Dark matter in ghost-free bigravity theory:
From a galaxy scale to the universe

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We study the origin of dark matter based on the ghost-free bigravity theory with twin matter fluids. The present cosmic acceleration can be explained by the existence of graviton mass, while dark matter is required in several cosmological situations (the galactic missing mass, the cosmic structure formation and the cosmic microwave background observation). Assuming that the Compton wavelength of the massive graviton is shorter than a galactic scale, we show the bigravity theory can explain dark matter by twin matter fluid as well as the cosmic acceleration by tuning appropriate coupling constants.

I. INTRODUCTION

Whether a graviton has a mass or not is one of the most fundamental issues in physics. In general relativity (GR), it is well-known that the graviton is a massless spin-2 particle. However, Fierz and Pauli proposed a massive spin-2 particle theory, which is known as a unique ghost-free linear massive gravity theory [1]. The present experimental solid constraint on the graviton mass is $m < 7.1 \times 10^{-23}$ eV from the solar system experiment [2–4]. Although a simple nonlinear extension of the Fierz–Pauli massive gravity theory contains instabilities called the Boulware–Deser ghost [5], it was shown that the special choice of the interaction term can exclude such a ghost state by de Rham *et al.* [6,7]. However, this theory cannot describe the flat Friedmann universe if the fictitious metric for the Stückelberg field is Minkowski's one. One may consider an inhomogeneous metric or extend it to the de Sitter metric. When we discuss a curved fictitious geometry, it may be natural for it to be dynamical. In fact, the de Rham–Gabadadze–Tolley massive gravity theory has been generalized to such a bigravity theory, which is still ghost free. It contains a massless spin-2 particle and a massive spin-2 particle [8].

A phenomenological motivation to consider such theories relates to the discovery of dark energy and dark matter. The cosmological parameters are now determined very precisely [9]. Although standard big bang cosmology explains many observed data, those observations reveal new unsolved mysteries in cosmology, i.e., the accelerated expansion of the Universe and the existence of dark matter. Among them, the observed cosmic acceleration is one of the most mysterious problems in modern cosmology [10]. The acceleration might predict the existence of the dark energy (either just a vacuum energy fine-tuned by the 120 orders of magnitude or some mysterious unknown matter

with a strange equation of state) or might be due to a modification of gravitational theory. If we adopt an appropriate gravitational theory different from GR, there is a possibility to explain the accelerating expansion without dark energy. As for the ghost-free massive gravity or bigravity theory, many studies addressed the possibility to explain cosmic acceleration by the "mass" term [11–24].

In contrast to the massive gravity theory, bigravity theories also have a possibility to explain the origin of dark matter [25]. It is because there are two types of matter fields in a bigravity theory. If a matter field interacts with both metrics [23], it will violate the equivalence principle, which must hold in very high accuracy [3,33]. Furthermore, the reappearance of the ghost for the case of matter coupled to both metrics is also problematic. Although some ghost-free matter coupling was discussed at a low-energy scale or in a homogeneous Friedmann background universe, a ghost mode appears in a generic situation such as an inhomogeneous background [34–36]. Hence, we have to discuss two different matter fields, which are decoupled each other and interact only through two metric interactions. We then call them twin matter fields [37].

In the previous paper [24], we found that both dark matter and dark energy components in the Friedmann equation can be obtained by modification of gravitational theory in the ghost-free bigravity theory. However, dark matter is required not only in the big bang scenario but also in the cosmological structure formation and as dark matter halos existing around galaxies. This paper will show a possibility to explain the origin of dark matter in such situations.

The bigravity theory includes GR with/without a cosmological constant as a special case. If both metrics are proportional, which we call a homothetic solution, the basic equations are reduced to two sets of the Einstein equations with cosmological constants, which originate from the interaction terms of two metrics [38]. Although two matters must satisfy a fine-tuned condition in a

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homothetic solution, such a solution is an attractor and is obtained asymptotically from more generic initial conditions [24].

The linear perturbations around a homothetic solution are decomposed into two eigenstates: the massless and massive graviton modes. Note that these are the mass eigenstates, whereas they are mixed up in the physical frame described by two metrics. That is, the massless and massive modes couple to both twin matter fluids. As a result, the perturbations of our spacetime are described by the linear combinations of the massless and massive modes. Our spacetime is affected by another one of the twin matter fluids via the massless and massive graviton modes, and then there is a possibility such that a dark matter component is originated by another twin matter. The purpose of this paper is to investigate such a possibility. Since dark matter is required in many situations, we shall discuss three typical evidences of dark matter: the content of the Universe, a galactic halo, and the cosmic structure formation.

The paper is organized as follows. Introducing the ghost-free bigravity, we summarize the basic equations and present a homothetic solution in Sec. II. In Sec. III, we then perform the perturbations around a homothetic solution. We show that dark matter can be obtained from another one of the twin matter fluids from a galactic scale to a cosmological scale in Sec. IV. Assuming the Compton wavelength of the massive graviton is shorter than a galactic scale, another twin matter can play the role of dark matter in our world for all scales. We summarize our results and give some remarks in Sec. V. In Appendix A, we evaluate the values of the graviton mass and a cosmological constant for given coupling parameters. We also present the basic equations for the gauge-invariant perturbations in a homothetic background solution.

II. BIGRAVITY THEORY

A. Hassan–Rosen bigravity model

In the present papers, we focus on the ghost-free bigravity theory proposed by Hassan and Rosen, of which the action is given by

$$S = \frac{1}{2\kappa_g^2}\int d^4x\sqrt{-g}R(g) + \frac{1}{2\kappa_f^2}\int d^4x\sqrt{-f}\mathcal{R}(f)$$
$$+ S^{[\mathrm{m}]}(g,f,\psi_g,\psi_f) - \frac{m^2}{\kappa^2}\int d^4x\sqrt{-g}\mathcal{U}(g,f), \quad (2.1)$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are two dynamical metrics and $R(g)$ and $\mathcal{R}(f)$ are those Ricci scalars, respectively. $\kappa_g^2 = 8\pi G$ and $\kappa_f^2 = 8\pi\mathcal{G}$ are the corresponding gravitational constants, while κ is defined by $\kappa^2 = \kappa_g^2 + \kappa_f^2$. We assume that the matter action $S^{(\mathrm{m})}$ is divided into two parts,

$$S^{(\mathrm{m})}(g,f,\psi_g,\psi_f) = S_g^{[\mathrm{m}]}(g,\psi_g) + S_f^{[\mathrm{m}]}(f,\psi_f); \quad (2.2)$$

i.e., matter fields ψ_g and ψ_f are coupled only to the g metric and to the f metric, respectively. This restriction guarantees the weak equivalence principle as well as the ghost-free condition. We call the g matter ψ_g and the f matter ψ_f twin matter fluids.

The ghost-free interaction term between two metrics is given by

$$\mathcal{U}(g,f) = \sum_{k=0}^{4} b_k \mathcal{U}_k(\gamma), \quad (2.3)$$

$$\mathcal{U}_0(\gamma) = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma},$$

$$\mathcal{U}_1(\gamma) = -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\nu\rho\sigma}\gamma^\mu{}_\alpha,$$

$$\mathcal{U}_2(\gamma) = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta,$$

$$\mathcal{U}_3(\gamma) = -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\rho{}_\gamma,$$

$$\mathcal{U}_4(\gamma) = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\rho{}_\gamma\gamma^\sigma{}_\delta, \quad (2.4)$$

where b_k are coupling constants, while $\gamma^\mu{}_\nu$ is defined by

$$\gamma^\mu{}_\rho\gamma^\rho{}_\nu = g^{\mu\rho}f_{\rho\nu}. \quad (2.5)$$

Taking the variation of the action with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, we find two sets of the Einstein equations,

$$G^\mu{}_\nu = \kappa_g^2(T^{[\gamma]\mu}{}_\nu + T^{[\mathrm{m}]\mu}{}_\nu), \quad (2.6)$$

$$\mathcal{G}^\mu{}_\nu = \kappa_f^2(\mathcal{T}^{[\gamma]\mu}{}_\nu + \mathcal{T}^{[\mathrm{m}]\mu}{}_\nu), \quad (2.7)$$

where $G^\mu{}_\nu$ and $\mathcal{G}^\mu{}_\nu$ are the Einstein tensors for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. The matter energy-momentum tensors are given by

$$T^{[\mathrm{m}]}{}_{\mu\nu} = -2\frac{\delta S_g^{[\mathrm{m}]}}{\delta g^{\mu\nu}}$$

$$\mathcal{T}^{[\mathrm{m}]}{}_{\mu\nu} = -2\frac{\delta S_f^{[\mathrm{m}]}}{\delta f^{\mu\nu}}. \quad (2.8)$$

The γ-"energy-momentum" tensors from the interaction term are given by

$$T^{[\gamma]\mu}{}_\nu = \frac{m^2}{\kappa^2}(\tau^\mu{}_\nu - \mathcal{U}\delta^\mu{}_\nu), \quad (2.9)$$

$$\mathcal{T}^{[\gamma]\mu}{}_\nu = -\frac{\sqrt{-g}}{\sqrt{-f}}\frac{m^2}{\kappa^2}\tau^\mu{}_\nu, \quad (2.10)$$

with

$$\tau^{\mu}{}_{\nu} = \{b_1 \mathcal{U}_0 + b_2 \mathcal{U}_1 + b_3 \mathcal{U}_2 + b_4 \mathcal{U}_3\}\gamma^{\mu}{}_{\nu}$$
$$- \{b_2 \mathcal{U}_0 + b_3 \mathcal{U}_1 + b_4 \mathcal{U}_2\}(\gamma^2)^{\mu}{}_{\nu}$$
$$+ \{b_3 \mathcal{U}_0 + b_4 \mathcal{U}_1\}(\gamma^3)^{\mu}{}_{\nu} - b_4 \mathcal{U}_0 (\gamma^4)^{\mu}{}_{\nu}.$$

The energy momenta of matter fields are assumed to be conserved individually as

$$\overset{(g)}{\nabla}_{\mu} T^{[\mathrm{m}]\mu}{}_{\nu} = 0, \qquad \overset{(f)}{\nabla}_{\mu} \mathcal{T}^{[\mathrm{m}]\mu}{}_{\nu} = 0, \qquad (2.11)$$

where $\overset{(g)}{\nabla}_{\mu}$ and $\overset{(f)}{\nabla}_{\mu}$ are covariant derivatives with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$. From the contracted Bianchi identities for (2.6) and (2.7), the conservation of the γ-energy momenta is also guaranteed as

$$\overset{(g)}{\nabla}_{\mu} T^{[\gamma]\mu}{}_{\nu} = 0, \qquad \overset{(f)}{\nabla}_{\mu} \mathcal{T}^{[\gamma]\mu}{}_{\nu} = 0. \qquad (2.12)$$

