95), (96) in each one of the mth diagonal block. Thus, the combination of the 5D-traceless and 5D-transverse conditions allows us to determine an eigenmode completely. This mode turns out to be $(n,l,m)=(0,0,m)$ (for $0\le m\le j$). Put differently, the eigenmodes with the eigenvalue $\mathcal{E}_{n,l}=\mathcal{E}_{0,0}=(j+4+\nu^2)$ are characterized by the traceless and transverse conditions on AdS_5.

The eigenmode wave functions of the 5D-traceless transverse modes $(n,l,m)=(0,0,m)$ are (see Appendix A.3)

$$\Psi^{(j);s,N}_{i\nu;0,0,m}(-\Delta^2,z) = \sum_{a=0}^{N}(-)^a{}_NC_a\left(\frac{z^3\partial_z z^{-3}}{\Delta}\right)^{s-2a}[(z\Delta)^m\Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2,z)]\times N_{j,m}. \tag{97}$$

$N_{j,m}$ is a dimensionless normalization constant. We choose it to be[24]

$$N^{-2}_{j,m} = {}_jC_m\frac{\Gamma(j+1-i\nu)}{\Gamma(j+1-m-i\nu)}\frac{\Gamma(j+1+i\nu)}{\Gamma(j+1-m+i\nu)}\frac{\Gamma(3/2+j-m)}{2^m\Gamma(3/2+j)}\frac{\Gamma(2+2j)}{\Gamma(2+2j-m)}, \tag{98}$$

so that the eigenmode wave functions are normalized as in

$$\int d^4x\int_0 dz\sqrt{-g(z)}g^{m_1n_1}\cdots g^{m_jn_j}A^{n,l,m;\Delta,\nu}_{m_1\cdots m_j}(x,z)A^{n',l',m';\Delta',\nu'}_{n_1\cdots n_j}(x,z) = (2\pi)^4\delta^4(\Delta+\Delta')\delta(\nu-\nu')\delta_{n,n'}\delta_{l,l'}\delta_{m,m'}[\epsilon^{(n,l,m)}(\Delta)]\cdot[\epsilon^{(n',l',m')}(\Delta')]. \tag{99}$$

Here, $[\epsilon^{(n,l,m)}]\cdot[\epsilon'^{(n,l,m)}] := \epsilon^{(n,l,m)}_{\mu_1\cdots\mu_{j-m}}\epsilon'^{(n,l,m)}_{\nu_1\cdots\nu_{j-m}}\eta^{\hat\mu_1\hat\nu_1}\cdots\eta^{\hat\mu_{j-m}\hat\nu_{j-m}}$.

4. Propagator

The propagator of the totally symmetric rank-j tensor field (respectively, the spin-j field) on AdS_5 is given by summing up propagators of the (n,l,m) modes [respectively, the (n,l,m) modes with $l=0$]. For the purpose of writing down the propagator of a given (n,l,m) eigenmode, it is convenient to introduce the following notation:

$$A^{n,l,m;\Delta,\nu}_{m_1\cdots m_j}(x,z) = [A^{n,l,m;\Delta,\nu}_{m_1\cdots m_j}(x,z)]^{\hat\kappa_1\cdots\hat\kappa_{j-m}}\epsilon^{(n,l,m)}_{\kappa_1\cdots\kappa_{j-m}} = e^{i\Delta\cdot x}[A^{n,l,m}_{m_1\cdots m_j}(z;\Delta^\mu,\nu)]^{\hat\kappa_1\cdots\hat\kappa_{j-m}}\epsilon^{(n,l,m)}_{\kappa_1\cdots\kappa_{j-m}}. \tag{100}$$

With this notation, the propagator of the (n,l,m) mode is given by

[24]Note that $N_{j,m}=1$, if $m=0$.

$$G(x,z;x',z')^{(n,l,m)}_{m_1\cdots m_j;n_1\cdots n_j} = \int \frac{d^4\Delta}{(2\pi)^4}\int_0^\infty d\nu \frac{-iP^{(j-m)}_{\rho_1\cdots\rho_{j-m};\sigma_1\cdots\sigma_{j-m}}}{\frac{\mathcal{E}_{n,l}+c}{\sqrt{\lambda}}+N_{\rm eff}-i\epsilon}\frac{\alpha' R^3}{t_y} \times [A^{n,l,m;\Delta,\nu}_{m_1\cdots m_j}(x,z)]^{\hat\rho_1\cdots\hat\rho_{j-m}}[A^{n,l,m;-\Delta,\nu}_{n_1\cdots n_j}(x',z')]^{\hat\sigma_1\cdots\hat\sigma_{j-m}}. \quad (101)$$

Here, $P^{(j-m)}_{\rho_1\cdots\rho_{j-m};\sigma_1\cdots\sigma_{j-m}}$ is a polarization tensor generalizing $\eta_{\rho\sigma}-\partial_\rho\partial_\sigma/\partial^2$; when an orthogonal basis $\epsilon_a(q)\cdot\epsilon_b(-q)=\delta_{a,b}D_a$ of rank-r 4D-traceless 4D-transverse tensors is given,

$$P^{(r)}_{\mu_1\cdots\mu_r;\nu_1\cdots\nu_r} \coloneqq \sum_a \frac{1}{D_a}\epsilon(q)_{a;\mu_1\cdots\mu_r}\epsilon(-q)_{a;\nu_1\cdots\nu_r}. \quad (102)$$

An alternative characterization of this $P^{(r)}_{\mu_1\cdots\mu_r;\nu_1\cdots\nu_r}$ is given by a combination of the following two conditions: one is

$$P^{(r)}_{\mu_1\cdots\mu_r;\nu_1\cdots\nu_r}\epsilon_a^{\hat\nu_1\cdots\hat\nu_r} = \epsilon_{a;\mu_1\cdots\mu_r}, \quad (103)$$

and the other is that $P^{(r)}_{\mu_1\cdots\mu_r;\nu_1\cdots\nu_r}$ also be a totally symmetric 4D-transverse 4D-traceless tensor with respect to $(\mu_1\cdots\mu_r)$ for any choice of $(\nu_1\cdots\nu_r)$. Its explicit form (A74), given in the Appendix, is useful for practical computations.

C. Representation in the dilatation eigenbasis

It is an essential process in the application of the AdS/CFT correspondence to classify solutions to the equation of motions on the gravity dual background (AdS_5) into irreducible representations of the conformal group SO (4,2) (or possibly its supersymmetric extension). In the CFT description, primary operators are in one-to-one correspondence with (highest weight) irreducible representations of the conformal group, and it is believed that one can establish a one-to-one correspondence between (i) a primary operator in the CFT description and (ii) a group of solutions to the equation of motion forming an irreducible representation in the gravity dual description. Once this correspondence is given, hadron matrix elements of the primary operators in a (nearly conformal) field theory can be calculated by using the corresponding solutions to the equations of motion (the wave functions) on AdS_5. Note that the hadron matrix elements of the primary operators are all that remain unknown in the formulation of conformal operator product expansion (26).

Let P_μ, K_μ, $L_{\mu\nu}$, and D denote the generators of the unitary operators of the conformal group transformation on the Hilbert space. They satisfy the following commutation relations:

$$[D,P_\mu]=iP_\mu, \qquad [P_\rho,L_{\mu\nu}]=i(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu), \quad (104)$$

$$[D,K_\mu]=-iK_\mu, \quad [K_\rho,L_{\mu\nu}]=i(\eta_{\rho\mu}K_\nu-\eta_{\rho\nu}K_\mu), \quad (105)$$

$$[P_\mu,K_\nu]=-2i(\eta_{\mu\nu}D+L_{\mu\nu}), \quad (106)$$

$$[L_{\mu\nu},L_{\rho\sigma}]=i(\eta_{\nu\rho}L_{\mu\sigma}-\eta_{\nu\sigma}L_{\mu\rho}-\eta_{\mu\rho}L_{\nu\sigma}+\eta_{\mu\sigma}L_{\nu\rho}). \quad (107)$$

When such a conformal symmetry exists in a conformal field theory in $3+1$ dimensions, these generators have a representation as differential operators on fields on $\mathbb{R}^{3,1}$; these differential operators are denoted by $\mathcal{P}_\mu$, $\mathcal{K}_\mu$, $\mathcal{L}_{\mu\nu}$, and $\mathcal{D}$. The generators and the differential operators on a CFT are in the following relation:

$$[\mathcal{O}(x),P_\mu]=\mathcal{P}_\mu\mathcal{O}(x), \qquad [\mathcal{O}(x),K_\mu]=\mathcal{K}_\mu\mathcal{O}(x), \quad [\mathcal{O}(x),D]=\mathcal{D}\mathcal{O}(x),\ldots, \quad (108)$$

and these differential operators acts on primary operators as follows:

$$\mathcal{D}\overline{\mathcal{O}}_n(x)=-i(x\cdot\partial+l_n)\overline{\mathcal{O}}_n(x), \quad (109)$$

$$\mathcal{L}_{\mu\nu}\overline{\mathcal{O}}_n(x)=(i(x_\mu\partial_\nu-x_\nu\partial_\mu)+[S_{\mu\nu}])\overline{\mathcal{O}}_n(x), \quad (110)$$

$$\mathcal{P}_\mu\overline{\mathcal{O}}_n(x)=-i\partial_\mu\overline{\mathcal{O}}_n(x), \quad (111)$$

$$\mathcal{K}_\mu\overline{\mathcal{O}}_n(x)=(-i(2x_\mu x\cdot\partial-x^2\partial_\mu)-i2l_nx_\mu-x^\nu[S_{\mu\nu}])\overline{\mathcal{O}}_n(x), \quad (112)$$

where l_n is the scaling dimension of the operator $\overline{\mathcal{O}}_n$ and $[S_{\mu\nu}]$ a finite dimensional representation of SO(3,1) generators satisfying the same commutation relation as $L_{\mu\nu}$'s. Thus, for a primary operator $\overline{\mathcal{O}}_n(x)$, $\overline{\mathcal{O}}_n(x=0)$ plays the role of the highest weight state

$$[\overline{\mathcal{O}}_n(0),K_\mu]=0, \qquad [\overline{\mathcal{O}}_n(0),D]=-il_n\overline{\mathcal{O}}_n(0); \quad (113)$$

all other states in the highest weight state representation—descendants—are generated by applying $[\bullet,P_\mu]$ multiple times; the whole representation, therefore, is spanned by a collection of

$$\{\overline{\mathcal{O}}_n(0),\partial_\mu\overline{\mathcal{O}}_n(0),\partial_\mu\partial_\nu\overline{\mathcal{O}}_n(0),\ldots\}; \quad (114)$$

it is also equivalent to a collection of $\overline{\mathcal{O}}(x=x_0)$, with arbitrary $x_0\in\mathbb{R}^{3,1}$.

In the preceding sections, we have worked on solutions to the eigenmode equation on AdS_5; once the mass-shell condition (66) is imposed, they become solutions to the equation of motion. They are obtained as an eigenmode of the spacetime translation in $3+1$ dimensions, $(-i\partial^\mu)=\Delta^\mu$. Under the conformal group SO(4,2), which contains Lorentz SO(3,1) symmetry, however, an irreducible representation has to include solutions with all kinds of eigenvalues Δ^μ.

In the case of a scalar field on AdS_5, one can think of the following linear combination $G(x,z;x_0;R_0)$ (for some $R_0\ll\Delta^{-1}$):

$$G(x,z;x_0)=\frac{i}{\pi^2}\frac{\Gamma(l_n)}{\Gamma(l_n-2)}R_0^{l_n-4}\left(\frac{z}{z^2+(x-x_0)^2}\right)^{l_n}$$
$$=\int\frac{d^4\Delta}{(2\pi)^4}e^{i\Delta\cdot(x-x_0)}\frac{(\Delta z)^2K_{l_n-2}(\Delta z)}{(\Delta R_0)^2K_{l_n-2}(\Delta R_0)}. \quad (115)$$

The factor $[e^{i\Delta\cdot x}(\Delta z)^2K_{l_n-2}(\Delta z)]$ in the integrand on the right-hand side is a solution to the equation of motion of a scalar field on AdS_5 whose mass square $M_{\rm eff}^2$ is given by $l_n-2=i\nu=\sqrt{4+M_{\rm eff}^2R^2}$. The coefficient of the linear combination, $e^{-i\Delta\cdot x_0}[(\Delta R_0)^2K_{l_n-2}(\Delta R_0)]^{-1}$, is chosen so that the integrand behaves as

$$e^{i\Delta\cdot(x-x_0)}\left(\frac{z}{R_0}\right)^{4-l_n} \quad (116)$$

at $0\leq z\ll\Delta^{-1}$. The space of solutions to the equation of motion $G(x,z;x_0)$ parametrized by $x_0\in\mathbb{R}^{3,1}$ is alternatively spanned by derivatives of $G(x,z;x_0)$ with respect to x_0^μ at $x_0^\mu=0$. It is easy to see that this basis,

$$\{G(x,z;0),\partial_\mu^{(x_0)}G(x,z;0),\partial_\mu^{(x_0)}\partial_\nu^{(x_0)}G(x,z;0),\ldots\}, \quad (117)$$

is an eigenbasis under the action of dilatation, $\mathcal{D}\coloneqq i(z\partial_z+x\cdot\partial)$, and their weights are $-il_n,-i(l_n+1),-i(l_n+2),\ldots$, respectively. Correspondence between scalar field wave functions on AdS_5 and scalar primary operators of the dual CFT is established in this way [34].

Let us now generalize the discussion above slightly to construct an analogue of $G(x,z;x_0)$ for a spin-j field $A_{m_1\cdots m_j}$ on AdS_5, from which the dilatation eigenbasis is constructed. To this end, note that all of the $(0,0,m)$ modes ($m=0,\ldots,j$) have the leading $z^{2-j-i\nu}$ term in the power series expansion only in the $A_{z^0\mu_1\cdots\mu_j}$ component, not in any other $A_{z^k\mu_1\cdots\mu_{j-k}}$ components[25] with $k>0$. It is possible to choose $\epsilon^{(0,0,m)}(\Delta^\mu)$ properly so that

$$\sum_{m=0}^{j}[A^{0,0,m;\Delta,\nu}_{\mu_1\cdots\mu_j}(x,z)]^{\hat{\kappa}_1\cdots\hat{\kappa}_{j-m}}\epsilon^{(0,0,m)}_{\kappa_1\cdots\kappa_{j-m}}e^{-i\Delta\cdot x_0}$$
$$\simeq e^{i\Delta\cdot(x-x_0)}\left(\frac{z}{R_0}\right)^{2-j-i\nu}\epsilon_{\mu_1\cdots\mu_j} \quad (118)$$

in the region near the UV boundary $z\ll\Delta^{-1}$, where $\epsilon_{\mu_1\cdots\mu_j}$ is a Δ^μ-independent 4D-traceless totally symmetric rank-j tensor of SO(3,1); the condition on $\epsilon^{(0,0,m)}(\Delta^\mu)$ is

$$\epsilon_{\mu_1\cdots\mu_j}=\left(\frac{R_0}{R}\right)^{2-j}K_{i\nu}(\Delta R_0)\frac{2}{\pi}\sqrt{\frac{\nu\sinh(\pi\nu)}{2R}}$$
$$\times\sum_{m=0}^{j}\frac{N_{j,m}\Gamma(m-j-i\nu)}{\Gamma(-j-i\nu)}$$
$$\times\sum_{N=0}^{[m/2]}\frac{b^{(j-m)}_{m,N}}{\Delta^{m-2N}}(\tilde{E}^ND^{m-2N}[\epsilon^{(0,0,m)}])_{\mu_1\cdots\mu_j}. \quad (119)$$

[25] Use (97).

It is possible to invert this relation by using (A37) and writing down $\epsilon^{(0,0,m)}(\Delta^\mu)$ in terms of $\epsilon_{\mu_1\cdots\mu_j}$, though we will not present the result here. What really matters to us is that $\epsilon^{(0,0,m)}(\lambda\Delta)=\epsilon^{(0,0,m)}(\Delta)\lambda^{i\nu}$. With $\epsilon^{(0,0,m)}$'s satisfying the condition above, one can see that the following linear combination of solutions to the equation of motion,

$$G_{m_1\cdots m_j}(x,z;x_0)$$
$$\coloneqq\int\frac{d^4\Delta}{(2\pi)^4}\sum_{m=0}^{j}[A^{0,0,m;\Delta,\nu}_{m_1\cdots m_j}(x,z)]^{\hat{\kappa}_1\cdots\hat{\kappa}_{j-m}}\epsilon^{(0,0,m)}$$
$$\times(\Delta)_{\kappa_1\cdots\kappa_{j-m}}e^{-i\Delta\cdot x_0}, \quad (120)$$

has a property

$$G_{m_1\cdots m_j}(\lambda x,\lambda z;\lambda x_0)=\lambda^{-(2+j+i\nu)}G_{m_1\cdots m_j}(x,z;x_0). \quad (121)$$

$i\nu$ is determined by the mass parameter on AdS_5 once the mass-shell condition (66) is imposed. Therefore, $G_{m_1\cdots m_j}(x,z;0)$ is an eigenstate of dilatation, and so are the derivatives of $G_{m_1\cdots m_j}(x,z;x_0)$ with respect to x_0^μ at $x_0^\mu=0$. All of the derivatives combined forms of a dilatation eigenbasis in the space of solutions with the equation of motion of a spin-j field.

It is now clear that the eigenmodes with $(n,l,m)=(0,0,m)$ $(0\leq m\leq j)$ and arbitrary Δ^μ as a whole—modes that satisfy the 5D-traceless and 5D-transverse conditions (95), (96)—form an irreducible representation of the conformal group. If one is interested purely in the matrix element of a spin-j primary operator $\overline{\mathcal{O}}_n(x_0=0)$ of an approximately conformal gauge theory, then the matrix element can be calculated by using the wave function $G_{m_1\cdots m_j}(x,z;0)$. Note that the $m=0$ mode alone—where the Pomeron/Reggeon wave function has a single component, as in [3]—cannot reproduce all of the matrix elements associated with matrix elements of spin-j primary operators.

D. Confinement effect

1. Top-down approach

QCD in the real world is not a conformal gauge theory, but it has a mass gap in the hadron spectrum due to confinement. Confinement of a nearly conformal strongly coupled gauge theory is realized in its gravitational dual description in the form of a nearly AdS geometry with a minimum value in the warp factor.

Klebanov-Strassler geometry of type IIB string theory [27] will be one of the most popular background geometries of this kind. The Klebanov-Strassler geometry is not dual to a confining gauge theory that is asymptotically free, however; it is dual to a gauge theory that is confining in the infrared, but its 't Hooft couplings become stronger and stronger toward ultraviolet. Such geometries as Klebanov-Strassler are not truly dual to the QCD of the real world, but one will still be able to learn a lot from studying the mode decomposition on such geometries.

Mode decomposition can be carried out once we know the background configuration and the action of the bilinear fluctuations around the background; we do not need interactions of stringy fields. Thus, it will be a doable task, at least at the supergravity level. Reduction over the $W_5 = T^{1,1}$ geometry has been worked out in the literature, and one is left to translate the smoothness condition of mode functions at the tip of the deformed warped conifold into the language of boundary conditions on a warped $(4+1)$-dimensional spacetime.[26] The authors do not find a reason not to work on it, except that it will take extra time to do so.

In this article, however, we set a higher priority in getting a broader perspective on the subject ranging from string theory to hadron physics, and we avoid taking too much time to solve technical problems in string theory. Instead, we discuss, in the following, two temporary approaches of implementing the confinement effects; one is an effective-theory model building approach and the other is a phenomenological approach. We will proceed with the phenomenological approach in the following sections, although we understand that the topdown approach above will eventually replace/back up/verify the phenomenological approach to be adopted in this article. The following "effective theory model building approach" is not used in this article, but we present it here because it helps us understand the physical meaning (the hidden assumptions) of the phenomenological approach.

2. Effective theory model building approach

The hard wall model and its variations are introduced in order to mimic the presence of a minimum value of the warped factor, mass gap, and nearly AdS background geometry. It remains simple enough so that analytic results are obtained in a relatively short amount of time, though we cannot discuss the stability of the geometry or the theoretical consistency of string theory.

With this philosophy in mind, one could think of implementing the confining effect in the form of

$$S = \int d^4x \int_0^{1/\Lambda} dz \sqrt{-g(z)}\mathcal{L}_{\rm bulk} + \int d^4x \sqrt{-g}|_{z=1/\Lambda}\mathcal{L}_{\rm bdry}, \tag{122}$$

where the background geometry remains AdS_5 and the holographic radius z is cut off at $z=\Lambda^{-1}$. Note that different choices of $\mathcal{L}_{\rm bdry}$ lead to a different physics; to be more precise, different choices of $(\mathcal{L}_{\rm bulk}, \mathcal{L}_{\rm bdry})$ modulo partial integration should be regarded as different models. It is reasonable to have such freedom in the choice of effective-theory models because we know that there is more than one holographic background of type IIB string theory that is dual to confining gauge theories. Such constraints as SO(3,1) symmetry unbroken global symmetry of a strongly coupled gauge theory, however, are very weak in constraining $\mathcal{L}_{\rm bdry}$.

Once a model is fixed, the Euler-Lagrange equation of this theory includes not only the equation of motion in the bulk (64)=(65), (66), but also the boundary conditions at $z = 1/\Lambda$. Different models (i.e., different $\mathcal{L}_{\rm bdry}$) predict different Pomeron/Reggeon wave functions.

We require that SO(3,1) symmetry is preserved even in $\mathcal{L}_{\rm bdry}$. Boundary conditions might introduce mixing between the eigenmode decomposition determined in the bulk, in principle, but the unbroken SO(3,1) symmetry excludes mixing between SO(3,1)-irreducible tensors of different ranks. This observation still does not exclude mixing among (n,l,m) modes of a spin-j totally symmetric field on AdS_5 with a common m, but different (n,l)'s.

3. Phenomenological approach

As an alternative approach, one can think of a phenomenological approach, which is to start from a small number of parameters and let the physical consequences constrain those parameters. When one finds that reasonable physical consequences cannot be available under a given set of parameters, then a few more parameters will be introduced so that more freedom is available.

As one of the simplest trial parametrizations of the confining effect, we make the following changes in the mode functions $\Psi^{(j);0,0}_{i\nu;0,0,m}(\Delta,z)$:

$$K_{i\nu}(\Delta z) \to \left[K_{i\nu}(\Delta z) + \frac{\pi}{2}\frac{c^{(j)}_{i\nu;0,0,m}}{\sin(\pi i\nu)} I_{i\nu}(\Delta z)\right] =: \text{"}K_{i\nu}(\Delta z)\text{"}. \tag{123}$$

$c^{(j)}_{i\nu;0,0,m}$'s, which may depend on Δ^2 and Λ, are the parameters we introduce. An implicit assumption here is that the confining effect does not introduce mixing among modes with different (n,l,m)'s. Under this assumption, however, the parametrization above does not lose any generality; once the ratio between the $K_{i\nu}(\Delta z)$ wave and $I_{i\nu}(\Delta z)$ is given for $\Psi^{(j);0,0}_{i\nu;0,0,m}(-\Delta^2,z)$, there is no freedom left for the other $\Psi^{(j);s,N}_{i\nu;0,0,m}(-\Delta^2,z)$ functions ($(s,N)\neq(0,0)$) of the same $(n,l,m)=(0,0,m)$ mode because the relation among them is completely fixed by the equation of motion in the bulk. In Sec. VII A, we will carry out a test of whether this simple parametrization works well or not.

