PHYSICAL REVIEW D **90,** 124078 (2014)

Mass functional for initial data in $4+1$-dimensional spacetime

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(Received 13 November 2014; published 29 December 2014)

We consider a broad class of asymptotically flat, maximal initial data sets satisfying the vacuum constraint equations, admitting two commuting rotational symmetries. We construct a mass functional for "$t-\phi^i$"-symmetric data which evaluates to the Arnowitt-Deser-Misner mass. We then show that $\mathbb{R}\times \mathrm{U}(1)^2$-invariant solutions of the vacuum Einstein equations are critical points of this functional amongst this class of data. We demonstrate the positivity of this functional for a class of rod structures which include the Myers-Perry initial data. The construction is a natural extension of Dain's mass functional to $D=5$, although several new features arise.

DOI: 10.1103/PhysRevD.90.124078 PACS numbers: 04.50.-h, 04.50.Gh, 97.60.Lf

I. INTRODUCTION

Dain has established the remarkable inequality $m \geq \sqrt{|J|}$, for complete asymptotically flat, axisymmetric maximal initial data (Σ, h, K) of the vacuum Einstein equations in four dimensions [1–4]. Here m is the Arnowitt-Deser-Misner (ADM) mass of the Riemannian manifold and J is the conserved angular momenta, defined from the existence of the U(1) isometry. This result was subsequently strengthened to a more general class of metrics, multiple asymptotic ends and weaker asymptotic falloff conditions [5]. Dain's equality is saturated if and only if (Σ, h, K) is that of a constant Boyer-Lindquist time hypersurface of the extreme Kerr black hole.

An important step in the proof of this inequality was the construction of a well-defined mass functional $\mathcal{M}$, defined for $t-\phi$-symmetric, asymptotically flat maximal initial data. $\mathcal{M}=\mathcal{M}(v,Y)$ depends on two scalar functions v and Y which can be shown to fully specify the initial data set. The proof shows that $m=\mathcal{M}(v,Y)$ for $t-\phi$-symmetric maximal initial data, and that $m\geq\mathcal{M}$ for arbitrary axisymmetric maximal data. $\mathcal{M}(v,Y)$ can be shown to be positive definite and the unique minimizer is extreme Kerr, completing the elegant argument.

It is natural to expect that an analogous inequality would hold in $D=5$ dimensions, under suitable restrictions on the initial data. The situation is particularly interesting as there are potentially two candidates for minimizers: extreme Myers-Perry black holes with S^3 horizon topology [6], and extreme black rings with $S^1\times S^2$ horizon topology. The masses of these solutions satisfy

$$M^3=\frac{27\pi}{32}(|J_1|+|J_2|)^2, \qquad \text{(Myers-Perry)} \qquad (1)$$

$$M^3=\frac{27\pi}{32}|J_1|(|J_1|+|J_2|), \qquad \text{(black ring)} \qquad (2)$$

where J_i are conserved angular momenta computed in terms of Komar integrals. Of course it is not manifestly clear how an expression which is derived from the ADM mass (i.e. evaluated at spatial infinity) would capture information on the topology of the horizon; indeed, at the level of the initial data, the horizon is a minimal surface in the interior. It is worth noting that another, related class of geometric inequalities relating the *area* of marginally outer trapped surfaces to the angular momenta (and charge) have also been established in three spatial dimensions [7,8]. Once again the geometries which uniquely saturate the bound were the horizon geometries corresponding to the extreme Kerr geometry. Recently, Hollands has derived an area-angular momenta inequality in general dimension D, for spaces admitting a $\mathrm{U}(1)^{D-3}$ action as isometries [9]. In this case, the inequality depends on the topology of the marginally outermost trapped surface.

As a first step towards establishing a mass-angular momenta inequality in five dimensions, in this work we investigate a generalization of Dain's mass functional $M(v,Y)$ to $D>4$ for maximal spatial slices of five-dimensional vacuum spacetimes with $\mathrm{U}(1)^2$ isometry. Note that most of the local analysis works equally well for D-dimensional spacetimes with $\mathrm{U}(1)^{D-3}$ isometry. However, such spacetimes could only be asymptotically flat for $D=5$ (there might be a useful extension to spaces that are asymptotically Kaluza-Klein). Hence we will focus on this case, although it will be sometimes convenient to leave D as a free parameter.

Our first goal is to construct a positive-definite functional which evaluates to the mass for a broad class of asymptotically flat, maximal initial data with "$t-\phi^i$" symmetry ($i=1\ldots D-3$). Such data can be thought of as data that is "stationary at a moment in time." In particular, it allows us to specify the extrinsic curvature in terms of $D-3$ twist

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potentials, using the construction of transverse traceless tensors given in Ref. [10]. The functional is defined over functions on the orbit space $\mathcal{B} \equiv \Sigma/\mathrm{U}(1)^{D-3}$ and depends on a matrix λ'_{ij} specifying the metric on surfaces of transitivity of the $\mathrm{U}(1)^{D-3}$ action (often called the "Gram" matrix) and $D-3$ twist potentials. Setting $D=5$, this amounts to five independent functions.

Carter has established a variational formulation of the stationary, axisymmetric vacuum Einstein equations [11]. Our main result is to demonstrate that the mass functional we have defined, when integrated over an appropriate domain, is the same as Carter's functional up to (divergent) boundary terms. Therefore $\mathbb{R}\times\mathrm{U}(1)^2$-invariant vacuum solutions arise as critical points of the mass amongst all asymptotically flat, $t-\phi^i$-symmetric initial data (see Bardeen's result [12] for the $3+1$-dimensional case). In this sense our proposed functional is an extension of Dain's functional $\mathcal{M}(v,Y)$, which also has this property. However, there are a number of important differences. As we will elaborate, our functional contains boundary terms which encode the "rod structure" of the initial data. In particular, the initial data may contain 2-cycles ("bubbles") which also contribute to the mass. The rod structure plays an important role in the black hole uniqueness theorem [13,14] in five-dimensional vacuum gravity. In the case of stationary black holes containing additional 2-cycles, the usual laws of black hole mechanics have been shown to be modified [15]. More recently, an explicit example of a black hole spacetime containing a 2-cycle in the exterior region was constructed in supergravity [16].

This paper is organized as follows. In Sec. II we introduce a broad class of $\mathrm{U}(1)^2$-invariant maximal initial data $(\Sigma, h_{ab}K_{ab})$ for the vacuum Einstein equations and discuss the particular case of $t-\phi^i$-symmetric data, which allows us to construct a general class of transverse-traceless tensors in the geometry. The resulting data is parametrized by functions on the orbit space and we discuss in detail their various boundary and asymptotic conditions that we must impose. In Sec. III we introduce a functional defined for asymptotically flat initial data which evaluates to the ADM mass and discuss some of its properties. Section IV investigates the relationship of this functional to Carter's variational formulation for stationary, axisymmetric vacuum solutions. We conclude with an argument that demonstrates the positivity of this functional for a particular class of rod structure which includes Myers-Perry black hole initial data.

II. INITIAL DATA WITH ROTATIONAL ISOMETRIES

A. Conformal data

Consider a stationary vacuum solution of the Einstein equations with the $\mathrm{U}(1)^{D-3}$ isometry group. It is a well-known result [13,17] that Frobenius' theorem and the vacuum equations imply orthogonal transitivity of the $\mathbb{R}\times\mathrm{U}(1)^{D-3}$ action, and the metric can be written in the form

$$g = G_{\alpha\beta}\mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta + g_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{3}$$

where $\mathrm{d}/\mathrm{d}\xi^\alpha$ generate the isometry group $(\alpha,\beta=0,1,2)$ and x^A are coordinates on the two-dimensional surfaces orthogonal to the surfaces of transitivity. We may write this explicitly as

$$g = -H\mathrm{d}t^2 + \frac{\lambda'_{ij}}{H^{1/N}}(\mathrm{d}\phi^i - w^i dt)(\mathrm{d}\phi^j - w^j\mathrm{d}t) + e^{2\nu}(\mathrm{d}\rho^2+\mathrm{d}z^2) \tag{4}$$

where $N=D-1$ and $\rho^2=\det\lambda'$ is harmonic on the spacetime orbit space $\tilde{\mathcal{B}}\equiv M/\mathrm{U}(1)^{D-3}$ and $\mathrm{d}z$ is the harmonic conjugate of $\mathrm{d}\rho$. Further details on the functions λ_{ij} and one-forms ω^i, and an analysis of $\tilde{\mathcal{B}}$, are given in Refs. [13,14]. Note that constant time slices in this spacetime have the induced metric $h=H^{-1/N}\tilde{h}$ where

$$\tilde{h} = \lambda'_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j + e^{2U}(\mathrm{d}\rho^2+\mathrm{d}z^2) \tag{5}$$

where $e^{2U}=e^{2\nu}H^{1/N}$. If we consider five-dimensional asymptotically flat black hole solutions, it is known that in an appropriate coordinate system, the spacetime metric takes the form [13]

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1-\frac{\mu}{R^2}+\mathcal{O}(R^{-2})\right)\mathrm{d}t^2 + \left(\frac{2\mu a_1(R^2+a_1^2)}{R^4}\sin^2\theta + \mathcal{O}(R^{-3})\right)\mathrm{d}t\mathrm{d}\phi_1 \\ &+ \left(\frac{2\mu a_2(R^2+a_2^2)}{R^4}\cos^2\theta+\mathcal{O}(R^{-3})\right)\mathrm{d}t\mathrm{d}\phi_2 + \left(1+\frac{\mu}{2R^2}+\mathcal{O}(R^{-3})\right) \\ &\times\left[\frac{R^2+a_1^2\cos^2\theta+a_2^2\sin^2\theta}{(R^2+a_1^2)(R^2+a_2^2)}R^2\mathrm{d}R^2 + (R^2+a_1^2\cos^2\theta+a_2^2\sin^2\theta)\mathrm{d}\theta^2\right. \\ &\left.+(R^2+a_1^2)\sin^2\theta\mathrm{d}\phi_1 + (R^2+a_2^2)\cos^2\theta\mathrm{d}\phi_2\right]\end{aligned} \tag{6}$$

where $R\to\infty$ corresponds to spatial infinity and (μ, a_i) are parameters related to the mass M and angular momenta J_i of the black hole respectively. It is clear that $t=$ constant slices in the above asymptotic geometry can be written $h=\Phi^2\tilde{h}$, where $\tilde{h}$ has vanishing ADM mass.