B. Homothetic solution

First, we give one simple solution, in which we assume that two metrics are proportional,

$$f_{\mu\nu} = K^2 g_{\mu\nu}, \qquad (2.13)$$

where K is a scalar function. In this case, since we find the tensor $\gamma^{\mu}{}_{\nu} = K\delta^{\mu}{}_{\nu}$, the γ-energy momentum is given by

$$\kappa_g^2 T^{[\gamma]\mu}{}_{\nu} = -\Lambda_g(K)\delta^{\mu}{}_{\nu},$$
$$\kappa_f^2 \mathcal{T}^{[\gamma]\mu}{}_{\nu} = -\Lambda_f(K)\delta^{\mu}{}_{\nu},$$

where

$$\Lambda_g(K) = m^2 \frac{\kappa_g^2}{\kappa^2}(b_0 + 3b_1 K + 3b_2 K^2 + b_3 K^3),$$

$$\Lambda_f(K) = m^2 \frac{\kappa_f^2}{\kappa^2}(b_4 + 3b_3 K^{-1} + 3b_2 K^{-2} + b_1 K^{-3}). \qquad (2.14)$$

From the energy-momentum conservation (2.12), we find that K is a constant. As a result, we find two sets of the Einstein equations with cosmological constants Λ_g and Λ_f:

$$G_{\mu\nu}(g) + \Lambda_g g_{\mu\nu} = \kappa_g^2 T^{[\mathrm{m}]}{}_{\mu\nu}, \qquad (2.15)$$

$$\mathcal{G}_{\mu\nu}(f) + \Lambda_f f_{\mu\nu} = \kappa_f^2 \mathcal{T}^{[\mathrm{m}]}{}_{\mu\nu}. \qquad (2.16)$$

Since two metrics are proportional, we have the constraints on the cosmological constants and matter fields as

$$\Lambda_g(K) = K^2 \Lambda_f(K), \qquad (2.17)$$

$$\kappa_g^2 T^{[\mathrm{m}]\mu}{}_{\nu} = K^2 \kappa_f^2 \mathcal{T}^{[\mathrm{m}]\mu}{}_{\nu}. \qquad (2.18)$$

The quartic equation (2.17) for K has at most four real roots, which give four different cosmological constants. The basic equations (2.15) [or (2.16)] are just the Einstein equations in GR with a cosmological constant. Hence, any solutions in GR with a cosmological constant are always the solutions in the present bigravity theory. We shall call these solutions homothetic solutions because of the proportionality of two metrics.

III. LINEARIZATION OF THE BIGRAVITY THEORY

A. Perturbations around a homothetic solution

The bigravity theory contains both massless and massive spin-2 particles. It becomes clear when we discuss the linear perturbations around a homothetic solution. Note that a homothetic solution is an attractor in a cosmological setting [24].

The unperturbed solution is assumed to be homothetic, i.e.,

$$\overset{(0)}{g}_{\mu\nu} \quad \text{and} \quad \overset{(0)}{f}_{\mu\nu} = K^2 \overset{(0)}{g}_{\mu\nu}, \qquad (3.1)$$

which is the solution of two Einstein equations:

$$\overset{(0)}{G}{}^{\mu}{}_{\nu}(\overset{(0)}{g}) = -\Lambda_g(K)\delta^{\mu}{}_{\nu} + \kappa_g^2 \overset{(0)}{T}{}^{[\mathrm{m}]\mu}{}_{\nu}, \qquad (3.2)$$

$$\overset{(0)}{\mathcal{G}}{}^{\mu}{}_{\nu}(\overset{(0)}{f}) = -\Lambda_f(K)\delta^{\mu}{}_{\nu} + \kappa_f^2 \overset{(0)}{\mathcal{T}}{}^{[\mathrm{m}]\mu}{}_{\nu}. \qquad (3.3)$$

A constant K is determined by the quartic equation (2.17), and the matter energy momenta satisfy the following condition:

$$\kappa_f^2 \overset{(0)}{\mathcal{T}}{}^{[\mathrm{m}]\mu}{}_{\nu} = \frac{1}{K^2} \kappa_g^2 \overset{(0)}{T}{}^{[\mathrm{m}]\mu}{}_{\nu}. \qquad (3.4)$$

We then consider the perturbations

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + h^{[g]}_{\mu\nu}, \qquad (3.5)$$

$$f_{\mu\nu} = \overset{(0)}{f}_{\mu\nu} + K^2 h^{[f]}_{\mu\nu} = K^2(\overset{(0)}{g}_{\mu\nu} + h^{[f]}_{\mu\nu}) \qquad (3.6)$$

where $|h^{[g]}_{\mu\nu}|, |h^{[f]}_{\mu\nu}| \ll |\overset{(0)}{g}_{\mu\nu}|$. The suffixes of $h^{[g]}_{\mu\nu}$ as well as $h^{[f]}_{\mu\nu}$ are raised and lowered by the background metric $\overset{(0)}{g}_{\mu\nu}$.

The energy-momentum tensors of twin matter fluid and γ-energy-momentum ones from the interaction terms can be expanded as

$$\kappa_g^2 T^{[\mathrm{m}]\mu}{}_{\nu} = \kappa_g^2 [\overset{(0)}{T}{}^{[\mathrm{m}]\mu}{}_{\nu} + \overset{(1)}{T}{}^{[\mathrm{m}]\mu}{}_{\nu}], \qquad (3.7)$$

$$K^2 \kappa_f^2 \mathcal{T}^{[m]\mu}{}_\nu = K^2 \kappa_f^2 [\overset{(0)}{\mathcal{T}}{}^{[m]\mu}{}_\nu + \overset{(1)}{\mathcal{T}}{}^{[m]\mu}{}_\nu] \qquad (3.8)$$

and

$$\kappa_g^2 T^{[\gamma]\mu}{}_\nu = -\Lambda_g \delta^\mu{}_\nu + \frac{m_g^2}{2}(h^{[-]\mu}{}_\nu - h^{[-]}\delta^\mu{}_\nu), \qquad (3.9)$$

$$K^2 \kappa_f^2 T^{[\gamma]\mu}{}_\nu = -K^2 \Lambda_f \delta^\mu{}_\nu - \frac{m_f^2}{2}(h^{[-]\mu}{}_\nu - h^{[-]}\delta^\mu{}_\nu), \qquad (3.10)$$

respectively, where

$$m_g^2 := \frac{m^2 \kappa_g^2}{\kappa^2}(b_1 K + 2b_2 K^2 + b_3 K^3), \qquad (3.11)$$

$$m_f^2 := \frac{m^2 \kappa_f^2}{K^2 \kappa^2}(b_1 K + 2b_2 K^2 + b_3 K^3). \qquad (3.12)$$

Here, we have introduced new variables $h_{\mu\nu}^{[-]}$ and $h_{\mu\nu}^{[+]}$ from two metric perturbations as

$$h_{\mu\nu}^{[-]} = h_{\mu\nu}^{[g]} - h_{\mu\nu}^{[f]},$$

$$h_{\mu\nu}^{[+]} = \frac{m_f^2}{m_{\text{eff}}^2} h_{\mu\nu}^{[g]} + \frac{m_g^2}{m_{\text{eff}}^2} h_{\mu\nu}^{[f]} \qquad (3.13)$$

with

$$m_{\text{eff}}^2 := m_g^2 + m_f^2$$
$$= \frac{m^2}{\kappa^2}\left(\kappa_g^2 + \frac{\kappa_f^2}{K^2}\right)(b_1 K + 2b_2 K^2 + b_3 K^3). \qquad (3.14)$$

The first-order perturbation equations are then given by

$$\overset{(0)}{g}{}^{\mu\rho}\overset{(1)}{R}{}_{\rho\nu}(h^{[+]}) - \overset{(0)}{R}{}^{\rho(\mu}h_{\nu)\rho}^{[+]} = \overset{(1)}{M}{}^{[+]\mu}{}_\nu, \qquad (3.15)$$

$$\overset{(0)}{g}{}^{\mu\rho}\overset{(1)}{R}{}_{\rho\nu}(h^{[-]}) - \overset{(0)}{R}{}^{\rho(\mu}h_{\nu)\rho}^{[-]}$$
$$+ \frac{m_{\text{eff}}^2}{4}(2h^{[-]\mu}{}_\nu + h^{[-]}\delta^\mu{}_\nu) = \overset{(1)}{M}{}^{[-]\mu}{}_\nu, \qquad (3.16)$$

where $\overset{(1)}{R}{}_{\mu\nu}$ denotes the linearized Ricci tensor, which is defined for a metric perturbation $h_{\mu\nu}$ by

$$\overset{(1)}{R}{}_{\mu\nu}(h) := \frac{1}{2}\left[-\overset{(0)}{\nabla}{}_\mu \overset{(0)}{\nabla}{}_\nu h - \overset{(0)}{\Box} h_{\mu\nu}\right.$$
$$\left. + \overset{(0)}{\nabla}{}^\alpha(\overset{(0)}{\nabla}{}_\nu h_{\alpha\mu}) + \overset{(0)}{\nabla}{}^\alpha(\overset{(0)}{\nabla}{}_\mu h_{\alpha\nu})\right], \qquad (3.17)$$

and the matter perturbations $\overset{(1)}{M}{}^{[\pm]\mu}{}_\nu$ are defined by

$$\overset{(1)}{M}{}^{[-]\mu}{}_\nu := \kappa_g^2\left[\overset{(1)}{T}{}^{[m]\mu}{}_\nu - \frac{1}{2}\overset{(1)}{T}{}^{[m]}\delta^\mu{}_\nu\right]$$
$$- K^2 \kappa_f^2\left[\overset{(1)}{\mathcal{T}}{}^{[m]\mu}{}_\nu - \frac{1}{2}\overset{(1)}{\mathcal{T}}{}^{[m]}\delta^\mu{}_\nu\right],$$

$$\overset{(1)}{M}{}^{[+]\mu}{}_\nu := \frac{m_f^2}{m_{\text{eff}}^2}\kappa_g^2\left[\overset{(1)}{T}{}^{[m]\mu}{}_\nu - \frac{1}{2}\overset{(1)}{T}{}^{[m]}\delta^\mu{}_\nu\right]$$
$$+ \frac{m_g^2}{m_{\text{eff}}^2}K^2 \kappa_f^2\left[\overset{(1)}{\mathcal{T}}{}^{[m]\mu}{}_\nu - \frac{1}{2}\overset{(1)}{\mathcal{T}}{}^{[m]}\delta^\mu{}_\nu\right], \qquad (3.18)$$

which are linear combinations of g- and f-matter perturbations. Equations (3.15) and (3.16) are decoupled, and then they provide two mass eigenstates. We find that $h_{\mu\nu}^{[+]}$ and $h_{\mu\nu}^{[-]}$ describe massless and massive modes, respectively, and m_{eff} denotes a graviton mass of the massive mode in the homothetic background spacetime.