When the infrared boundary is introduced in the holographic background geometry, the normalization of the

[26]Such geometries typically are in the form of $\mathbb{R}^{3,1}\times W'_n$, which nearly remains constant around the tip of the throat $r=0$, and a shrinking $(5-n)$ cycle with the metric $ds^2 = dr^2 + r^2(d\Omega_{5-n})^2$. For simplicity, let $n=4$ and $d\Omega_1 = d\theta$. A scalar field $\phi(r,\theta)$ with smooth configuration in the coordinate $(r\cos\theta, r\sin\theta)$ is decomposed into $\sum_k e^{ik\theta}\phi_k(r)$ when the mode $\phi_k(r)$ needs to be in the form of $r^k \times \mathrm{fcn}(r^2)$. Thus, $\partial_r[r^{-k}\phi_k(r)] = 0$ at $r=0$.

Pomeron/Reggeon wave function also needs to be changed. In the case of the $(n,l,m) = (0,0,0)$ mode, with the Dirichlet boundary condition at the IR boundary $z = 1/\Lambda$, for example, the wave function $\Psi^{(j);0,0}_{i\nu;0,0,0} = \Psi^{(j)}_{i\nu}(-\Delta^2, z)$ was given the following normalization [3,7]:

$$\Psi^{(j)}_{i\nu}(-\Delta^2, z) = e^{(j-2)A}\frac{2}{\pi}\sqrt{\frac{\nu\sinh(\pi\nu)}{2R}}\sqrt{\frac{I_{i\nu}(x_0)}{I_{-i\nu}(x_0)}} \times\left[K_{i\nu}(\Delta z) - \frac{K_{i\nu}(x_0)}{I_{i\nu}(x_0)}I_{i\nu}(\Delta z)\right], \qquad (124)$$

with an extra factor $\sqrt{I_{i\nu}(x_0)/I_{-i\nu}(x_0)}$, where $x_0 \coloneqq \Delta/\Lambda$. This result is generalized as follows. By repeating the same argument as in Appendix A.3.a, one finds that the normalization factor $N_{j,m}$ should be replaced by

$$N_{j,m} \to N_{j,m} \times \frac{1}{\sqrt{1 - c^{(j)}_{i\nu;0,0,m}}}. \qquad (125)$$

The Dirichlet boundary condition for the $m = 0$ mode above corresponds to $(1 - c^{(j)}_{i\nu;0,0,0}) = [I_{-i\nu}(x_0)/I_{i\nu}(x_0)]$; the modified normalization (124) is a special case of (125). The mode functions are defined, so far, for $\nu \geq \mathbb{R}$ since the eigenvalue $\mathcal{E}_{0,0} = 4 + j + \nu^2$ depends only on ν^2. When the mode function is analytically continued to the $\nu < 0$ region, the mode function for $-\nu$ should be the same as $+\nu$. From this observation, it follows that

$$(1 - c^{(j)}_{-i\nu;0,0,m}) = (1 - c^{(j)}_{i\nu;0,0,m})^{-1}. \qquad (126)$$

VI. ORGANIZING THE SCATTERING AMPLITUDE ON AdS_5

A. "Effective" string field action on AdS_5

If we are to start from type IIB string theory in ten dimensions with a background that is approximately $AdS_5 \times W_5$ (except near the infrared boundary), one can think of an effective theory on AdS_5 after carrying out spherical harmonics mode decomposition on W_5. As we have already discussed in Sec. V how to construct propagators in such an effective theory, we would now like to construct the scattering amplitude.

For this purpose, we need interaction among string fields, and we turn to cubic string field theory, which we reviewed already in Sec. IV. This allows us to write down a concrete expression for the scattering amplitude. Clearly the biggest drawback of this approach is in the fact that no stable background geometry $AdS_5 \times W_{21}$ is known in bosonic string theory for a 21-dimensional internal manifold W_{21}. In the following, we will construct an effective action on AdS_5 by carrying out dimensional reduction of the cubic string field theory action, as if there exists an $AdS_5 \times W_{21}$ solution to bosonic string theory. This is not meant to claim that we obtain such an action as an effective theory of bosonic string theory, but to use it as a starting point in constructing a toy-model scattering amplitude of a hadron and a (virtual) photon that may still carry some fragrance of interaction structure in superstring theory.

Let us start off by clarifying the relation between the normalization of string component fields in (38), (41), (42) and that of the component fields in (62). All of the component fields in (38) are normalized so that they have canonically normalized kinetic terms in the action in the 26-dimensional spacetime. Now, we make them dimensionless by the redefinition $\phi \to g_o^{-1}\phi$, $A_M \to g_o^{-1}A_M$, etc. All of the terms in the cubic string field theory—both the kinetic terms and the interactions—will then have $(1/g_o^2)$ as an overall factor. When a mode decomposition of the following form is assumed for the component in this new normalization,

$$\phi(x,z,\theta) = \sum_y \phi^{(y)}(x,z)Y_y(\theta),$$
$$A_M(x,z,\theta) = \begin{cases} \sum_y A^{(y)}_m(x,z)Y_y(\theta) & M = m = 0,\ldots,3,z \\ 0 & M = 5,\ldots,25. \end{cases} \qquad (127)$$

Similarly decomposition holds for spin-h_a fields $A_{M_1\cdots M_{h_a}}(x,z,\theta)$; we take spherical harmonics $Y_y(\theta)$ (labeled by y) to be dimensionless, so that the component fields on AdS_5 such as $\phi^{(y)}(x,z)$, $A^{(y)}_m(x,z)$, $A^{(y)}_{m_1\cdots m_{h_a}}(x,z)$ are also dimensionless.

The overall coefficient of the effective action on AdS_5 then becomes a dimension-$(+3)$ parameter

$$\frac{\mathrm{vol}(W_{21})}{2(g_o)^2} \times \mathcal{O}(1), \qquad (128)$$

which is to be identified with the overall coefficient $t_y/(2R^3)$ in (62). Reduction of interaction terms (43), (45), (46) also yields the same overall factor (128), apart from possibly one order factor coming from the overlap integration of spherical harmonics over the internal manifold. Because the amplitudes from exchanging states with higher spherical harmonics are suppressed in small-x DIS and DVCS (e.g., [7]), we will be interested only in the interactions involving $\phi^{(y)}$-$\phi^{(y)}$–(intermediate states) and $A^{(y)}_m$-$A^{(y)}_m$–(intermediate states) cubic couplings, with the intermediate states having spherical harmonics $Y(\theta) = 1$. The overall factor of the cubic interactions then becomes precisely the same as that of the kinetic terms of $\phi^{(y)}$ and $A^{(y)}_m$.

For this reason, we write down the following interaction terms for the effective action on AdS_5:

$$S_{\text{eff int}} = -\frac{t_{\phi y}\lambda_{\text{sft}}}{3\alpha' R^3}\int d^4x dz\sqrt{-g(z)}\hat{E} \times \left(3\,\text{tr}[\phi_y^2\phi] + \sqrt{\frac{8\alpha'}{3}}\text{tr}[(-iA_m)(\phi_y\overleftrightarrow{\nabla}^m\phi_y)] - \frac{8\alpha'}{9\sqrt{2}}\text{tr}[f_{mn}(\phi_y\overleftrightarrow{\nabla}^m\overleftrightarrow{\nabla}^n\phi_y)] - \frac{5}{9\sqrt{2}}\text{tr}[f^m_m\phi_y^2] + \frac{2\sqrt{\alpha'}}{3}\text{tr}[(\nabla_m g^m)\phi_y^2] - \frac{11}{9}\text{tr}[h\phi_y^2]\right) + \cdots. \tag{129}$$

Fields without a label y are to be used for the intermediate states exchanged in the t channel (in the sense that we explained in Sec. IV B); ϕ_y are for the incoming and outgoing states. Partial derivatives have been replaced by covariant derivatives on AdS_5. Similarly, all other interactions, such as (45), (46) in 26 dimensions, also give rise to their corresponding cubic interactions on AdS_5. Certainly such a choice of effective action on AdS_5 will be one of the most likely (and simple) setups that may still maintain some aspects of scattering amplitude in string theory, although top-down justification is not given.

We will only sum t-channel amplitudes where $Y_y(\theta) = 1$ modes of the stringy states in the leading Reggeon/Pomeron trajectory are exchanged, because that constitutes the dominant contribution in small-x scattering. Thus, three-point interactions of such modes with incoming and outgoing tachyon states are necessary, which we write down as follows,

$$\Delta S_{\text{eff int}} = -\frac{t_h\lambda_{\text{sft}}}{R^3\alpha'}\int d^4x dz\sqrt{-g(z)}\hat{E} \times \text{tr}[A^{(Y)}_{m_1\cdots m_N}(\phi\overleftrightarrow{\nabla}^{m_1}\cdots\overleftrightarrow{\nabla}^{m_N}\phi)]\left(\frac{8\alpha'}{27}\right)^{\frac{N}{2}}\frac{(-i)^N}{\sqrt{N!}}, \tag{130}$$

by keeping only the $Y_y(\theta) = 1$ modes and replacing the derivatives in (45) by covariant derivatives. The normalization constant $t_{\phi y}$ for the target hadron kinetic term is now simply written as t_h, as we will have to pay attention only to the individual choices of target hadrons [the individual choices of $Y_y(\theta)$] in the external states. Similarly, we also need interaction of the same group of modes with the incoming and outgoing photon states, which we write down as follows:

$$\Delta S_{\text{eff int}} = -\frac{t_\gamma\lambda_{\text{sft}}}{R^3\alpha'}\int d^4x dz\sqrt{-g(z)}\hat{E} \times \text{Tr}\left[A^{(N)}_{m_1\cdots m_N}(A_l(-i\overleftrightarrow{\nabla}^{m_1}) \cdots(-i\overleftrightarrow{\nabla}^{m_N})A_k)\left(\frac{8\alpha'}{27}\right)^{\frac{N}{2}}\frac{g^{kl}\frac{16}{27}}{\sqrt{N!}} + \cdots\right], \tag{131}$$

following the same procedure by starting from (46). We have retained only the terms that have N derivatives and are proportional to η^{kl}, as they are necessary in determining the twist-2 contributions to the structure function V_1. Since we need the normalization constant t_{Ay} of the kinetic term of the external state only for the spherical harmonics $Y(\theta) = 1$, we no longer need to refer to the choice of spherical harmonics; t_{Ay} is therefore rewritten as t_γ.

B. External states wave function

The vertex operator insertions in the world-sheet calculation are replaced by appropriate external state wave functions in amplitude calculations based on string field theories.

First, the insertions of a vertex operator of the form (33) for the U(1) currents on flavor D7-branes are replaced by wave functions for the massless vector field in bosonic string theory. We use the wave functions for the incoming state $\gamma^*(q_1)$ and the outgoing state $\gamma^{(*)}(q_2)$:

$$A^{\text{in}}_m(x_\gamma, z_\gamma) = R\int\frac{d^4q_1}{(2\pi)^4}e^{iq_1\cdot(x_\gamma-(\bar{x}-(\Delta x)/2)}A_m(z_\gamma; q_1), \tag{132}$$

$$A^{\text{out}}_m(x_\gamma, z_\gamma) = R\int\frac{d^4q_2}{(2\pi)^4}e^{-iq_2\cdot(x_\gamma-(\bar{x}+(\Delta x)/2))}A_m(z_\gamma; q_2), \tag{133}$$

where $A_m(z; q)$ on the right-hand sides are the wave functions given in (35). A factor R is inserted here because we adopted a normalization convention, so that $A^{(\text{in/out})}_m(x, z)$ on AdS_5 is dimensionless.[27] The arguments of the electromagnetic current insertions $T\{J^\nu(x)J^\mu(y)\}$—coordinates in boundary theory x and $y \in \mathbb{R}^{3,1}$—are now denoted by $\bar{x} + (\Delta x)/2$ and $\bar{x} - (\Delta x)/2$, respectively.

[27] $A_m(x, z)$ is often normalized so that it has mass dimension $(+1)$, and hence this factor R is not then necessary. In a case in which the gauging of a global symmetry of a strongly coupled gauge theory is realized in the form of a flavor D7-brane, the natural reduction of the 7-brane action on a three-cycle leads to the form of

$$S_{\text{eff}} \sim -\frac{N_c}{R}\int d^4x dz\sqrt{-g(z)}F_{mn}F^{mn}; \tag{134}$$

the external state wave function (132), (133) without the factor R can be used in such cases. In the presentation adopted in this section, where a bosonic string is used and the gauge field is assigned zero mass dimension (like other higher spin fields), the factor R is included in (132), (133) and the kinetic term of $F_{mn}F^{mn}$ has the coefficient t_γ/R^3 instead. Thus, we can think of t_γ as something like N_c.

The vertex operators (30) for the target hadron are replaced by wave functions of the form

$$\phi^{\rm in}(x_h, z_h) = e^{ip_1\cdot x_h}\Phi(z_h; m_n),$$
$$\phi^{\rm out}(x_h, z_h) = e^{-ip_2\cdot x_h}\Phi(z_h; m_n), \qquad (135)$$

where $\Phi(z;m)$'s on the right-hand sides are the wave function given by (31). The first one is for the incoming state and the second for the outgoing hadron.

C. Leading trajectory contribution to the Compton tensor

When the target hadron is to be identified with some Kaluza-Klein state of the tachyon of bosonic string theory, then $\ell_\phi - 2 = \sqrt{4 + M_{\rm eff}^2 R^2} = \sqrt{4 + c - \sqrt{\lambda}}$ is not real valued for $\lambda \gg 1$. We treat this $\ell_\phi - 2$ as if it were real valued, until the last moment. Since our true interest is in the scattering amplitude in type IIB string theory, or in hadron scattering in the real world, this problem is absent in such situations, and we do not bother about this issue.

Let us combine all the pieces together to organize an amplitude of photon-tachyon scattering given by a t-channel exchange of a leading trajectory spin-j state reduced to AdS_5, with $Y_y(\theta) = 1$. Such an amplitude—denoted by $i\mathcal{M}^{(t)}_{(N_{\rm eff}=j,j)}$—consists of a t-channel exchange of all the eigenmodes labeled by (n,l,m). We will further focus on contributions from $(n,l,m) = (0,0,m)$. It is given by

$$i\mathcal{M}^{(t)}_{(j,j);(0,0,m)} \simeq \frac{-it_\gamma}{R^3\alpha'}\int d^4x_\gamma dz_\gamma \sqrt{-g(z_\gamma)} J^{\gamma\gamma}_{k_1\cdots k_j;pq} g^{pq}(g^{k_1r_1}\cdots g^{k_jr_j})(z_\gamma)\left(\frac{\alpha'}{2}\right)^{j/2} e^{-2A(z_\gamma)}$$
$$\times \frac{-it_h}{R^3\alpha'}\int d^4x_h dz_h \sqrt{-g(z_h)} J^{hh}_{l_1\cdots l_j}(g^{l_1s_1}\cdots g^{l_js_j})(z_h)\left(\frac{\alpha'}{2}\right)^{j/2} e^{-2A(z_h)}$$
$$\times \frac{1}{j!}\left(\frac{27}{16}\right)^{\alpha' t-(j-1)}[e^{2A(z_\gamma)}e^{2A(z_h)}G^{(0,0,m)}(x_\gamma, z_\gamma; x_h, z_h)_{r_1\cdots r_j; s_1\cdots s_j}]; \qquad (136)$$

just like in the amplitude calculation in Sec. IV B, this amplitude is meant to be the coefficient of $\mathrm{Tr}[\lambda^{\gamma 2}\lambda^{\gamma 1}\lambda^{h1}\lambda^{h2}]$. $J^{\gamma\gamma}$ and J^{hh} (above) are given by the external state wave functions as follows:

$$J^{\gamma\gamma}_{k_1\cdots k_j;pq}(x_\gamma, z_\gamma) = (-i)^j[A^{\rm out}_p \overleftrightarrow{\nabla}_{k_1}\cdots\overleftrightarrow{\nabla}_{k_j} A^{\rm in}_q](x_\gamma, z_\gamma), \quad (137)$$

$$J^{hh}_{l_1\cdots l_j}(x_h, z_h) = (-i)^j[\phi^{\rm in}\overleftrightarrow{\nabla}_{l_1}\cdots\overleftrightarrow{\nabla}_{l_j}\phi^{\rm out}](x_h, z_h). \quad (138)$$

Here, $\phi^{\rm in/out}(x_h, z_h)$ are both of mass dimension (-1), and $A^{\rm in/out}_m(x_\gamma, z_\gamma)$ is of mass dimension $(+3) + \dim[\epsilon_\mu]$. From this expression, one can see that the first line has mass dimension $(+6) + 2\times\dim[\epsilon_\mu]$, the second line (-2), and the last line 0. Thus, $i\mathcal{M}^{(t)}_{(j,j);(0,0,m)}$ is a function of p_1^κ, p_2^κ, $\bar{x}^\kappa$ and Δx^κ of mass dimension $4 + 2\times\dim[\epsilon_\mu]$. This is precisely the property expected for

$$(i)^2\langle h(p_2)|T\{J^\nu(\bar{x}+(\Delta x)/2)J^\mu(\bar{x}-(\Delta x)/2)\}|h(p_1)\rangle\epsilon^1_\mu\epsilon^{2*}_\nu. \qquad (139)$$

Its Fourier transform with respect to $(\Delta x)^\mu$ becomes $(iT^{\mu\nu})\times e^{-i\bar{x}\cdot(p_2-p_1)}$.

If we carry out an integration over d^4x_γ, d^4x_h, and $d^4(\Delta x)$ first, then the three integration variables Δ^μ in (101) and $q_{1,2}$ in (132), (133) are determined in terms of the input $p^\mu_{1,2}$ and q^μ; we have $\Delta^\mu \coloneqq (p_2 - p_1)^\mu$, $q_2^\mu = (q - \Delta/2)^\mu$ and $q_1 \coloneqq (q + \Delta/2)^\mu$. As a result, it follows that

$$[T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu]^{(t)} = \int d^4(\Delta x)e^{-iq\cdot(\Delta x)}\mathcal{M}^{(t)}_{(j,j);(0,0,m)}|_{\bar{x}=0}$$
$$\simeq \frac{t_\gamma}{R^3\alpha'}\int dz_\gamma\sqrt{-g(z_\gamma)}\bar{J}^{\gamma\gamma}_{k_1\cdots k_j;pq}R^2 g^{pq}(g^{k_1r_1}\cdots g^{k_jr_j})\left(\frac{\alpha'}{2}\right)^{j/2}\frac{t_h}{R^3\alpha'}\int dz_h\sqrt{-g(z_h)}\bar{J}^{hh}_{l_1\cdots l_j}(g^{l_1s_1}\cdots g^{l_js_j})\left(\frac{\alpha'}{2}\right)^{j/2}$$
$$\times\frac{1}{j!}\left(\frac{27}{16}\right)^{\alpha' t-(j-1)}\frac{R^3\alpha'}{t_{(j,j,1)}}\int_0^\infty d\nu\frac{P^{(j-m)}_{\rho_1\cdots\rho_{j-m};\sigma_1\cdots\sigma_{j-m}}}{\frac{\mathcal{E}_{0,0}+c_y}{\sqrt{\lambda}} + N_{\rm eff} - i\epsilon}[A^{0,0,m}_{r_1\cdots r_j}(z_\gamma; -\Delta, \nu)]^{\hat\rho_1\cdots\hat\rho_{j-m}}[A^{0,0,m}_{s_1\cdots s_j}(z_h; \Delta, \nu)]^{\hat\sigma_1\cdots\hat\sigma_{j-m}}, \quad (140)$$

where

$$\bar{J}^{\gamma\gamma}_{k_1\cdots k_j;pq}(z_\gamma) = (-i)^j[A_p(z_\gamma; -q_2)\overleftrightarrow{\nabla}_{k_1}\cdots\overleftrightarrow{\nabla}_{k_j}A_q(z_\gamma; q_1)], \qquad (141)$$

$$\bar{J}^{hh}_{l_1\cdots l_j}(z_h) = (-i)^j[\Phi(z_h;p_1)\overset{\leftrightarrow}{\nabla}_{l_1}\cdots\overset{\leftrightarrow}{\nabla}_{l_j}\Phi(z_h;-p_2)]. \tag{142}$$

Although momentum vectors are used in the second arguments of the external state wave functions A and Φ here, instead of their Lorentz-invariant momentum square, this is only to remind ourselves of the sign when ∇'s act on the wave functions.

The expression (140) is meant to be a part of the t-channel contribution to the Compton tensor $[T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu]^{(t)}$, and we should obtain the full contribution to the Compton tensor $[T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu]$ after employing the prescription (52). At least this prescription tells us to set the factor $(27/16)^{[\alpha' t-(j-1)]}$ in the fourth line to $(27/16)^{\mathcal{O}(1/\sqrt{\lambda})}\simeq 1$. Now, we claim that this is the only necessary change under this prescription, so far as the amplitude of $(0,0,m)$-mode exchange is concerned.

To see this, remember that, prior to applying the prescription (52), we need to rewrite the residues of the t-channel poles in terms only of the Mandelstam variables s and t, not of u. Let us take the expression $[\Phi_h\overset{\leftrightarrow}{\nabla}_m\Phi_h]g^{mn}[A_\gamma\overset{\leftrightarrow}{\nabla}_n A_\gamma]$ as an example which captures the feature of contraction of SO(4,1) indices in (140). In the scattering $\phi(P_1)+A(Q_1)\to\phi(P_2)+A(Q_2)$, with $P_{1,2}$ and $Q_{1,2}$ "momenta" $\sim$ derivatives in five dimensions, $(s-u)\sim(P_1+P_2)\cdot(Q_1+Q_2)$ is converted to $(2s+t)$ in the following steps:

$$\begin{aligned}(P_1+P_2)\cdot(Q_1+Q_2) &= (2P_1+(P_2-P_1))\cdot(Q_1+Q_2),\\ &= (2P_1)\cdot(Q_1+Q_2)+(Q_1-Q_2)\cdot(Q_1+Q_2) = (2P_1)\cdot(2Q_1+(Q_2-Q_1))+(Q_1)^2-(Q_2)^2,\\ &= (2P_1)\cdot(2Q_1+(P_1-P_2))+(Q_1)^2-(Q_2)^2 = (4P_1\cdot Q_1)+(-2P_1\cdot P_2)+2(P_1)^2+(Q_1)^2-(Q_2)^2;\end{aligned}$$

each one of the steps above is regarded as either one of partial integration in $dx_\gamma dz_\gamma$, one in $dx_h dz_h$, or a rewriting of (P_2-P_1) by (Q_1-Q_2) or vice versa. The last procedure is to pass a derivative on one side of the propagator to the other. Because of the 5D-transverse condition characterizing the $(0,0,m)$ modes, such terms proportional to ∇ drop out from the amplitudes exchanging the $(0,0,m)$ modes. Noting that the prescription (52) modifies the $-2(P_1\cdot P_2)\sim t$ term above into the propagator mass, and that this term appeared only after passing a derivative ∇ through the propagator, we see that the term which would have been affected by the prescription (52) has indeed already dropped out.