We now focus attention on a general class of vacuum initial data. It is important to note that the results discussed in this work apply to spacetimes which will evolve from this data. In particular, the evolution need not be stationary, and so the results apply to dynamical spacetimes. We will consider solutions of the vacuum constraint equations in N space dimensions (latin indices $a, b=1\ldots N$)

$$R_h - K^{ab}K_{ab} + (\mathrm{Tr}_h K)^2 = 0,\qquad \nabla^b(K_{ab} - \mathrm{Tr}_h K h_{ab}) = 0 \tag{7}$$

where (Σ, h_{ab}, K_{ab}) refer to an asymptotically flat Riemannian manifold (Σ, h_{ab}) with second fundamental form K_{ab} in spacetime. This initial data set is assumed to be *maximal* ($\mathrm{Tr}_h K=0$) and invariant under a $\mathrm{U}(1)^{N-2}$ isometry group generated by commuting Killing vector fields m_i, that is

$$\mathcal{L}_{m_i} h_{ab} = 0, \qquad \mathcal{L}_{m_i} K_{ab} = 0. \tag{8}$$

Motivated by the above asymptotic geometry of black hole slices, we will focus on the case $N=4$ and h_{ab} is *conformal* to a $\mathrm{U}(1)^2$ invariant metric $\tilde{h}$ of the form (5) with vanishing ADM mass. This encompasses a broad class of initial data (we have checked this explicitly for the initial data for Myers-Perry black holes and the extreme doubly spinning black ring). For generic initial data, of course, one need not have orthogonal transitivity of the $\mathrm{U}(1)^{N-2}$ action, and, indeed, $\rho=\sqrt{\det\lambda'}$ need not be harmonic and so the two-dimensional metric will not take the conformally flat form above (i.e. $h_{\rho z}\neq 0$). We expect, however, that these restrictions can be removed (see e.g. Ref. [18]).

By Frobenius' theorem the two-dimensional subspace of the tangent space at each point that is spanned by vectors orthogonal to m_1 and m_2 is integrable (tangent to two-dimensional surfaces) if and only if

$$\nabla_{[a}\eta_{b]} = l_{[a}\eta_{b]} + s_{[a}\gamma_{b]}, \tag{9}$$

$$\nabla_{[a}\gamma_{b]} = p_{[a}\eta_{b]} + q_{[a}\gamma_{b]} \tag{10}$$

where we have set $\eta^a\equiv m_1^a$, $\gamma^a\equiv m_2^a$ and $\eta=\eta_a\eta^a$ and $\gamma=\gamma_a\gamma^a$, and l_a, s_a, p_a, and q_a are arbitrary one-forms. It is straightforward to verify that these imply the following identities:

$$\begin{aligned}(\det\lambda')\nabla_a\eta_b &= \gamma\eta_{[b}\nabla_{a]}\eta - 2L\gamma^c\eta_{[b}\nabla_{a]}\eta_c - L\gamma_{[b}\nabla_{a]}\eta \\ &\quad + 2\eta\gamma^c\gamma_{[b}\nabla_{a]}\eta_c,\\ (\det\lambda')\nabla_a\gamma_b &= -L\eta_{[b}\nabla_{a]}\gamma + 2\gamma\gamma^c\eta_{[b}\nabla_{a]}\gamma_c + \eta\gamma_{[b}\nabla_{a]}\gamma \\ &\quad - 2L\gamma^c\gamma_{[b}\nabla_{a]}\gamma_c\end{aligned} \tag{11}$$

where $L=\eta_a\gamma^a$. We will use these shortly.

The Ricci tensor for the metric (5) is straightforward to compute [17]. We will divide the indices a, b along $A, B = 1\ldots 2$ and $i, j = 1\ldots N-2$. Our main interest is the scalar curvature. Since $\lambda'^{ij}R_{ij}=0$ we have

$$\begin{aligned}\tilde{R} &= g^{AB}R_{AB}\\ &= e^{-2U}\left[-2\nabla^2 U + \frac{1}{\rho^2} - \frac{1}{4}\lambda'^{ik}\nabla_A\lambda'_{kj}\lambda'^{jm}\nabla_B\lambda'_{mi}\delta^{AB}\right]\end{aligned} \tag{12}$$

where ∇ refers to the flat partial derivative operator ∂_A. We will also denote by $\cdot$ the scalar product with respect to the flat metric δ_{AB}. The last term can also be written in the compact form

$$-\frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2]. \tag{13}$$

If $N=3$, the final two terms in Eq. (12) cancel, and $\tilde{R}$ takes a particularly simple form. This fact is crucial to establish the positive definiteness of the mass functional for three-dimensional initial data.

We consider solutions (Σ, h_{ab}, K_{ab}) of Eq. (7) expressed by the conformal rescaling

$$h_{ab} = \Phi^2\tilde{h}_{ab}, \qquad K_{ab} = \Phi^{-2}\tilde{K}_{ab} \tag{14}$$

in terms of which the constraint equations for maximal slices become (note $\mathrm{Tr}_{\tilde{h}}\tilde{K}=0$)

$$\Delta_{\tilde{h}}\Phi - \frac{1}{6}R_{\tilde{h}}\Phi + \frac{1}{6}\tilde{K}_{ab}\tilde{K}^{ab}\Phi^{-5} = 0, \tag{15}$$

$$\tilde{\nabla}_b\tilde{K}^{ab} = 0. \tag{16}$$

The Lichnerowicz equation (15) is a second-order nonlinear elliptic partial differential equation for the conformal factor Φ on a fixed Riemannian manifold $(\Sigma, \tilde{h}_{ab})$ with a given symmetric, traceless, divergenceless rank-two tensor field $\tilde{K}_{ab}$. The existence and uniqueness of solutions of Eq. (15) is guaranteed by the results of Ref. [19] (see Sec. VIII) under suitable regularity conditions, i.e. $\tilde{K}_{ab}\tilde{K}^{ab}$ belongs to a particular weighted Sobolev space and $(\Sigma, \tilde{h})$ is in the positive Yamabe class. Clearly, the latter condition is true, because $\tilde{h}$ is conformal to h, which must have positive scalar curvature. The former condition is an additional condition we impose on the data.

We now focus attention on a class of axisymmetric initial data sets $(\Sigma, \tilde{h}_{ab}, \tilde{K}_{ab})$ for which the extrinsic curvature can be specified completely from scalar potentials. The construction of transverse, traceless tensors under these conditions is given in Ref. [10] and we will only briefly review it here. Let ϕ^i be angular coordinates adapted to the commuting Killing fields m_i. Following Ref. [20] we define an initial data set to be $t-\phi^i$*-symmetric* if (i) $\partial/\partial\phi^i$ generate a $\mathrm{U}(1)^2$ isometry and (ii) $\phi^i \to -\phi^i$ is a diffeomorphism that preserves $\tilde{h}$ but reverses the sign of $\tilde{K}_{ab}$. Condition (i) is obviously trivially satisfied by construction. In terms of the Weyl coordinate system used above, condition (ii) implies $K_{ij} = K_{AB} = 0$ (initial data with this property arise naturally within the context of slices of stationary, axisymmetric black holes for $N = 3, 4$). Note that $t-\phi^i$ symmetry implies that the $\mathrm{U}(1)^2$ action is orthogonally transitive, i.e. the identities (11) hold.

As a consequence of this symmetry, K_{ab} is automatically traceless. Using the divergenceless condition and the property that Σ is simply connected [10], we can express $\tilde{K}_{ab}$ in a compact form. We define two scalar potentials Y_i and one-forms

$$S^i = \frac{1}{2\det\lambda'} i_{m_1} i_{m_2} \star \mathrm{d}Y^i. \tag{17}$$

Note that $\mathrm{d}\star S^i = 0$. Then an *arbitrary* divergenceless $t-\phi^i$-symmetric extrinsic curvature can be expressed as [10]

$$\tilde{K}_{ab} = \frac{2}{\det\lambda'}[(\lambda'_{22}S^1{}_{(a}m_{1b)} - \lambda'_{12}S^2{}_{(a}m_{1b)}) + (\lambda'_{11}S^2{}_{(a}m_{2b)} - \lambda'_{12}S^1{}_{(a}m_{2b)})]. \tag{18}$$

The vanishing of the trace of Eq. (18) is obvious since S^i and the m_i are orthogonal. The divergenceless condition is more involved and requires the use of the identities (11). Hence for $t-\phi^i$-symmetric initial data, the extrinsic curvature is completely characterized by the scalar potentials Y^i as well as the metric functions λ'_{ij}. One can show [10] that these potentials are simply the pull-backs of the spacetime twist potentials defined in the usual way, i.e. $\mathrm{d}Y^i = \star_5(m_1\wedge m_2\wedge \mathrm{d}m_i)$.

Further, it is useful to note that the full contraction of this tensor is

$$\tilde{K}_{ab}\tilde{K}^{ab} = e^{-2U}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\cdot\mathrm{d}Y^t)}{2\det\lambda'} \tag{19}$$

where for simplicity we use the notation $\mathrm{d}Y = (\mathrm{d}Y^1, \mathrm{d}Y^2)^t$ to define a column vector. If we consider the extrinsic curvature $\bar{K}_{ab}$ of a $\mathrm{U}(1)^2$-invariant, non-$t-\phi^i$-symmetric initial data set, one can show that

$$\bar{K}_{ab}\bar{K}^{ab} = \tilde{K}_{ab}\tilde{K}^{ab} + K_{AB}K_{CD}g^{AC}g^{BD} + K_{ij}K_{kl}\lambda'^{ik}\lambda'^{jl} \geq \tilde{K}_{ab}\tilde{K}^{ab} \tag{20}$$

where $g_{AB} = e^{2U}\delta_{AB}$. In particular there is equality if and only if $t-\phi^i$ symmetry holds.