The Bianchi identity ($\overset{(g)}{\nabla}{}_\mu G^\mu{}_\nu = 0$) gives the conservation of γ-energy-momentum tensor, i.e.,

$$\overset{(g)}{\nabla}{}_\mu T^{[\gamma]\mu}{}_\nu = 0, \qquad (3.19)$$

of which the perturbation gives the constraint on the massive mode $h_{\alpha\beta}^{[-]}$:

$$\overset{(0)}{\nabla}{}_\mu(\kappa_g^2 \overset{(1)}{T}{}^{[\gamma]\mu}{}_\nu) = \frac{m_g^2}{2}[-\overset{(0)}{\nabla}{}_\mu h^{[-]\mu}{}_\nu + \overset{(0)}{\nabla}{}_\nu h^{[-]}] = 0. \qquad (3.20)$$

Since $m_g^2 \neq 0$, we find

$$\overset{(0)}{\nabla}{}_\mu h^{[-]\mu}{}_\nu = \overset{(0)}{\nabla}{}_\nu h^{[-]}. \qquad (3.21)$$

From another conservation equation, $\overset{(f)}{\nabla}{}_\mu \mathcal{T}^{[\gamma]\mu}{}_\nu = 0$ gives the same constraint equation.

Taking a trace of Eq. (3.16) and using Eq. (3.21), we find

$$(3m_{\text{eff}}^2 - 2\Lambda_g)h^{[-]} = \kappa_g^2(2\overset{(0)}{T}{}^{[m]}_{\alpha\beta}h^{[-]\alpha\beta} - \overset{(0)}{T}{}^{[m]}h^{[-]}) + 2\overset{(1)}{M}{}^{[-]\alpha}{}_\alpha. \qquad (3.22)$$

Equations (3.21) and (3.22) give five constraint equations on $h_{\alpha\beta}^{[-]}$. There is no gauge freedom because $h_{\alpha\beta}^{[-]}$ is a gauge-invariant tensor. This is consistent with the fact that a massive graviton has five degrees of freedom.

Using these constraints, we rewrite the above perturbation equations as

$$-\overset{(0)}{\nabla}{}_\mu \overset{(0)}{\nabla}{}_\nu h^{[+]} - \overset{(0)}{\Box} h_{\mu\nu}^{[+]} + 2\overset{(0)}{\nabla}{}_{(\nu}[\overset{(0)}{\nabla}{}^\alpha h_{\mu)\alpha}^{[+]}]$$
$$- 2\overset{(0)}{R}{}_\mu{}^\alpha{}_\nu{}^\beta h_{\alpha\beta}^{[+]} = 2\overset{(1)}{M}{}^{[+]}_{\mu\nu}, \qquad (3.23)$$

$$- \overset{(0)}{\nabla}_\mu \overset{(0)}{\nabla}_\nu h^{[-]} - \overset{(0)}{\Box} h^{[-]}_{\mu\nu} - 2 \overset{(0)}{R}{}_\mu{}^\alpha{}_\nu{}^\beta h^{[-]}_{\alpha\beta}$$
$$+ m^2_{\mathrm{eff}} \left(h^{[-]}_{\mu\nu} + \frac{1}{2} h^{[-]} \overset{(0)}{g}_{\mu\nu} \right) = 2 \overset{(1)}{M}{}^{[-]}_{\mu\nu}, \qquad (3.24)$$

where we have used

$$\overset{(0)}{\nabla}{}^\alpha (\overset{(0)}{\nabla}_\nu \chi_{\mu\alpha}) = \overset{(0)}{\nabla}_\nu (\overset{(0)}{\nabla}{}^\alpha \chi_{\mu\alpha}) + \overset{(0)}{R}{}_\mu{}^{\alpha\beta}{}_\nu \chi_{\alpha\beta} + \overset{(0)}{R}{}^\rho{}_\nu \chi_{\mu\rho}. \tag{3.25}$$

Since two modes are decoupled, we shall analyze them separately and then discuss the physical perturbations in the g and f worlds, which are represented as

$$h^{[g]}_{\mu\nu} = h^{[+]}_{\mu\nu} + \frac{m^2_g}{m^2_{\mathrm{eff}}} h^{[-]}_{\mu\nu},$$
$$h^{[f]}_{\mu\nu} = h^{[+]}_{\mu\nu} - \frac{m^2_f}{m^2_{\mathrm{eff}}} h^{[-]}_{\mu\nu}. \tag{3.26}$$

Since the massive mode $h^{[-]}_{\mu\nu}$ does not propagate beyond the scale of the Compton wavelength of a massive graviton, the spacetime perturbations are dominated by the massless mode $h^{[+]}_{\mu\nu}$ in a large scale system beyond the Compton wavelength. The massless mode couples to both twin matter fluids. As a result, there exists a possibility that the f-matter fluid behaves like a dark matter component in g world via a massless graviton mode, which we will discuss in what follows.

IV. ORIGIN OF DARK MATTER

In this section, we will analyze whether the f-matter field can be dark matter in our g world. We believe from observation that the evidence of dark matter appears in three situations: (A) dark matter in the Friedmann equation, (B) a dark halo at a galaxy scale, and (C) cold dark matter (CDM) in cosmic structure formation. So, we discuss them in order.

A. Cosmic pie

First, we discuss the pie chart of the content of the Universe. The amount of dark matter is about five times as large as the baryonic matter. Since we discussed the details of the dynamics of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime and the possibility to explain the dark matter component by the f-matter fluid in Ref. [24], we give a brief overview here.

To explain the cosmic pie, we consider the homogeneous and isotropic spacetime, of which the metrics are given by

$$ds^2_g = -N^2_g dt^2 + a^2_g \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{4.1}$$

$$ds^2_f = -N^2_f dt^2 + a^2_f \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{4.2}$$

where N_g and N_f are the lapse function, while a_g and a_f are scale factors for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Using the gauge freedom, we can set $N_g = 1$ without loss of generality.

For generic initial data, the ratios N_f/N_g and a_f/a_g can approach to the same constant K given by (2.17), as the universe expands; i.e., the homothetic solution is an attractor in the present system. The dynamical time scale is about m^{-1}_{eff}. We can approximate the cosmological solution by small deviation of metrics from a homothetic background when

$$3m^2_{\mathrm{eff}} - 2\Lambda_g \gg |\kappa^2_g \rho_g - K^2 \kappa^2_f \rho_f|. \tag{4.3}$$

As a result, near the attractor, i.e., near the present stage of the Universe, the evolution of the Universe is described by the effective Friedmann equation

$$H^2_g + \frac{k}{a^2_g} = \frac{\Lambda_g}{3} + \frac{\kappa^2_{\mathrm{eff}}}{3} [\rho_g + \rho_D], \tag{4.4}$$

where

$$\kappa^2_{\mathrm{eff}} = \kappa^2_g \left[1 - \frac{3m^2_g}{3m^2_{\mathrm{eff}} - 2\Lambda_g} \right], \tag{4.5}$$

$$\rho_D = \frac{3m^2_f}{3m^2_f - 2\Lambda_g} K^4 \rho_f, \tag{4.6}$$

and ρ_g and ρ_f are energy densities of g and f matter, respectively [24]. $H_g = \dot{a}_g/a_g$ is the Hubble parameter where a dot denotes the derivative with respect to t. κ^2_{eff} is the effective gravitational constant, and ρ_D is regarded as the energy density of a dark component in the g world; i.e., another one of twin matter fluids works as dark matter through the interaction term between two metrics. Note that the matter densities can deviate largely from those of the homothetic solution such that $\rho_D \geq \rho_g$, although the metric deviation is still small enough as long as $m^2_{\mathrm{eff}} \geq \kappa^2_{\mathrm{eff}} \rho_D$. See also Ref. [24], where we showed the full numerical result of evolutions of density parameters without the above approximation. It also confirms the above approximation is valid.

If both matter components are dominated by nonrelativistic matter,

$$\rho_g = \frac{\rho_{g,0}}{a^3_g}, \qquad \rho_f = \frac{\rho_{f,0}}{a^3_f}, \tag{4.7}$$

the density of the dark component is approximated by

$$\rho_{\mathrm{D}} = \frac{3m_f^2}{3m_f^2 - 2\Lambda_g} \frac{K^4 \rho_{f,0}}{a_f^3}$$

$$\approx \frac{3m_f^2}{3m_f^2 - 2\Lambda_g} \frac{K\rho_{f,0}}{a_g^3} + \mathcal{O}(a_g^{-6}). \qquad (4.8)$$

Hence, if $3m_f^2 > 2\Lambda_g$, ρ_{D} behaves as a dark matter component in the g world. If ρ_g consists just of baryonic matter, in order to explain the observed amount of dark matter, we have to require

$$\frac{\rho_{\mathrm{D}}}{\rho_g} = \frac{3m_f^2}{3m_f^2 - 2\Lambda_g} \frac{K\rho_{f,0}}{\rho_{g,0}} \sim 5. \qquad (4.9)$$

With an appropriate choice of the coupling parameters, we find the above value, which may explain dark matter by the f-matter fluid.

B. Dark matter halo

Next, we discuss how to explain a dark matter halo around a galaxy by another one of the twin matter fluids. The existence of the dark matter halo is confirmed by observations such as a flat rotation curve of a galaxy [39].