1. Casting the amplitude into the form of OPE

So far, the (virtual) photon and the target hadron have been treated equally in the scattering amplitude. We are interested, however, in the $h+\gamma^*\to h+\gamma^{(*)}$ scattering in the regime of generalized Bjorken scaling, where

$$|(q^2)|,(q\cdot p),|(q_1\cdot\Delta)|,|(q_2\cdot p)|\gg|\Delta^2|,m_h^2,\Lambda^2, \tag{143}$$

while the ratio among $(q\cdot p)$, (q^2), and $(q\cdot\Delta)$—namely, x and η—is kept finite. It is, thus, desirable to rewrite the scattering amplitude (the structure functions) in a form that fits to the conformal OPE. To do this, we follow a prescription that has been used in the study of DIS in holographic models.

Let us focus on the following factors that appear in the third and fourth lines of (140):

$$\int_0^\infty d\nu[A^{0,0,m}_{r_1\cdots r_j}(z_\gamma;-\Delta,\nu)]^{\hat{\rho}_1\cdots\hat{\rho}_{j-m}} \times[A^{0,0,m}_{s_1\cdots s_j}(z_h;\Delta,\nu)]^{\hat{\sigma}_1\cdots\hat{\sigma}_{j-m}}\times[\cdots]. \tag{144}$$

The last factor $[\cdots]$ denotes the remaining ν dependence (denominator) in the integrand; we need to remember only that $\mathcal{E}_{0,0}=(4+j+\nu^2)$, and hence it is even under the change $\nu\to-\nu$.

We begin with the case $m=0$. The expression (144) for the $m=0$ case becomes

$$\begin{aligned}&\int_0^\infty d\nu[\Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2;z_\gamma)][\Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2;z_h)]\times[\cdots],\\ &=\frac{2}{\pi^2R}\int_0^\infty d\nu\frac{\nu\sinh(\pi\nu)}{(1-c^{(j)}_{i\nu;0,0,0})}[e^{(j-2)A(z_\gamma)}\text{“}K_{i\nu}(\Delta z_\gamma)\text{”}]\\ &\quad\times[e^{(j-2)A(z_h)}\text{“}K_{i\nu}(\Delta z_h)\text{”}]\times[\cdots]\end{aligned} \tag{145}$$

multiplied by a factor $[\delta^{\hat{\rho}_1}_{r_1}\cdots\delta^{\hat{\rho}_j}_{r_j}\delta^{\hat{\sigma}_1}_{s_1}\cdots\delta^{\hat{\sigma}_j}_{s_j}]$. Using the fact that $K_{i\nu}(x)=i\pi/2\times(I_{i\nu}(x)-I_{-i\nu}(x))/[\sinh(\pi\nu)]$, the ν integral above can be rewritten as

$$\frac{1}{\pi R}\int_{-\infty}^{+\infty}d\nu i\nu[e^{(j-2)A(z_\gamma)}I_{i\nu}(\Delta z_\gamma)][\text{“}K_{i\nu}(\Delta z_h)\text{”}e^{(j-2)A(z_h)}] \times[\cdots], \tag{146}$$

where we used the relation (126). This expression is more convenient than (145); this is because (i) the z_γ integration is dominated in the region $qz_\gamma\lesssim 1$ because of the photon external state wave functions containing $K_1(q_{1,2}z)$, (ii) $I_{i\nu}(\Delta z_\gamma)$ decreases rapidly toward positive $i\nu$ for $qz_\gamma\lesssim 1$ and $q\gg\Delta$ [a generalized Bjorken scaling (143)], and (iii) the rapidly decreasing $I_{i\nu}(\Delta z_\gamma)$ in the lower half of the complex ν plane allows us to close the ν integration contour through the large-radius lower half complex ν plane (see [7] and the literature therein).

It is straightforward to generalize this treatment for all other $m \neq 0$ modes. Note that the Pomeron/Reggeon wave function $[A^{0,0,m}_{m_1\cdots m_j}(z;\Delta,\nu)]^{\hat\rho_1\cdots\hat\rho_{j-m}}$ for $m \neq 0$ is obtained from that of $m=0$ by multiplying $(\Delta z)^m$ and $N_{j,m}$ (which is even in ν), applying differential operators in z and manipulating Lorentz indices. Obviously the order of such manipulations on the wave function and the procedure from (145) to (146) can be exchanged.

Therefore, the contribution to the Compton tensor from the leading trajectory spin-j state $(0,0,m)$ mode is

$$(T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu)_{(j,j);(0,0,m)} \simeq \frac{1}{j!}\frac{t_\gamma\sqrt{\lambda}}{t_y\pi}\left(\frac{\alpha'}{2}\right)^j \int_{-\infty}^{+\infty} d\nu \frac{P^{(j-m)}_{\rho_1\cdots\rho_{j-m};\sigma_1\cdots\sigma_{j-m}}}{\frac{\mathcal{E}_{0,0}+c_y}{\sqrt{\lambda}}+N_{\rm eff}-i\epsilon} i\nu$$
$$\times\frac{R^2}{R^3}\int dz_\gamma\sqrt{-g(z_\gamma)}\overline{J}^{\gamma\gamma}_{k_1\cdots k_j;pq}g^{pq}(g^{k_1r_1}\cdots g^{k_jr_j})[\overline{\overline{A}}{}^{0,0,m}_{r_1\cdots r_j}(z_\gamma;-\Delta,\nu)]^{\hat\rho_1\cdots\hat\rho_{j-m}}$$
$$\times\frac{t_h}{R^3}\int dz_h\sqrt{-g(z_h)}\overline{J}^{hh}_{l_1\cdots l_j}(g^{l_1s_1}\cdots g^{l_js_j})[\overline{A}^{0,0,m}_{s_1\cdots s_j}(z_h;\Delta,\nu)]^{\hat\sigma_1\cdots\hat\sigma_{j-m}}, \tag{147}$$

where $\overline{A}$ and $\overline{\overline{A}}$ are obtained from A by removing the factor $(2/\pi)\sqrt{[\nu\sinh(\pi\nu)/2R]}$ in (93) first, then replacing $K_{i\nu}(\Delta z_h)$ by "$K_{i\nu}(\Delta z_h)$" in $\overline{A}(z_h)$, while replacing $K_{i\nu}(\Delta z_\gamma)$ by $I_{i\nu}(\Delta z_\gamma)$ in $\overline{\overline{A}}(z_\gamma)$. Short distance (stringy) parameters such as AdS radius R and string length $\sqrt{\alpha'}$ can be eliminated from this expression of the Compton tensor so that it is written purely in terms of parameters of strongly coupled gauge theory/hadron physics;

$$(T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu)_{(j,j);(0,0,m)} \simeq \frac{1}{j!}\frac{t_\gamma\sqrt{\lambda}}{t_y\pi}\left(\frac{1}{2\sqrt{\lambda}}\right)^j \int_{-\infty}^{+\infty} d\nu \frac{P^{(j-m)}_{\rho_1\cdots\rho_{j-m};\sigma_1\cdots\sigma_{j-m}}}{\frac{\mathcal{E}_{0,0}+c_y}{\sqrt{\lambda}}+N_{\rm eff}-i\epsilon} i\nu$$
$$\times\int_0\frac{dz_\gamma}{z_\gamma}[A_p(z_\gamma;q_2)(-i\overset{\leftrightarrow}{\nabla})^j_{k_1\cdots k_j}A_q(z_\gamma;q_1)]\delta^{\hat p\,\hat q}z^j[\delta^{\hat k_1\hat r_1}\cdots\delta^{\hat k_j\hat r_j}][e^{(2-j)A}\overline{\overline{A}}{}^{0,0,m}_{r_1\cdots r_j}(z_\gamma;-\Delta,\nu)]^{\hat\rho's}$$
$$\times t_h\int_0\frac{dz_h}{z_h^3}[\Phi(-i\overset{\leftrightarrow}{\nabla})^j_{l_1\cdots l_j}\Phi]z^j[\delta^{\hat l_1\hat s_1}\cdots\delta^{\hat l_j\hat s_j}][e^{(2-j)A}\overline{A}^{0,0,m}_{s_1\cdots s_j}(z_h;\Delta,\nu)]^{\hat\sigma's}. \tag{148}$$

Each line of this expression has zero mass dimension, hence $T^{\mu\nu}$ is also of zero mass dimension, as expected from the Fourier transform of the matrix element (139).

The leading twist contribution to the Compton tensor $T^{\mu\nu}$ should be obtained by summing up the amplitudes of exchanging the spin-j field in the leading trajectory, with $m=0,\ldots,j$ also being summed. It is known in the literature that, for each spin j, the second line of (149) becomes something close to the Wilson coefficient of the OPE, and the third line of (149) something close to the operator matrix element. We will elaborate more on it, with a particular emphasis on the role played by the summation over m. For now, we define

$$C^{0,0,m} \coloneqq \int_0\frac{dz}{z}\left[A_p(z;-q_2)(-i\overset{\leftrightarrow}{\nabla})^j_{k_1\cdots k_j}A_q(z;q_1)\right]\times\left[\frac{(2\Lambda)^{i\nu-j}}{\Delta^{i\nu}}\Gamma(i\nu+1)\right]$$
$$\times\delta^{\hat p\,\hat q}z^j\left[\delta^{\hat k_1\hat r_1}\cdots\delta^{\hat k_j\hat r_j}\right]\left[e^{(2-j)A}\overline{\overline{A}}{}^{0,0,m}_{r_1\cdots r_j}(z;-\Delta,\nu)\right]^{\hat\rho_1\cdots\hat\rho_{j-m}}\epsilon^{(0,0,m)}_{\rho_1\cdots\rho_{j-m}}$$

and

$$\Gamma^{0,0,m} \coloneqq t_h\int_0^{1/\Lambda}\frac{dz}{z^3}\left[\Phi(-i\overset{\leftrightarrow}{\nabla})^j_{l_1\cdots l_j}\Phi\right]z^j\times\left[\left(\frac{\Delta}{2\Lambda}\right)^{i\nu}\frac{\Lambda^j}{\Gamma(i\nu)}\right]\times\left[\delta^{\hat l_1\hat s_1}\cdots\delta^{\hat l_j\hat s_j}\right]\left[e^{(2-j)A}\overline{A}^{0,0,m}_{s_1\cdots s_j}(z;\Delta,\nu)\right]^{\hat\sigma_1\cdots\hat\sigma_{j-m}}\epsilon^{(0,0,m)}_{\sigma_1\cdots\sigma_{j-m}} \tag{149}$$

separately. The factor $[\Gamma(i\nu+1)(2\Lambda)^{i\nu-j}/\Delta^{i\nu}]$ in $C^{0,0,m}$ and a similar factor in $\Gamma^{0,0,m}$ are introduced so that $C^{0,0,m}$ and $\Gamma^{0,0,m}$ correspond to the OPE Wilson coefficients and hadron matrix elements, respectively, renormalized at $\mu_F\sim\Lambda$, as we will see later.

We will focus on the spin-even contribution to a flavor-nonsinglet component of the structure function V_1 in (7). The V_1 structure function is picked up here, only because it is computed a little more easily than other structure functions. We will not touch flavor-singlet components in this article, apart from a brief discussion in Sec. VII C; this is because the cubic SFT with a Chan-Paton factor in Sec. IV is not an adequate tool to study the singlet components. The coefficient $C^{0,0,m}$ above is decomposed, just like $T^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu$ is; the spin-j (with $j \in 2\mathbb{Z}$) contribution to the structure function $V_1^{+,\alpha}$—spin even (+) and flavor nonsinglet (α)—is denoted by $C^{0,0,m}_{V_1;+,\alpha}$.

2. Amplitude of the ($m = 0$)-mode exchange

We first study $V_1^{+,\alpha}$ from the $m = 0$-mode exchange. With the Reggeon wave function given by $\Psi^{(j);0,0}_{i\nu;0,0,0}(t,z) = \Psi^{(j)}_{i\nu}(t,z)$ in (93), this $m = 0$ contribution is expected to be the closest to what has been studied in the literature (such as [3,4,6,7]). Indeed, we reproduce the expression known in the literature, but with a little refinement, in (163).

Note first that the Reggeon wave functions $\overline{\overline{A}}{}^{0,0,m=0}_{r_1\cdots r_j}$ and $\overline{A}{}^{0,0,m=0}_{s_1\cdots s_j}$ are nonzero only when all the r_i's and s_i's are in the $3+1$ Minkowski directions $(r_1\cdots r_j) = (\rho_1\cdots\rho_j)$ and $(s_1\cdots s_j) = (\sigma_1\cdots\sigma_j)$; furthermore, the wave function is 4D-transverse and 4D-traceless totally symmetric tensors of SO(3,1).

This makes it much easier to evaluate the matrix element $\Gamma^{0,0,m=0}$. Because

$$(\nabla^k\Phi)_{\sigma_1\cdots\sigma_k} = \partial_{\sigma_1}\cdots\partial_{\sigma_k}\Phi + [\text{terms proportional to}\, \eta_{\sigma_a\sigma_b}], \tag{150}$$

only

$$\begin{aligned}[\Phi(z;p_1)(-i\overleftrightarrow{\nabla})^j\Phi(z;-p_2)]_{\sigma_1\cdots\sigma_j} &\coloneqq \sum_{k=0}^{j} {}_jC_k[(i\nabla)^{j-k}\Phi(z;p_1)]_{\sigma_{k+1}\cdots\sigma_j}[(-i\nabla)^k\Phi(z,-p_2)]_{\sigma_1\cdots\sigma_k} \\ &\to (-1)^j(p_1+p_1)_{\sigma_1}\cdots(p_1+p_2)_{\sigma_j}\Phi(z;p_1)\Phi(z;-p_2)\end{aligned} \tag{151}$$

contributes to $\Gamma^{0,0,m=0}$:

$$\Gamma^{0,0,m=0} = \left[\epsilon^{(0,0,0)}_{\sigma_1\cdots\sigma_j}(-1)^j(p_1+p_2)^{\hat\sigma_1}\cdots(p_1+p_2)^{\hat\sigma_j}\right]\overline{g}^{0,0,0}(j,i\nu,\Delta), \tag{152}$$

$$\overline{g}^{0,0,0}(j,i\nu,\Delta) \coloneqq \int_0^{1/\Lambda}\frac{dz}{z^3}(\Lambda z)^j t_h(\Phi(z;m_h))^2\frac{\{``K_{i\nu}(\Delta z)"\}}{[(\frac{\Delta}{2\Lambda})^{-i\nu}\Gamma(i\nu)]}; \tag{153}$$

note here that the confinement effect has been included in the form of (i) introducing a cut in the holographic radius $z_h \le 1/\Lambda$ and (ii) $K_{i\nu}(\Delta z_h)$ is modified to $K_{i\nu}(\Delta z_h)$ in (123). The expression of $\overline{g}^{0,0,0}$ here, or that of $\Gamma^{0,0,m}$ in (149), implicitly ignores the possibility of $\mathcal{L}_{\text{bdry}} \neq 0$. For practical purposes, though, this may not be a big deal, since Ref. [6] reports that such confinement effects do not play a significant role quantitatively for most of the kinematical region.

Let us also evaluate the Wilson coefficient $C^{0,0,m=0}$. The expression

$$[A_p(z;-q_2)(-i\overleftrightarrow{\nabla})^jA_q(z;q_1)_q]_{\rho_1\cdots\rho_j}\delta^{\hat p\hat q} \coloneqq \sum_{k=0}^{j}{}_jC_k[(i\nabla)^{j-k}A(z;-q_2)]_{\rho_{k+1}\cdots\rho_j p}[(-i\nabla)^kA(z;q_1)]_{\rho_1\cdots\rho_k q}\delta^{\hat p\hat q} \tag{154}$$

appearing in $C^{0,0,m=0}$ can be evaluated by using the fact that

$$(\nabla^kA)_{\rho_1\cdots\rho_k\kappa} \equiv (\partial_{\rho_1}\cdots\partial_{\rho_k}A_\kappa) - \sum_{a=1}^{k}\frac{\eta_{\mu_a\kappa}}{z}(\partial_{\rho_1}\cdots\check\partial_{\rho_a}\cdots\partial_{\rho_k}A_z) - \sum_{1\le a<b\le k}\frac{\eta_{\rho_a\kappa}}{z^2}(\partial_{\rho_1}\cdots\check\partial_{\rho_a}\cdots\check\partial_{\rho_b}\cdots\partial_{\rho_k}A_{\rho_b}), \tag{155}$$

$$(\nabla^kA)_{\rho_1\cdots\rho_k z} \equiv (\partial_{\rho_1}\cdots\partial_{\rho_k}A_z) + \frac{1}{z}\sum_{a=1}^{k}(\partial_{\rho_1}\cdots\check\partial_{\rho_a}\cdots\partial_{\rho_k}A_{\rho_a}) \tag{156}$$

modulo terms proportional to $\eta_{\rho_c\rho_d}$. As we will focus only on the structure function $V_1^{+,\alpha}$, we can further drop the terms with A_z in (155), (156). Then the expression above becomes

$$\begin{aligned}&[\eta^{\mu\nu}\epsilon^1_\mu\epsilon^{2*}_\nu](q_1+q_2)_{\rho_1}\cdots(q_1+q_2)_{\rho_j} \\ &+\frac{2}{z^2}\sum_{a\neq b}\epsilon(-q_2)_{\rho_a}\epsilon(q_1)_{\rho_b}(q_1+q_2)_{\rho_1}\cdots_{\check\rho_a\check\rho_b}\cdots(q_1+q_2)_{\rho_j}\end{aligned} \tag{157}$$

multiplied by $[(q_1z)K_1(q_1z)][(q_2z)K_1(q_2z)]$.

There are two remaining tasks in evaluating the $(m=0)$-mode contribution to the $V_1^{+,\alpha}$ structure function: (a) one is to carry out the z_γ integral and (b) the other is to sum $C^{0,0,0}\Gamma^{0,0,0}$ for different polarizations of $\epsilon^{(0,0,0)}$. As for the z_γ integral, the integrand sharply falls off[28] at $z_\gamma \approx q^{-1}$ because of the photon wave functions of the form $[(q_i z)K_{i\nu}(q_i z)]$. The z_γ integral in $C^{0,0,m}$ over the holographic radius $z_\gamma \in [0, \Lambda^{-1}]$ therefore comes mainly from a very small fraction of it, $\Lambda/q \ll 1$, in the regime of generalized Bjorken scaling (143). It is then all right to make an approximation that

$$I_{i\nu}(\Delta z_\gamma) \approx \frac{1}{\Gamma(i\nu+1)}\left(\frac{\Delta z_\gamma}{2}\right)^{i\nu}[1+\mathcal{O}(\Delta/q)]$$
$$\text{when } (\Delta z_\gamma) \lesssim \Delta/q \ll 1, \tag{158}$$

and also to replace the range of integral $z_\gamma \in [0, \Lambda^{-1}]$ to $[0, +\infty)$, as in the literature; the error caused by this approximation is only in the higher order in (Δ/q), and the twist-$(2+\gamma(j))$ contribution is still obtained properly. The integral is then cast into the form of (B1) with $\delta = j + i\nu$ for the first line of (157) [respectively $\delta = j + i\nu - 2$ for the second line of (157)] and $\vartheta = \eta/x$; thus we can use the analytic expression (B3), (B5) in the Appendix.

The other task, (b) tensor computations, is carried out in Appendix A.6. Using the results of (A77) and (A81), one finds that the contribution to $(C^{0,0,0}_{V_1;+,\alpha})_{m=0}$ from the second line of (157) is roughly

$$\frac{q^2\Delta^2}{(q\cdot\Delta)(p\cdot q)} \ll 1 \tag{159}$$

times smaller than the contribution from the first line of (157) in the generalized Bjorken scaling regime (143), and hence it is ignored, when only the twist-$[2+\gamma(j)]$ contributions are retained.

Combining all of the above, the spin-$j \in 2\mathbb{Z}$ contribution is

$$(V_1^{+,\alpha})_{j,m=0} \approx \frac{\sqrt{\lambda}}{\Gamma(j+1)\pi}\frac{t_\gamma}{t_y}\int_{-\infty}^{+\infty} d\nu \frac{1}{\frac{4+j+\nu^2+c_j}{\sqrt{\lambda}}+j-1-i\epsilon}$$
$$\times C_1(j+i\nu,\vartheta)\left(\frac{\Lambda}{q}\right)^{i\nu-j}$$
$$\times\left(\frac{1}{\sqrt{\lambda}x}\right)^j \overline{g}^{0,0,0}(j,i\nu,\Delta)\hat{d}_j([\eta]), \tag{160}$$

where C_1 is given in (B5) and $\hat{d}_j$ is a polynomial of degree j in the argument

$$[\eta] \coloneqq \eta \times \sqrt{\frac{-4p^2}{\Delta^2}} = \eta\sqrt{\frac{4m_h^2+\Delta^2}{\Delta^2}} \tag{161}$$

and is given in terms of Legendre polynomial, as in (A79).

Now that all of the factors of the spin-j contribution to $V_1^{+,\alpha}$ are given as analytic functions of j, it is possible to convert the sum over the (spin-$j \in 2\mathbb{N}$) string states in the leading trajectory to a contour integral in the complex angular momentum plane:

$$(V_1^{+,\alpha})_{m=0} = -\int \frac{dj}{4i}\frac{1+e^{\pi ij}}{\sin(\pi j)}(V_1^{+,\alpha})_{j,m=0}, \tag{162}$$

with the contour in the j plane moving just below the real positive axis toward the left, and then just above the real positive axis toward the right. The integration contour in the ν plane is deformed so that it picks up the residue of the pole in the lower complex ν plane coming from the t-channel propagation of strings. Thus,

$$(V_1^{+,\alpha})_{m=0} \approx -\int \frac{dj}{4i}\frac{1+e^{\pi ij}}{\sin(\pi j)}\frac{t_\gamma/t_y}{\Gamma(j+1)}\frac{\lambda}{i\nu_j}$$
$$\times C_1\left(j+i\nu_j,\frac{\eta}{x}\right)\left(\frac{\Lambda}{q}\right)^{\gamma(j)}$$
$$\times\left(\frac{1}{\sqrt{\lambda}x}\right)^j \overline{g}^{0,0,0}(j,i\nu_j,\Delta)\hat{d}_j([\eta]), \tag{163}$$

where $\gamma(j) = i\nu_j - j$ and $i\nu_j \geq 0$ is a function of j determined by the on-shell condition

$$j-1+\frac{4+j+\nu^2+c_j}{\sqrt{\lambda}} = 0. \tag{164}$$

This is the result known in [2–7], etc.; under an assumption that $\overline{g}^{0,0,0}(j,i\nu_j,\Delta)$ does not grow too rapidly for a large $\mathrm{Re}(j)$ to cancel the large factor $\Gamma(j+1)$ in the denominator, the integration contour in the j plane can be deformed toward the left in the j plane, as in the classical Watson-Sommerfeld transformation; this is how the non-converging $j \in 2\mathbb{N}$ sum of the OPE is rendered well defined for physical kinematics $x<1$. The integrand forms a saddle point due to the two factors $(1/x)^j$ and $(\Lambda/q)^{\gamma(j)}$;

[28] In DVCS, VMP, and TCS, the incoming photon has spacelike momentum q_1, although the outgoing photon may be either on shell or timelike. The sharp cutoff in the z_γ integral comes from the wave function of the incoming photon. Since the wave function of the outgoing photon remains a Hankel function even for VMP and TCS, rather than $I_{i\nu}(qz_\gamma)$, the approximation of replacing the integration interval $z_\gamma \in [0,\Lambda^{-1}]$ by $z_\gamma \in [0,\infty)$ remains valid. In such applications, $\vartheta > 1$, and the expressions (B1) and (B5) should be understood through analytic continuation. The authors thank the Physical Review D referee for bringing this issue to our attention.

let j_* in the complex j plane be where the saddle point is.[29] The integrand also has poles in the j plane. The hadron matrix element $\overline{g}^{0,0,0}$ contains $c^{(j)}_{i\nu_j;0,0,0}$ in its definition, and $c^{(j)}_{i\nu;0,0,0}$ may have a pole in the j plane [3].[30] The saddle point value j_* has a larger real part than any one of the poles, where $\ln(q/\Lambda)$ is large relatively to $\ln(1/x)$; the j integral is well approximated by the saddle point value of the integrand and yields the DGLAP regime. When $\ln(1/x)$ is large relatively to $\ln(q/\Lambda)$, however, one of the poles may have a real part larger than $\mathrm{Re}(j_*)$. Then the integral is approximated by the residue at such a leading pole. In this way, the string-theory result $(V_1^{+,\alpha})_{m=0}$ goes back and forth between the DGLAP phase and Regge phase, depending on the kinematical variables $x, (q^2/\Lambda^2)$ and $t = -\Delta^2$ [3,7].