In summary, we are considering the class of $\mathrm{U}(1)^2$-invariant maximal initial data sets (h_{ab}, K_{ab}) of the form (14) satisfying Eq. (8). The conformal metric (5) has vanishing ADM mass and is specified by the functions U and λ'_{ij}. Finally, if we impose in addition that the initial data be $t-\phi^i$ symmetric, then the extrinsic curvature is fully characterized by specifying in addition to the other data, two twist potentials Y^i.

B. Geometry of Σ

To conclude this section we discuss general properties of the manifold (Σ, h) and its $\mathrm{U}(1)^2$ action. We assume Σ is complete and simply connected and it may have several asymptotic ends, each asymptotically flat or asymptotically cylindrical. As a simple example,the maximal initial data slices of Schwarzschild have this property; the topology of the slice is $\mathbb{R}\times S^3$ which has two asymptotic flat ends. Asymptotically cylindrical ends arise in the context of initial data for extreme black holes.

Holland, Hollands and Ishibashi [21] have shown that for asymptotically flat spacetime metrics with $\mathbb{R}\times\mathrm{U}(1)$ isometry we must have

$$\Sigma \cong (\mathbb{R}^4 \# p(S^2\times S^2)\# p'\mathbb{CP}^2)\backslash B \tag{21}$$

where B is the black hole region and p, p' are non-negative integers. For spherical topology, this is equivalent to removing a point from the manifold. However, if the black hole horizon is $S^1\times S^2$, then $B = S^1\times\mathbb{B}^3$, where $\mathbb{B}^n$ is an n-ball [22]. We will assume that our spatial slices have a similar form, as explained below.

We consider a Σ which possesses an arbitrary number of asymptotic ends and an arbitrary number of 2-cycles (we assume $p' = 0$ above). The invariance of the functions under the $\mathrm{U}(1)^2$ isometry implies we need only consider them as functions on the two-dimensional orbit space $\mathcal{B} \equiv \Sigma/U(1)^2$. As in the spacetime case, $\mathcal{B}$ is a noncompact, simply connected manifold with one-dimensional boundary $\partial\mathcal{B}\cong\mathbb{R}$ and corners [23]. Since Σ is asymptotically flat, $\mathcal{B}$ has an asymptotic end corresponding to this spatial infinity. In the interior of $\mathcal{B}$, the matrix λ'_{ij} has rank two, whereas on $\partial\mathcal{B}$ and the corners it has rank one and rank zero respectively. $\partial\mathcal{B}$ consists of finite spatial intervals ("rods"), and two semi-infinite intervals [13]. These represent codimension-two surfaces upon which an integer linear combination of the rotational Killing fields vanish, and the two rotation axes which extend to spatial infinity, respectively. As we discuss in more detail below, the finite-length rods correspond to 2-cycles in the spacetime.

Thus in the absence of any additional asymptotic ends this situation represents initial data on $\mathbb{R}^4$ with "bubbles," and is qualitatively different to the situation in $\mathbb{R}^3$, where topological censorship rules out the existence of 2-cycles.

If we are considering initial data for spacetimes containing black holes, then Σ will have regions removed (i.e. $\Sigma \cong \mathbb{R}^4 - pt$ for a spatial slice of Myers-Perry that does not intersect the event horizon). Hence the orbit space will have, in addition to a boundary consisting of intervals and an asymptotic boundary, additional points removed from it. By axisymmetry, these points must lie on the axis $\rho = 0$. Of course, these removed points represent entire asymptotic regions that are an infinite proper distance from other points in Σ. Note that a similar situation in the Lorentzian setting occurs when analyzing the rod structure of extreme black holes, when the rod corresponding to a timelike Killing field becoming null shrinks to zero size [14]. In summary, the boundary $\rho = 0$ of the orbit space for more general initial data will consist of two semi-infinite rods, possibly finite rods, and points removed between the rods. We illustrate this in Fig. 1, which shows the orbit space of a black hole spacetime and its associated standard constant-time maximal slice for the Myers-Perry black hole [6] and a black ring [24].

We now discuss the behavior of the functions appearing in the class of conformal metrics (5) on the boundary and asymptotic regions of the orbit space. These will be required to analyze properties of the mass functional to be defined in the next section.

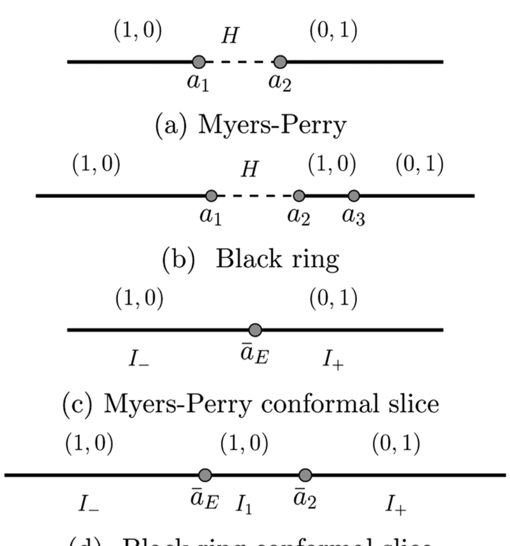


FIG. 1. (a) and (b) are spacetime rod structures for the Myers-Perry black hole and Emparan-Reall black ring. (c) and (d) are rod structures of conformal slice metrics for initial data with a spherical minimal surface and a ring-topology minimal surface respectively. The rod point $\bar{a}_E$ in both rod structures represents another asymptotic end of the slice.

1. Asymptotic behavior

First of all, note that $\mathbb{R}^4$ in the ρ, z chart given in Eq. (5) is

$$\delta_4 = \frac{d\rho^2 + dz^2}{2\sqrt{\rho^2+z^2}} + \left(\sqrt{\rho^2+z^2} - z\right)d\phi^2 + \left(\sqrt{\rho^2+z^2} + z\right)d\psi^2 \tag{22}$$

where $\rho \in \mathbb{R}^+ \cup \{0\}$ and $z \in \mathbb{R}$. This can be put in a more familiar chart by setting

$$\rho = \frac{r^2}{2}\sqrt{1-x^2}, \qquad z = \frac{r^2}{2}x \tag{23}$$

and noting that $r^2 = 2\sqrt{\rho^2+z^2}$, the metric is

$$\delta_4 = dr^2 + \frac{r^2 dx^2}{4(1-x^2)} + \frac{r^2}{2}((1-x)d\phi^2 + (1+x)d\psi) \tag{24}$$

where $r \geq 0$ and $-1 \leq x \leq 1$ and ϕ, ψ have period 2π. Hence our asymptotically flat metrics must approach δ_4 with appropriate falloff conditions. Note that asymptotic infinity corresponds to $r \to \infty$ so that $\rho, z \to \infty$ with $z(\rho^2+z^2)^{-1/2}$ fixed and the axes of rotation $x = \pm 1$ lie on the axis $\rho = 0$ with finite z [14]. In particular, the boundary of the orbit space for $(\mathbb{R}^4, \delta_4)$ consists of the semi-infinite rods $I_-: -\infty < z < 0$ and $I_+: 0 < z < \infty$ where ∂_ψ and ∂_ϕ vanish respectively (see Fig. 2).

Let us now consider our class of asymptotically flat conformal metrics $\tilde{h}$. We will consider U and λ'_{ij} as functions on the orbit space $\mathcal{B}$. First of all, asymptotic flatness implies $e^{-2U} \to 2\sqrt{\rho^2+z^2}$. Since we assume the conformal metric has zero ADM mass, it is convenient to decompose U as

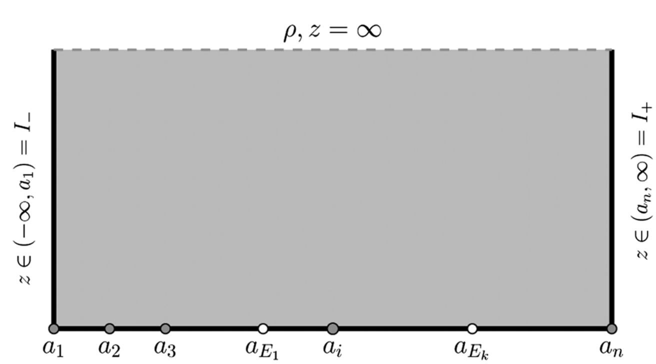


FIG. 2. The region is the orbit space $\mathcal{B}$. The thick line is the axis $\rho = 0$ and the dashed line is infinity $z = \rho = \infty$. Some of these rod points a_{E_k} may represent another asymptotic end.

$$U = V - \frac{1}{2}\log\left(2\sqrt{\rho^2 + z^2}\right) \tag{25}$$

where $V = O(r^{-2})$, that is

$$V = \frac{\bar{V}(x)}{r^2} + o(r^{-2}), \qquad r \to \infty \tag{26}$$

and $\bar{V}$ satisfies the condition that the integral given in Eq. (A3) vanishes. As shown in Appendix A, this is equivalent to the requirement that $\tilde{h}$ has vanishing ADM mass. Next, we take the falloff conditions of the Killing metric λ'_{ij} to be

$$\begin{aligned} \lambda'_{11} &= \frac{r^2}{2}(1-x)\left[1 + \frac{f(x)}{r^2} + o(r^{-2})\right], \\ \lambda'_{22} &= \frac{r^2}{2}(1+x)\left[1 + \frac{g(x)}{r^2} + o(r^{-2})\right], \\ \lambda'_{12} &= (1-x^2)o(r^{-2}) \end{aligned} \tag{27}$$

with $f(x) + g(x) = 0$ because $\det \lambda' = \rho^2$ where ρ is given by Eq. (23). We also assume the following fall off at infinity $r \to \infty$:

$$\begin{aligned} Y^1 &= y_1 - \frac{J_1(x+1)^2}{\pi} + \mathcal{O}(r^{-2}), \\ Y^2 &= y_2 - \frac{J_2(3-x)(x+1)}{\pi} + \mathcal{O}(r^{-2}) \end{aligned} \tag{28}$$

where J_i are angular momenta and y_i are constants [14]. Therefore we have

$$\tilde{K}_{ab} = o(1/r^3). \tag{29}$$

Finally, we have assumed

$$\Phi - 1 = \mathcal{O}(1/r^2), \qquad \Phi_{,r} = \mathcal{O}(1/r^3) \tag{30}$$

which is sufficient to ensure a finite ADM mass of (Σ, h).