Since we analyze a galactic scale, the background spacetime is well approximated by the Minkowski metric $(\overset{(0)}{g}_{\mu\nu} \simeq \eta_{\mu\nu})$, ignoring the effect of a cosmological constant Λ_g. The gravitational phenomena can be analyzed by the linear perturbations around the Minkowski spacetime. The equations of the massive mode are given by

$$\overset{(0)}{\nabla}_\mu \overset{(0)}{\nabla}_\nu h^{[-]} - \overset{(0)}{\Box} h_{\mu\nu}^{[-]} + m_{\mathrm{eff}}^2 \left(h_{\mu\nu}^{[-]} + \frac{1}{2} h^{[-]}\eta_{\mu\nu} \right) = 2\overset{(1)}{M}{}_{\mu\nu}^{[-]},$$

$$\overset{(0)}{\nabla}_\mu h^{[-]\mu}{}_\nu = \overset{(0)}{\nabla}_\nu h^{[-]},$$

$$3m_{\mathrm{eff}}^2 h^{[-]} = 2\overset{(1)}{M}{}^{[-]\mu}{}_\mu. \qquad (4.10)$$

Substituting the third equation into first one, we obtain

$$\overset{(0)}{\nabla}_\mu \overset{(0)}{\nabla}_\nu h^{[-]} - \overset{(0)}{\Box} h_{\mu\nu}^{[-]} + m_{\mathrm{eff}}^2 h_{\mu\nu}^{[-]} = 2\overset{(1)}{M}{}_{\mu\nu}^{[-]} - \frac{1}{3}\overset{(1)}{M}{}^{[-]\rho}{}_\rho \eta_{\mu\nu}. \qquad (4.11)$$

To analyze the gravitational fields of a galaxy, we consider static Newtonian potentials Φ_g and Φ_f formed by nonrelativistic mass densities ρ_g and ρ_f. From the 0-0 component of Eq. (4.11), we obtain the Poisson equation for the massive mode as

$$(\Delta - m_{\mathrm{eff}}^2)\Phi_- = \frac{4}{3}(4\pi G\rho_g - 4\pi\mathcal{G}K^2\rho_f), \qquad (4.12)$$

where $\Delta = \partial^i \partial_i$ is the usual three-dimensional Laplacian operator and $\Phi_- = -h_{00}^{[-]}/2$ is the gravitational potential of the massive mode. The factor 4/3 comes from van Dam–Veltmann–Zakharov (vDVZ) discontinuity [40]. Note that the source term is described by the difference of two mass densities, and then it can be negative.

For the massless mode, we obtain the ordinary form of the Poisson equation,

$$\Delta\Phi_+ = 4\pi G \frac{m_f^2}{m_{\mathrm{eff}}^2}\rho_g + 4\pi\mathcal{G}K^2 \frac{m_g^2}{m_{\mathrm{eff}}^2}\rho_f, \qquad (4.13)$$

where $\Phi_+ = -h_{00}^{[+]}/2$ is the gravitational potential of the massless mode. This source term is positive definite.

We find that both gravitational potentials are affected by both g- and f-matter fluids. This is the main difference from the Newtonian gravity theory. It may make it possible for the f matter to behave as dark matter in the g worlds.

In a small scale such as the solar system, however, GR must be restored because GR has been well confirmed by the experiments and observations [3]. The restoration can be realized via the so-called Vainshtein mechanism [41]. In this range (below the Vainshtein radius), the linear perturbation approach is broken down, and then nonlinear effects must be taken into account. However, when GR is restored from the bigravity theory, the effect on the g world from the f-matter fluid must be screened [42]. It indicates that the f matter cannot be dark matter below the Vainshtein radius. Since we are interested in whether the f matter plays a role of dark matter in the g world, we shall only analyze the linear perturbations. The evaluation of the Vainshtein radius will be given in the last part of this subsection.

For a simplest case in which matter fluids are localized spherically, the Newtonian potentials outside matter distributions are solved as

$$\Phi_- = \frac{4}{3}\left(\frac{GM_g}{r} e^{-m_{\mathrm{eff}}r} - \frac{K^2\mathcal{G}\mathcal{M}_f}{r} e^{-m_{\mathrm{eff}}r} \right), \qquad (4.14)$$

$$\Phi_+ = \frac{m_f^2}{m_{\mathrm{eff}}^2} \frac{GM_g}{r} + \frac{m_g^2}{m_{\mathrm{eff}}^2} \frac{K^2\mathcal{G}\mathcal{M}_f}{r}, \qquad (4.15)$$

where the gravitational masses are defined by

$$M_g = \int 4\pi\rho_g r^2 dr, \qquad \mathcal{M}_f = \int 4\pi\rho_f r^2 dr. \qquad (4.16)$$

From (3.26), the Newtonian potentials in the g and f worlds are described as

$$\Phi_g = \Phi_+ + \frac{m_g^2}{m_{\rm eff}^2}\Phi_-$$

$$= -\frac{GM_g}{r}\left(\frac{m_f^2}{m_{\rm eff}^2} + \frac{4m_g^2}{3m_{\rm eff}^2}e^{-m_{\rm eff}r}\right)$$

$$-\frac{m_g^2}{m_{\rm eff}^2}\frac{K^2\mathcal{G}\mathcal{M}_f}{r}\left(1 - \frac{4}{3}e^{-m_{\rm eff}r}\right), \qquad (4.17)$$

$$\Phi_f = \Phi_+ - \frac{m_f^2}{m_{\rm eff}^2}\Phi_-$$

$$= -\frac{K^2\mathcal{G}\mathcal{M}_f}{r}\left(\frac{m_g^2}{m_{\rm eff}^2} + \frac{4m_f^2}{3m_{\rm eff}^2}e^{-m_{\rm eff}r}\right)$$

$$-\frac{m_f^2}{m_{\rm eff}^2}\frac{GM_g}{r}\left(1 - \frac{4}{3}e^{-m_{\rm eff}r}\right), \qquad (4.18)$$

where $\Phi_g = -h_{00}^{[g]}/2$, $\Phi_f = -h_{00}^{[f]}/2$.

Let us consider the Newtonian potential in the g world. Below the Compton wavelength of the massive graviton ($r < m_{\rm eff}^{-1}$), the potential becomes

$$\Phi_g = -\frac{GM_g}{r}\left(1 + \frac{m_g^2}{3m_{\rm eff}^2}\right) + \frac{m_g^2}{3m_{\rm eff}^2}\frac{K^2\mathcal{G}\mathcal{M}_f}{r}. \qquad (4.19)$$

Note that the second term is positive definite. It means that the f matter acts as a repulsive force in the g world. It comes from the factor 4/3 in (4.17). To explain dark matter, of course, the gravitational force must be attractive. Therefore, the f matter cannot behave as dark matter when the size of the localized system is smaller than the Compton wavelength.

The origin of this repulsive force is the massive mode, which cannot propagate in the large system such that $m_{\rm eff}r \gg 1$. In fact, beyond the Compton wavelength ($r > m_{\rm eff}^{-1}$), the potential is approximated by

$$\Phi_g = -\frac{G_{\rm eff}}{r}(M_g + K^4\mathcal{M}_f), \qquad (4.20)$$

where

$$G_{\rm eff} = \frac{m_f^2}{m_{\rm eff}^2}G \qquad (4.21)$$

is the local effective gravitational constant. This potential is formed by the f matter as well as the g matter. Hence, it is possible to explain dark matter by another one of the twin matter fluids.

Inside the Vainshtein radius, the gravitational constant is restored to the Newtonian gravitational constant [42]. The effective gravitational constants at a galactic scale is the same as the effective one in the Friedmann equation with $m_{\rm eff}^2 \gg \Lambda_g$, which we assume to explain dark matter as well as dark energy. Since the difference between the effective

gravitational constant in the Friedmann equation and the Newtonian one should not be so large from the cosmic microwave background observation [43], we find a constraint such that

$$\frac{m_g^2}{m_f^2} = \frac{K^2\kappa_g^2}{\kappa_f^2} \ll 1. \qquad (4.22)$$

Now, we return to the Poisson equations, (4.12) and (4.13), and study numerically whether f matter can provide a flat rotation curve of a galaxy in the g world or not. We assume the dark matter halo is composed of only f matter, and the distribution of ρ_f is assumed to be proportional to r^{-2} around the galactic disk. Although the rotation curve is sensitive to the matter distributions of ρ_f as well as ρ_g, for simplicity, we assume a spherically symmetric matter distribution as

$$\rho_g(r) = \rho_g(0)\exp[-r/r_{\rm gal}],$$

$$\rho_f(r) = \frac{\rho_f(0)}{1 + (r/r_{\rm halo})^2}. \qquad (4.23)$$

We show the resulting rotation curves for several values of $m_{\rm eff}$ in Fig. 1. The rotation velocity V is evaluated as $V^2 = rd\Phi_g/dr$. We find a flat rotation curve if $m_{\rm eff}^{-1} \sim$ kpc. Note that since

$$m_{\rm eff}^{-1} = 6.4 \times \left(\frac{m_{\rm eff}}{10^{-33}\ {\rm eV}}\right)^{-1}\ {\rm Gpc}, \qquad (4.24)$$

we have the solid limit on the Compton wave length as
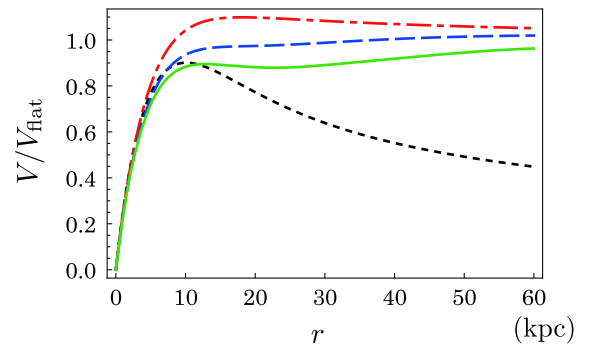
$$m_{\rm eff}^{-1} > 0.091\ {\rm pc} \qquad (4.25)$$

FIG. 1 (color online). The rotation curve in the g worlds. We plot three cases of $m_{\rm eff}^{-1} = 5$ (the red dashed-dotted curve), 10 (the blue dashed curve), and 15 kpc (the green solid curve). Matter distributions are given by $\rho_g(r) = \rho_g(0)\exp[-r/r_{\rm gal}]$, $\rho_f(r) = \rho_f(0)(1 + (r/r_{\rm halo})^2)^{-1}$, where we set $r_{\rm gal} = r_{\rm halo} = 3$ kpc and $\rho_g(0) = \rho_f(0)$. The effective gravitational constant is $G_{\rm eff}/G = 0.961538$ ($m_g/m_f = 0.2$). The black dotted curve is the rotation curve without f matter.

from the experimental constraint on the graviton mass [2–4].

We then conclude that the f matter behaves as dark matter in the g world if the Compton wavelength of the massive graviton is less than a galaxy scale such as $m_{\text{eff}}^{-1} \sim 1$ kpc. When the mass becomes lighter, then the rotation velocity decreases. It is due to a "repulsive force" induced by the massive mode because the Compton wavelength becomes larger. Note that in the shorter range than $r \sim 10$ the rotational velocity with the f matter (the green curve) is smaller than that without the f matter (the black dotted curve), which is the evidence that the f matter acts as a repulsive force.