The derivation of (163) was not just a review of preceding works, however. First, the integration over z_γ yields a function $C_1(j + i\nu_j, \eta/x)$, which has precisely the same form as the one expected from the conformal OPE; comparing (25), (26) and (B3), (B5), one finds that they agree, under the identification

$$[(l_n + j_n - 2) = 2j_n + (\tau_n - 2)] \Leftrightarrow [(j + i\nu_j) = 2j + \gamma(j)]. \tag{165}$$

The expression (163) is indeed regarded as conformal OPE contributions from twist-$\tau_n = (2 + \gamma(j))$ operators.

Second, the η dependence of the $m = 0$ contribution is now worked out. As we will see later in Sec. VII, it comes in a form that fits very well with what has been known as dual parametrization of GPD [15]. One will also notice that the argument of the degree-j polynomial $\hat{d}_j([\eta])$ is $[\eta]$ in (161), rather than η. This means that the coefficients of the η^2 term and higher diverge in the $t = -\Delta^2 \to 0$ limit. This indicates that it is essential to sum the $m \neq 0$ modes to obtain results that are physically sensible. We will address this issue in Sec. VII A.

We have considered amplitudes from a t-channel exchange of states that (a) are in the leading trajectory and (b) have 5D-traceless and 5D-transverse polarizations, and that have used the prescription (52), so that we obtained the contributions that correspond to the twist-$[2 + \gamma(j)]$ operators in the conformal OPE. When the amplitudes with a t-channel exchange of other modes are included, cancellation due to BRST symmetry is at work among some of them, but other physical contributions remain. Computation of those contributions will shed a light on the higher twist contributions to the DVCS amplitudes. To do this, however, we need wave functions of modes other than the $(n, l, m) = (0, 0, m)$ modes, and detailed knowledge on the interaction terms in the string field theory more than (130), (131); furthermore, the prescription (52)—the process carried out just before Sec. VI C 1—becomes more complicated for modes other than the $(0, 0, m)$ modes. After all of this, one then has to work out which operators in QCD correspond to which group of modes in the t-channel exchange in the gravity dual calculation. Although this is an interesting question, we do not address that problem in this article.

3. Preparation

Let us move on to the amplitudes of $m \geq 1$-mode exchange. We begin with deriving a few general properties of those amplitudes, which makes the subsequent computations less tedious.

First, we observe that the hadron matrix element $\Gamma^{0,0,m}$ vanishes for any odd value of m. To see that this statement is true, we use the following property of $J^{hh}_{l_1 \cdots l_j}$:

$$\Phi(z, p_1) \overleftrightarrow{\nabla}_{\{l_1} \ldots \overleftrightarrow{\nabla}_{l_j\}} \Phi(z, -p_2) = (-1)^j \Phi(z, -p_2) \overleftrightarrow{\nabla}_{\{l_1} \ldots \overleftrightarrow{\nabla}_{l_j\}} \Phi(z, p_1); \tag{166}$$

this is true in a process where the initial state hadron $h(p_1)$ continues to be the same hadron $h(p_2)$ in the final state, so that $-(p_1)^2 = -(p_2)^2 = m_h^2$. This property is used below to study when $J^{hh}_{z^k \lambda_{k+1} \cdots \lambda_j} \overline{A}^{z^k \lambda_{k+1} \cdots \lambda_j}$ vanishes for various $k = 0, \ldots, m$.

For an even j, the SO(3,1) indices of $J^{hh}_{z^k \lambda_{k+1} \cdots \lambda_j}$ are provided by an even number of $(p_1 + p_2)_\lambda$'s and even (respectively odd) number of Δ_λ's when k is even (respectively odd). The hadron matrix element $\Gamma^{0,0,m}$ receives a nonvanishing contribution from $J^{hh}_{z^k \lambda_{k+1} \cdots \lambda_j} \overline{A}^{z^k \hat{\lambda}_{k+1} \cdots \hat{\lambda}_j}$ (no sum in k) only when the D operator (71) is used for an even (respectively odd) number of times in the Reggeon wave function (89). This means that s is even (respectively odd), and hence $\Gamma^{0,0,m}$ can be nonzero only when $m = k + s$ is even.

For an odd j, the SO(3,1) indices of $J^{hh}_{z^k \lambda_{k+1} \cdots \lambda_j}$ are provided by an odd number of $(p_1 + p_2)_\lambda$'s and an even (respectively odd) number of Δ_λ's when k is even (respectively odd). Thus, the matrix element $\Gamma^{0,0,m}$ receives a nonzero contribution only when an even (respectively odd) number of the D operator is used in (89). This means, once again, that s is even (respectively odd), and hence $\Gamma^{0,0,m}$ can be nonzero only when $m = k + s$ is even. This statement for an odd j is not more than a side remark, though, since we focus on the spin-even contribution $\propto [1 + e^{-\pi i j}]/\sin(\pi j)$ in this article.

[29] The saddle point value j_* is determined by $\frac{\partial \gamma(j)}{\partial j}\big|_{j=j_*} = \frac{\ln(1/x)}{\ln(q/\Lambda)}$.

[30] For example, imagine a case $(1 - c^{(j)}_{i\nu;0,0,0}) = [I_{-i\nu}(\Delta/\Lambda)/I_{i\nu}(\Delta/\Lambda)]$; the factor $c^{(j)}_{i\nu_j;0,0,0}$ has poles $j = \alpha_{\mathbb{R},n}(t)$ $(n = 1, 2, \ldots)$ in the j plane given by the condition $j_{i\nu_j,n} = \sqrt{t}/\Lambda$; $j_{\mu,n}$'s are the nth zero of the Bessel function J_μ.

Second, $\Gamma^{0,0,m}$ can always be written in the form of

$$\Gamma^{0,0,m} = [(-2)^{j-m}(p^{\hat{\sigma}_1}\cdots p^{\hat{\sigma}_{j-m}})\cdot \epsilon^{(0,0,m)}_{\sigma_1\cdots\sigma_{j-m}}] \times \overline{g}^{0,0,m}(j, i\nu, \Delta^2), \tag{167}$$

and $\overline{g}^{0,0,m}$ is an SO(3,1) scalar of mass dimension m; we have encountered a special case of this statement in (152), (153). This statement itself is understood as follows. When we write down the covariant derivatives in $\overline{J}^{hh}_{z^k\lambda_1\cdots\lambda_{j-k}}$ explicitly, the SO(3,1) indices—there are $(j-k)$ of them—are one of either p_λ, Δ_λ, and $\eta_{\lambda\lambda'}$; $\eta_{\lambda\lambda'}$ can be further rewritten as $\eta_{\lambda\lambda'} - \Delta_\lambda\Delta_{\lambda'}/\Delta^2$ and $\Delta_\lambda\Delta_{\lambda'}$. Suppose that there are N_p of the SO(3,1) indices from $\{p_\lambda\}$'s, N_Δ indices from $\{\Delta_\lambda\}$'s, and $N_{\tilde{\eta}}$ from $\tilde{\eta}_{\lambda\lambda'}$'s in a given term; $N_p + N_\Delta + 2N_{\tilde{\eta}} = (j-k)$. When such an SO(3,1) tensor is contracted with $\sum_N^{[(m-k)/2]} \tilde{E}^N D^{m-k-2N}[\epsilon^{(0,0,m)}]$ in the Reggeon wave function $\overline{A}^{0,0,m}_{z^k\lambda_1\cdots\lambda_{j-k}}$, it remains nonzero only when $(m-k-2N) = N_\Delta$ and $N \geq N_{\tilde{\eta}}$ because of the relation (A37). It is not hard now to see that all of the remaining terms are proportional to the prefactor of $\overline{g}^{0,0,m}$ in (167); the mass dimension of the remaining scalar factor (the reduced matrix element) $\overline{g}^{0,0,m}$ follows from the fact that $\Gamma^{0,0,m}$ is defined to be of mass dimension j.

Finally, we note that the twist-$[2+\gamma(j)]$ contribution to the coefficient $C^{0,0,m}$ arises only from the contraction $\overline{J}^{\gamma\gamma}_{z^k\kappa_{k+1}\cdots\kappa_j}\overline{\overline{A}}^{z^k\hat{\kappa}_{k+1}\cdots\hat{\kappa}_j}$, with $k=0$. We have already seen an example of this in the $m=0$ amplitude; the first term of (157) contributes to (163), while the second term does not because of (159), and the first term came from the $k=0$ contraction.

In order to verify the claim above, note first that both an extra ∂_z and an extra power of $1/z$ virtually change the integral of $C^{0,0,m}$ by about an extra power of $q \sim q_1 \sim q_2 \gg \Lambda, \Delta$. Explicitly writing down covariant derivatives in $\overline{J}^{\gamma\gamma}_{z^k\kappa_1\cdots\kappa_{j-k}}$ and evaluating the integrals only by the order of magnitudes, one can see that

$$(C^{0,0,m}_{V_1;+,\alpha})_k \sim \sum_M^{[\frac{j-k}{2}]}\left(\frac{\Lambda}{q}\right)^{i\nu-j}\frac{q^{k+2M}}{(q^2)^j}\left[\overbrace{(q_\kappa\cdots q_\kappa)}^{j-k-2M}(\eta_{\kappa\kappa}q^2)^M\right] \cdot \sum_N^{[\frac{s}{2}]}\frac{1}{\Delta^{s-2N}}\tilde{E}^N D^{s-2N}[\epsilon^{(0,0,m)}]. \tag{168}$$

The $M=0$ contribution above is further evaluated by using the definition of $\tilde{E}$ and D operators. Details of computation are found partially in (A82); we find that

$$(C^{0,0,m}_{V_1;+,\alpha})_{k,M=0} \sim \left(\frac{\Lambda}{q}\right)^{i\nu-j}\frac{\overbrace{(q_\kappa\cdots q_\kappa)}^{j-m}\epsilon^{(0,0,m)}}{(q^2)^j}\left(\frac{(q\cdot\Delta)}{\Delta}\right)^s q^k. \tag{169}$$

Keeping the relation $m = k+s$ and also the result (167) in mind, we obtain

$$C^{0,0,m}_{k,M=0}\cdot\Gamma^{0,0,m} \sim \left(\frac{\Lambda}{q}\right)^{i\nu-j}\frac{(q\cdot p)^{j-m}}{(q^2)^j}\left(\frac{(q\cdot\Delta)}{\Delta}\right)^s q^k \times \overline{g}^{0,0,m} \sim \left(\frac{\Lambda}{q}\right)^{i\nu-j}\left(\frac{1}{x}\right)^j \eta^{m-k}\left(\frac{(q^2)(\Delta^2)}{(q\cdot p)^2}\right)^{\frac{k}{2}}\frac{\overline{g}^{0,0,m}}{\Delta^m}. \tag{170}$$

Therefore, this is regarded as a twist-$[2+\gamma(j)+k/2]$ contribution in the generalized Bjorken scaling regime. Thus, only the $k=0$ term remains a twist-$[2+\gamma(j)]$ contribution, and the terms with $k>0$ are irrelevant to GPD.

The analysis becomes a little more complicated when $M>0$ terms are also included, but not in an essential way. Contributions with some (k,M) correspond to twist-$(2+\gamma+M+k/2)$, and only the $k=M=0$ terms contribute to GPD. This means that $C^{0,0,m}$ can be evaluated under the following approximation:

$$[A_p(z;-q_2)(-i\overleftrightarrow{\nabla})^j A_q(z;q_1)]^{\hat{m}_1\cdots\hat{m}_j}\delta^{\hat{p}\,\hat{q}}\overline{\overline{A}}_{m_1\cdots m_j} \to [A_\mu(z;-q_2)(-i\overleftrightarrow{\partial})^j A_\nu(z;q_1)]^{\hat{\kappa}_1\cdots\hat{\kappa}_j}\eta^{\mu\nu}\overline{\overline{A}}_{\kappa_1\cdots\kappa_j}. \tag{171}$$

4. Wilson coefficients, conformal OPE and hadron matrix elements

The twist-$[2+\gamma(j)]$ contribution to $C^{0,0,m}_{V_1;+,\alpha}$ can be determined completely, using the approximations above.

$$C^{0,0,m}_{V_1;+,\alpha} \simeq \left(\frac{2\Lambda}{\Delta}\right)^{i\nu}\frac{\Gamma(i\nu+1)}{(2\Lambda)^j}\int\frac{dz}{z}[(q_1z)K_1(q_1z)][(q_2z)K_1(q_2z)]z^j \times\sum_{N=0}^{[\frac{m}{2}]}[2^j(q^{\hat{\rho}_1}\cdots q^{\hat{\rho}_j})\cdot\tilde{E}^N D^{m-2N}[\epsilon^{(0,0,m)}]_{\rho_1\cdots\rho_j}]\frac{b^{(j-m)}_{m,N}}{\Delta^{m-2N}}[e^{(2-j)A}\overline{\overline{\Psi}}^{(j);m,N}_{i\nu;0,0,m}]. \tag{172}$$

The product of rank-j SO(3,1) tensors in the second line is reduced to a product of rank-$(j-m)$ tensors by the computation in (A82). The Reggeon wave function $\overline{\overline{\Psi}}$ is also rewritten by using the small $(\Delta z_\gamma) \lesssim (\Delta/q)$ approximation (158):

$$[e^{(2-j)A}\overline{\overline{\Psi}}^{(j);m,N}_{i\nu;0,0,m}] \simeq \sum_{a=0}^{N}(-1)^a{}_N C_a(\zeta^{j+1}\partial_\zeta^{m-2a}[\zeta^{-1-j+m}(\zeta/2)^{i\nu}])_{\zeta\to(\Delta z)}\frac{N_{j,m}}{\Gamma(i\nu+1)}. \quad (173)$$

The $a=0$ term in this expression has the lowest dimension in $\zeta = \Delta z_\gamma \lesssim (\Delta/q)$, and hence we only need to retain the $a=0$ term for a given N for the twist-$[2+\gamma(j)]$ contribution. Thus,

$$[e^{(2-j)A}\overline{\overline{\Psi}}^{(j);m,N}_{i\nu;0,0,m}] \simeq 2^{-i\nu}\frac{(-1)^m\Gamma(j+1-i\nu)}{\Gamma(j+1-i\nu-m)}(\Delta z_\gamma)^{i\nu}\frac{N_{j,m}}{\Gamma(i\nu+1)}. \quad (174)$$

Using this expression and (A82) in (172), we obtain

$$C^{0,0,m}_{V_1;+,\alpha} \simeq \frac{\Lambda^{i\nu-j}}{q^{i\nu+j}}\frac{j!}{(j-m)!}\sum_{N=0}^{[m/2]}\left[\frac{(q\cdot\Delta)^2}{\Delta^2}\right]^N(-i)^m\frac{(q\cdot\Delta)^{m-2N}}{\Delta^{m-2N}}b^{(j-m)}_{m,N}[(q_{\mu_1}\cdots q_{\mu_{j-m}})\cdot\epsilon^{(0,0,m)}]$$
$$\times\int\frac{dz}{z}[(q_1z)K_1(q_1z)][(q_2z)K_1(q_2z)](qz)^{j+i\nu}\frac{(-1)^m\Gamma(j+1-i\nu)}{\Gamma(j+1-i\nu-m)}N_{j,m}, \quad (175)$$

$$= \left(\frac{\Lambda}{q}\right)^{i\nu-j}\frac{[(q_{\mu_1}\cdots q_{\mu_{j-m}})\cdot\epsilon^{(0,0,m)}]}{(q^2)^j}\frac{(q\cdot\Delta)^m}{\Delta^m}C_1\left(j+i\nu,\frac{\eta}{x}\right)$$
$$\times\, i^m\frac{j!}{(j-m)!}N_{j,m}\frac{\Gamma(j+1-i\nu)}{\Gamma(j+1-i\nu-m)}\left(\sum_{N=0}^{[m/2]}b^{(j-m)}_{m,N}\right), \quad (176)$$

$$= \left(\frac{\Lambda}{q}\right)^{i\nu-j}\frac{[(q_{\mu_1}\cdots q_{\mu_{j-m}})\cdot\epsilon^{(0,0,m)}]}{(q^2)^j}\frac{(q\cdot\Delta)^m}{\Delta^m}C_1\left(j+i\nu,\frac{\eta}{x}\right)$$
$$\times\, i^m\frac{\Gamma(j+1+i\nu-m)}{N_{j,m}\Gamma(j+1+i\nu)}, \quad (177)$$

where (B1), (B5) is used for the equality in the middle, while (A67) is used for the last one.

Repeating the same argument as in Sec. VI C 2, we thus arrive at

$$(V_1^{+,\alpha})_m \simeq -\int\frac{dj}{4i}\frac{1+e^{-\pi ij}}{\sin(\pi j)}\frac{t_\gamma/t_y}{\Gamma(j+1)}\frac{\lambda}{i\nu_j}C_1(j+i\nu_j)\left(\frac{\Lambda}{q}\right)^{\gamma(j)}$$
$$\times\left(\frac{1}{\sqrt{\lambda}x}\right)^j\eta^m\hat{d}_{j-m}([\eta])\frac{\overline{g}^{0,0,m}}{\Delta^m}\frac{i^m}{N_{j,m}}$$
$$\times\frac{\Gamma(j+1+i\nu-m)}{\Gamma(j+1+i\nu)}; \quad (178)$$

the computation in (A77), (A79) for an even j and m was used once again. Similar to the case of the $m=0$ amplitude, this expression is in the form of the conformal OPE and inverse Mellin transformation in (20). It should be noted that the integrand can be defined as a holomorphic function of j (apart from poles and cuts), using the definition of C_1 in (B1) and that of $\hat{d}_{j-m}$ in (A79), not just for an integer-valued j; at the same time, $\eta^m\hat{d}_{j-m}([\eta])$ becomes a polynomial of η of degree j for $j\in 2\mathbb{N}$, which is one of the important properties expected for the hadron matrix element [9].

The integration contour of (178) is chosen so that it circles around the pole at $j=m$ after running just below the real positive axis in the j plane and before running just above the real positive axis. Only spin-j stringy states with $m\le j$ contribute then. It is not obvious whether the contour can be deformed so that it encircles $j=0,2,\ldots,m$ without changing $(V_1^{+,\alpha})_m$, and we leave it an open question. $\hat{d}_{j-m}$ in (178) is given by a Legendre polynomial of degree $(j-m)$ when $j-m$ is an even positive integer, but otherwise it is defined by the hypergeometric function, as in (A79), and it may or may not have a zero at negative even integer $(j-m)$ so that the pole from $\sin(\pi j)$ is canceled. Similarly, $\overline{g}^{0,0,m}(j,i\nu_j,\Delta)/N_{j,m}$ may or may not have a zero at negative integer $(j-m)$. The authors have not found a reason to believe that they have a zero, but we may be wrong.

The twist-$[2+\gamma(j)]$ contribution to the structure function $V_1^{+,\alpha}$ is obtained by summing $(V_1^{+,\alpha})_m$ from the $(n,l,m)=(0,0,m)$ modes with $m=0,2,\ldots$:

$$V_1^{+,\alpha} = \sum_{m=0,2,\ldots}^{\infty}(V_1^{+,\alpha})_m. \quad (179)$$

Combining (163), (178) with (179), a holographic version of (20) is obtained. It is not obvious, though, whether or not the integration variable j in (178) for all the different m's should be identified. If we are to define $j' := (j-m)$ and use it as a new variable of integration, then the integration contour of (178) would be the same for all different m's; the cost of doing so, however, is this:

$$C_1(j+i\nu_j,\vartheta)\left(\frac{\Lambda}{q}\right)^{\gamma(j)}\frac{1}{x^j}\eta^m\hat{d}_{j-m} = [C_1(j'+m+i\nu_{j'+m},\vartheta)\times\vartheta^m]\left(\frac{\Lambda}{q}\right)^{\gamma(j'+m)}\frac{1}{x^{j'}}\hat{d}_{j'}. \tag{180}$$

Certainly $\hat{d}_{j'}$ still remains to be a polynomial of degree at most j', but the expression no longer fits into the form of conformal OPE. For this reason, we identify the integration variable j in (178) for all $m = 0, 2, \ldots$ with that (complex angular momentum) of the inverse Mellin transformation (20). This implies that the reduced hadron matrix element of the spin-j primary operator is given a holographic expression

$$\overline{A}_j^{+,\alpha}(\eta,t) \propto \sum_{m=0}^{j}\frac{(-1)^{m/2}}{\sqrt{\lambda}^j\Gamma(j+1)}\frac{\overline{g}^{0,0,m}(j,i\nu_j,\Delta^2)}{N_{j,m}\Delta^m} \times\frac{\Gamma(j+1+i\nu_j-m)}{\Gamma(j+1+i\nu_j)}\times\eta^m\hat{d}_{j-m}([\eta]). \tag{181}$$

5. The $(m=2)$-mode hadron matrix element

Most aspects of the expression (178) are dictated by basic principles of field theory, such as (conformal) OPE. Additional information from the holographic setup is found primarily in the hadron matrix element $\overline{g}^{0,0,m}(j,i\nu,\Delta)$, apart from the anomalous dimension $\gamma(j) = i\nu_j - j$ of the twist-$[2+\gamma(j)]$ operators. Now we have seen that $\overline{g}^{0,0,0}(j,i\nu_j,\Delta)$ is not the only hadron matrix element contributing to the nonperturbative information of $h+\gamma^*\to h+\gamma^{(*)}$; let us take a moment here to have a closer look at one of the new hadron matrix elements we encounter, $\overline{g}^{0,0,2}(j,i\nu,\Delta)$.

The hadron matrix element $\Gamma^{0,0,m}$ receives contributions from $\overline{J}^{hh}_{z^k\lambda_{k+1}\cdots\lambda_j}\overline{A}^{\hat{z}^k\hat{\lambda}_{k+1}\cdots\hat{\lambda}_j}$'s, with $k = 0,1,\ldots,m$. The contribution from each k can be written in the form of (167), and hence $(\overline{g}^{0,0,m}(j,i\nu,\Delta))_k$ is defined ($k\le m$). We compute $(\overline{g}^{0,0,2})_k$ explicitly for $k=0,1,2$.