2. Boundary conditions on the axis

The boundary of the orbit space $\partial\mathcal{B}$ lies on the z axis $\rho = 0$. We know from Ref. [17] that the eigenspace for the eigenvalue zero of the matrix λ'_{ij} for a given z is one-dimensional, except for isolated values of z. These isolated points are denoted $a_1, \ldots, a_n$ and we can divide the axis into subintervals $(-\infty, a_1), (a_1, a_2), \ldots, (a_n, \infty)$ (see Fig. 2). On each interval a particular integer linear combination of the m_i vanishes. The semi-infinite rods $I_\pm$ at the ends correspond to the axes of rotation of the asymptotic $\mathbb{R}^4$ region. Without loss of generality, we can choose m_1, m_2 to vanish on I_+ and I_- respectively.

The finite-length rods, on the other hand, correspond to 2-cycles in Σ. Suppose that on a particular rod I_i we have

$$\lambda'_{ij} v^j = 0, \qquad v = v^i \frac{\partial}{\partial \phi^i}, \qquad v^i \in \mathbb{Z}. \tag{31}$$

By an $SL(2,\mathbb{Z})$ change of basis $(m_1, m_2) \to (m'_1, m'_2)$ of the Killing fields, one can always choose $v = m'_1$ and another basis vector $w = m'_2$ such that w is nonvanishing on I_i except at its two end points a_i and a_{i+1}. Hence the interval I_i is topologically an S^2 submanifold of Σ.

The functions V and λ'_{ij} must satisfy regularity requirements as $\rho \to 0$. We will review these briefly here. Let ψ be a coordinate such that $\partial/\partial\psi = v$. Then in order to ensure the absence of conical singularities of the metric we impose that

$$\Delta\psi = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 e^{2U}}{\lambda'_{ij} v^i v^j}} = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 e^{2V}}{2\sqrt{\rho^2 + z^2}\lambda'_{ij} v^i v^j}} = 2\pi, \quad z \in (a_i, a_{i+1})$$

and hence

$$V = \frac{1}{2}\log\left(\frac{2\sqrt{\rho^2 + z^2}\lambda'_{ij} v^i v^j}{\rho^2}\right) = \frac{1}{2}\log V_i, \quad \text{where } z \in (a_i, a_{i+1}) \quad \text{and} \quad \rho \to 0 \tag{32}$$

where $V_i = V_i(z)$ is a bounded function.

Now consider the behavior of V on the semi-infinite rods $I_\pm$ defined above. The metric must take the asymptotic form of the flat metric. Then on either axis, we have, as $z \to \pm\infty$,

$$\lambda'_{ij} v^i v^j = \frac{1}{2|z|}\rho^2 + O(\rho^4) \tag{33}$$

where v is taken to be m_1 or m_2. Thus we see that $V_\pm \to 1$ as $z \to \pm\infty$, where $V_\pm$ refers to the function V_i evaluated on $I_\pm$.

Moreover, consider the rod I_i (either finite or semi-infinite). Near I_i, in the adapted basis (m'_1, m'_2) discussed above we have the following behavior for λ'_{ij}:

$$\lambda'_{ij} = \begin{pmatrix} \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2) \\ \mathcal{O}(\rho^2) & \mathcal{O}(1) \end{pmatrix}, \qquad \rho \to 0. \tag{34}$$

In addition, we will require that near I_i, the twist potentials in this basis behave as

$$Y^1 = C_1 + \mathcal{O}(\rho^4), \qquad Y^2 = C_2 + \mathcal{O}(\rho^2), \qquad \rho \to 0 \tag{35}$$

where C_1 and C_2 are constants. The falloff of Y^1 is more restrictive than simply requiring $\mathrm{d}Y^1 = \mathcal{O}(\rho^2)$ along a rod where $v = m_1'$ vanishes, but this condition is satisfied by the twist potentials of the Myers-Perry and black ring solutions. The condition (35) will be needed to show that certain terms in the mass functional are finite as $\rho \to 0$.

Finally, on the axis $\rho = 0$, apart from isolated points corresponding to asymptotic ends, we require

$$\Phi = \mathcal{O}(1). \tag{36}$$

3. Behavior near asymptotic ends

As discussed above, Σ may have additional asymptotic ends. Note that to have nontrivial angular momenta, Σ must have nontrivial topology. More precisely, if S represents the sphere at infinity, then since $\mathrm{d}\star\mathrm{d}m_i = 0$ by virtue of the vacuum spacetime equations, it follows that the Komar angular momenta J_i vanishes unless S is *not* the boundary of some compact domain contained in Σ. Such a situation arises when isolated points are removed from Σ, yielding additional asymptotic ends. By the $\mathrm{U}(1)^2$ symmetry assumption, these lie on a point on the axis $\rho = 0$. For example, in the case that the initial data arises from a stationary black hole, the location of the removed point corresponds to the location of the event horizon. We will allow for both asymptotically flat ends and cylindrical ends, the latter of which arise in the context of initial data for extreme black holes [10]. We impose singular boundary conditions on the conformal factor Φ:

$$\Phi = \mathcal{O}(r_i^{-2}), \qquad \partial_{r_i}\Phi = \mathcal{O}(r_i^{-3}), \qquad \text{asymptotically flat,} \tag{37}$$

$$\Phi = \mathcal{O}(r_i^{-1}), \qquad \partial_{r_i}\Phi = \mathcal{O}(r_i^{-2}), \qquad \text{cylindrical end} \tag{38}$$

where r_i represents the distance to the asymptotic end. We assume that the conformal metric $\tilde{h}$ approaches the flat metric on $\mathbb{R}^4$ in the former case, whereas in the latter case, the conformal metric $\tilde{h}$ will be asymptotically cylindrical, so $\tilde{h} = \Omega^2(\mathrm{d}r_i^2 + r_i^2\gamma)$ where γ is a metric on a compact manifold (the example of extreme Myers-Perry is discussed in detail in Appendix B). This assumption is most easily illustrated by considering the example of a maximal constant-t slice of the five-dimensional Schwarzschild geometry,

$$h = \left(1 + \frac{\mu}{2r^2}\right)^2 \delta_4 \tag{39}$$

where $\tilde{h} = \delta_4$ is the flat metric on $\mathbb{R}^4$ given in Eq. (22) and $r = 2\sqrt{\rho^2 + z^2}$. One easily sees that the point $r = 0$ corresponds to another asymptotically flat end and the conformal factor Φ has the singular behavior (37). In this simple case, this asymptotic region corresponds to the corner point $(\rho, z) = (0, 0)$ on the axis of the orbit space where both Killing fields vanish.

For the initial data of the Myers-Perry black hole, the conformal factor Φ diverges in the same way and the removed point is again located at a corner [13] of the orbit space [the point $\bar{a}_E$ in Fig. 1(c)]. However, for the black ring, the removed point is *not* at a corner, but instead lies on the rod corresponding to the Killing field which vanishes on the S^2 of the horizon. One can again verify that Φ has the above singular behavior at this point, and is in fact regular at the point $\bar{a}_2$ [see Fig. 1(d)]. Finally, we impose the same condition (26) on V with r replaced by r_i and $\bar{V} \to \bar{V}_i$.

III. A MASS FUNCTIONAL FOR INITIAL DATA

We now follow the approach of Dain [25] to construct a proposal for a mass functional $\mathcal{M}$. This functional depends on the functions (λ'_{ij}, Y^i) and should evaluate to the mass for the class of initial data we considered in the previous section when $t - \phi^i$ symmetry holds. Our starting point is the ADM mass for the asymptotically flat, complete Riemannian manifold (Σ, h_{ab}):

$$M_{\mathrm{ADM}} = \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r^3} (\partial_a h_{ac} - \partial_c h_{aa}) n^c ds_h \tag{40}$$

where S_r^3 refers to a three-sphere of coordinate radius r with volume element ds_h in the Euclidean chart outside a large compact region and n is the unit normal. If we evaluate the ADM mass in terms of the conformally scaled initial data and by the assumptions in the previous section, one has

$$M_{\mathrm{ADM}} = -\frac{3}{8\pi}\lim_{r\to\infty}\int_{S_r^3} n^c \tilde{\nabla}_c \Phi ds_{\tilde{h}} \tag{41}$$

where $\tilde{\nabla}$ refers to the covariant derivative with respect to $\tilde{h}_{ab}$. Using Eq. (7) and the fact that $\Phi \to 1$ as $r \to \infty$, we define

$$\begin{aligned} m &\equiv -\frac{3}{8\pi}\lim_{r\to\infty}\int_{S_r^3} \frac{\tilde{\nabla}_c\Phi}{\Phi} n^c ds_{\tilde{h}} \\ &= -\frac{3}{8\pi}\int_\Sigma \tilde{\nabla}^c\left(\frac{\tilde{\nabla}_c\Phi}{\Phi}\right) d\Sigma_{\tilde{h}} + \frac{3}{8\pi}\lim_{r_i\to 0}\int_{S_i^3} \frac{\tilde{\nabla}_c\Phi}{\Phi} n^c ds_{\tilde{h}} \\ &= \frac{3}{8\pi}\int_\Sigma \left(-\frac{\tilde{R}}{6} + \frac{\tilde{K}_{ab}\tilde{K}^{ab}}{6\Phi^6} + \frac{\tilde{\nabla}^c\Phi\tilde{\nabla}_c\Phi}{\Phi^2}\right) d\Sigma_{\tilde{h}} \\ &\quad + \frac{3}{8\pi}\lim_{r_i\to 0}\int_{S_i^3} \frac{\tilde{\nabla}_c\Phi}{\Phi} n^c ds_{\tilde{h}} \end{aligned} \tag{42}$$

where in passing from the first line to the second line, we have used the divergence theorem. Provided the behavior of Φ at the asymptotic ends is given by Eq. (37) the last term in m is zero.