To justify the above analysis, we have to evaluate the Vainshtein radius below which the linear approximation is broken. Performing the same method as Ref. [42], we find the linear perturbation analysis for a spherically symmetric system is valid only when

$$m_{\text{eff}}^2 \gg \frac{GM_-(r)}{r^3}, \qquad (4.26)$$

where

$$GM_-(r) := \left| G \int_0^r 4\pi \tilde{r}^2 \rho_g(\tilde{r}) d\tilde{r} - K^2 \mathcal{G} \int_0^r 4\pi \tilde{r}^2 \rho_f(\tilde{r}) d\tilde{r} \right|. \qquad (4.27)$$

Here, we have ignored a cosmological constant. The mass of the galaxy is dominated by the dark matter component, and we have the constraint (4.22); we find $K^2 \mathcal{G} \mathcal{M}_f \gg GM_g$, where M_g and $\mathcal{M}_f$ are total masses of the g- and f-matter fluids, respectively. Hence, the right-hand side is bounded from the above as

$$GM_-(r) \leq K^2 \mathcal{G} \mathcal{M}_f.$$

As a result, we conclude that the linear perturbation analysis is valid for

$$r \gg r_V := \left(\frac{K^2 \mathcal{G} \mathcal{M}_f}{m_{\text{eff}}^2} \right)^{1/3}. \qquad (4.28)$$

From Eq. (4.20), we find the effective mass of a galaxy in the g world is

$$M_{\text{gal}} \approx \frac{m_f^2}{m_{\text{eff}}^2} K^4 \mathcal{M}_f. \qquad (4.29)$$

For $M_{\text{gal}} \sim 10^{12} M_\odot$, we can evaluate the Vainshtein radius as

$$r_V \sim 0.04 \text{ kpc} \left(\frac{m_{\text{eff}}^{-1}}{1 \text{ kpc}} \right)^{2/3} \left(\frac{1}{1 - G_{\text{eff}}/G} \right)^{1/3}. \qquad (4.30)$$

It guarantees that the linear perturbation approximation is valid in a galactic scale if $m_{\text{eff}}^{-1} \lesssim$ kpc.

Such a galactic scale graviton mass as well as a cosmological constant to explain dark energy can be obtained if the ratio of two gravitational constants is given by $\kappa_f^2/\kappa_g^2 \sim 10^{12}$ as shown in Appendix A. However, the linear perturbation approximation may not be justified because we find

$$r_V \sim 0.4 \text{ Mpc} \times \left(\frac{m_{\text{eff}}^{-1}}{1 \text{ kpc}} \right)^{2/3} K^{-3/2}, \qquad (4.31)$$

by using $(1 - G_{\text{eff}}/G)^{-1} = 1 + K^{-2} \kappa_f^2/\kappa_g^2$. It may give the larger Vainshtein radius such as $r_V \sim 1$ Mpc unless $K \gg O(1)$. We may have to fine-tune the coupling constants $\{b_i\}$ as shown in Appendix A.

C. Cosmic structure formation

Finally, we discuss the evolution of cosmological density perturbations based on the linear perturbation theory [44]. The basic equations for scalar-mode perturbations are summarized in Appendix B. For simplicity, we assume that the background flat FLRW spacetimes are given by the homothetic solution. In this subsection, we calculate the evolution of the density perturbations δ_g and δ_f in the g and f worlds, respectively. Those perturbations are given by the linear combinations of δ_+ and δ_-, which are the density perturbations in the equations for the massless and the massive graviton modes, respectively. Since two modes are decoupled, we first solve numerically the perturbation equations for the massive graviton mode.

1. Numerical solutions

Since we are interested in formation of galaxies, we discuss only subhorizon scale perturbations, $a/k \ll H^{-1}$. In this subsection, we first analyze the linear perturbation equations numerically. We assume that the matter component is dominated by nonrelativistic matter ($w = 0$). Since there is another scale of length, i.e., the Compton wavelength of the massive graviton m_{eff}^{-1}, we can classify those three scales into three possibilities:

Case (a) $a/k \ll H^{-1} \ll m_{\text{eff}}^{-1}$,
Case (b) $a/k \ll m_{\text{eff}}^{-1} \ll H^{-1}$,
Case (c) $m_{\text{eff}}^{-1} \ll a/k \ll H^{-1}$.

Assuming the initial data at the decoupling time are given in each case, we solve numerically the perturbation equations (B25)–(B30) for the massive graviton mode. We show the results for one metric component $\beta_-^{(L)}$ and the density perturbation δ_- in Fig. 2, where we have chosen the initial data as (a) $a_{\text{in}}/k = 10^{-4} \times m_{\text{eff}}^{-1}$, $H_{\text{in}}^{-1} = 10^{-2} \times m_{\text{eff}}^{-1}$, (b) $a_{\text{in}}/k = 10^{-2} \times m_{\text{eff}}^{-1}$, $H_{\text{in}}^{-1} = 10^2 \times m_{\text{eff}}^{-1}$, and (c) $a_{\text{in}}/k = 10^2 \times m_{\text{eff}}^{-1}$, $H_{\text{in}}^{-1} = 10^4 \times m_{\text{eff}}^{-1}$. In the calculation, we have ignored the terms with the sound
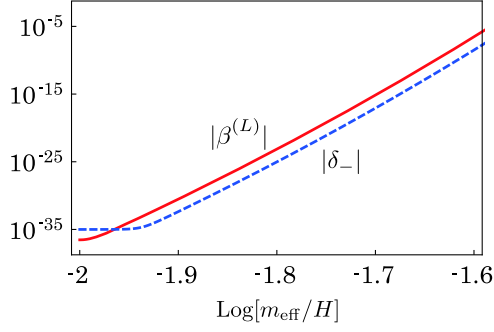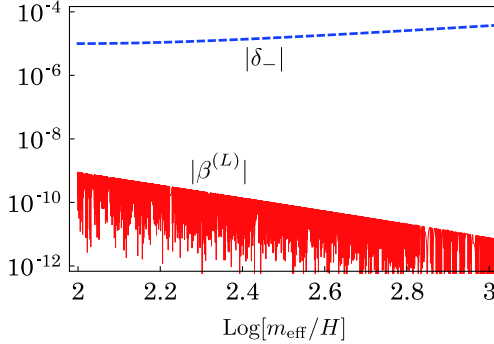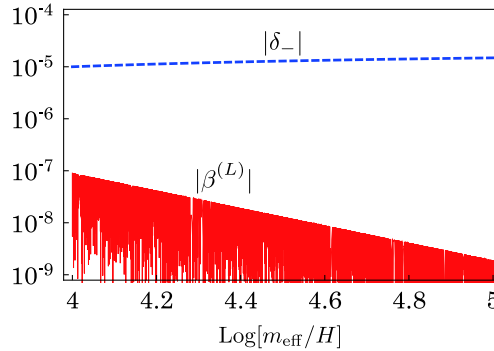
(a) $a_{\rm in}/k \ll H_{\rm in}^{-1} \ll m_{\rm eff}^{-1}$

(b) $a_{\rm in}/k \ll m_{\rm eff}^{-1} \ll H_{\rm in}^{-1}$

(c) $m_{\rm eff}^{-1} \ll a_{\rm in}/k \ll H_{\rm in}^{-1}$

FIG. 2 (color online).   The time evolution of $\beta_-^{(L)}$ and δ_-. The background spacetime is the dust-dominant universe ($a \propto t^{2/3}$). We choose the initial data (a) $a_{\rm in}/k = 10^{-4} \times m_{\rm eff}^{-1}$, $H_{\rm in}^{-1} = 10^{-2} \times m_{\rm eff}^{-1}$, (b) $a_{\rm in}/k = 10^{-2} \times m_{\rm eff}^{-1}$, $H_{\rm in}^{-1} = 10^{2} \times m_{\rm eff}^{-1}$, and (c) $a_{\rm in}/k = 10^{2} \times m_{\rm eff}^{-1}$, $H_{\rm in}^{-1} = 10^{4} \times m_{\rm eff}^{-1}$. The perturbations grow exponentially for (a). For (b) and (c), the metric perturbation $\beta_-^{(L)}$ decays with oscillations, while the density perturbation δ_- increases slowly without oscillation.

speed because we consider the perturbations larger than the Jeans length, i.e., $k \ll k_J = a\sqrt{4\pi G \bar\rho}/c_s$.

For case (a), both perturbation variables ($\beta_-^{(L)}, \delta_-$) grow exponentially. Hence, the linear perturbation is unstable. On the other hand, for cases (b) and (c), the metric perturbation $\beta_-^{(L)}$ decays with oscillations, of which the frequency is about $\sqrt{(k/a)^2 + m_{\rm eff}^2}$, while the density

perturbation δ_- increases monotonically without oscillations. The increase rates are evaluated numerically by power-law functions of the scale factor a as $\delta_- \propto a^{1.176}$ and $a^{0.1077}$ for (b) and (c), respectively.

The Compton wavelength $m_{\rm eff}^{-1}$ is larger than the horizon scale H^{-1} for (a), while the relation is opposite for (b) and (c). Hence, the above result concludes that if $m_{\rm eff}^{-1} > H^{-1}$ [case (a)] the perturbative approach is no longer valid. Note that it was shown that in the bigravity theory there exists a gradient instability against linear cosmological perturbations in the massless limit [45–47]. The nonlinear effect must be taken into account.

When $m_{\rm eff}^{-1} < H^{-1}$ [cases (b) and (c)], there are two important time scales: One is the Hubble expansion time H^{-1}, and the other is the oscillation time scale of the massive graviton $m_{\rm eff}^{-1}$. We find that the metric variables $\{\alpha_-, \beta_-^{(L)}, h_-^{(L)}, h_-^{(T)}\}$ are divided into two parts: the monotonically growing part and the oscillating part. The former part changes in the Hubble expansion time H^{-1}, while the latter part with the high frequency $\sqrt{(k/a)^2 + m_{\rm eff}^2}$ is always decaying. The metric component $\beta_-^{(L)}$ has no former part and then eventually vanishes as shown in Fig. 2. On the other hand, the matter perturbations $\{\delta_-, v_-^{(L)}\}$ grow slowly in the Hubble time scale H^{-1} without oscillation.

As a result, all variables asymptotically approach monotonic functions increasing in the Hubble time scale H^{-1}. There seems to exist an asymptotic solution that changes monotonically in the Hubble time scale H^{-1}. We then assume that the perturbation variables change in the Hubble time scale H^{-1}, i.e., $|\dot X_-| \sim |HX_-|$, which provides the above asymptotic solution. We call such an approach an adiabatic potential approximation [48], since we ignore the oscillation parts of the metric that correspond to the scalar gravitational waves.