For this purpose, we need the following technical results:

$$(\nabla^l\Phi)_{\lambda_1\cdots\lambda_l} \equiv (\partial_{\lambda_1}\cdots\partial_{\lambda_l}\Phi) - \sum_{1\le a<b\le l}\frac{\eta_{\lambda_a\lambda_b}}{z}\left(\partial_{\lambda_1}\cdots_{\check{\lambda}_a\check{\lambda}_b}\cdots\partial_{\lambda_l}\left(\partial_z+\frac{l-a-1}{z}\right)\Phi\right) \tag{182}$$

modulo terms proportional to $\eta_{\lambda_a\lambda_b}\eta_{\lambda_c\lambda_d}$ instead of (150), and

$$(\nabla^l\Phi)_{\lambda_1\cdots z\cdots\lambda_l} \equiv \left(\partial_z+\frac{l-1}{z}\right)\partial_{\lambda_1}\cdots_{\check{\lambda}_a}\cdots\partial_{\lambda_l}\Phi,$$
$$(\nabla^l\Phi)_{\lambda_1\cdots z\cdots z\cdots\lambda_l} \equiv \left[\left(\partial_z+\frac{l-1}{z}\right)\left(\partial_z+\frac{l-2}{z}\right)+\frac{a-1}{z^2}\right]\times\partial_{\lambda_1}\cdots_{\check{\lambda}_a\check{\lambda}_b}\cdots\partial_{\lambda_l}\Phi, \tag{183}$$

modulo terms proportional to $\eta_{\lambda_c\lambda_d}$.

It is now a straightforward computation to use the relations above as well as the explicit Reggeon wave functions $\overline{A}$ determined in Sec. V to derive the following:

$$\frac{\overline{g}^{0,0,2}_{k=2}}{N_{j,2}\Delta^2} = \frac{j(j-1)}{2}\int_0^{1/\Lambda}\frac{dz}{z^3}(\Lambda z)^j\frac{\{z^2\text{“}K_{i\nu}(\Delta z)\text{”}\}}{[(\frac{\Delta}{2\Lambda})^{-i\nu}\Gamma(i\nu)]}t_h\left[2\{-\Phi(\partial_z^2\Phi)+(\partial_z\Phi)^2\}-\frac{2}{z}\Phi(\partial_z\Phi)-\frac{4(j-2)}{3z^2}\Phi^2\right],$$
$$\frac{\overline{g}^{0,0,2}_{k=1}}{N_{j,2}\Delta^2} = \frac{j(j-1)}{2}\int_0^{1/\Lambda}\frac{dz}{z^3}(\Lambda z)^j\frac{\{z^{j+1}\partial_z(z^{1-j}\text{“}K_{i\nu}(\Delta z)\text{”})\}}{[(\frac{\Delta}{2\Lambda})^{-i\nu}\Gamma(i\nu)]}\left[\frac{-2t_h}{z}\Phi^2\right],$$
$$\frac{\overline{g}^{0,0,2}_{k=0}}{N_{j,2}\Delta^2} = j(j-1)\int_0^{1/\Lambda}\frac{dz}{z^3}(\Lambda z)^j\left(\frac{\{[z^{j+1}\partial_z^2z^{1-j}-(z\Delta)^2]\text{“}K_{i\nu}(\Delta z)\text{”}\}}{[(\frac{\Delta}{2\Lambda})^{-i\nu}\Gamma(i\nu)]}\times\left[\frac{p^2}{(j-\frac{1}{2})\Delta^2}t_h\Phi^2\right] + \frac{\{-z^2\text{“}K_{i\nu}(\Delta z)\text{”}\}}{[(\frac{\Delta}{2\Lambda})^{-i\nu}\Gamma(i\nu)]}\times t_h\left[\frac{1}{z}\Phi(\partial_z\Phi)+\frac{j-2}{3z^2}\Phi^2\right]\right). \tag{184}$$

These results are used in the study below.

VII. A HOLOGRAPHIC MODEL OF GPD

The differential cross section of the DVCS process involves an integral of GPD; GPD needs to be parametrized first, and then the parameters are determined by fitting the data [10]. The idea of dual parametrization of GPD [15]—also known as collinear factorization approach [17,18]—is to expand the reduced hadron matrix element $\overline{A}_j^{+,\alpha}(\eta,t)$ as

$$\overline{A}_j^{+,\alpha}(\eta,t)=\sum_{m=0}^{j}\overline{\Gamma}_m^{+,\alpha}(j,t)\eta^m\times[\eta^{j-m}d_{j-m}(1/\eta)], \tag{185}$$

where $d_\ell(\cos\theta)$'s are polynomials of degree ℓ in the argument $(\cos\theta)$; Legendre polynomials, Gegenbauer polynomials, or Jacobi polynomials are used, depending on the helicity change of the target hadron h in the scattering process [9]. When the target hadron is a scalar, as in the study of this article, a Legendre polynomial is chosen for d_ℓ [15]. With no ambiguity introduced in the polynomials $d_{j-m}(x)$, $\overline{\Gamma}_m^{+,\alpha}(j,t)$'s are the fully general, yet nonredundant parametrization for the reduced hadron matrix element for GPD.

At the end of the study in the preceding sections, we arrived at a holographic model of GPD, with the reduced hadron matrix element given by (181) for the flavor-nonsinglet sector. String theory—the descendant of the dual resonance model—yields a result that fits straightforwardly with the format of the dual parametrization (185); this should not be a surprise, but rather must be something the authors of [15] have anticipated. With the string-theory implementation provided, one can now move forward; now

$$\overline{\Gamma}_m^{+,\alpha}(j,t)\sim(-1)^{m/2}\frac{\overline{g}^{0,0,m}(j,i\nu_j,\Delta)}{N_{j,m}\Delta^m} \tag{186}$$

can be computed using holographic backgrounds, independently from experimental data. Certainly the matrix elements $[\overline{g}^{0,0,m}/\Delta^m]$ will depend on holographic backgrounds to be used for computation, and predictions from individual holographic backgrounds should not be taken seriously at the quantitative level. But it is still worth looking closely into qualitative features of the holographic hadron matrix elements $\overline{g}^{0,0,m}/\Delta^m$ to learn nonperturbative aspects of $\overline{\Gamma}_m^{+,\alpha}(j,t)$.

A. $\Delta^2\to 0$ limit

As we have already remarked earlier in this article, the holographic result (181) is not precisely in the same form of parametrization as in (185); the argument of the polynomial $\hat{d}_{j-m}$ is [η] as defined in (161), rather than η. This difference itself does not raise an issue immediately; [η] is the same as η in the hard scattering regime, $\Delta^2\gg m_h^2$.

Let us study how the hadron matrix element behaves in the $t=-\Delta^2\to 0$ limit, however. The matrix element $\overline{g}^{0,0,0}(j,i\nu_j,\Delta)$ has already been studied in the literature and is known not to diverge or vanish in the $\Delta^2\to 0$ limit. The polynomial $\hat{d}_j([\eta])$ to be multiplied with this $\overline{g}^{0,0,0}(j,i\nu_j,\Delta)$, however, has diverging coefficients in all of the terms $\eta^2,\eta^4,\ldots$ except the η^0 term. Therefore, the $m=0$ contribution (163) alone does not have a physically reasonable behavior in the $\Delta^2\to 0$ limit. A natural expectation will be that the hadron matrix element $\overline{A}_j^{+,\alpha}(\eta,t)$ still has a reasonable behavior after summing up $m=0,2,\ldots,j$.

To get started, we focus on the η^2 term. It is generated from the $(m=0)$-mode exchange, and also from the $(m=2)$-mode exchange. There is a $(p^2)/\Delta^2$ factor both in $\overline{g}^{0,0,0}\times\hat{d}_j([\eta])|_{\eta^2}$ and $\overline{g}^{0,0,2}/\Delta^2\times\eta^2$, and hence both diverge in the $\Delta^2\to 0$ limit. When they are summed, however, the divergence may cancel, as we see in the following. Let us study the coefficient of the η^2 term

$$-\int\frac{dj}{4i}\frac{1+e^{-\pi ij}}{\sin(\pi j)}\left(\frac{\Lambda}{q}\right)^{i\nu-j}\left(\frac{1}{\sqrt{\lambda}x}\right)^{j}C_1\left(j+i\nu,\frac{\eta}{x}\right)\times\frac{\lambda}{i\nu_j}\frac{t_\gamma/t_y}{\Gamma(j+1)}\times\eta^2 \tag{187}$$

in the $\Delta^2\to 0$ limit, picking up a contribution to the integral $\overline{g}^{0,0,0}$ and $\overline{g}^{0,0,2}$ from the $I_{-i\nu}(\Delta z_h)$ component in $K_{i\nu}(\Delta z_h)$ first.[31] Then in that limit, the coefficient of the expression (187) becomes

$$\frac{p^2}{\Delta^2}\lim_{\Delta^2\to 0}\left[\overline{g}^{0,0,0}(j,i\nu_j,\Delta)\frac{j(j-1)}{(j-\frac{1}{2})}-\frac{\overline{g}^{0,0,2}(j,i\nu_j,\Delta)/(p^2)}{(j-1+i\nu_j)(j+i\nu_j)N_{j,2}}\right]+\mathcal{O}(\Delta^0). \tag{189}$$

The two terms in $\lim_{\Delta^2\to 0}[\cdots]$ cancel each other, as one can see by using the approximation in footnote 31. Thus, the η^2 term in $\overline{A}_j^{+,\alpha}(\eta,t)$ also has a finite limit value in the $\Delta^2\to 0$ limit.

It is quite likely, however, that the $I_{i\nu}(\Delta z)$ component in $K_{i\nu}(\Delta z)$ has just as important a contribution as the $I_{-i\nu}(\Delta z)$ component does in the $\Delta^2\to 0$ limit to the hadron matrix elements $\overline{g}^{0,0,0}$ and $\overline{g}^{0,0,2}$; the coefficient $(1-c^{(j)}_{i\nu;0,0,m})$ may behave as $(\Delta/\Lambda)^{-2i\nu}$ in the $\Delta^2\to 0$ limit. Because we have seen above that the divergence (p^2/Δ^2) cancels when only the $I_{-i\nu}(\Delta z)$ component is taken into account, the contributions from the $I_{i\nu}(\Delta z)$ should also have some cancellation mechanism. Using an approximation for the $I_{i\nu}(\Delta z)$ components in $K_{i\nu}(\Delta z)$ similar to the one in footnote 31, one finds that the (p^2/Δ^2) divergence cancels in the η^2 coefficient, if and only if

[31] The leading divergence in the $\Delta^2\to 0$ limit comes from

$$K_{i\nu}(\Delta z)\sim\left(\frac{\pi}{2}\right)\frac{I_{-i\nu}(\Delta z)}{\sin(\pi i\nu)}\simeq\left(\frac{\pi}{2}\right)\frac{(\Delta z/2)^{-i\nu}}{\sin(\pi i\nu)\Gamma(-i\nu+1)}=\frac{\Gamma(i\nu)}{2}(\Delta z/2)^{-i\nu}. \tag{188}$$

$$\lim_{\Delta^2/\Lambda^2\to 0}\left[\left(\frac{\Delta}{2\Lambda}\right)^{2i\nu}\left\{(1-c^{(j)}_{i\nu_j;0,0,0})-(1-c^{(j)}_{i\nu_j;0,0,2})\times\frac{(j-1-i\nu_j)(j-i\nu_j)}{(j-1+i\nu_j)(j+i\nu_j)}\right\}\right]=0. \tag{190}$$

The coefficients $c^{(j)}_{i\nu;0,0,m}$ are functions of Δ/Λ, rather than complex numbers. The discussion above shows that physically sensible implementations of the confining effect require one of the conditions above between the two functions $c^{(j)}_{i\nu;0,0,0}$ and $c^{(j)}_{i\nu;0,0,2}$.

The η^{2M} term with $M=2,\ldots$, instead of the η^2 term in (187), also receives divergent contributions from amplitudes of the $(m=0,2,\ldots,2M)$-mode exchange. There will be an apparent divergence of order $(p^2/\Delta^2)^M$, $(p^2/\Delta^2)^{M-1},\ldots,(p^2/\Delta^2)$. The cancellation of divergence in the $\Delta^2\to 0$ limit will set M conditions on the $\Delta^2/\Lambda^2\to 0$ limit of $(1-c^{(j)}_{i\nu_j;0,0,2M})$.

In a phenomenological approach of implementing the confining effect, that is all we can say for now. With a little more of a model building mind-set, however, we can find some solutions to the conditions above. It is not hard to verify that the combination of

$$[\partial_z(\Psi^{(j);0,0}_{i\nu;0,0,0}(t,z))]|_{z\Lambda=1}=0,$$
$$[\partial_z(\Psi^{(j);2,0}_{i\nu;0,0,2}(t,z))]|_{z\Lambda=1}=0 \tag{191}$$

results in $c^{(j)}_{i\nu;0,0,0}$ and $c^{(j)}_{i\nu;0,0,2}$ satisfying the condition (190). It is tempting to generalize this and impose the boundary condition $\partial_z[\Psi^{(j);2M,0}_{i\nu;0,0,2M}]=0$ to determine $c^{(j)}_{i\nu;0,0,2M}$, though we do not know whether all of the m_h^2/Δ^2 divergences above are removed under this boundary condition. The top-down approach is much more authentic and well motivated than such a hand-waving and wishful approach, and we do not try to speculate beyond that; we use this implementation of the confining effect, (191), only to "get the feeling" in the numerical presentation in Sec. VII D.

B. Large Δ^2 behavior

Certainly the holographic model of GPD yields a result of the reduced hadron matrix element that fits perfectly with the dual parametrization. The holographic result, however, turns out to be a little more complicated than the models that have often been explored for the purpose of phenomenological fit of the DVCS data. An example of a model for phenomenological fit (see, e.g., [18]) was to introduce an ansatz that

$$\overline{\Gamma}^{+,\alpha}_m(j,t)=f_{j,m}\Sigma_{j-m}(t), \tag{192}$$

where only one $(t=-\Delta^2)$-dependent *function* is involved in the form of a "form factor" $\Sigma_{j-m}(t)$ for some spin $(j-m)$, and all the remaining nonperturbative information is reduced to some *numbers* $f_{j,m}\in\mathbb{R}$. The function $\Sigma_J(t)$ may also be parametrized by an ansatz like

$$\Sigma_J(t)=\frac{1}{J-\alpha_0-\alpha'_{\rm eff}t}\frac{1}{[1-\frac{t}{m^2(J)}]^p}, \tag{193}$$

in order to implement both the Regge behavior and the power-law form factor in the hard regime $1\ll -t/\Lambda^2$. To fit the data in practice, it is certainly unavoidable to reduce the unknown information into a finite set of real numbers.

A theoretical picture based on the holographic model, on the other hand, suggests that the $t=-\Delta^2$ dependence is more complicated than this. If we strictly stick to the expansion (185), then individual $\overline{\Gamma}^{+,\alpha}_m(j,t)$'s may diverge at $t=-\Delta^2=0$, as we have seen above, and are not like form factors. The $\overline{\Gamma}^{+,\alpha}_m(j,t)$ would not depend only on the difference $(j-m)$, as in (192), either; we have already seen that $\overline{\Gamma}^{+,\alpha}_{m=2}(j,t)\propto\overline{g}^{0,0,m=2}/\Delta^2$ diverges at $t=-\Delta^2\to 0$ for an arbitrary j, but there is no such divergence in $\overline{\Gamma}^{+,\alpha}_{m=0}(j,t)\propto\overline{g}^{0,0,0}$, for example. Therefore, holographic models of GPD might be used as a theoretical guide to think of parametrization (for fitting) that is different from (192).

The holographic model provided by the calculation in the previous section involves infinitely many spin-dependent form factors, $\overline{g}^{0,0,m}(j,i\nu_j,\Delta)/\Delta^m$. We can still find that they share a common behavior at large $\Delta^2=-t$. To see this, note that $K_{i\nu}(\Delta z_h)$ in the Reggeon wave function effectively cuts off the integral over the holographic radius z_h at $z_h\lesssim 1/\Delta$ in the regime

$$\Lambda^2, m_h^2\ll\Delta^2\ll|q^2|,(p\cdot q),|(q\cdot\Delta)|. \tag{194}$$

The explicit form of $\Psi^{(j);s,N}_{i\nu;0,0,m}(z;\Delta)$ in (97) is not more than a modification of $K_{i\nu}(\Delta z)$ by a function of Δz_h, and hence they still play just the role of cutting off the integral at $z_h\Delta\lesssim 1$. The "current" $\overline{J}^{hh}_{z^k\lambda_{k+1}\cdots\lambda_j}$ provides extra mth powers of either $1/z$ or ∂_z and $(j-m)$-momenta p_λ, in addition to $[\Phi]^2$, which behaves like

$$[\Phi]\sim z(\Lambda z)^{\ell_\phi-1} \tag{195}$$

in the region $z\lesssim 1/\Delta\ll 1/\Lambda$; ℓ_ϕ is the conformal dimension of an operator in a strongly coupled gauge theory dual to the holographic model, which is a property of the target hadron h. The $\tilde{E}^N D^{s-2N}[\epsilon]/\Delta^{2-2N}$ operation on the SO(3,1) tensor in (89) does not introduce any power of (Δ/Λ) or (Λz). Therefore, we find in the hard scattering regime (194) that

$$\frac{\overline{g}^{0,0,m}}{\Delta^m} \sim \left(\frac{\Delta}{\Lambda}\right)^{i\nu} \times (\Lambda/\Delta)^{j+2(\ell_\phi-1)} \times \Delta^m/\Delta^m \sim \frac{1}{(\Delta/\Lambda)^{2\ell_\phi-2-\gamma(j)}}. \quad (196)$$

Interestingly, the reduced hadron matrix elements $\overline{g}^{0,0,m}/\Delta^m$ for (j,m) have the large Δ^2 power-law behavior that is independent of m; $2\Delta_\phi$ reflects a property of the target hadron h, and $-(2+\gamma(j)) = -\tau_n$ is j dependent, but the power does not depend on m.[32] Holographic models suggest this j-dependent $p = \text{const} - \gamma(j)/2$ scaling behavior as an alternative to the fixed-power $p = \text{const.}$ scaling of (193).

We have chosen a factorization into the Wilson coefficient and the matrix element that corresponds to renormalization at $\mu = \Lambda$; this choice was made implicitly when we chose a factor $[\Delta^{i\nu}/\Lambda^{i\nu-j}]^{\pm 1}$ at the time the amplitude was factorized into $C^{0,0,m}$ and $\Gamma^{0,0,m}$ in (149). When we keep the renormalization scale μ arbitrary (e.g., taking μ higher than Δ when $\Delta \gg \Lambda$), the Wilson coefficient contains a factor $(\mu/q)^{\gamma(j)}$ instead of $(\Lambda/q)^{\gamma(j)}$, and the reduced matrix element also has the following large Δ^2 behavior,

$$\frac{\overline{g}^{0,0,m}}{\Delta^m} \sim \frac{1}{(\Delta/\Lambda)^{2\ell_\phi-2}} \times \frac{1}{(\mu/\Delta)^{\gamma(j)}}. \quad (197)$$

C. Pomeron and superstring

We have so far talked about Reggeon and the flavor-nonsinglet sector in Secs. VI and VII, instead of Pomeron. Since the flavor-singlet sector ($\approx$ gluon) dominates in small-x physics, that was not a desired choice.

This is due to technical limitations in string theory at this moment. In order to deal with the propagation of string states on a curved spacetime, vertex operators and L_0 (the Virasoro generator) need to be defined properly as composite operators; the nonlinear σ model for $AdS_5 \times W_5$ on the world sheet becomes conformal and the renormalization of the composite operators well defined, however, only after the Ramond-Ramond background is also implemented (e.g., [35]). Presumably an option in the future will be to implement the Klebanov-Strassler model and its variations in the Green-Schwarz formalism. One then computes the spectrum of stringy excited states, and further works out the world-sheet OPE, in the form of

$$V^{(q_1)}(z)V^{(-q_2)}(-z) \sim \sum_I C_I(z)\mathcal{O}_I(0), \quad (198)$$

using operators $\mathcal{O}_I(0)$ at the middle point, where $V^{(q_1)}$ and $V^{(-q_2)}$ are the vertex operators corresponding to the incoming and outgoing photons (32). In this way, we would not have to use string field theory.

It may also be possible to use bosonic string field theory for closed string theory, instead of the bosonic open string field theory we used in Sec. IV of this article. Bosonic closed string field theory is also well understood already [36]. Certainly the bosonic closed string field theory is not for type IIB superstring theory, but it will still allow us to get the feeling of how much open string (flavor-nonsinglet sector) and closed string (flavor-singlet sector) theories are different, from theoretical perspectives, as well as in phenomenological consequences. At least it is known that the Virasoro-Shapiro amplitude is generated, not just by the one string exchange in the t channel, the s channel, and the u channel, but also a four-point contact interaction vertex in string field theory [37]. The Virasoro-Shapiro amplitude does not have a simple s-t duality of the Veneziano amplitude, either. Certainly it is possible to write it down in the form of "t-channel" expansion only (cf. [3] and [7]), but we also need to be aware that the discussion in these two references did not use the OPE at the middle point as in (198), but instead used an OPE of the form $V(z)V(0) \sim \sum_I C_I(z)\mathcal{O}_I(0)$. To get the skewness dependence right, this difference really matters. Thus, an analogue of the prescription (52) needs to be worked out separately for the closed string amplitude.

Orthodox approaches such as those above are way beyond the scope of this article. One can hardly overestimate the importance of such a solid approach, but at the same time, very few would find the following guess terribly wrong. For practical purposes, therefore, one can live with that for the time being. First of all, the on-shell relation for the bosonic open string in (164) will be replaced by

$$\frac{j}{2} - 1 + \frac{4 + j + \nu^2 + c_j}{4\sqrt{\lambda}} = 0, \quad (199)$$

with the constraint $c_{j=1} = -4$ for the bosonic open string replaced by $c_{j=2} = -2$. Interaction vertices should also be different; looking at the difference between the Veneziano amplitude and the Virasoro-Shapiro amplitude, one finds that the following replacements should be made:

$$\frac{t_\gamma/t_y}{\Gamma(j+1)} \to \frac{t_\gamma/t_y}{[\Gamma(j/2)]^2}, \qquad \left(\frac{1}{\sqrt{\lambda}x}\right)^j \to \left(\frac{1}{4\sqrt{\lambda}x}\right)^j. \quad (200)$$

The overall normalization t_γ/t_y is like $N_c/N_c^{-2} \sim N_c^{-1}$ now, when the Pomeron (closed string) contribution is used in the t channel, and the source field for the "QED current" is implemented in the form of the D7-brane gauge field; the $1/N_c$ scaling (see footnotes 14 and 27) is also the natural expectation in the large N_c argument.

[32] This scaling was known already for $\overline{g}^{0,0,0}$ [7].

D. Numerical results

At the end of this article, we leave a few plots of numerical evaluation of various results that have been obtained. We do not intend to provide a quantitative (precise) prediction from holography, as we have repeatedly emphasized our perspective on this issue in this article; the holographic approach to GPD will provide at best a qualitatively new way to think of how to parametrize the matrix elements for GPD. Having said that, it is still desirable to grasp various expressions in a more intuitive form and bring them down to more practical situations. This section serves this purpose.