This form of expressing the mass as a bulk integral is important for defining the functional $\mathcal{M}$. For our class of initial data, we can reduce the integral to one over the orbit space $\mathcal{B}$. Note that $\det \tilde{h} = e^{4U}\rho^2$. Performing the trivial integrals over the angles gives

$$m = \frac{3\pi}{2}\int_{\mathcal{B}}\left(-\frac{\tilde{R}e^{2U}}{6} + \frac{\tilde{K}_{ab}\tilde{K}^{ab}e^{2U}}{6\Phi^6} + \frac{\partial_A\Phi\partial^A\Phi}{\Phi^2}\right)\rho \mathrm{d}\rho \mathrm{d}z. \tag{43}$$

Equivalently, in terms of the flat metric on $\mathbb{R}^3$ in cylindrical coordinates

$$\delta = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2 \mathrm{d}\varphi^2 \tag{44}$$

where φ is an auxiliary angular coordinate with period 2π, we can write

$$m = \frac{3}{4}\int_{\mathbb{R}^3}\left(-\frac{\tilde{R}e^{2U}}{6} + \frac{\tilde{K}_{ab}\tilde{K}^{ab}e^{2U}}{6\Phi^6} + \frac{\partial_A\Phi\partial^A\Phi}{\Phi^2}\right)\mathrm{d}\mu_0$$

and $\mathrm{d}\mu_0$ is the volume element of δ.

We are now in the position to define our mass functional for arbitrary $t-\phi^i$-symmetric initial data. Recall that for this data, the extrinsic curvature is specified in terms of twist potentials Y^i as given by Eq. (18) and its square is given by the contraction (19). Using the expression for the scalar curvature $\tilde{R}$ [Eq. (12)] we have

$$m = \frac{1}{8}\int_{\mathbb{R}^3}\left(2\Delta_2 U - \frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] + e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\det\lambda'} + 6(\mathrm{d}v)^2\right)\mathrm{d}\mu_0 \tag{45}$$

where $v = \log\Phi$. As an integral over $\mathbb{R}^3$, this expression appears to be similar to the analogous formula for m when $N=3$ first given in Ref. [25]. However, in $N=4$ there are a number of key qualitative differences.

First, consider the terms

$$-\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2]. \tag{46}$$

In $N=3$, it is easily seen that the above expression vanishes identically. This is no longer the case in $N=4$. We note for later use the identity

$$\begin{aligned} &-\frac{1}{\rho^2} + \frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] \\ &= -\frac{1}{4}(\mathrm{Tr}(\lambda'^{-1}\mathrm{d}\lambda'))^2 + \frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] \\ &= -\frac{1}{2}\frac{\det \mathrm{d}\lambda'}{\det\lambda'} \quad \text{for } 2\times 2 \text{ matrices} \end{aligned} \tag{47}$$

where we are using the notation $\det \mathrm{d}\lambda' = \frac{1}{2}\epsilon^{ik}\epsilon^{jl}\mathrm{d}\lambda'_{ij}\cdot\mathrm{d}\lambda'_{kl}$.

A second important difference is that, unlike in $N=3$, the integral over $\Delta_2 U$ does *not* vanish. Indeed, in terms of the rod structure formalism, the class of three-dimensional initial data studied in Ref. [25] has $\partial\mathcal{B}$ consisting of a single rod [the rotation axis of the generator of the U(1) symmetry] with points removed corresponding to asymptotic ends. As we shall now explain, this is sufficient, along with appropriate falloff conditions, to prove that the first term in m does not contribute either. However, we shall see that the situation is more complicated in our present case. Note that

$$\Delta_2 U = \frac{\partial^2 U}{\partial\rho^2} + \frac{\partial^2 U}{\partial z^2}. \tag{48}$$

Using our expression for U, we have

$$\begin{aligned} \int_{\mathcal{B}}\Delta_2 U\rho\mathrm{d}\rho\mathrm{d}z &= \int_{\mathcal{B}}\left(\Delta_2 V - \frac{1}{2}\Delta_2\log(2\sqrt{\rho^2+z^2})\right)\rho\mathrm{d}\rho\mathrm{d}z \\ &= \int_{\mathcal{B}}\Delta_2 V\rho\mathrm{d}\rho\mathrm{d}z = \int_{\mathcal{B}}\mathrm{d}\alpha \end{aligned} \tag{49}$$

where we have defined the one-form

$$\alpha \equiv (\rho V_{,\rho} - V)\mathrm{d}z - \rho V_{,z}\mathrm{d}\rho. \tag{50}$$

Recall that the boundary of the orbit space consists of the asymptotic region $\mathcal{B}_\infty \equiv \{z,\rho\to\infty, z(\rho^2+z^2)^{-1/2}\text{ finite}\}$, i.e. $r\to\infty$, and the axis $\rho=0$ is denoted by $\partial\mathcal{B}$. Using the asymptotic condition (26), we find to leading order that $\alpha = -\bar{V}(x)\mathrm{d}x$ as $r\to\infty$. Hence by Stokes' theorem we have

$$\begin{aligned} \int_{\mathcal{B}}\mathrm{d}\alpha &= \int_{\partial\mathcal{B}\cup\mathcal{B}_\infty}\alpha = \int_{\partial\mathcal{B}}\alpha = \int_{I_-\cup I_1\cup\cdots\cup I_+}\alpha \\ &= \int_{I_-\cup I_1\cup\cdots\cup I_+}(\rho V_{,\rho} - V)|_{\rho=0}\mathrm{d}z \\ &= -\int_{I_-\cup I_1\cup\cdots\cup I_+} V|_{\rho=0}\mathrm{d}z \\ &= -\frac{1}{2}\sum_{\text{rods}}\int_{I_i}\log V_i\mathrm{d}z. \end{aligned}$$

Note that the integral over $\mathcal{B}_\infty$ vanishes due to the fact that $\tilde{h}$ has vanishing ADM mass [Eq. (A3)].

The above considerations lead us to define the following mass functional for $t-\phi^i$-symmetric, maximal asymptotically flat vacuum data:

$$\mathcal{M} \equiv \frac{\pi}{4}\int_{\mathcal{B}}\left(-\frac{1}{\rho^2}+\frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] + e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\det\lambda'}+6(\mathrm{d}v)^2\right)\rho\mathrm{d}\rho\mathrm{d}z - \frac{\pi}{4}\sum_{\mathrm{rods}}\int_{I_i}\log V_i\mathrm{d}z \tag{51}$$

where $\mathcal{M} = \mathcal{M}(\lambda'_{ij}, Y^i, v)$. Note that if we consider maximal, $\mathrm{U}(1)^2$-invariant data without $t-\phi^i$ symmetry, we have $m \geq \mathcal{M}$ as a consequence of Eq. (20).

The mass functional depends on the matrix λ'_{ij}, the twist potentials Y^i, and the conformal factor v. It also depends upon the *boundary* values of the λ'_{ij} along finite rods on the axis, via the functions V_i. If we define $\mathcal{A} \equiv \{(\lambda', Y, v)\colon \mathcal{M}(\lambda', Y, v)$ is bounded$\}$, then $\mathcal{M}$ will be well defined on $\mathcal{A}$. Of course, not all elements belonging to $\mathcal{A}$ will represent the mass of some initial data set. There are three functions in λ'_{ij} with a constraint $\det\lambda' = \rho^2$ so there are two independent functions in λ'_{ij}, two independent potentials Y^i and the conformal factor Φ (or $v = \log\Phi$). We have seen that all axisymmetric and $t-\phi^i$-symmetric data can be generated by six functions (U, λ', Y, v). These functions are coupled by the Lichnerowiscz equation (15) which can be rewritten as

$$\Delta_3 v + \frac{1}{3}\Delta_2 U - \frac{1}{6\rho^2} + \frac{1}{24}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] = e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{12\det\lambda'} \tag{52}$$

where Δ_3 is the three-dimensional Laplace operator with respect to the metric δ. Now for a given (λ'_{ij}, Y, v), we have a linear two-dimensional Poisson equation for U. Then by Eq. (25) we have a linear two-dimensional Poisson equation for V:

$$\Delta_2 V = F(v, \lambda', Y) \tag{53}$$

with the boundary (26) at infinity and we have $V = O(1)$ at $\rho = 0$. Now let $\mathcal{A}_1$ be a solution of Eq. (53) with appropriate falloff conditions. Then $\mathcal{M}(\lambda'_{ij}, Y^i, v)$ will give us the mass of an initial data set only if the given data is selected from $\mathcal{A}_1 \subset \mathcal{A}$.

Finally, using the asymptotic and boundary conditions on the orbit-space functions, we now show that $\mathcal{M}$ is finite. By the asymptotic condition (27) the behavior of the first two terms of $\mathcal{M}$ near infinity is

$$-\frac{1}{\rho^2}+\frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] = \mathcal{O}(r^{-8}) \quad \text{as} \quad r\to\infty. \tag{54}$$

Thus it is bounded at infinity. Near the axis, we must analyze the behavior of these terms near each rod. One can check that

$$-\frac{1}{\rho^2}+\frac{1}{4}\mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2] = \mathcal{O}(1) \quad \text{as} \quad \rho\to 0. \tag{55}$$

The third term in the mass functional from Eqs. (27) and (28) has the following behavior at infinity:

$$e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\det\lambda'} = \mathcal{O}(r^{-10}) \quad \text{as} \quad r\to\infty. \tag{56}$$

Near the axis one has, using Eqs. (34) and (35),

$$e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\det\lambda'} = \mathcal{O}(1) \quad \text{as} \quad \rho\to 0. \tag{57}$$

Finally, if one uses the conditions near additional asymptotically flat ends, one can ensure that $\mathcal{M}$ is finite, assuming continuity of the functions in the interior of $\mathcal{B}$.