2. Adiabatic potential approximation

Under the adiabatic potential approximation, we look for a solution for subhorizon scale perturbations. From the perturbation equations for the massive mode, (B25), (B29), and (B30), we find

$$-\left(2\frac{k^2}{a^2} + 3m_{\rm eff}^2\right)\alpha_- = \kappa_g^2 \bar\rho_g \delta_- + 3m_{\rm eff}^2 h_-^{(L)}, \quad (4.32)$$

$$\beta_-^{(L)} = 0, \quad h_-^{(T)} = -3\left(\frac{\alpha_-}{2} + h_-^{(L)}\right). \quad (4.33)$$

Substituting (B31) into (4.32), we obtain

$$-\left(\frac{k^2}{a^2} + m_{\rm eff}^2\right)\alpha_- = \frac{4}{3} \times \frac{\kappa_g^2 \bar\rho_g}{2}\delta_-, \quad (4.34)$$

where we have ignored a cosmological constant compared with the graviton mass term because we are interested in the case with a rather large value of m_{eff}. This equation is interpreted as the massive Poisson equation. The factor $4/3$ comes from the vDVZ discontinuity. Using Eq. (4.34) and ignoring the sound velocity term, the equation for the density perturbation δ_- is described as

$$\ddot{\delta}_- + 2H\dot{\delta}_- - \frac{4k^2/a^2}{3(k^2/a^2 + m_{\text{eff}}^2)}\frac{\kappa_g^2\bar{\rho}_g}{2}\delta_- = 0. \qquad (4.35)$$

As we showed numerically, the solution of this equation is found as an attractor for generic initial data if $m_{\text{eff}}^{-1} < H^{-1}$ is satisfied initially. However, the condition $m_{\text{eff}}^{-1} < H^{-1}$ is not always true. In fact, when we go back to the past, since $H^{-1} \sim t$, then the condition is broken in the past epoch.

When we start from the epoch of $m_{\text{eff}}^{-1} > H^{-1}$, which corresponds to the case (a), the linear perturbation is unstable, and then nonlinear effect must be taken into account. We can see this fact from the constraint equation. For the small scale such that $a/k \ll m_{\text{eff}}^{-1}(< H^{-1})$, we find

$$[3m_{\text{eff}}^2 - 2\Lambda_g - (1-w)\kappa_g^2\bar{\rho}_g]h_-^{(L)} = \frac{\kappa_g^2\bar{\rho}_g}{3}(\delta_- - 3w\pi_-^{(L)})$$
$$(4.36)$$

from (B31). Note that $(3m_{\text{eff}}^2 - 2\Lambda_g)$ is a positive constant if the Higuchi bound is satisfied, while $-(1-w)\kappa_g^2\bar{\rho}_g$ for the ordinary matter is negative definite, and its magnitude decreases in time. Hence, the coefficient of the left-hand side of (4.36) eventually vanishes when we go back to the past, while the right-hand side does not usually vanish simultaneously. It indicates that the linear perturbation approximation is broken at the time when the coefficient of the left-hand side vanishes because $h_-^{(L)}$ must diverge. In this epoch, to answer the question of whether there still exists an adiabatic potential solution as an attractor, we have to analyze the full dynamical equations with inhomogeneities, which is quite difficult without heavy numerical simulation. However, there is some hope from Eq. (4.36), which shows a possibility that the density perturbation is still small enough to be treated as linear perturbation even when the metric perturbations become nonlinear. In a spherically symmetric case, we find an adiabatic potential solution with nonlinear metric perturbations but with linear matter perturbations [49]. In this solution, we claim that the Vainshtein mechanism is working even in a cosmological context, and the solution can be described by GR.

Hence, we may conceive the following scenario, although the present analysis is based on the perturbations around a homothetic solution, and an extended analysis with more generic background such as that in Ref. [46] will be required. In the early stage of the Universe, because of the Vainshtein mechanism, gravity is described by GR, and
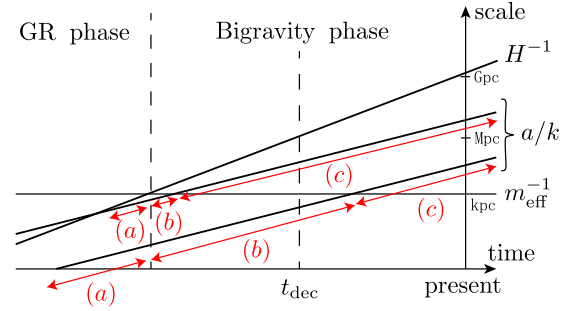

FIG. 3 (color online). The schematic diagram of the growth history. In the early stage of the Universe ($H^{-1} < m_{\text{eff}}^{-1}$), because of the Vainshtein mechanism, the standard big bang universe is recovered. However, the Universe eventually evolves into the bigravity phase, in which there are two cases (b) and (c) depending on the perturbation scale compared with m_{eff}^{-1}.

then the standard big bang scenario is found. However, the Universe eventually evolves into the bigravity phase at $H^{-1} \sim m_{\text{eff}}^{-1}$ as shown in Fig. 3. When the Universe reaches the decoupling time, we find the case (b) or (c) for the perturbations, in which the adiabatic potential approximation becomes valid as an attractor. Hence, we analyze whether the f matter can be dark matter in the cosmic structure formation, using the above adiabatic potential approximation.

3. Growth history of density perturbation

The evolution equation of density perturbation for the massless mode in a subhorizon scale is given from Eq. (B24) as

$$\ddot{\delta}_+ + 2H\dot{\delta}_+ - \frac{\kappa_g^2\bar{\rho}_g}{2}\delta_+ = 0, \qquad (4.37)$$

where we have ignored a cosmological constant and the term with a sound velocity as before. This equation for δ_+ is the same as that in GR. On the other hand, as found in Eq. (4.35), the evolution of the massive-mode variable δ_- depends on the Compton wavelength of the massive graviton as well as the scale of the perturbations.

From Eqs. (4.35) and (4.37) for δ_+ and δ_-, we obtain the equations for the physical density perturbations (δ_g and δ_f) as

$$\ddot{\delta}_g + 2H\dot{\delta}_g - 4\pi G_{\text{eff}}(\bar{\rho}_g\delta_g + \bar{\rho}_D\delta_D) = 0,$$
$$\ddot{\delta}_f + 2H\dot{\delta}_f - 4\pi \mathcal{G}_{\text{eff}}(\bar{\rho}_f\delta_f + \bar{\rho}_G\delta_G) = 0, \qquad (4.38)$$

where

$$G_{\text{eff}} = G\frac{m_f^2}{m_{\text{eff}}^2}\left(1 + \frac{m_g^2}{m_f^2}F\right), \qquad (4.39)$$

$$\bar{\rho}_D = K^4 \bar{\rho}_f, \qquad (4.40)$$

$$\delta_D = \frac{1 - F}{1 + \frac{m_g^2}{m_f^2} F} \delta_f, \qquad (4.41)$$

and

$$\mathcal{G}_{\text{eff}} = \mathcal{G} \frac{K^2 m_g^2}{m_{\text{eff}}^2} \left(1 + \frac{m_f^2}{m_g^2} F \right), \qquad (4.42)$$

$$\bar{\rho}_G = K^{-4} \bar{\rho}_g, \qquad (4.43)$$

$$\delta_G = \frac{1 - F}{1 + \frac{m_f^2}{m_g^2} F} \delta_g, \qquad (4.44)$$

with

$$F := \frac{4 m_{\text{eff}}^{-2}}{3 (m_{\text{eff}}^{-2} + a^2/k^2)}. \qquad (4.45)$$

Beyond the Compton wavelength of the massive graviton, the effective gravitational constant becomes $\mathcal{G}_{\text{eff}}/\mathcal{G} \approx m_f^2/m_{\text{eff}}^2$. It is the same not only as the cosmological value but also as the local one if the graviton mass is large ($m_{\text{eff}}^2 \gg \Lambda_g$). The perturbation of the dark matter component coincides with that of the f matter, i.e.,

$$\delta_D \approx \delta_f, \qquad (4.46)$$

for $a/k \gg m_{\text{eff}}^{-1}$. Therefore, the f-matter perturbation behaves as the dark matter component in the g world as in Secs. IV A and IV B.

Inside the Compton wavelength, the f matter acts as a repulsive force as shown in Sec. IV B. In the present case, it can be seen explicitly from the relation

$$\delta_D \sim -\frac{1}{3 + 4 \frac{m_g^2}{m_f^2}} \delta_f \qquad (4.47)$$

for $a/k \ll m_{\text{eff}}^{-1}$. It indicates that the g matter accumulates in a low-density region of the f matter.

We show the numerical result of the evolution of density perturbations for two different scales [$k^{-1} = 10$ Mpc and 100 kpc at the present ($a = 1$)] in Fig. 4. We assume $\delta_g = 10^{-5}$ and $\delta_f = 10^{-1}$ at the decoupling time ($a = 10^{-3}$). For the large-scale perturbation, its scale is always larger than the Compton wavelength after the decoupling time. Hence, the f matter plays the role of dark matter in the g world and helps small baryon perturbation δ_g to grow rapidly as shown in Fig. 4(a). The evolution of δ_g is similar to the growth of density perturbations with CDM in GR.

(a) $k^{-1} = 10$ Mpc
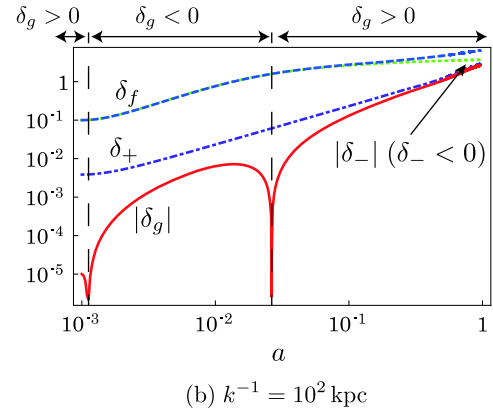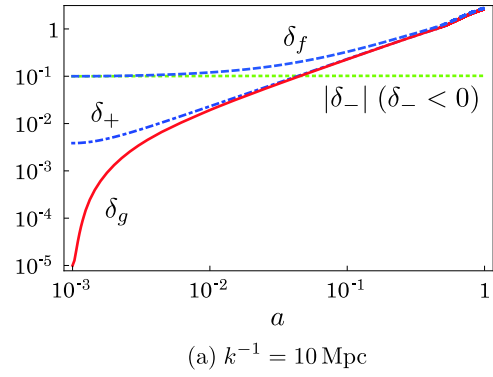

(b) $k^{-1} = 10^2$ kpc

FIG. 4 (color online). The evolution of density perturbations for two scales [(a) 10 Mpc and (b) 100 kpc at the present ($a = 1$)]. We assume $\delta_g = 10^{-5}$ and $\delta_f = 10^{-1}$ at the decoupling time ($a = 10^{-3}$). The blue dashed curve indicates the evolution of δ_f, while the red solid curve indicates that of δ_g. The δ_+ and δ_- are the source terms for the massless graviton mode and the massive graviton mode, respectively. We set $m_{\text{eff}}^{-1} = 1$ kpc and $m_g/m_f = 0.2$. The background spacetime is the dust-dominant universe ($a \propto t^{2/3}$).