There are a couple of parameters that need to be specified in order to obtain numerical outputs in a few summary plots. We used the on-shell relation (164), which means that we should understand the numerical results to be that of the Reggeon contribution. We adopted $c_j + 4 = 0$ for all j, although there is no rationale to specify the j dependence in this way (see [12,38] and the literature therein for how to work out the j dependence of c_j). The confining effect was implemented in the form of the boundary condition (191) for the Reggeon wave function. As for the target hadron, we set the mass term of the scalar field to be $5/R^2$ (i.e., $c_y = 5$), just like the lowest nontrivial spherical harmonics on $W_5 = S^5$ for the type IIB dilaton field [39]. The operator dimension in the dual CFT becomes $\ell_\phi = 2 + \sqrt{4 + R^2 M_{\rm eff}^2} = 2 + \sqrt{4 + c_y} = 5$.

Figure 6 shows the reduced matrix element $\overline{g}^{0,0,0}(j, i\nu_j, \Delta)$ for the ($m = 0$)-mode exchange; the results for different values of spin $j = 1$, 1.5, 2, 2.5 are shown in the figure. Lattice computation can be used to determine matrix elements at integer-valued spins, but the analytic expression (153) allows us to determine the matrix element even for noninteger spin, so that the inverse Mellin transformation is possible, and we can also talk of the matrix elements evaluated at the saddle point value of spin $j = j^*$. Panel (b) in Fig. 6 is essentially the same as that of Fig. 5 in [7], while panel (a) shows $\overline{g}^{0,0,0}$ without normalizing the matrix element by its value at $t = -\Delta^2 = 0$. Since they are not the matrix element of a "conserved current" for $j \neq 1$, the matrix element does not necessarily approach 1 in the $\Delta^2 \to 0$ limit. Panel (b) has a property such that $\overline{g}^{0,0,0}$ is soft ($\overline{g}^{0,0,0}$ gets smaller slowly in Δ^2) for a larger j; this is consistent with the observation in (196) because $\partial\gamma(j)/\partial j > 0$.

A numerical result for the η^2 term in $\overline{A}_j^{+,\alpha}$, which is proportional to

$$\overline{g}^{0,0,0}(j, i\nu_j, \Delta) \times \left[\frac{p^2}{\Delta^2}\frac{j(j-1)}{j-\frac{1}{2}}\right] + \frac{\overline{g}^{0,0,2}(j, i\nu_j, \Delta)}{N_{j,2}\Delta^2} \times \frac{-1}{(j + i\nu_j)(j - 1 + i\nu_j)}, \tag{201}$$

is shown in Fig. 7, using $j = 2$. The first and second terms of (201) both diverge at the $\Delta^2 \to 0$ limit, as we saw in Sec. VII A, but their sum has a finite value at $\Delta^2 = 0$, as one can see in the figure. It is worth mentioning that this finite limit value ≈ -700 is much larger than that of $\overline{g}^{0,0,0}$. This is likely due, at least partially, to the hadron mass m_h value in this case; for the value of parameters we chose, $m_h = j_{\ell_\phi - 2,1}\Lambda$, $j_{3,1} \simeq 6.4$, and $m_h^2/\Lambda^2 \approx 40$. An extra derivative ∂_z in the matrix elements $\overline{g}_k^{0,0,2}$ is more like m_h than Λ, and hence the second term can be larger than the first term by about $(m_h/\Lambda)^2$. The factor $(m_h/\Lambda)^2 \approx 40$ does not explain all of the moderately large value -700, however. The $t = -\Lambda^2$ dependence of the η^0 term [i.e., $\overline{g}^{0,0,0}(j, i\nu_j, \Delta)$] is quite different from that of the coefficient of the η^2, at least at small Δ^2.

In the DGLAP phase, a crude approximation of the GPD is given by

$$\overline{H}^{+,\alpha}(x, \eta, t; q^2) \approx \left(\frac{1}{x}\right)^{j_*}\left(\frac{\Lambda}{q}\right)^{\gamma(j_*)}\overline{A}_{j*}^{+,\alpha}(\eta, t), \tag{202}$$

where j^* is the saddle point value of j depending primarily on $\ln(1/x)$, $\ln(q/\Lambda)$, and $t = -\Lambda^2$. Apart from applications

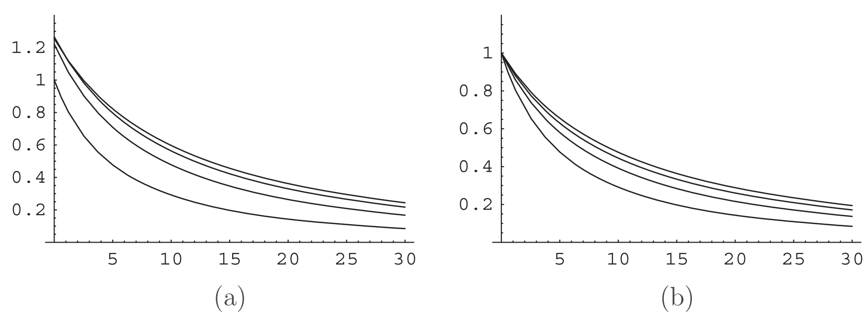


FIG. 6. Panel (a) shows $\bar{g}^{0,0,0}(j, i\nu_j, \Delta)$ as a function of Δ^2/Λ^2. The curve at the bottom is for $j = 1$, while the one at the top is for $j = 2.5$; the two in the middle correspond to $j = 1.5$ and $j = 2$. Panel (b) shows $\bar{g}^{0,0,0}(\Delta)/\bar{g}^{0,0,0}(\Delta = 0)$, i.e., $\bar{g}^{0,0,0}(j, i\nu_j, \Delta)$ normalized at the value of $\Delta^2 = 0$. The curve at the bottom is for $j = 1$, and the curve goes up for $j = 1.5$, 2, and 2.5; this softer behavior for larger j is consistent with (196).

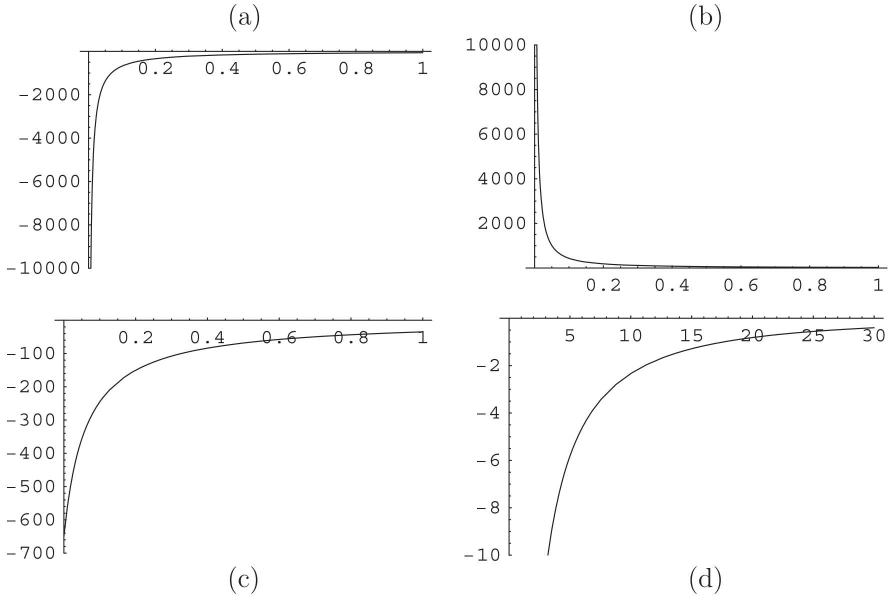


FIG. 7. The first and second term of (201) are plotted in (a) and (b), respectively, as functions of Δ^2/Λ^2; parameters are set to the values described in the text, and we used $j = 2$ in these figures. Although both (a) and (b) diverge at $\Delta^2 \to 0$, they add up to be (c), where the $\Delta \to 0$ limit is finite. The large Δ^2 behavior is seen better in panel (d).

to the timelike Compton scattering with very large (positive) lepton invariant mass square, the relevant range of $|\eta|$ is not much more than x in such processes as TCS, DVCS, and VMP. Suppose, in the power series expansion of $\overline{A}^{+,\alpha}_{j*}$ in η, that all of the terms with a different power of η have a (t-dependent) coefficient at most of $\mathcal{O}(1)$. Then the GPD [or $\overline{A}^{+,\alpha}_{j^*}(\eta, t)$] in the small-$x$ regime would not have skewness dependence very much in the range of interest, $|\eta| \lesssim x$, because η^2 and higher-order terms are small relative to the η^0 term. The coefficient of the η^2 term, however, turns out to be of $\mathcal{O}(-700)$ for $\Delta^2 \approx 0$, which at least contains a factor m_h^2/Λ^2. Thus, for the range of moderately small x's, such as $x \sim 10^{-1}$ and $|\eta| \lesssim x$, the η^2 term in $\overline{A}^{+,\alpha}_{j^*}(\eta, t)$ can be just as important as the η^0 term for small Δ^2. Consequently the prediction/fit of the slope parameter

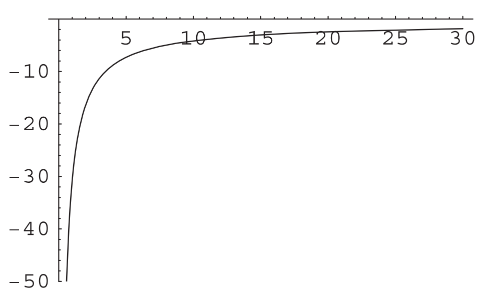


FIG. 8. The ratio of the coefficient of the η^2 term of $\bar{A}^{+,\alpha}_{j}(\eta, t)$ to that of the η^0 term, as a function of $-t/\Lambda^2 = \Delta^2/\Lambda^2$. We used $j = 2$ and other parameters described in the text. This is the ratio of Fig. 7(d) to Fig. 6(a).

(the t dependence) may also be affected since the η^2 term with a steeper t dependence is involved. Toward higher Δ^2, however, the ratio of the coefficient of the η^2 term to that of the η^0 term changes as in a numerical computation shown in Fig. 8. Since the η^2-term coefficient becomes not more than 10 times the η^0 term for $5\Lambda^2 \lesssim (\Delta^2 = -t)$ at $j = 2$ in this numerical computation, the η^0 term alone will become a good enough approximation in this range of t, even for the moderately small $|\eta| \lesssim x \approx \mathcal{O}(10^{-1})$; for an even smaller x, the η^2 term can be negligible for a broader range of $t = -\Lambda^2$. We have nothing more to say about the η^4 term and higher at this moment, or whether this moderately large value ≈ 700 is an artifact of a specific implementation of confining effects we adopted for the numerical presentation in this section. If this relatively large coefficient of the η^2 term (and also higher-order terms) turns out to be a robust consequence of holographic models, that may be regarded as an unexpected lesson from holography to phenomenology.

ACKNOWLEDGMENTS

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APPENDIX A: MORE ON THE MODE DECOMPOSITION ON AdS_5

For convenience, let us copy here the eigenmode equation (65) for a totally symmetric rank-j tensor field on AdS_5; the equation consists of the following equations labeled by $k = 0, \ldots, j$:

$$\begin{aligned}((R^2\Delta_j) - [(2k+1)j - 2k^2 + 3k])A_{z^k\mu_1\cdots\mu_{j-k}} \\ + 2zk\partial^{\hat\rho}A_{z^{k-1}\rho\mu_1\cdots\mu_{j-k}} + k(k-1)A^{\hat\rho}_{z^{k-2}\rho\mu_1\cdots\mu_{j-k}} \\ - 2z(D[A_{z^{k+1}\ldots}])_{\mu_1\cdots\mu_{j-k}} + (E[A_{z^{k+2}\ldots}])_{\mu_1\cdots\mu_{j-k}} \\ = -\mathcal{E}A_{z^k\mu_1\cdots\mu_{j-k}}.\end{aligned} \tag{A1}$$

1. Eigenvalues and eigenmodes for $\Delta^\mu = 0$

a. Block diagonal decomposition

In the main text, we considered a decomposition of the rank-j totally symmetric tensor field with $(-i\partial_\mu) = \Delta_\mu = 0$ in the form of

$$A_{z^k\mu_1\cdots\mu_{j-k}}(z;\Delta^\mu = 0) = \sum_{N=0}^{[(j-k)/2]}(E^N[a^{(k,N)}])_{\mu_1\cdots\mu_{j-k}},$$

where $a^{(k,N)}$'s are z-dependent rank-$(j-k-2N)$ totally symmetric tensor fields of SO(3, 1), satisfying the 4D-traceless condition (76). This is indeed a decomposition, in that all of the degrees of freedom in $A_{z^k\mu_1\cdots\mu_j-k}(z;\Delta^\mu = 0)$ are described by $a^{(k,N)}(z)_{\mu_1\cdots\mu_{j-k-2N}}$, with $0 \le N \le [(j-k)/2]$ without redundancy. To see this, one needs only to note that there is a relation[33] that, for a totally symmetric 4D-traceless rank-r SO(3, 1) tensor a,

$$\eta^{\hat\rho\hat\sigma}E^N[a]_{\rho\sigma\mu_1\cdots\mu_{r+2N-2}} = 4N(r+N+1)E^{N-1}[a]_{\mu_1\cdots\mu_{r+2N-2}}. \tag{A2}$$

Using this relation, $a^{(k,N)}_{\mu_1\cdots\mu_{j-k-2N}}$ can be retrieved from $A_{z^k\mu_1\cdots\mu_{j-k}}$, progressing from ones with a larger N to ones with a smaller N.

Let us now see that the eigenmode equation (65), (69), (A1) can be made block diagonal by using this decomposition. The eigenmode equation (A1) with the label k for $\Delta^\mu = 0$ can be rewritten by using this relation (A2) as follows:

$$\begin{aligned}\sum_N[(R^2\Delta_j - [(2k+1)j - 2k^2 + 3k] + \mathcal{E})E^N[a^{(k,N)}] \\ + k(k-1)[4(N+1)(j-k-N+2)]E^N[a^{(k-2,N+1)}] \\ + E^N[a^{(k+2,N-1)}]] = 0.\end{aligned}$$

[33]This relation can be verified recursively in N.

Although this equation has to hold only after the summation in N, it actually has to be satisfied separately for different N's. To see this, let us first multiply $\eta^{\hat\rho\hat\sigma}$ for $[(j-k)/2]$ times and contract indices just like in (A2); we obtain an equation that involves only $a^{(k,[(j-k)/2])}$, $a^{(k-2,[(j-k)/2]+1)}$, and $a^{(k+2,[(j-k)/2]-1)}$. Next, multiply $\eta^{\hat\rho\hat\sigma}$ for $[(j-k)/2]-1$ times to obtain another equation involving $a^{(k,[(j-k)/2]-1)}$, $a^{(k-2,[(j-k)/2])}$, and $a^{(k+2,[(j-k)/2]-2)}$. In this way, we obtain

$$\begin{aligned}(R^2\Delta_j - [(2k+1)j - 2k^2 + 3k] + \mathcal{E})a^{(k,N)} \\ + k(k-1)[4(N+1)(j-k-N+2)]a^{(k-2,N+1)} \\ + a^{(k+2,N-1)} = 0 \quad (\text{for } {}^\forall k, N).\end{aligned} \tag{A3}$$

Fields $a^{(k,N)}$'s with the same $k + 2N = n$ form a system of coupled equations, but those with different $n = k + 2N$ do not mix. Thus, the eigenmode equation for $\Delta^\mu = 0$ is decomposed into sectors labeled by n. The nth sector consists of z-dependent fields that are all in the $(j-n) = (j-k-2N)$ totally symmetric tensor of SO(3, 1).

b. Classification of eigenmodes for $\Delta^\mu = 0$

Let us now study the eigenmode equations in more detail for the separate diagonal blocks we have seen. Simultaneous treatment is possible for all of the nth sectors with an even n and for all of the sectors with an odd n.

Let us first look at the nth sector of the eigenmode problem for an $n = 2\bar n \le j$. In the eigenmode equation of $\Delta^\mu = 0$, we can assume[34] the same z dependence for all of the fields in this diagonal block:

$$a^{(k,N)}(z)_{\mu_1\cdots\mu_{j-n}} = \bar a^{(k,N)}_{\mu_1\cdots\mu_{j-n}}z^{2-j-i\nu}, \qquad k + 2N = n, \tag{A4}$$

where $\bar a^{(k,N)}$'s are (x,z)-independent 4D-traceless rank-$(j-n)$ tensors of SO(3, 1). The eigenmode equations with the label $(k,N) = (2\bar k, \bar n - \bar k)$, with $\bar k = 0, \ldots, \bar n$, are relevant to the $n = 2\bar n$ sector and are now written in a matrix form:

$$\sum_{\bar k'=0}^{\bar n}\mathcal{D}_{2\bar k,2\bar k'}\bar a^{(2\bar k',\bar n-\bar k')} = ((4+\nu^2) - \mathcal{E})\bar a^{(2\bar k,\bar n-\bar k)}, \tag{A5}$$

where

(i) diagonal $(k,k') = (2\bar k, 2\bar k)$ entry: $\mathcal{D}_{2\bar k,2\bar k} = -[(2k+1)j - 2k^2 + 3k]$,
(ii) diagonal$^+$; $(k,k') = (2\bar k, 2\bar k+2)$ entry: $\mathcal{D}_{2\bar k,2\bar k+2} = 1$,
(iii) diagonal$^-$; $(k,k') = (2\bar k, 2\bar k-2)$ entry: $\mathcal{D}_{2\bar k,2\bar k-2} = k(k-1)\times 4(\bar n - \bar k + 1)(j - \bar n - \bar k + 2)$.

[34]This is because, in the absence of $z^2\partial^2$ term, the operator Δ_j becomes a constant multiplication when it acts on a simple power of z. Upon $z^{2-j-i\nu}$, for example, $R^2\Delta_j$ returns $-(4+\nu^2)$.

There must be $(\overline{n}+1)$ independent eigenmodes in this $(\overline{n}+1)\times(\overline{n}+1)$ matrix equation. Let $\mathcal{E}_{n,l}$ denote the collection of eigenvalues in this $n=2\overline{n}$th diagonal block, and $l=0,\ldots,\overline{n}=n/2$ label distinct eigenmodes. The corresponding eigenmode wave function for the $(n=2\overline{n},l)$ mode is in the form of

$$a^{(k,N)}(z;\Delta^\mu=0)=a^{(2\overline{k},\overline{n}-\overline{k})}=c_{2\overline{k},l,n}\epsilon^{(n,l)}z^{2-j-i\nu}, \tag{A6}$$

where $\epsilon^{(n,l)}$ is an (x,z)-independent 4D-traceless totally symmetric rank-$(j-n)=(j-2\overline{n})$ tensor of SO(3, 1), and $c_{2\overline{k},l,n}$ are (x,z)-independent constants determined as the eigenvector corresponding to the eigenvalue $\mathcal{E}_{n,l}$.

Similarly, in the $n=2\overline{n}+1\le j$th sector of the eigenmode problem, with an odd n, we can assume a simple power law for all of the component fields involved in this sector;

$$a^{(k,N)}(z)_{\mu_1\cdots\mu_{j-n}}=\overline{a}^{(k,N)}_{\mu_1\cdots\mu_{j-n}}z^{2-j-i\nu},\qquad k+2N=n, \tag{A7}$$

where $\overline{a}^{(k,N)}$ are (x,z)-independent 4D-traceless totally symmetric tensors of SO(3, 1). The eigenmode equation with the label $(k,N)=(2\overline{k}+1,\overline{n}-\overline{k})$, with $\overline{k}=0,\ldots,\overline{n}$, are relevant to this sector, and in the matrix form, the eigenmode equation now looks like

$$\sum_{\overline{k}'=0}^{\overline{n}}\mathcal{D}_{2\overline{k}+1,2\overline{k}'+1}\overline{a}^{(2\overline{k}'+1,\overline{n}-\overline{k})}=((4+\nu^2)-\mathcal{E})\overline{a}^{(2\overline{k}+1,\overline{n}-\overline{k})}, \tag{A8}$$

where

(i) diagonal $(k,k')=(2\overline{k}+1,2\overline{k}+1)$ entry: $\mathcal{D}_{2\overline{k}+1,2\overline{k}+1}=-[(2k+1)j+(-2k^2+3k)]$,

(ii) diagonal$^+$ $(k,k')=(2\overline{k}+1,2\overline{k}+3)$ entry: $\mathcal{D}_{2\overline{k}+1,2\overline{k}+3}=1$,

(iii) diagonal$^-$ $(k,k')=(2\overline{k}+1,2\overline{k}-1)$ entry: $\mathcal{D}_{2\overline{k}+1,2\overline{k}-1}=k(k-1)\times 4(\overline{n}-\overline{k}+1)(j-\overline{n}-\overline{k}+1)$.

From here, $\overline{n}+1$ independent modes arise; their eigenvalues are denoted by $\mathcal{E}_{n,l}$, and $l=\{0,\ldots,\overline{n}\}$ is the label distinguishing different modes. The eigenmode labeled by $(n=2\overline{n}+1,l)$ has a wave function

$$a^{(k,N)}(z;\Delta^\mu=0)=a^{(2\overline{k}+1,\overline{n}-\overline{k})}=c_{2\overline{k}+1,l,n}\epsilon^{(n,l)}z^{2-j-i\nu}, \tag{A9}$$

where $\epsilon^{(n,l)}$ is an (x,z)-independent 4D-traceless rank-$(j-n)$ totally symmetric tensor of SO(3, 1), and $c_{2\overline{k}+1,l,n}$ is the eigenvector for the (n,l) eigenmode determined in the matrix equation above.

c. Explicit examples

Let us take a moment to see how the general theory above works in practice.