IV. STATIONARY, BIAXISYMMETRIC DATA

Let us return to vacuum solutions with the $\mathbb{R}\times\mathrm{U}(1)^2$ isometry group. As discussed above, the metric takes the canonical form

$$g = -H\mathrm{d}t^2 + \frac{\lambda'_{ij}}{H^{1/2}}(\mathrm{d}\phi^i - w^i\mathrm{d}t)(\mathrm{d}\phi^j - w^j\mathrm{d}t) + e^{2\nu}(\mathrm{d}\rho^2+\mathrm{d}z^2) \tag{58}$$

where $\rho^2 = \det\lambda'$ is harmonic on the orbit space. Remarkably, the vacuum field equations for this spacetime can be derived from the critical points of the following functional, as first discussed by Carter for $D = 4$ in Ref. [11] (see Ref. [13] for a general dimension):

$$\mathcal{M}' = \frac{\pi}{16}\int_{\tilde{\mathcal{B}}}\mathrm{Tr}(\mathcal{V}^{-1}\mathrm{d}\mathcal{V})^2\rho\mathrm{d}\rho\mathrm{d}z \tag{59}$$

where $\tilde{\mathcal{B}}$ is the orbit space of *spacetime*, $\mathcal{V}$ is the 3×3 unimodular matrix

$$\mathcal{V} = \begin{pmatrix} \frac{1}{\det\lambda} & -\frac{Y_i}{\det\lambda} \\ -\frac{Y_j}{\det\lambda} & \lambda_{ij}+\frac{Y_iY_j}{\det\lambda}\end{pmatrix} \tag{60}$$

where

$$\lambda_{ij} = \frac{\lambda'_{ij}}{H^{1/2}} \tag{61}$$

and Y are the *spacetime* twist potentials. Note that it follows that $H = \rho^2(\det\lambda)^{-1}$. That is, the Euler-Lagrange equations for $\mathcal{M}'$ are precisely those for the vacuum field equations for the above form of the metric. Once Φ is determined, the remaining functions H and conformal factor ν are determined by quadrature.

Expanding out the Lagrangian gives

$$\mathrm{Tr}[(\mathcal{V}^{-1}\mathrm{d}\mathcal{V})^2] = \left(\frac{\mathrm{d}\det\lambda}{\det\lambda}\right)^2 + \mathrm{Tr}[(\lambda^{-1}\mathrm{d}\lambda)^2] + 2\frac{\mathrm{Tr}(\lambda^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{\det\lambda}. \tag{62}$$

We wish to express the action in terms of λ'_{ij}. Since

$$\mathrm{d}\lambda = \frac{1}{2}\left(\frac{\det\lambda}{\rho^2}\right)^{-\frac{1}{2}}\left(\frac{\mathrm{d}\det\lambda}{\rho^2} - 2\frac{\det\lambda\mathrm{d}\rho}{\rho^3}\right)\lambda' + \left(\frac{\det\lambda}{\rho^2}\right)^{\frac{1}{2}}\mathrm{d}\lambda', \tag{63}$$

a calculation yields

$$\mathrm{Tr}[(\lambda^{-1}\mathrm{d}\lambda)^2] = \frac{1}{2}\left(\frac{\mathrm{d}\det\lambda}{\det\lambda}\right)^2 - 2\left(\frac{\mathrm{d}\rho\cdot\mathrm{d}\rho}{\rho^2}\right) + \mathrm{Tr}[(\lambda'^{-1}\mathrm{d}\lambda')^2].$$

Note that $\mathrm{d}\rho\cdot\mathrm{d}\rho = 1$.

Consider a constant-time spatial slice of the stationary, axisymmetric metric (58). The metric can be placed in our general form for our initial data provided

$$\Phi^2 = e^{2v} = \frac{1}{H^{1/2}} = \left[\frac{\det\lambda}{\det\lambda'}\right]^{1/2} \tag{64}$$

which implies

$$v = \frac{1}{4}\log(\det\lambda) - \frac{\log\rho}{2}. \tag{65}$$

We then deduce that

$$\mathrm{d}v\cdot\mathrm{d}v = \left(\frac{\mathrm{d}\det\lambda}{4\det\lambda} - \frac{\mathrm{d}\rho}{2\rho}\right)^2 = \frac{1}{16}\left(\frac{\mathrm{d}\det\lambda}{\det\lambda}\right)^2 - \frac{1}{4}\mathrm{d}(\log\rho)\cdot\mathrm{d}\log\left(\frac{\rho}{\det\lambda}\right). \tag{66}$$

Using Eqs. (61) and (65) one can replace the independent variables v and λ'_{ij} by $\det\lambda$ and λ_{ij} in the mass functional. Then $\det\lambda$, λ_{ij}, and Y^i are taken to be independent, and we have

$$\begin{aligned}\mathcal{M}_{\tilde{\mathcal{B}}} &= \mathcal{M}' + \frac{\pi}{4}\int_{\tilde{\mathcal{B}}}\left[\frac{\mathrm{d}\rho\cdot\mathrm{d}\rho}{\rho} - \frac{3}{2}\frac{\mathrm{d}\rho\cdot\mathrm{d}\det\lambda}{\det\lambda}\right]\mathrm{d}\rho\wedge\mathrm{d}z + \frac{\pi}{2}\int_{\partial\tilde{\mathcal{B}}}\alpha \\ &= \mathcal{M}' + \frac{\pi}{4}\int_{\tilde{\mathcal{B}}}\mathrm{d}\left[\log\left(\frac{\rho}{(\det\lambda)^{3/2}}\right)\mathrm{d}z\right] + \frac{\pi}{2}\int_{\partial\tilde{\mathcal{B}}}\alpha \\ &= \mathcal{M}' + \frac{\pi}{4}\int_{\partial\tilde{\mathcal{B}}\cup\tilde{\mathcal{B}}_\infty}\left(2\alpha + \log\left(\frac{\rho}{(\det\lambda)^{3/2}}\right)\mathrm{d}z\right)\end{aligned} \tag{67}$$

where α is the one-form defined in Eq. (50). Note that we have taken the domain of integration in $\mathcal{M}$ to be over $\tilde{\mathcal{B}}$ when demonstrating this equivalence. This is an important point, because $\partial\tilde{\mathcal{B}}$ will, in general, contain additional finite timelike rods on the axis $\rho = 0$ corresponding to Killing horizons (i.e. where a timelike Killing vector field becomes null) which are not present on $\partial\mathcal{B}$. The domain of integration of $\mathcal{M}'$ covers only the region exterior to the black hole, with an inner boundary representing the horizon. In contrast, our mass functional is naturally defined over $\mathcal{B}$ and covers a complete manifold with no inner boundary, and in particular may have additional asymptotic regions. In general, $\mathcal{M}_{\tilde{\mathcal{B}}}$ will be singular because it may diverge on the horizon rod, whereas, $\mathcal{M}$ is finite. In the special case of *extreme* horizons, however, the orbit spaces coincide, because the timelike horizon rod shrinks to a point and corresponds to an asymptotically cylindrical region [14].

In summary, we have shown that over an appropriate domain, $\mathcal{M}$ equals Carter's functional, up to a divergent boundary term. Equivalently, we have proved that if one considers the change of variables $(v, \lambda'_{ij}) \to \lambda_{ij}$ given by Eqs. (61) and (65), then $\mathcal{M}$ is precisely the same as $\mathcal{M}'$ up to a boundary term. It follows that they have the same Euler-Lagrange equations, provided we consider variations which are fixed on $\partial\tilde{\mathcal{B}}$. Hence the critical points of Carter's functional i.e. the stationary, axisymmetric vacuum solutions, are also critical points of the mass functional.

It is interesting to directly compute the Euler-Lagrange equations of $\mathcal{M}$. The details are tedious and we simply summarize the result here. First we vary the mass functional with respect to functions on the orbit space $\bar{\lambda}'$, $\bar{v}$ and $\bar{Y}$, which have compact support on the interior of $\mathcal{B}$ (and in particular vanish on $\partial\mathcal{B}$ and $\mathcal{B}_\infty$). Therefore, by Ref. [10] the angular momenta will be preserved. We define

$$\mathcal{E}(\epsilon) = \mathcal{M}(v + \epsilon\bar{v}, \lambda' + \epsilon\bar{\lambda}', Y + \epsilon\bar{Y}). \tag{68}$$

Then we have

$$\delta\mathcal{E}(0) = \frac{\pi}{4}\int_{\mathcal{B}}\left[12\mathrm{d}v\cdot\mathrm{d}\bar{v} - \frac{1}{2\rho^2}(\mathrm{d}\bar{\lambda}'_{11}\cdot\mathrm{d}\lambda'_{22} + \mathrm{d}\lambda'_{11}\cdot\mathrm{d}\bar{\lambda}'_{22} - 2\mathrm{d}\bar{\lambda}'_{12}\cdot\mathrm{d}\lambda'_{12}) + e^{-6v}\left((\det\bar{\lambda}')\frac{\mathrm{Tr}(\bar{\lambda}'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\rho^4} + \frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}\bar{Y}\mathrm{d}Y^t)}{\rho^2} - 3\bar{v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{\rho^2}\right)\right]\rho\mathrm{d}\rho\mathrm{d}z. \tag{69}$$

Performing the appropriate integration by parts and imposing $\delta\mathcal{E}(0) = 0$ yields

$$4\Delta_3 v + e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\nabla Y\cdot\nabla Y^t)}{\rho^2} = 0, \tag{70}$$

$$\nabla\cdot\left(\frac{\nabla\lambda'}{\rho^2}\right) + \frac{e^{-6v}}{\rho^4}\nabla Y\cdot\nabla Y^t = 0,$$
$$\nabla\cdot\left(\frac{e^{-6v}}{\rho^2}\lambda'^{-1}\nabla Y\right) = 0 \tag{71}$$

where Δ_3 is the Laplacian and ∇ is the usual gradient with respect to the metric (44). Consider the critical points of $\mathcal{M}$ that are extreme, stationary, axisymmetric vacuum solutions. We can use the above to show that the mass functional is positive definite for these data. We have $h = e^{2v}\tilde{h}$ and thus

$$\tilde{R} = Re^{2v} + 6\Delta_{\tilde{h}} v + 6e^{-2U}(\mathrm{d}v)^2. \tag{72}$$

Since $\Delta_{\tilde{h}} = e^{-2U}\Delta_3$ on $\mathrm{U}(1)^2$-invariant functions, using Eqs. (7), (19) and (70) yields

$$\tilde{R} = e^{-2U}\left(-2e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\rho^2} + 6(\mathrm{d}v)^2\right). \tag{73}$$

Substituting this into the expression (43) gives

$$\mathcal{M}_{\mathrm{cp}} = \frac{3\pi}{4}\int_{\mathcal{B}} e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\mathrm{d}Y^t)}{2\det\lambda'}\rho\mathrm{d}\rho\mathrm{d}z \tag{74}$$

where $\mathcal{M}_{\mathrm{cp}}$ is the restriction of $\mathcal{M}$ to these critical points. Clearly this is positive definite.