On the other hand, for the small scale perturbation, its scale is shorter than the Compton wavelength at the decoupling time. During the period of $a/k < m_{\text{eff}}^{-1}$, the f matter acts as a repulsive source in the g world. Then, the evolution of δ_g is quite different due to the appearance of a repulsive force by the f-matter perturbations as shown in Fig. 4(b). δ_g changes its sign and then decreases to a negative value in the early stage. But the perturbation scale eventually exceeds m_{eff}^{-1} as the scale factor increases. In fact, the perturbation scale becomes larger than the Compton wavelength after $a = k/\sqrt{3} \times m_{\text{eff}}^{-1}$, when δ_D changes its sign. After then, the f matter begins to act as dark matter. As shown in Fig. 4(b), δ_g changes its sign again to be positive. δ_g then grow into a nonlinear regime via large density perturbations of the f-matter fluid.

We set $m_g/m_f = 0.2$, which satisfies the constraint (4.22). From Eq. (B19), we find that the perturbations of the g variables are dominated by the massless mode, while those of the f variables have a significant influence by the

massive mode. Since the massive mode can grow only when $a/k \ll m_{\text{eff}}^{-1}$, δ_f grows first, and then δ_g follows as shown in Fig. 4(b). On the other hand, as shown in Fig. 4(a), δ_f cannot grow at first because the massive mode cannot grow for $a/k \gg m_{\text{eff}}^{-1}$. δ_f starts to grow after the perturbation of the massless mode catches up to that of the massive mode. δ_g grows rapidly by the increase of the massless mode even when δ_f does not grow.

We conclude that the cosmic structure formation can also be explained by another one of the twin matter fluids.

V. CONCLUDING REMARKS

We have studied a possibility to explain a dark matter component by another one of the twin matter fluids in the ghost-free bigravity theory. We have analyzed from a galactic scale to a cosmological scale. If we assume the Compton wavelength of the massive graviton is shorter than a galactic scale, i.e., a graviton mass is rather heavy ($m_{\text{eff}} \sim 10^{-27}$ eV), we find a dark matter component can be explained by another twin matter for all scales. For such a model, our matter field consists just of baryons. The origin of the dark matter field is another matter field that couples only to another metric.

For such a model, at first glance, it seems that the cosmic acceleration cannot be explained by the interaction term because the expected cosmological constant is also large. As shown in Appendix A (see Table I), however, we can find a large graviton mass and a small cosmological constant although we need a fine-tuning of the coupling parameters. For such fine-tuned coupling parameters, the ghost-free bigravity theory could explain dark matter as well as dark energy.

Our analysis is valid only for the late stage of the Universe because the background space is assumed to be homothetic. To find a whole history of the Universe, we have to analyze more generic background spacetime with perturbations. We also have to show that the Vainshtein mechanism does really work in the early stage of the Universe as we conceived. Those are in progress.

Our result shows the graviton mass is phenomenologically significant. The bigravity theory can explain only dark energy for $m_{\text{eff}}^{-1} \sim$ Gpc, while if $m_{\text{eff}}^{-1} \lesssim$ kpc it could explain dark matter (as well as dark energy). Therefore, an important remaining question is how large graviton mass is possible from the theoretical and observational points of

view. From the theoretical point of view, we should start from more fundamental theory in which a bigravity theory is reduced as a low-energy effective theory [50–52]. We hope that the hierarchy between the graviton mass and an effective cosmological constant to explain both dark sectors will be solved in such a fundamental theory. From the observational point of view, the evidence of a graviton mass could be detected by gravitational waves [3,53]. Furthermore, comparing some bigravity phenomena with the observational data in a galactic scale as well as in a cosmological scale, we may find the constraint of the graviton mass (e.g., gravitational lensing by galaxies [54]). To clarify whether the graviton really has a mass and gives a constraint on its value, further studies are required.

APPENDIX A: EVALUATION OF THE EFFECTIVE COSMOLOGICAL CONSTANT AND THE GRAVITON MASS

The effective cosmological constant and the graviton mass are given by (2.14) and (3.14), which contain many unknown or unfixed values of coupling constants. To discuss the evolution of the Universe, we have first to evaluate the values of the graviton mass and the cosmological constant for given coupling constants.

For this purpose, it is more convenient to introduce another set of coupling constants $\{c_k\}(k = 0, 1, \ldots, 4)$ by rewriting the interaction term in terms of another tensor defined by $\mathcal{K}_\nu^\mu = \delta_\nu^\mu - \gamma^\mu{}_\nu$ as

$$\mathcal{U}(g, f) = \sum_{k=0}^{4} c_k \mathcal{U}_k(\mathcal{K}). \tag{A1}$$

The relations between $\{b_k\}$ and $\{c_k\}$ are given by

$$c_0 = b_0 + 4b_1 + 6b_2 + 4b_3 + b_4,$$
$$c_1 = -(b_1 + 3b_2 + 3b_3 + b_4),$$
$$c_2 = b_2 + 2b_3 + b_4,$$
$$c_3 = -(b_3 + b_4),$$
$$c_4 = b_4. \tag{A2}$$

We assume that a flat Minkowski spacetime exists in the present bigravity model. Then, we impose the following conditions:

TABLE I. The ratios of the cosmological constant to the graviton mass square. We assume $c_3 = -1$.

κ_g^2/κ_f^2	$2c_3^2 + 3c_4$	K_{dS}	$\Lambda_g/m_{\text{eff}}^2$	m_g^2/m_f^2
1	1	5.08	0.0815	25.8
10^{-12}	1	8.85	5.11×10^{-11}	7.84×10^{-11}
1	10^{-12}	4.00	9.34×10^{-14}	16.0
10^{-6}	10^{-6}	4.00	8.10×10^{-11}	1.60×10^{-5}

$$c_0 = c_1 = 0. \tag{A3}$$

If m is assumed to be the graviton mass in the Minkowski background in the massive gravity limit, we should set

$$c_2 = -1. \tag{A4}$$

As a result, $\{b_k\}$ are described by two free coupling constants c_3 and c_4 as

$$
\begin{aligned}
b_0 &= 4c_3 + c_4 - 6, \\
b_1 &= 3 - 3c_3 - c_4, \\
b_2 &= 2c_3 + c_4 - 1, \\
b_3 &= -(c_3 + c_4), \\
b_4 &= c_4.
\end{aligned}
\tag{A5}
$$

These coupling constants guarantee

$$m_{\text{eff}}^2|_{K=1} = m^2, \qquad \Lambda_g|_{K=1} = \Lambda_f|_{K=1} = 0, \tag{A6}$$

for the Minkowski background with $K = 1$.

To explain dark energy, de Sitter spacetime must be an attractor solution. As shown in Refs. [24,55], the quartic equation (2.17) gives one de Sitter solution with $K = K_{\text{dS}}$ as well as two anti-de Sitter solutions if

$$2c_3^2 + 3c_4 > 0. \tag{A7}$$

Since the Higuchi bound must be satisfied [56], the lower bound of the graviton mass is given by the cosmological constant as

$$m_{\text{eff}}^2 > \frac{2}{3}\Lambda_g.$$

If we consider a simple and natural case, i.e., $b_k \sim O(1)$ (or $c_k \sim O(1)$) and $\kappa_g \sim \kappa_f$, we find the cosmological constant and the graviton mass as

$$\Lambda_g \sim m^2$$

$$m_{\text{eff}} \sim m \tag{A8}$$

for $K = K_{\text{dS}}$, assuming no fine-tuning of the coupling constants.

In this case, dark energy fixes the value of Λ_g, and then m^{-1} (the Compton wavelength) must be the cosmological horizon scale H^{-1}. As a result, the massive mode becomes important for a subhorizon scale such as a galaxy. In this case, the f matter does not explain dark matter because it is in the GR phase. For the f matter to be dark matter, $m_{\text{eff}} \sim m$ is too light. As we show in Sec. IV, if $m_{\text{eff}} \sim 1$ kpc, the f matter can play the role of dark matter. However, in that case, Λ_g is too large to explain the cosmic

acceleration, except for the $K = 0$ branch with a different origin of dark energy.

Is there any possibility such that $\Lambda_g \sim H^{-1}$ but $m_{\text{eff}} \sim 1$ kpc? We then look for the possibility of a heavy graviton mass, i.e., $\Lambda_g \ll m_{\text{eff}}^2$. One way to get a heavy graviton mass as well as a small cosmological constant is to assume $\kappa_g^2 \gg \kappa_f^2$ or $\kappa_f^2 \gg \kappa_g^2$. If we have such a hierarchy between two gravitational constants, we find $\Lambda_g \ll m_{\text{eff}}^2$ without fine-tuning of coupling constants $\{c_i\}$. Otherwise, we have to fine-tune the coupling constants. Fine-tuning the coupling constants such that

$$0 < 2c_3^2 + 3c_4 \ll 1,$$

we find a small effective cosmological constant ($\Lambda_g \ll m_{\text{eff}}^2 \sim m^2$). We show some examples in Table I.

As a result, although the graviton mass square and the cosmological constant are ordinarily the same as $m_{\text{eff}}^2 \sim \Lambda_g$, it is possible to find a much heavier graviton mass compared with the observed cosmological constant.

APPENDIX B: COSMOLOGICAL LINEAR PERTURBATIONS

In this Appendix, we shortly summarize the linear perturbations of a flat FLRW universe in the bigravity theory [44]. Just for simplicity, we assume that the background metrics are given by the homothetic flat FLRW spacetimes. The detail analysis for more generic background spacetimes including vector and tensor modes was discussed in Ref. [46].