The easiest of all is the $n=0$ sector, which contains only one rank-j 4D-traceless field, $a^{(0,0)}$. The eigenmode equation is

$$\left[\Delta_j-\frac{[(2k+1)j-2k^2+3k]_{k=0}}{R^2}\right]a^{(0,0)}=\left[\Delta_j-\frac{j}{R^2}\right]a^{(0,0)}=-\frac{\mathcal{E}_{0,0}}{R^2}a^{(0,0)}. \tag{A10}$$

The eigenmode wave function has the form

$$a^{(0,0)}(z)_{\mu_1\cdots\mu_j}=\epsilon^{(0,0)}_{\mu_1\cdots\mu_j}z^{2-j-i\nu}, \tag{A11}$$

and the eigenvalue $\mathcal{E}_{n,l}$ is

$$\mathcal{E}_{0,0}=(j+4+\nu^2). \tag{A12}$$

Also to the $n=1$ sector, only one rank-$(j-1)$ 4D-traceless tensor field contributes. That is $a^{(1,0)}$. The eigenmode equation becomes

$$[R^2\Delta_j-[(2k+1)j-2k^2+3k]|_{k=1}]a^{(1,0)}=[R^2\Delta_j-(3j+1)]a^{(1,0)}=-\mathcal{E}_{1,0}a^{(1,0)}. \tag{A13}$$

The solution is

$$a^{(1,0)}(z)_{\mu_1\cdots\mu_{j-1}}=\epsilon^{(1,0)}_{\mu_1\cdots\mu_{j-1}}z^{2-j-i\nu},\qquad \mathcal{E}_{1,0}=(3j+5+\nu^2). \tag{A14}$$

In the $n=2$ sector, two rank-$(j-2)$ 4D-traceless fields are involved. They are $a^{(0,1)}$ and $a^{(2,0)}$. After introducing the z dependence $\propto z^{2-j-i\nu}$, the eigenmode equation (A5) in the $n=2$ sector becomes

$$\begin{bmatrix}-j & 1\\ 8j & -(5j-2)\end{bmatrix}\begin{pmatrix}\overline{a}^{(0,1)}\\ \overline{a}^{(2,0)}\end{pmatrix}=((4+\nu^2)-\mathcal{E})\begin{pmatrix}\overline{a}^{(0,1)}\\ \overline{a}^{(2,0)}\end{pmatrix}. \tag{A15}$$

One of the two eigenmodes is

$$\mathcal{E}_{2,0}=(4+5j+\nu^2),\begin{pmatrix}a^{(0,1)}(z)_{\mu_1\cdots\mu_{j-2}}\\ a^{(2,0)}(z)_{\mu_1\cdots\mu_{j-2}}\end{pmatrix}=\begin{pmatrix}1\\ -4j\end{pmatrix}\epsilon^{(2,0)}_{\mu_1\cdots\mu_{j-2}}z^{2-j-i\nu}, \tag{A16}$$

and the other is

$$\mathcal{E}_{2,1} = (2+j+\nu^2), \begin{pmatrix} a^{(0,1)}(z)_{\mu_1\cdots\mu_{j-2}} \\ a^{(2,0)}(z)_{\mu_1\cdots\mu_{j-2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \epsilon^{(2,1)}_{\mu_1\cdots\mu_{j-2}} z^{2-j-i\nu}. \quad (A17)$$

In the $n=3$ sector, two rank-$(j-3)$ 4D-traceless tensor fields are involved: $a^{(1,1)}$ and $a^{(3,0)}$. The eigenmode equations (A8) become

$$\begin{bmatrix} -(3j+1) & 1 \\ 24(j-1) & -(7j-9) \end{bmatrix} \begin{pmatrix} \overline{a}^{(1,1)} \\ \overline{a}^{(3,0)} \end{pmatrix} = ((4+\nu^2)-\mathcal{E}) \begin{pmatrix} \overline{a}^{(1,1)} \\ \overline{a}^{(3,0)} \end{pmatrix}. \quad (A18)$$

So, one of the two eigenmodes is

$$\mathcal{E}_{3,0} = (7j+1+\nu^2), \begin{pmatrix} a^{(1,1)}(z)_{\mu_1\cdots\mu_{j-3}} \\ a^{(3,0)}(z)_{\mu_1\cdots\mu_{j-3}} \end{pmatrix} = \begin{pmatrix} 1 \\ -4(j-1) \end{pmatrix} \epsilon^{(3,0)}_{\mu_1\cdots\mu_{j-3}} z^{2-j-i\nu}, \quad (A19)$$

and the other one is

$$\mathcal{E}_{3,1} = (3j-1+\nu^2), \begin{pmatrix} a^{(1,1)}(z)_{\mu_1\cdots\mu_{j-3}} \\ a^{(3,0)}(z)_{\mu_1\cdots\mu_{j-3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \epsilon^{(3,1)}_{\mu_1\cdots\mu_{j-3}} z^{2-j-i\nu}. \quad (A20)$$

Finally, in the $n=4$ sector, the eigenmode equation (A5) is given by

$$\begin{bmatrix} -j & 1 & 0 \\ 16(j-1) & -(5j-2) & 1 \\ 0 & 48(j-2) & -(9j-20) \end{bmatrix} \begin{pmatrix} \overline{a}^{(0,2)} \\ \overline{a}^{(2,1)} \\ \overline{a}^{(4,0)} \end{pmatrix} = ((4+\nu^2)-\mathcal{E}) \begin{pmatrix} \overline{a}^{(0,2)} \\ \overline{a}^{(2,1)} \\ \overline{a}^{(4,0)} \end{pmatrix}. \quad (A21)$$

There are three solutions. First,

$$\mathcal{E}_{4,0} = (9j-4+\nu^2), \quad (A22)$$

$$(a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, -8(j-1), 32(j-1)(j-2))\epsilon^{(4,0)} z^{2-j-i\nu}, \quad (A23)$$

second,

$$\mathcal{E}_{4,1} = (5j-6+\nu^2), \quad (A24)$$

$$(a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, -(4j-10), -48(j-2))\epsilon^{(4,1)} z^{2-j-i\nu}, \quad (A25)$$

and, finally,

$$\mathcal{E}_{4,2} = (j+\nu^2), \quad (A26)$$

$$(a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, 4, 24)\epsilon^{(4,2)} z^{2-j-i\nu}. \quad (A27)$$

An empirical relation is observed in the j dependence of the eigenvalues we have worked out so far. The eigenvalues in the nth sector are in the form of $\mathcal{E}_{n,l} = \nu^2 + (2n+1-4l)j + \mathcal{O}(1)$ for $0 \le l \le [n/2]$.

d. 5D-traceless modes: the $l = 0$ modes

Although the precise expressions for the eigenvalues $\mathcal{E}_{n,l}$ and the eigenvectors $c_{k,l,n}$ are not given for all of the eigenmodes, there is a class of eigenmodes whose eigenvalues and eigenvectors (wave functions) are fully understood.

As we discussed in Sec. V B 3, it is possible to require both that a field is an eigenmode and that it satisfies the 5D-traceless condition (95) at the same time. In the $n=(k+2N)$th sector, the 5D-traceless condition becomes

$$\begin{aligned} 0 &= (E^N[a^{(k,N)}])^{\hat{\rho}}_{\rho\mu_3\cdots\mu_{j-n}} + (E^{N-1}[a^{(k+2,N-1)}])_{\mu_3\cdots\mu_{j-n}}, \\ &= E^{N-1}[4N(j-n+N+1)a^{(k,N)} + a^{(k+2,N-1)}] \\ &\quad \times \begin{cases} k = 0, 2, \ldots, 2(\overline{n}-1) & (\text{even } n), \\ k = 1, 3, \ldots, 2\overline{n}-1 & (\text{odd } n). \end{cases} \end{aligned} \quad (A28)$$

Thus, the 5D-traceless condition uniquely determines one eigenmode in each one of the nth sectors.

$$\mathcal{E}_{n,0} = (2n+1)j + 2n - n^2 + 4 + \nu^2 \quad (A29)$$

and

$$\begin{aligned} c_{2\overline{k},0,2\overline{n}} &= (-)^{\overline{k}} 4^{\overline{k}} \frac{\overline{n}!}{(\overline{n}-\overline{k})!} \frac{(j-\overline{n}+1)!}{(j-\overline{n}-\overline{k}+1)!}, \\ c_{2\overline{k}+1,0,2\overline{n}+1} &= (-)^{\overline{k}} 4^{\overline{k}} \frac{\overline{n}!}{(\overline{n}-\overline{k})!} \frac{(j-\overline{n})!}{(j-\overline{n}-\overline{k})!}. \end{aligned} \quad (A30)$$

2. MODE DECOMPOSITION FOR NONZERO Δ_μ

a. Diagonal block decomposition for the $\Delta^\mu \neq 0$ case

Let us now turn our attention to the eigenmode equations (65), (69) with $\Delta^\mu \neq 0$. Because of the second and fourth terms in (69), the eigenmode problem becomes much more complicated. We begin by finding a diagonal block decomposition suitable for the case with $\Delta^\mu \neq 0$.

In the main text, we introduced a decomposition of a totally symmetric rank-j tensor field $A_{m_1\cdots m_j}$ of SO(4, 1) into a collection of totally symmetric 4D-traceless 4D-transverse tensor fields of SO(3, 1). Instead of (75), a new decomposition is given by (85), (A31):

$$A_{z^k\mu_1\cdots\mu_{j-k}}(z;\Delta^\mu) = \sum_{s=0}^{j-k}\sum_{N=0}^{[s/2]}(\tilde{E}^N D^{s-2N}[a^{(k,s,N)}])_{\mu_1\cdots\mu_{j-k}}, \tag{A31}$$

where $a^{(k,s,N)}$ are totally symmetric 4D-traceless 4D-transverse rank-$(j-k-s)$ tensor fields of SO(3, 1). An operation $a\mapsto\tilde{E}[a]$ on a totally symmetric SO(3, 1) tensor a is given by (86).

In order to see that the parametrization of $A_{z^k\mu_1\cdots\mu_{j-k}}$ by $(a^{(k,s,N)})_{\mu_1\cdots\mu_{j-k-s}}$'s above is indeed a decomposition, one needs to see that $a^{(k,s,N)}$'s can be retrieved from $A_{z^k\mu_1\cdots\mu_{j-k}}$ so that the degrees of freedom $a^{(k,s,N)}$ are independent. For this purpose, it is convenient to derive some relations analogous to (A2). First of all, note that $E[D[a]] = D[E[a]]$ and[35] $\tilde{E}[D[a]] = D[\tilde{E}[a]]$ for a totally symmetric SO(3, 1) tensor a. If the rank-r tensor a is also 4D transverse and 4D traceless, one can then derive the following relations:

$$\partial^{\hat\rho}(E^tD^{s-2t}[a])_{\rho\mu_2\cdots\mu_{r+s}} = -\Delta^2(s-2t)E^tD^{s-2t-1}[a] + (2t)E^{t-1}D^{s-2t+1}[a], \tag{A33}$$

$$\eta^{\hat\rho\hat\sigma}(E^tD^{s-2t}[a])_{\rho\sigma\mu_3\cdots\mu_{r+s}} = -\Delta^2(s-2t)(s-2t-1)E^tD^{s-2t-2}[a] + 4t(r+s-t+1)E^{t-1}D^{s-2t}[a], \tag{A34}$$

$$\partial^{\hat\rho}(\tilde{E}^ND^{s-2N}[a])_{\rho\mu_2\cdots\mu_{r+s}} = -(s-2N)\Delta^2\tilde{E}^ND^{s-2N-1}[a], \tag{A35}$$

$$\left(\eta^{\hat\rho\hat\sigma} - \frac{\partial^{\hat\rho}\partial^{\hat\sigma}}{\partial^2}\right)(\tilde{E}^ND^{s-2N}[a])_{\rho\sigma\mu_3\cdots\mu_{r+s}} = 4N(r+N+1/2)\tilde{E}^{N-1}D^{s-2N}[a]. \tag{A36}$$

With the relations above, it is now possible to compute

$$\left(\eta^{\hat\mu_1\hat\mu_2} - \frac{\partial^{\hat\mu_1}\partial^{\hat\mu_2}}{\partial^2}\right)\cdots\left(\eta^{\hat\mu_{2p-1}\hat\mu_{2p}} - \frac{\partial^{\hat\mu_{2p-1}}\partial^{\hat\mu_{2p}}}{\partial^2}\right)\frac{\partial^{\hat\mu_{2p+1}}}{\partial^2}\cdots\frac{\partial^{\hat\mu_{2p+q}}}{\partial^2}(\tilde{E}^ND^{s-2N}[a])_{\mu_1\cdots\mu_{r+s}} = \begin{cases}\frac{b^{(r)}_{s-2p-q,N-p}}{b^{(r)}_{s,N}}(\tilde{E}^{N-p}D^{s-2N-q}[a])_{\mu_{2p+q+1}\cdots\mu_{r+s}} & \text{if } p\le N \quad\text{and}\quad q\le s-2N,\\ 0 & \text{otherwise,}\end{cases} \tag{A37}$$

where we assume that a is a totally symmetric 4D-traceless 4D-transverse rank-r tensor of SO(3, 1). In the last line,

$$b^{(r)}_{s,N} := \frac{1}{4^NN!(s-2N)!}\frac{\Gamma(r+3/2)}{\Gamma(r+N+3/2)}. \tag{A38}$$

It is now clear how to retrieve $a^{(k,s,N)}$ from the $A_{z^k\mu_1\cdots\mu_{j-k}}$ given by (85), (A31). First, one has to multiply $\eta^{\hat\rho\hat\sigma} - \partial^{\hat\rho}\partial^{\hat\sigma}/\partial^2$ and $\partial^{\hat\sigma}/\partial^2$ by $A_{z^k\mu_1\cdots\mu_{j-k}}$ as many times as possible in order to obtain $a^{(k,s,N)}$ with a larger N and $(s-2N)$. Then $a^{(k,s,N)}$'s with smaller N or $(s-2N)$ can be determined by multiplying $\eta^{\hat\rho\hat\sigma} - \partial^{\hat\rho}\partial^{\hat\sigma}/\partial^2$ and $\partial^{\hat\sigma}/\partial^2$ fewer times.

Let us now return to the eigenmode equation for the cases with $\Delta^\mu \neq 0$. Following precisely the same argument as in Sec. A. 1, one can see that the eigenmode equation can be separated into the following independent equations labeled by k, s, and N:

$$[R^2\Delta_j - [(2k+1)j - 2k^2 + 3k] + \mathcal{E}]a^{(k,s,N)} + 2zk(s+1-2N)(\partial^2)a^{(k-1,s+1,N)} + k(k-1)(s+2-2N)(s+1-2N)(\partial^2)a^{(k-2,s+2,N)} + 4k(k-1)(N+1)(j-m+N+3/2)a^{(k-2,s+2,N+1)} - 2za^{(k+1,s-1,N)} + a^{(k+2,s-2,N-1)} + (\partial^2)^{-1}a^{(k+2,s-2,N)} = 0 \quad \text{for } {}^\forall k, s, N. \tag{A39}$$

The relations (A33), (A34) were used to evaluate the second–fourth terms of (A1). One can see that $a^{(k,s,N)}$'s with a common value of $m := k+s$ form coupled eigenmode equations, but those with different m's do not. Thus, $a^{(k,s,N)}(z;\Delta^\mu)$'s with $k+s=m$ form the mth subspace of $A_{m_1\cdots m_j}(z;\Delta^\mu)$, and the eigenmode equation becomes block diagonal in the decomposition into the subspaces labeled by $m = 0,\ldots,j$.

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$$E^tD^{s-2t}[a] = \sum \eta_{\mu_{p_1}\mu_{p_2}}\cdots\eta_{\mu_{p_{2t-1}}\mu_{p_{2t}}}\partial_{\mu_{p_{2t+1}}}\cdots\partial_{\mu_{p_s}}[a]_{\mu_1\cdots;\cdots\mu_{r+s}}, \tag{A32}$$

where the sum is taken over all possible ordered choices of $p_1, p_2, \ldots, p_s \in \{1,\ldots,j\}$ such that $p_i \neq p_j$ for $i \neq j$.

The eigenmode equation on the mth subspace is given by the equation above, with $0 \leq k = (m-s) \leq m$ and $0 \leq N \leq [s/2]$. Thus, the total number of equations is

$$\sum_{s=0}^{m} ([s/2]+1), \tag{A40}$$

and the same number of eigenvalues should be obtained from the mth sector.

b. Examples

The sector m = 0.—There is only one field $a^{(0,0,0)}$ in this sector, and the eigenmode equation is

$$\left[\Delta_j - \frac{j}{R^2}\right] a^{(0,0,0)}(z;\Delta^\mu) = -\frac{\mathcal{E}}{R^2} a^{(0,0,0)}(z;\Delta^\mu). \tag{A41}$$

Assuming a power series expansion for the solution to this equation, beginning with some power $z^{2-j-i\nu}$, the eigenvalue is determined as a function of $(i\nu)$:

$$\mathcal{E}_{0,0} = (j+4+\nu^2),$$

and the wave function can be chosen as

$$a^{(0,0,0)}(z;\Delta^\mu)_{\mu_1\cdots\mu_j} = \epsilon^{(0,0,0)}_{\mu_1\cdots\mu_j} \Psi^{(j)}_{i\nu}(-\Delta^2, z), \tag{A42}$$

$$\Psi^{(j)}_{i\nu}(\Delta^2, z) := \frac{2}{\pi}\sqrt{\frac{\nu\sinh(\pi\nu)}{2R}} e^{(j-2)A} K_{i\nu}(\Delta z). \tag{A43}$$

The sector m = 1.—The eigenmode equation in this sector becomes

$$\begin{bmatrix} R^2\Delta_j - j & -2z \\ -2z\Delta^2 & R^2\Delta_j - (3j+1) \end{bmatrix} \begin{pmatrix} a^{(0,1,0)} \\ a^{(1,0,0)} \end{pmatrix} = -\mathcal{E}\begin{pmatrix} a^{(0,1,0)} \\ a^{(1,0,0)} \end{pmatrix}. \tag{A44}$$

Assuming the power series expansion in z, beginning with $z^{2-j-i\nu}$ terms, we obtain two eigenvalues depending on $i\nu$. They are given by evaluating $R^2\Delta_j - j$ and $R^2\Delta_j - (3j+1)$ on $z^{2-j-i\nu}$:

$$\mathcal{E}_{0,0} = (j+4+\nu^2) \quad \text{and} \quad \mathcal{E}_{1,0} = (3j+5+\nu^2). \tag{A45}$$

The sector m = 2.—The eigenmode equation becomes

$$\left((R^2\Delta_j + \mathcal{E})\mathbf{1}_{4\times4} + \begin{bmatrix} -j & & -2z & 1/\partial^2 \\ & -j & & 1 \\ 4z\partial^2 & & -(3j+1) & -2z \\ 4\partial^2 & 8j-4 & 4z\partial^2 & -(5j-2) \end{bmatrix}\right)\begin{pmatrix} a^{(0,2,0)} \\ a^{(0,2,1)} \\ a^{(1,1,0)} \\ a^{(2,0,0)} \end{pmatrix} = 0. \tag{A46}$$

The indicial equation relating the exponent $(2-j-i\nu)$ at $z=0$ and the eigenvalues split into two parts; three eigenvalues of this matrix,

$$\begin{pmatrix} -j & & 1 \\ & -j & 1 \\ 4 & (8j-4) & -(5j-2) \end{pmatrix}, \tag{A47}$$

determine $-\mathcal{E} - (4+\nu^2)$ for the three eigenmodes, and $-(\mathcal{E} - (4+\nu^2)) = -(3j+1)$ for the last eigenmode. Therefore, the four eigenvalues in the $m=2$ sector are

$$\mathcal{E}_{0,0} = (j+4+\nu^2), \qquad \mathcal{E}_{1,0} = (3j+5+\nu^2),$$
$$\mathcal{E}_{2,0} = (5j+4+\nu^2), \qquad \mathcal{E}_{2,1} = (j+2+\nu^2). \tag{A48}$$

In all of the examples above, the mth sector consists of eigenmodes with eigenvalues $\mathcal{E}_{n,l}$ for $0 \leq n \leq m$, $0 \leq l \leq [n/2]$. The number of eigenmodes is, of course, the same as in (A40).

3. Wave functions of 5D-traceless 5D-transverse modes

As we discussed toward the end of Sec. V B, it is possible to require for a rank-j totally symmetric tensor field configuration $A_{m_1\cdots m_j}(z;\Delta^\mu)$ to be an eigenmode and to be 5D traceless and 5D transverse (95), (96) at the same time. We will see in the following that these two extra conditions (95), (96) leave precisely one eigenmode in each one of the block diagonal sectors labeled by $m = 0, \ldots, j$. We will further determine the wave function profile of such eigenmodes.

Let us first rewrite the 5D-traceless condition (95) in a more convenient form:

$$\eta^{\hat\rho\hat\sigma} A_{z^{k-2}\rho\sigma\mu_1\cdots\mu_{j-k}} + A_{z^k\mu_1\cdots\mu_{j-k}} = 0, \tag{A49}$$

which, in the mth sector, means

$$a^{(k,s,N)} = (s+2-2N)(s+1-2N)\Delta^2 a^{(k-2,s+2,N)} + 4(N+1)(j-m+N+3/2)a^{(k-2,s+2,N+1)} \tag{A50}$$

for $N = 0, \ldots, [s/2]$; $k + s = m$ is understood. Under the 5D-traceless condition, the 5D-transverse condition

$$(k-1)\eta^{\hat{\rho}\hat{\sigma}}A_{z^{k-2}\rho\sigma\mu_1\cdots\mu_{j-k}} + z\partial^{\hat{\rho}}A_{z^{k-1}\rho\mu_1\cdots\mu_{j-k}} + (z\partial_z + (k-4))A_{z^k\mu_1\cdots\mu_{j-k}} = 0 \tag{A51}$$

becomes

$$z\partial^{\hat{\rho}}A_{z^{k-1}\rho\mu_1\cdots\mu_{j-k}} + (z\partial_z - 3)A_{z^k\mu_1\cdots\mu_{j-k}} = 0. \tag{A52}$$

In the mth sector ($k + s = m$), therefore,

$$(s+1-2N)\Delta^2 a^{(k-1,s+1,N)} = z^3\partial_z z^{-3}a^{(k,s,N)} \tag{A53}$$

for $N = 0, \ldots, [s/2]$. Hereafter, we use a simplified notation $\mathcal{D} \coloneqq z^3\partial_z z^{-3}$. One can see that all of the $a^{(k,s,N)}$'s with $k + s = m$ and $N \leq [s/2]$ can be determined from $a^{(m,0,0)}$ by using the relations (A50), (A53). This observation already implies that there can be at most one eigenmode in a given mth sector that satisfies both the 5D-traceless and the 5D-transverse conditions.

For now, let us assume that there is one, and proceed to determine the wave function. The wave function—z dependence—of $a^{(m.0.0)}(z;\Delta^\mu)$ can be determined from the eigenmode equation (A39), with $k = m$, $s = N = 0$. Using (A50) and (A53), we can rewrite the equation as

$$[R^2\Delta_j - \{(2m+1)j - m^2 + 2m\} - 2m(z\partial_z - 3) + \mathcal{E}] \times a^{(m,0,0)}(z;\Delta) = 0. \tag{A54}$$

For this equation,

$$(a^{(m,0,0)}(z;\Delta))_{\mu_1\cdots\mu_{j-m}} = \epsilon_{\mu_1\cdots\mu_{j-m}}\left(\frac{z}{R}\right)^{2-j}(\Delta z)^m K_{i\nu}(\Delta z), \qquad \mathcal{E} = (j+4+\nu^2), \tag{A55}$$

is a solution, where $\epsilon_{\mu_1\cdots\mu_{j-m}}$ is a z-independent 4D-traceless 4D-transverse totally symmetric rank-$(j-m)$ tensor of SO (3, 1). From the value of the eigenvalue, it turns out that the 5D-traceless 5D-transverse mode in the mth sector corresponds to the $(n,l,m) = (0,0,m)$ mode. The z dependence we determined above implies that

$$\Psi^{(j);0,0}_{i\nu;0,0,m}(-\Delta^2, z) \propto (\Delta z)^m \Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2, z). \tag{A56}$$

This result corresponds to the $(s,N) = (0,0)$ case of (97). The normalization constant $N_{j,m}$ is determined later in this section.