V. POSITIVITY OF $\mathcal{M}$

In this section we investigate the positivity of $\mathcal{M}$. Positivity is a desirable property as it plays a key role in applications to geometric inequalities for three-dimensional initial data [1–3] and investigating the linear stability of extreme black holes[1] [28]. Gibbons and Holzegel [29] have generalized Brill's proof of positive mass for a restricted class of four-dimensional initial data with $\mathrm{U}(1)^2$ isometry, by expressing the mass in a manifestly positive-definite way. We will show that for a particular set of initial data, $\mathcal{M}$ can be expressed in a form such that the arguments of Ref. [29] can be adapted to demonstrate positivity. It is important to note that our boundary conditions are weaker than the ones used in Ref. [29]. A proof of positivity for arbitrary rod data remains to be found. In the following, we will consider asymptotically flat data with a single additional asymptotic end.

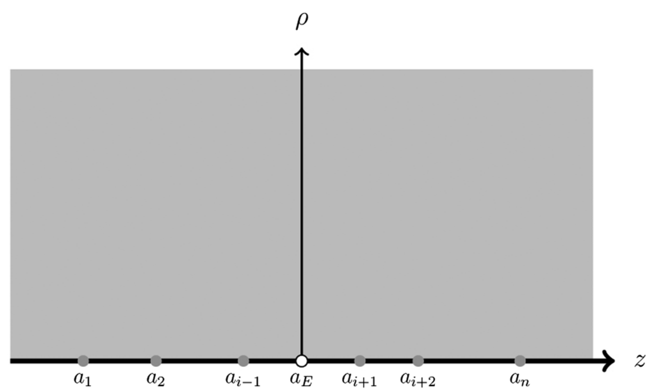
We introduce the coordinates (r,x) given by the transformation (23). This is equivalent to introducing a map from $\mathcal{B}\cong\mathbb{R}\times\mathbb{R}^+\backslash\{a_E\}$ to the infinite strip $\mathcal{B}\cong\mathbb{R}\times[-1,1]$ [14]. This map will divide the axis $\rho=0$ into two disconnected axes $\mathcal{I}^\pm = \{r>0, x=\pm1\}$ and another end, $a_E = \{r=0, |x|\le 1\}$. Note that the rod structure is contained on $\mathcal{I}^\pm$; see Fig. 3.

Consider the mass functional (45) and (51). The inner products are taken with respect to $\delta_2 = \mathrm{d}\rho^2 + \mathrm{d}z = r^2(\mathrm{d}r^2 + \frac{r^2\mathrm{d}x^2}{4(1-x^2)}) = r^2\delta_2'$. We will rewrite the functional with respect to δ_2' and as an integral over the infinite strip. Thus we have

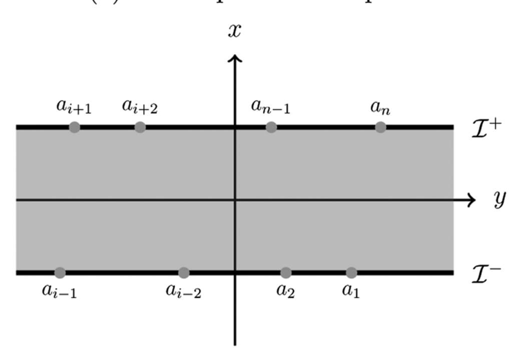
$$\mathcal{M} = \frac{1}{32}\int_{\mathcal{B}}\left(-\frac{\det\mathrm{d}\lambda'}{2\det\lambda'} + e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}\mathrm{d}Y\cdot\mathrm{d}Y^t)}{2\det\lambda'} + 6(\mathrm{d}v)^2\right)$$
$$\times r^3\mathrm{d}r\mathrm{d}x + \frac{1}{4}\int_{\partial\mathcal{B}\cup\mathcal{B}_\infty}\alpha \tag{75}$$

where all scalar products of one-forms are taken with respect to δ_2'. Note that (ρ,z) and (x,r) have positive orientation. The boundary is $\partial\mathcal{B}\cup\mathcal{B}_\infty = \mathcal{I}_E + \mathcal{I}_\infty + \mathcal{I}^+ + \mathcal{I}^-$ where $\mathcal{I}_E \equiv \{r=0, -1\le x\le 1\}$, $\mathcal{I}_\infty \equiv \{r=\infty, -1\le x\le 1\}$. In terms of the (r,x) chart, we have

$$\alpha = -(r(1-x^2)V_{,x} + rxV)\mathrm{d}r + \left(\frac{r^3}{4}V_{,r} - \frac{r^2}{2}V\right)\mathrm{d}x. \tag{76}$$



(a) Orbit space as half plane



(b) Orbit space as infinite strip

FIG. 3. The rod point $a_E = \{\rho, z=0\}$ is another end. (a) and (b) illustrate the map from the $z+i\rho$ complex plane to the $y+ix$ complex plane where $y=\log r$.

[1]The stability argument uses the positivity of the second variation of the mass functional about extreme Kerr. This energy is related to the recent construction of Hollands-Wald [26] of a canonical energy, which has recently been used to demonstrate the existence of instabilities of (near-)extreme black holes [27].

Then we have

$$\alpha|_{\mathcal{I}_E} = -\bar{V}_E(x)\mathrm{d}x, \qquad \alpha|_{\mathcal{I}_\infty} = -\bar{V}(x)\mathrm{d}x,$$
$$\alpha|_{\mathcal{I}^+} = -rV|_{x=1}\mathrm{d}r, \qquad \alpha|_{\mathcal{I}^-} = rV|_{x=-1}\mathrm{d}r.$$

Thus with the appropriate orientation

$$\int_{\partial\mathcal{B}\cup\mathcal{B}_\infty} \alpha = \int_0^\infty r(V|_{x=1} + V|_{x=-1})\mathrm{d}r = \sum_{i=0}^{n-1}\int_{b_i}^{b_{i+1}} r(V|_{x=1} + V|_{x=-1})\mathrm{d}r. \quad (77)$$

Consider the integral (75). We are given n rod points a_i. Subdivide the infinite strip into n rectangular columns A_i with

$$A_i = \{-1 \le x \le 1, b_i < r < b_{i+1}\}, \quad i = 0 \ldots n-1 \quad (78)$$

where b_i correspond to the locations of the rod points a_i after ordering along the $y = \log r$ axis (see Fig. 4). For convenience, we have chosen $b_1 < b_2 < \ldots < b_{n-1}$. We take $b_0 = 0$ to correspond to the asymptotic end a_E and b_n to correspond to the asymptotically flat end $r \to \infty$. We then express Eq. (75) as

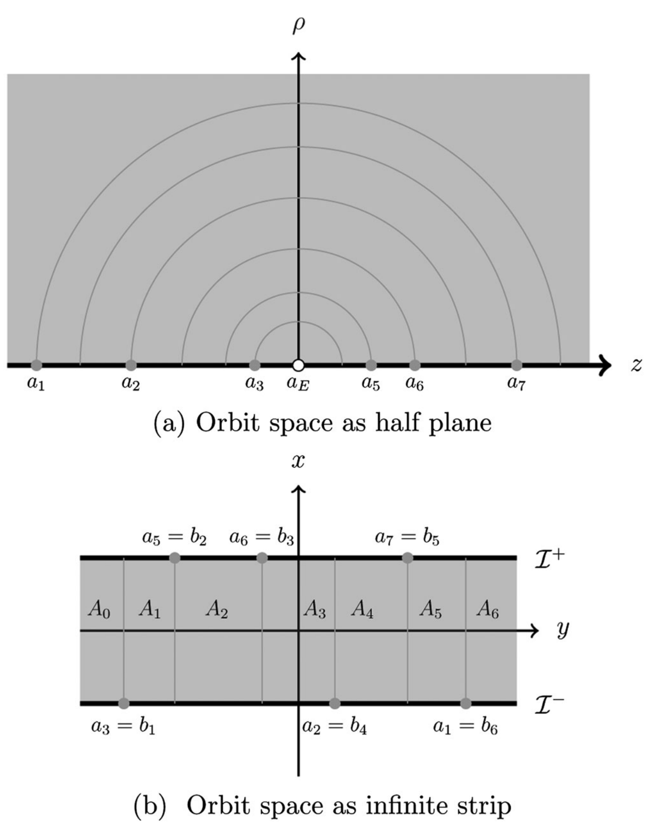


FIG. 4. The orbit space can be subdivided into subregions A_i which are half-annuli in the (ρ, z) plane and rectangles in the (y, x) plane. In this case $n = 7$.

$$\mathcal{M} = \sum_{i=0}^{n-1}\int_{A_i}\mathcal{M}_i \quad (79)$$

where $\mathcal{M}_i$ is the restriction of $\mathcal{M}$ to A_i.

We fix a region A_i. Then one of the following two possibilities must occur: (a) the distinct Killing fields $v_{(i)}$ and $w_{(i)}$ vanish on $A_i \cap \mathcal{I}^+$ and $A_i \cap \mathcal{I}^-$ respectively (in this case A_i is topologically $S^3 \times \mathbb{R}$), or (b) the same Killing field $v_{(i)} = v^i_{(i)} m_i$ vanishes on both of the disjoint sub intervals $A_i \cap \mathcal{I}^\pm$ (in this case A_i is topologically $S^2 \times D$ where D is a noncontractible disc). We can demonstrate positivity for case (a). In this case without loss of generality we can select the following parametrization of the three independent functions contained in λ'_{ij} and v:

$$\lambda'_{11} = \frac{r^2(1-x)}{2\sqrt{1-W^2}}e^{V_1 - V_2}, \qquad \lambda'_{22} = \frac{r^2(1+x)}{2\sqrt{1-W^2}}e^{V_2 - V_1},$$
$$\lambda'_{12} = \frac{r^2\sqrt{1-x^2}W}{2\sqrt{1-W^2}}, \qquad v = \frac{V_1 + V_2 + \log\sqrt{1-W^2}}{2} \quad (80)$$

where without loss of generality we have chosen $v_{(i)} = \partial_{\phi 1}$ and $w_{(i)} = \partial_{\phi 2}$. V_1, V_2 and W are C^1 functions whose boundary conditions on the axis are induced from those of λ'_{ij} and v [Eqs. (26) and (27)]. In particular, we have $\det\lambda' = \rho^2$ and to remove conical singularities on $\mathcal{I}^\pm$ [Eq. (32)] we require

$$2V - V_1 + V_2 = 0 \quad \text{on } \mathcal{I}^+, \quad 2V - V_2 + V_1 = 0 \quad \text{on } \mathcal{I}^-,$$
$$W = 0 \quad \text{on } \mathcal{I}^\pm. \quad (81)$$