The background homothetic flat FLRW spacetimes are given by

$$\overset{(0)}{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{B1}$$

$$\overset{(0)}{f}_{\mu\nu} = K^2 \overset{(0)}{g}_{\mu\nu}. \tag{B2}$$

This background solution is determined by the standard Friedmann equation with a cosmological constant, and the following constraints must be satisfied:

$$\kappa_g^2 \overset{(0)}{T}{}^{[\text{m}]\mu}{}_\nu = K^2 \kappa_f^2 \overset{(0)}{\mathcal{T}}{}^{[\text{m}]\mu}{}_\nu, \tag{B3}$$

$$\Lambda_g = K^2 \Lambda_f. \tag{B4}$$

We then consider the adiabatic scalar perturbations and ignore an anisotropic stress. The perturbed metrics are expressed as

$$g_{00} = -(1 + 2\alpha_g Y),$$

$$g_{0i} = -a\beta_g^{(L)} Y_i,$$

$$g_{ij} = a^2(\delta_{ij} + 2h_g^{(L)}\delta_{ij}Y + 2h_g^{(T)}Y_{ij}), \quad \text{(B5)}$$

$$f_{00} = -K^2(1 + 2\alpha_f Y),$$

$$f_{0i} = -K^2 a\beta_f^{(L)} Y_i,$$

$$f_{ij} = K^2 a^2(\delta_{ij} + 2h_f^{(L)}\delta_{ij}Y + 2h_f^{(T)}Y_{ij}), \quad \text{(B6)}$$

while the perturbed energy-momentum tensors are given by

$$T^0{}_0 = -\bar{\rho}_g(1 + \delta_g),$$

$$T^0{}_i = a(\bar{\rho}_g + \bar{P}_g)(v_g^{(L)} - \beta_g^{(L)})Y_i,$$

$$T^i{}_0 = -a^{-1}(\bar{\rho}_g + \bar{P}_g)v_g^{(L)}Y^i,$$

$$T^i{}_j = P_g(\delta^i{}_j + \pi_g^{(L)}\delta^i{}_j), \quad \text{(B7)}$$

$$\mathcal{T}^0{}_0 = -\bar{\rho}_f(1 + \delta_f),$$

$$\mathcal{T}^0{}_i = a(\bar{\rho}_f + \bar{P}_f)(v_f^{(L)} - \beta_f^{(L)})Y_i,$$

$$\mathcal{T}^i{}_0 = -a^{-1}(\bar{\rho}_f + \bar{P}_f)v_f^{(L)}Y^i,$$

$$\mathcal{T}^i{}_j = P_f(\delta^i{}_j + \pi_f^{(L)}\delta^i{}_j), \quad \text{(B8)}$$

where the scalar harmonic function Y is defined by

$$(\Delta + k^2)Y = 0, \quad \text{(B9)}$$

with $-k^2$ being an eigenvalue of the usual three-dimensional Laplacian operator Δ, and its vector and tensor harmonic functions are defined by

$$Y_i = -k^{-1}\partial_i Y,$$

$$Y_{ij} = k^{-2}\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^a\partial_a\right)Y, \quad \text{(B10)}$$

respectively. The perturbation variables $\{\alpha_{g/f}, \beta_{g/f}^{(L)}, h_{g/f}^{(L)}, h_{g/f}^{(T)}\}$ and $\{\delta_{g/f}, v_{g/f}^{(L)}, \pi_{g/f}^{(L)}\}$ depend only on time. The unperturbed energy densities and pressures, $\{\bar{\rho}_{g/f}, \bar{P}_{g/f}\}$, must satisfy

$$\kappa_g^2\bar{\rho}_g = K^2\kappa_f^2\bar{\rho}_f, \quad \kappa_g^2\bar{P}_g = K^2\kappa_f^2\bar{P}_f. \quad \text{(B11)}$$

For the perturbation variables in the g world, we can define the gauge-invariant variables as in GR,

$$\Phi_g = \alpha_g - \dot{\sigma}_g^{(L)},$$

$$\Psi_g = \mathcal{R}_g - H\sigma_g^{(L)},$$

$$\Delta_g = \delta_g + 3(1 + w)\frac{a}{k}H(\beta_g^{(L)} - v_g^{(L)})_g,$$

$$V_g = v_g^{(L)} + \frac{a}{k}\dot{h}_g^{(T)}, \quad \text{(B12)}$$

where

$$w = \bar{P}_g/\bar{\rho}_g, \quad c_s^2 = \dot{\bar{P}}_g/\dot{\bar{\rho}}_g. \quad \text{(B13)}$$

$\mathcal{R}_g$ and σ_g are the curvature and the shear perturbations, respectively, which are defined by

$$\mathcal{R}_g = h_g^{(L)} + \frac{1}{3}h_g^{(T)}, \quad \text{(B14)}$$

$$\sigma_g^{(L)} = \frac{a^2}{k^2}\dot{h}_g^{(T)} - \frac{a}{k}\beta_g^{(L)}. \quad \text{(B15)}$$

Similarly, we introduce the gauge-invariant variables in the f world, which are defined by those with the subscript f. We note w and c_s^2 coincide in the g and f worlds because of (B11).

The massless and massive mode perturbations, X_+ and X_-, are described by the linear combination of the perturbed variables in the g and f worlds, X_g and X_f, as

$$X_+ = \frac{m_f^2}{m_{\text{eff}}^2}X_g + \frac{m_g^2}{m_{\text{eff}}^2}X_f, \quad \text{(B16)}$$

$$X_- = X_g - X_f, \quad \text{(B17)}$$

or inversely

$$X_g = X_+ + \frac{m_g^2}{m_{\text{eff}}^2}X_-, \quad \text{(B18)}$$

$$X_f = X_+ - \frac{m_f^2}{m_{\text{eff}}^2}X_-. \quad \text{(B19)}$$

For the massless mode, there are four independent equations,

$$-\frac{k^2}{a^2}\Phi_+ = \frac{\kappa_g^2\bar{\rho}_g}{2}\Delta_+ \quad \text{(B20)}$$

$$\Phi_+ + \Psi_+ = 0, \quad \text{(B21)}$$

$$\dot{\Delta}_+ - 3wH\Delta_+ + (1 + w)\frac{k}{a}V_+ = 0, \quad \text{(B22)}$$

and

$$\dot{V}_+ + HV_+ - \frac{k}{a}\left[\frac{c_s^2 \Delta_+}{1+w} + \Phi_+\right] = 0, \qquad \text{(B23)}$$

for four perturbation variables $\{\Phi_+, \Psi_+, \Delta_+, V_+\}$.

If both background matter densities ($\bar{\rho}_g$ and $\bar{\rho}_f$) are dominated by nonrelativistic matter ($w = 0$), the equation for the density perturbation Δ_+ is given by

$$\ddot{\Delta}_+ + 2H\dot{\Delta}_+ + \left(\frac{k^2 c_s^2}{a^2} - \frac{\kappa_g^2 \rho_g}{2}\right)\Delta_+ = 0, \qquad \text{(B24)}$$

which is the same as that in GR. Then, we will not discuss it furthermore.

Unlike the massless mode, there are six independent equations of motion for the massive mode variables $\{\alpha_-, \beta_-^{(L)}, h_-^{(L)}, h_-^{(T)}, \delta_-, v_-^{(L)}\}$. By use of Φ_-, Ψ_-, which are given by the above six variables, we find the similar four equations to those of the massless mode as

$$-\frac{k^2}{a^2}\Phi_- + m_{\text{eff}}^2\left(\frac{3}{2}h_-^{(L)} + \frac{3}{4}\frac{a}{k}H\beta_-^{(L)} + h_-^{(T)}\right) = \frac{\kappa_g^2 \bar{\rho}_g}{2}\Delta_-, \qquad \text{(B25)}$$

$$\Phi_- + \Psi_- = m_{\text{eff}}^2 \frac{a^2}{k^2}h_-^{(T)}, \qquad \text{(B26)}$$

$$\dot{\Delta}_- - 3wH\Delta_- + (1+w)\frac{k}{a}V_- + \frac{3}{4}(1+w)m_{\text{eff}}^2\frac{a}{k}\beta_-^{(L)} = 0, \qquad \text{(B27)}$$

$$\dot{V}_- + HV_- - \frac{k}{a}\left[\frac{c_s^2 \Delta_-}{1+w} + \Phi_-\right] = 0, \qquad \text{(B28)}$$

in which the extra terms come from the interactions between two metrics. In addition, we have two more independent equations from (3.21) as

$$6\dot{h}_-^{(L)} + 6Hh_-^{(L)} - 6H\alpha_- + \frac{k}{a}\beta_-^{(L)} = 0, \qquad \text{(B29)}$$

$$\frac{a}{k}\left(\frac{3}{2}\dot{\beta}_-^{(L)} + 6H\beta_-^{(L)}\right) + 3\alpha_- + 6h_-^{(L)} + 2h_-^{(T)} = 0. \qquad \text{(B30)}$$

Note that, although the massive mode variables are gauge invariant in themselves, we also use Φ_-, Ψ_-, Δ_- and V_- just for the similar description to those of the massless mode.

Once the equations of state are given, since the above six dynamical equations are independent, we can solve the six variables $\{\alpha_-, \beta_-^{(L)}, h_-^{(L)}, h_-^{(T)}, \delta_-, v_-^{(L)}\}$ for given appropriate initial data.

To set up initial data, we have the additional constraint equations,

$$(3m_{\text{eff}}^2 - 2\Lambda_g)(\alpha_- + 3h_-^{(L)})$$
$$= \kappa_g^2 \bar{\rho}_g(\delta_- - 3w\pi^{(L)} - (1+3w)\alpha_- + 3(1-w)h_-^{(L)}), \qquad \text{(B31)}$$

$$-H\Phi_- + \dot{\Psi}_- = \frac{a}{k}\dot{H}V_- + \frac{1}{4}m_{\text{eff}}^2\frac{a}{k}\beta_-^{(L)}, \qquad \text{(B32)}$$

which are obtained from (3.22) and the 0-i component of the Einstein equations.

From the above basic equations, we find that the variables consist of two parts: One is an oscillating wave part, and the other is a monotonically changing part in time. As an example, we show the equation for $h_-^{(T)}$:

$$\ddot{h}_-^{(T)} + 3H\dot{h}_-^{(T)} + \left(\frac{k^2}{a^2} + m_{\text{eff}}^2\right)h_-^{(T)}$$
$$= -\frac{k^2}{a^2}(\alpha_- + 3h_-^{(L)}) + 12H(\dot{h}_-^{(L)} + Hh_-^{(L)} - H\alpha_-)$$
$$\approx -\frac{k^2}{a^2}(\alpha_- + 3h_-^{(L)})\left(\text{for } \frac{a}{k} \ll H^{-1}\right). \qquad \text{(B33)}$$

This equation naively shows that $h_-^{(T)}$ oscillates with the frequency $\omega \sim \sqrt{k^2/a^2 + m_{\text{eff}}^2}$ with the damping amplitude due to the expansion of the universe (H). Although the right-hand side may work as a source term, which could increase the amplitude, it is not the case as we show it numerically. As a result, the metric variable will approach a monotonically changing part with damping oscillations.

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