Let us now proceed to determine other $\Psi^{(j);s,N}_{i\nu;0,0,m}$, not just for $(s,N) = (0,0)$. Using the 5D-transverse condition, (A53), $a^{(m-1,1,0)}(z;\Delta)$ can be determined from $a^{(m,0,0)}(z;\Delta)$.

$$a^{(m-1,1,0)} = \frac{\mathcal{D}}{\Delta^2}a^{(m,0,0)}, \qquad \Psi^{(j);1,0}_{i\nu;0,0,m} = \frac{\mathcal{D}}{\Delta}\Psi^{(j);0,0}_{i\nu;0,0,m}. \tag{A57}$$

In order to determine the $s = 2$ components $a^{(m-2,2,N)}$ ($N = 0, 1$) of the $(n,l) = (0,0)$ mode in the mth sector, one has to use both the 5D-transverse condition and the 5D-traceless condition:

$$2\Delta^2 a^{(m-2,2,0)} = \mathcal{D}a^{(m-1,1,0)}, \tag{A58}$$

$$2\Delta^2 a^{(m-2,2,0)} - 4(j-m+3/2)a^{(m-2,2,1)} = a^{(m,0,0)}. \tag{A59}$$

Therefore,

$$a^{(m-2,2,0)} = \frac{1}{2\Delta^2}\left(\frac{\mathcal{D}}{\Delta}\right)^2 a^{(m,0,0)}, \quad a^{(m-2,2,1)} = \frac{1}{4(j-m+3/2)}\left\{\left(\frac{\mathcal{D}}{\Delta}\right)^2 - 1\right\}a^{(m,0,0)}. \tag{A60}$$

After factoring out the normalization factor $(b^{(j-m)}_{s,N}/\Delta^{s-2N})$ and the common 4D-tensor $\epsilon^{(0,0,m)}$, we obtain

$$\Psi^{(j);2,0}_{i\nu;0,0,m} = \left(\frac{\mathcal{D}}{\Delta}\right)^2\Psi^{(j);0,0}_{i\nu;0,0,m}, \quad \Psi^{(j);2,1}_{i\nu;0,0,m} = \left\{\left(\frac{\mathcal{D}}{\Delta}\right)^2 - 1\right\}\Psi^{(j);0,0}_{i\nu;0,0,m}. \tag{A61}$$

The 5D-transverse conditions (A53) determine the $s = 3$ components $a^{(m-3,3,N)}(z;\Delta)$ ($N = 0, 1$) from the $s = 2$ components:

$$a^{(m-3,3,0)} = \frac{1}{6\Delta^3}\left(\frac{\mathcal{D}}{\Delta}\right)^3 a^{(m,0,0)}, \quad a^{(m-3,3,1)} = \frac{1}{4(j-m+3/2)\Delta}\left\{\left(\frac{\mathcal{D}}{\Delta}\right)^3 - \left(\frac{\mathcal{D}}{\Delta}\right)\right\}a^{(m,0,0)}, \tag{A62}$$

and after factoring out the normalization factor $(b^{(j-m)}_{s,N}/\Delta^{s-2N})$ and $\epsilon^{(0,0,m)}$ as before, we obtain

$$\Psi^{(j);3,0}_{i\nu;0,0,m} = \left(\frac{\mathcal{D}}{\Delta}\right)^3\Psi^{(j);0,0}_{i\nu;0,0,m}, \quad \Psi^{(j);3,1}_{i\nu;0,0,m} = \left\{\left(\frac{\mathcal{D}}{\Delta}\right)^3 - \left(\frac{\mathcal{D}}{\Delta}\right)\right\}\Psi^{(j);0,0}_{i\nu;0,0,m}. \tag{A63}$$

The $s = 3$ components determined purely by the conditions (A53) satisfy the 5D-traceless condition (A50) with the $s = 1$ component:

$$6\Delta^2 a^{(m-3,3,0)} - 4(j-m+3/2)a^{(m-3,3,1)} = \frac{\mathcal{D}}{\Delta^2}a^{(m,0,0)} = a^{(m-1,1,0)}. \tag{A64}$$

In this way, the wave functions $\Psi^{(j);s,N}_{i\nu;0,0,m}(-\Delta^2,z)$ for all (s,N) are determined, and the result is

$$\Psi^{(j);s,N}_{i\nu;0,0,m}(-\Delta^2,z) = \sum_{a=0}^{N}(-)^a {}_N C_a\left(\frac{\mathcal{D}}{\Delta}\right)^{s-2a}[(z\Delta)^m\Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2,z)]\times N_{j,m}. \tag{A65}$$

The only remaining concern was that there are more conditions from (A50), (A53) than number of components $a^{(k,s,N)}$ in the mth sector; there can be at most one eigenmode satisfying these 5D-traceless 5D-transverse conditions, as we stated earlier, but there may be no eigenmode left if the conditions are overdetermining. We have confirmed, however, that the wave functions (97), (A65) satisfy all of the relations given by (A50), (A53).

a. Normalization

We have yet to determine the normalization factor $N_{j,m}$; as in the main text, we choose (99) to be the normalization condition. Orthogonal nature among the eigenmodes is guaranteed because of the Hermitian nature of the operator $\alpha'(\nabla^2-M^2)$. It is thus sufficient to focus only on the divergent part of the integral in the normalization condition in order to determine $N_{j,m}$.

The divergent part of the integral in (99) comes only from terms with $s=m$, $k=0$, $(0\le N\le[m/2])$, and $a=0$. For a given m,

$$[\epsilon\cdot\epsilon']\delta(\nu-\nu')\sim N^2_{j,m}\int_0 dz\sqrt{-g(z)}e^{-2jA} \times\left(\sum_{N=0}^{[m/2]}\tilde{E}^N D^{m-2N}[\epsilon^{(0,0,m)}]\frac{b^{(j-m)}_{m,N}}{\Delta^{m-2N}}\frac{z^3\partial_z^m z^{-3}}{\Delta^m}(z\Delta)^m\Psi^{(j);0,0}_{i\nu;0,0,m}(-\Delta^2,z)\right)_{\mu_1\cdots\mu_j} \times\left(\sum_{M=0}^{[m/2]}\tilde{E}^M D^{m-2M}[\epsilon^{(0,0,m)}]\frac{b^{(j-m)}_{m,M}}{\Delta^{m-2M}}\frac{z^3\partial_z^m z^{-3}}{\Delta^m}(z\Delta)^m\Psi^{(j);0,0}_{i\nu;0,0,m}(-\Delta^2,z)\right)^{\hat{\mu}_1\cdots\hat{\mu}_j}. \tag{A66}$$

The divergent part of the integral in this expression comes from

$$\left(\frac{2}{\pi}\right)^2\frac{\nu\sinh(\pi\nu)}{2}\int dx x^{2j-5}[x^3\partial_x^m x^{-1-j+m}K_{i\nu}(x)][x^3\partial_x^m x^{-1-j+m}K_{i\nu'}(x)] \simeq\prod_{p=1}^{m}[(j-p+1)^2+\nu^2]\delta(\nu-\nu')=\frac{\Gamma(j+1-i\nu)\Gamma(j+1+i\nu)}{\Gamma(j+1-m-i\nu)\Gamma(j+1-m+i\nu)}\delta(\nu-\nu').$$

Noting that

$$\left(\sum_{N=0}^{[m/2]}\tilde{E}^N D^{m-2N}[\epsilon^{(0,0,m)}]\frac{b^{(j-m)}_{m,N}}{\Delta^{m-2N}}\right)\left(\sum_{M=0}^{[m/2]}\tilde{E}^M D^{m-2M}[\epsilon'^{(0,0,m)}]\frac{b^{(j-m)}_{m,M}}{\Delta^{m-2M}}\right), =\frac{j!}{(j-m)!}\left(\sum_{N=0}^{[m/2]}b^{(j-m)}_{m,N}\right)\epsilon^{(0,0,m)}_{\mu_1\cdots\mu_{j-m}}\cdot\epsilon'^{(0,0,m)\hat{\mu}_1\cdots\hat{\mu}_{j-m}},$$

we find that (A66) implies

$$N^{-2}_{j,m}=\frac{\Gamma(j+1-i\nu)\Gamma(j+1+i\nu)}{\Gamma(j+1-m-i\nu)\Gamma(j+1-m+i\nu)}\frac{j!}{(j-m)!}\left(\sum_{N=0}^{[m/2]}b^{(j-m)}_{m,N}\right), =\frac{\Gamma(j+1-i\nu)}{\Gamma(j+1-m-i\nu)}\frac{\Gamma(j+1+i\nu)}{\Gamma(j+1-m+i\nu)}{}_jC_m\frac{\Gamma(3/2+j-m)}{2^m\Gamma(3/2+j)}\frac{\Gamma(2+2j)}{\Gamma(2+2j-m)}. \tag{A67}$$

4. A note on the wave function of the massless vector field

For a rank-1 tensor (vector) field on AdS_5, we can determine the wave function of the $(n,l,m)=(1,0,1)$ eigenmode, not just for the $(n,l,m)=(0,0,m)$ modes with $m=0,1$. With the eigenvalue $\mathcal{E}_{1,0}=(3j+5+\nu^2)|_{j=1}$,

$$a^{(0,1,0)} = \epsilon^{(1,0,1)} z^2 K_{i\nu}(\Delta z),$$
$$a^{(1,0,0)} = \epsilon^{(1,0,1)} \partial_z (z^2 K_{i\nu}(\Delta z)) \tag{A68}$$

is the eigenvector solution to (A44).

The $(n,l,m) = (0,0,1)$ mode and the $(n,l,m) = (1,0,1)$ mode are independent, even after the mass-shell condition (66) for generic vector fields in bosonic string theory. However, for the massless vector field A_m obtained by the simple dimensional reduction of the massless vector field $A_M^{(Y)}$ with $Y = \{1,0,0\}$, these two modes become degenerate. To see this, note that $c_y = -4$ for this mode, so that the mass-shell condition (66) implies,

$$(j + 4 + \nu^2 + c_y)|_{j=1} = 0 \ (0,0,1) \text{ mode},$$
$$(3j + 5 + \nu^2 + c_y)|_{j=1} = 0 \ (1,0,1) \text{ mode}, \tag{A69}$$

or, equivalently, $i\nu = 1$ and $i\nu = 2$, respectively, for these two modes. It is now obvious that the terms proportional to $(\epsilon \cdot q)$ in (35) are in the form of this $(n,l,m) = (1,0,1)$ mode. With the relations $x^3 \partial_x [x^{-3+2} K_1(x)] = -x^3 [x^{-1} K_2(x)]$ and $\partial_x [x^2 K_2(x)] = -x^2 K_1(x)$, one can also see that the wave function for the $(n,l,m) = (0,0,1)$ mode is also proportional to the form given in (35) when the on-shell condition is imposed.

5. Projection operator of SO(3, 1) tensors

Note first that

$$a = \sum_{s=0}^{r} \sum_{N=0}^{[s/2]} \tilde{E}^N D_\Delta^{s-2N} [a^{(s,N)}] \tag{A70}$$

is an *orthogonal* decomposition of a totally symmetric SO (3, 1) tensor a of rank r into totally symmetric 4D-traceless 4D-transverse SO(3, 1) tensors $a^{(s,N)}$ of rank $(r-s)$. Here, the metric is given by

$$[b_{(-\Delta)}] \cdot [a_{(\Delta)}] \coloneqq [b_{(-\Delta)}]_{\rho_1 \cdots \rho_r} [a_{(+\Delta)}]_{\sigma_1 \cdots \sigma_r} \eta^{\hat{\rho}_1 \hat{\sigma}_1} \cdots \eta^{\hat{\rho}_r \hat{\sigma}_r}, \tag{A71}$$

as in the main text. To see that the decomposition is orthogonal under this metric, one needs to use only (A37) to verify that

$$[\tilde{E}^M D_{-\Delta}^{t-2M} [b^{(t,M)}]] \cdot [\tilde{E}^N D_\Delta^{s-2N} [a^{(s,N)}]]$$
$$= \delta_{M,N} \delta_{t-2M,s-2N} \frac{\Delta^{2(s-2N)}}{b_{s,N}^{(r-s)}} [b^{(t,M)}] \cdot [a^{(s,N)}]. \tag{A72}$$

Using the fact that (A70) is an orthogonal decomposition, let us construct projection operators $\overline{P}^{(r;s,N)}$ that extract various components $a^{(s,N)}$ from a totally symmetric SO(3, 1) tensor a of rank r. We introduced an operator $P^{(r)}$ in (102) which acts on rank-r SO(3, 1) tensors. From what we have seen above, it can be used to extract the $a^{(s,N)=(0,0)}$ component from a rank-r tensor a. That is, $\overline{P}^{(r;0,0)} = P^{(r)}$. It is straightforward to see that the projection operator for other components $a^{(s,N)}$ with general (s,N) is given by

$$\overline{P}^{(r;s,N)} \coloneqq \sum_a \frac{b_{s,N}^{(r-s)}}{\Delta^{2(s-2N)}} \frac{1}{D_a} (\tilde{E}^N D_\Delta^{s-2N} [\epsilon_a])_{\rho_1 \cdots \rho_r}$$
$$\times (\tilde{E}^N D_{-\Delta}^{s-2N} [\epsilon_a])_{\sigma_1 \cdots \sigma_r}, \tag{A73}$$

where ϵ_a's are an orthogonal basis of totally symmetric 4D-traceless 4D-transverse SO(3, 1) tensors of rank $(r-s)$.

It is also useful to have a concrete form of the projection operator $P^{(r)}$, not just its abstract definition in (102). We find that it is given by

$$P^{(r)} \cdot a = \sum_{M=0}^{[\frac{j}{2}]} \frac{(-1)^M \Gamma(r + \frac{1}{2} - M)}{4^M M! \Gamma(r + \frac{1}{2})}$$
$$\times \sum_{k=0}^{r-2M} \frac{(-1)^k}{k!} [\tilde{E}^M D^k OP_{(p,q)=(M,k)}] \cdot a, \tag{A74}$$

where $OP_{(p,q)}$ is the operator given in (A37). A totally symmetric rank-r tensor a is converted once into rank-$(r - 2M - k)$ tensors, and then they are converted back to a rank-r tensor under the operator $P^{(r)}$. To see that all of the $\tilde{E}^N D^{s-2N} [a^{(s,N)}]$ components are projected out by $P^{(r)}$, one needs to use only the following formula [40]:

$$\sum_{M=0}^{N} (-1)^M {}_N C_M \frac{\Gamma(r - N + \frac{3}{2})}{\Gamma(r - N + \frac{3}{2} - M)} \frac{\Gamma(r + \frac{1}{2} - M)}{\Gamma(r + \frac{1}{2})}$$
$$= \frac{\Gamma(\frac{1}{2} - r)}{\Gamma(\frac{1}{2} - r + N) \Gamma(1 - N)}, \tag{A75}$$

which vanishes for an integer $N \geq 0$.

6. Some tensor computations

Let us derive a more concrete expression for the product $(q^{\mu_1} \ldots q^{\mu_r}) \cdot [P^{(r)}]_{\mu_1 \ldots \mu_r}^{\nu_1 \ldots \nu_r} \cdot (p_{\nu_1} \ldots p_{\nu_r})$ by using the explicit expression for the projection operator $P^{(r)}$ to the SO(3, 1)-transverse SO(3, 1)-traceless rank-r tensor:

$$(q^{\mu_1} \ldots q^{\mu_r}) \cdot [P^{(r)}]_{\mu_1 \ldots \mu_r}^{\nu_1 \ldots \nu_r} \cdot (p_{\nu_1} \ldots p_{\nu_r})$$
$$= \sum_{M=0}^{[r/2]} \frac{(-1)^M \Gamma(j + \frac{1}{2} - M)}{4^M M! \Gamma(j + \frac{1}{2})} \frac{r!}{(r-2M)!}$$
$$\times \left[q^2 - \frac{(q \cdot \Delta)^2}{\Delta^2} \right]^M (p^2)^M (q \cdot p)^{r-2M}, \tag{A76}$$

where we made $p \cdot \Delta = 0$. Within the regime of $q^2, (p \cdot q), (q \cdot \Delta) \gg \Lambda^2, \Delta^2, p^2$ that we have been interested in this article, $(q \cdot \Delta)^2 / \Delta^2 \gg q^2$. Thus, after ignoring q^2,

$$(q^{\mu_1}\ldots q^{\mu_r})\cdot[P^{(r)}]^{\nu_1\ldots\nu_r}_{\mu_1\ldots\mu_r}\cdot(p_{\nu_1}\ldots p_{\nu_r})$$
$$\approx (p\cdot q)^r\sum_{M=0}^{[r/2]}\frac{\Gamma(r+\frac{1}{2}-M)}{4^M M!\Gamma(r+\frac{1}{2})}\frac{r!}{(r-2M)!}\left[\left(\frac{q\cdot\Delta}{q\cdot p}\right)^2\frac{p^2}{\Delta^2}\right]^M$$
$$=: (p\cdot q)^r\times\hat{d}_r(\eta,\Delta^2). \tag{A77}$$

This introduces $\hat{d}_r$, which is a polynomial of skewness $(q\cdot\Delta)/(p\cdot q)=-2\eta$ of degree $2[r/2]$.

When r is even, this polynomial of η can also be rewritten by using a Legendre polynomial, $P_\ell(x)$, which is defined by ([41], page 82)

$$P_\ell(x) = {}_2F_1\left(-\ell,\ell+1,1;\frac{1-x}{2}\right)$$
$$=\frac{(2\ell-1)!!}{\ell!}x^\ell{}_2F_1\left(-\frac{\ell}{2},\frac{1-\ell}{2},\frac{1}{2}-\ell;\frac{1}{x^2}\right). \tag{A78}$$

For an even r,

$$\hat{d}_r(\eta,\Delta^2)=\sum_{M=0}^{r/2}\frac{(-\frac{r}{2})_M(\frac{1-r}{2})_M}{M!(\frac{1}{2}-r)_M}\left(-\frac{4p^2}{\Delta^2}\eta^2\right)^M={}_2F_1\left(-\frac{r}{2},\frac{1-r}{2},\frac{1}{2}-r;\frac{(4m_h^2+\Delta^2)\eta^2}{\Delta^2}\right)$$
$$=\frac{r!}{(2r-1)!!}\left[\sqrt{\frac{4m_h^2+\Delta^2}{\Delta^2}}\eta\right]^r P_r\left(\sqrt{\frac{\Delta^2}{4m_h^2+\Delta^2}}\frac{1}{\eta}\right)=:\hat{d}_r([\eta]), \tag{A79}$$

where we used the kinematical relation $4p^2=-(4m_h^2+\Delta^2)$.

Similarly, it is also necessary to compute the following expression in order to study the $m=0$ exchange amplitude in Sec. VI C 2:

$$\left[\sum_{a\neq b}\epsilon^{2*}_{\rho_a}\epsilon^1_{\rho_b}q_{\rho_1}\cdots\check{\rho_a}\check{\rho_b}\cdots q_{\rho_j}\right]\cdot[P^{(j)}]^{\hat{\rho}_1\cdots\hat{\rho}_j}_{\sigma_1\cdots\sigma_j}\cdot[p^{\hat{\sigma}_1}\cdots p^{\hat{\sigma}_j}], \tag{A80}$$

which is also evaluated as above. The term proportional to $\eta^{\hat{\mu}\hat{\nu}}\epsilon^{2*}_\nu\epsilon^1_\mu$ (the contribution to the structure function V_1) is

$$2^2\sum_{M=1}^{[j/2]}\frac{(-1)^M\Gamma(j+\frac{1}{2}-M)}{4^M M!\Gamma(j+\frac{1}{2})}\frac{j!}{(j-2M)!2!}\left[q^2-\frac{(q\cdot\Delta)^2}{\Delta^2}\right]^{M-1}(p^2)^M(q\cdot p)^{j-2M}$$
$$\approx -2\frac{\Delta^2}{(q\cdot\Delta)^2}\times(q\cdot p)^j\sum_{M=1}^{[j/2]}\frac{\Gamma(j+\frac{1}{2}-M)}{4^M M!\Gamma(j+\frac{1}{2})}\frac{j!}{(j-2M)!}\left[\left(\frac{q\cdot\Delta}{q\cdot p}\right)^2\frac{p^2}{\Delta^2}\right]^M. \tag{A81}$$

This expression is once again a polynomial of η of degree $2[j/2]-2$ and is roughly of order $\Delta^2/(q\cdot p)^2$ times the expression (A77).

We will also need the following computation in Secs. VI C 3 and VI C 4:

$$(q_{\mu_1}\cdots q_{\mu_{j-k}})\cdot(\tilde{E}^N D^{s-2N}_{-\Delta}[\epsilon^{(0,0,m)}])^{\hat{\mu}_1\cdots\hat{\mu}_{j-k}}=\frac{(j-k)!}{(j-m)!}\left[q^2-\frac{(q\cdot\Delta)^2}{\Delta^2}\right]^N(-iq\cdot\Delta)^{s-2N}[(q_{\mu_1}\cdots q_{\mu_{j-m}})\cdot\epsilon^{(0,0,m)}]. \tag{A82}$$

APPENDIX B: CONFORMAL OPE COEFFICIENTS FROM AdS INTEGRALS

Let us introduce an integral,

$$C_1(\delta,\vartheta) := (1-\vartheta^2)^{1/2}\int_0^\infty dy y^{1+\delta}K_1(y\sqrt{1+\vartheta})K_1(y\sqrt{1-\vartheta}), \tag{B1}$$

which we encounter as the photon-photon– Pomeron/Reggeon vertex on AdS_5. $\vartheta=\eta/x$ and $\delta=j+i\nu$ in that context.

It is known ([42], page 101), if $\mathrm{Re}(\alpha+\beta)>0$ and $\mathrm{Re}(1\pm\nu\pm\mu-\rho)>0$, that

$$\int_0^\infty dt t^{-\rho} K_\mu(\alpha t) K_\nu(\beta t) = 2^{-\rho-2}\alpha^{\rho-\nu-1}\beta^\nu[\Gamma(1-\rho)]^{-1} \times\Gamma\left(\frac{1+\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1+\nu-\mu-\rho}{2}\right)\Gamma\left(\frac{1-\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1-\nu-\mu-\rho}{2}\right) \times{}_2F_1\left(\frac{1+\nu+\mu-\rho}{2},\frac{1+\nu-\mu-\rho}{2};1-\rho;1-\frac{\beta^2}{\alpha^2}\right). \tag{B2}$$

Substituting in $\rho=-1-\delta$, $\mu=1$, $\nu=-1$, $\alpha=\sqrt{1-\vartheta}$, and $\beta=\sqrt{1+\vartheta}$, we obtain

$$C_1(\delta,\vartheta)=\frac{\Gamma(\frac{\delta}{2})(\Gamma(\frac{\delta}{2}+1))^2\Gamma(\frac{\delta}{2}+2)}{\Gamma(\delta+2)}2^{\delta-1}(1-\vartheta)^{-\frac{\delta}{2}}{}_2F_1\left(\frac{\delta}{2},\frac{\delta+2}{2};\delta+2;\frac{2\vartheta}{\vartheta-1}\right). \tag{B3}$$

An equivalent, but slightly different expression is also obtained by using the following relation ([41], page 60):

$${}_2F_1(\alpha,\beta,2\beta;2z)=(1-z)^{-\alpha}{}_2F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2},\beta+\frac{1}{2};\left(\frac{z}{1-z}\right)^2\right); \tag{B4}$$

namely,

$$C_1(\delta,\vartheta)=2^{\delta-1}\frac{\delta+2}{\delta}\frac{(\Gamma(\frac{\delta}{2}+1))^4}{\Gamma(\delta+2)}{}_2F_1\left(\frac{\delta}{4},\frac{\delta}{4}+\frac{1}{2};\frac{\delta}{2}+\frac{3}{2};\vartheta^2\right). \tag{B5}$$

As a function of $\vartheta=\eta/x$, (B3) and (B5) are precisely of the form (25) and (26), respectively, required in the conformal OPE coefficients.

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