Note that since λ'_{ij} and v are continuous across the boundary of A_i, this will impose boundary conditions on the parametrization functions in adjacent subregions. Second, we rewrite the second and fourth terms of $\mathcal{M}$ as functions of V_1, V_2, and W, yielding

$$\frac{\det d\lambda'}{2\det\lambda'} = \frac{-1}{2(1-W^2)} \times \left[(\mathrm{d}V_1 - \mathrm{d}V_2)^2 - \frac{8}{r^2}\partial_x(V_1 - V_2) + (\mathrm{d}W)^2 + \frac{W^2(\mathrm{d}W)^2}{1-W^2} + \frac{4W^2}{r^2(1-x^2)}\right] \quad (82)$$

and

$$6(\mathrm{d}v)^2 = \frac{3}{2}(\mathrm{d}V_1 + \mathrm{d}V_2)^2 + \frac{3}{2}\frac{W^2(\mathrm{d}W)^2}{(1-W^2)^2} - \frac{3W}{1-W^2}(\mathrm{d}V_1 \cdot \mathrm{d}W + \mathrm{d}V_2 \cdot \mathrm{d}W). \quad (83)$$

Therefore, we have

$$\mathcal{M}_i = \frac{1}{32}\int_{A_i}\Bigg(Re^{2v+2U} + (\mathrm{d}V_1+\mathrm{d}V_2)^2 + (\mathrm{d}V_1)^2 + (\mathrm{d}V_2)^2 + \frac{W^2}{2(1-W^2)}\left[(\mathrm{d}V_1-\mathrm{d}V_2)^2 - \frac{6}{W}(\mathrm{d}V_1\cdot\mathrm{d}W + \mathrm{d}V_2\cdot\mathrm{d}W)\right]$$
$$+\frac{W^2}{r^2(1-W^2)}\left[4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{(1-x^2)}\right] + \frac{(\mathrm{d}W)^2}{2(1-W^2)} + \frac{2W^2(\mathrm{d}W)^2}{(1-W^2)^2}\Bigg)r^3\mathrm{d}x\mathrm{d}r$$
$$+\frac{1}{8}\int_{b_i}^{b_{i+1}} r((V_1-V_2)|_{x=-1} - (V_1-V_2)|_{x=1})\mathrm{d}r + \frac{1}{4}\int_{b_i}^{b_{i+1}} r(V|_{x=1}+V|_{x=-1})\mathrm{d}r$$
$$=\frac{1}{32}\int_{A_i}\Bigg(Re^{2v+2U} + (\mathrm{d}V_1+\mathrm{d}V_2)^2 + (\mathrm{d}V_1)^2 + (\mathrm{d}V_2)^2 + \frac{W^2}{2(1-W^2)}\left[(\mathrm{d}V_1-\mathrm{d}V_2)^2 - \frac{6}{W}(\mathrm{d}V_1\cdot\mathrm{d}W + \mathrm{d}V_2\cdot\mathrm{d}W)\right]$$
$$+\frac{W^2}{r^2(1-W^2)}\left[4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{(1-x^2)}\right] + \frac{(\mathrm{d}W)^2}{2(1-W^2)} + \frac{2W^2(\mathrm{d}W)^2}{(1-W^2)^2}\Bigg)r^3\mathrm{d}x\mathrm{d}r. \tag{84}$$

Consider the first equality. The first term follows from the constraint equation for maximal slices, Eq. (19), and Eq. (14). The remaining bulk terms follow from Eqs. (82) and (83) while the first boundary term comes from Eq. (82) and the second from Eq. (77). The second equality is obtained by noting that the boundary contributions cancel by regularity on the axes (81). The remaining terms can be shown to be positive by a straightforward application of the arguments given in Sec. 4.3 of Ref. [29]. Therefore, $\mathcal{M}_i \geq 0$.

From this result it follows that provided *all* subregions A_i fall into class (a) then $\mathcal{M}$ is positive definite. In particular, for the rod structure of Myers-Perry initial data, there is only one region A_0 of class (a) and hence for any data with the same rod structure, $\mathcal{M} \geq 0$. One might expect a similar argument to hold for class (b). This case of course includes initial data for black rings (the same Killing vector field vanishes on either side of the asymptotic end). By choosing a general parametrization for the various functions in this region, one finds that the boundary term has an indefinite sign. However our strategy is merely sufficient to demonstrate positivity, and we expect that positivity will hold for a general rod structure. Interestingly, for the initial data for extreme black rings, the expression (74) shows $\mathcal{M} \geq 0$.

VI. DISCUSSION

We have constructed a mass functional $\mathcal{M}$ valid for a broad class of asymptotically flat $t-\phi^i$-symmetric maximal initial data for the vacuum Einstein equations in five dimensions. $\mathcal{M}$ can be considered an extension of a similar functional defined for three-dimensional initial data sets [25]. We can check that this mass functional is finite and evaluates to the ADM mass provided certain boundary and asymptotic conditions are met. These conditions encompass a large class of initial data, and in particular we have checked this explicitly for the usual maximal constant-time slices for the Myers-Perry black hole (see Appendix B) and the extreme vacuum black ring solution. Moreover, we proved that $\mathbb{R}\times \mathrm{U}(1)^2$-invariant solutions of the vacuum Einstein equations are critical points of this functional amongst this class of data. Finally, we have shown explicitly that the mass functional is positive for a particular class of rod structures as explained in detail above, although it remains to show this for arbitrary rod data. This property is relevant for investigating geometric inequalities for five-dimensional vacuum solutions. A start towards this goal is to show a *local* mass-angular momenta inequality along the lines of Ref. [1]. This problem is currently under investigation.

ACKNOWLEDGMENTS

We would like to thank S. Hollands for clarifying the relationship of mass functionals and his recent construction with R. Wald of a canonical energy. H. K. K. also thanks J. Lucietti for comments concerning uniqueness theorems for extreme black holes. A. A. is partially supported by a graduate scholarship from Memorial University. H. K. K. is supported by an National Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant.

APPENDIX A: MASS OF CONFORMAL METRIC

Assume that we have an asymptotically flat initial data set (Σ, h_{ab}, K_{ab}) of Einstein's equation. The ADM mass of this data is given by Eq. (40). But by a rescaling similar to Eq. (14) we have

$$M_{\mathrm{ADM}} = -\frac{3}{8\pi}\lim_{r\to\infty}\int_{S_r^3} n^c\tilde{\nabla}_c\Phi \mathrm{d}s_{\tilde{h}} + \tilde{M}_{\mathrm{ADM}} \tag{A1}$$

where $\tilde{M}_{\mathrm{ADM}}$ is the ADM mass of $\tilde{h}$. Now as in Sec. V we can introduce a chart with coordinates (r,x) such that the asymptotically flat conformal metric takes the form

$$\tilde{h} = e^{2V}\left(\mathrm{d}r^2 + \frac{r^2}{4(1-x^2)}\mathrm{d}x^2\right) + f_2\frac{r^2}{2}(1-x)\mathrm{d}\phi^2 + f_3\frac{r^2}{2}(1+x)\mathrm{d}\psi^2 + f_4 r^2(1-x^2)\mathrm{d}\phi\mathrm{d}\psi \tag{A2}$$

with the falloff conditions $e^{2V}-1, f_2-1$, and $f_3-1=\mathcal{O}(r^{-2})$ and $f_4=o(r^{-2})$ as $r\to\infty$. Then the ADM mass of the conformal metric is

$$\begin{aligned}\tilde{M}_{\rm ADM}&=-\frac{1}{16\pi}\lim_{r\to\infty}\int_{S^3}\left(r^2\partial_r[r(f_2+f_3-2)]+r^5\partial_r\left(\frac{e^{2V}-1}{r^2}\right)\right)\mathrm{d}\Omega_3\\&=\frac{1}{16\pi}\int_{S^3}(f(x)+g(x))\mathrm{d}\Omega_3+\frac{1}{2\pi}\int_{S^3}\bar{V}(x)\mathrm{d}\Omega_3\\&=\frac{\pi}{2}\int_{-1}^{1}\bar{V}(x)\mathrm{d}x.\end{aligned}\tag{A3}$$

The first equality is the definition of ADM mass applied to Eq. (A2). The second equality uses the expansion of λ'_{ij} and V at infinity [Eqs. (27) and (26)]. Therefore, we can see that the ADM mass of the conformal metric is zero if and only if the right-hand side of Eq. (A3) vanishes. It is trivially satisfied if $V=o(r^{-2})$. In general however, one may wish to consider weaker falloff conditions on V that still lead to a vanishing ADM mass of the conformal metric. In particular, we have checked explicitly for the general Myers-Perry black hole and for the extreme doubly spinning black ring that the right-hand side of Eq. (26) vanishes, although $\bar{V}(x)\neq 0$ in these cases.

APPENDIX B: MYERS-PERRY INITIAL DATA

Here we consider the Myers-Perry solution with coordinates $(t,\tilde{r},\theta,\phi_1,\phi_2)$ [30]. The ϕ_i have period 2π. Then we have following metric functions:

$$\begin{aligned}\omega^1&=\frac{\mu a\lambda_{22}\sin^2\theta-\mu b\lambda_{12}\cos^2\theta}{\Sigma\det\lambda},\\ \omega^2&=\frac{\mu b\lambda_{11}\cos^2\theta-\mu a\lambda_{12}\sin^2\theta}{\Sigma\det\lambda},\end{aligned}\tag{B1}$$

$$\begin{aligned}\lambda_{11}&=\frac{a^2\mu}{\Sigma}\sin^4\theta+(\tilde{r}^2+a^2)\sin^2\theta,\\ \lambda_{12}&=\frac{ab\mu}{\Sigma}\sin^2\theta\cos^2\theta,\end{aligned}\tag{B2}$$

$$\lambda_{22}=\frac{b^2\mu}{\Sigma}\cos^4\theta+(\tilde{r}^2+b^2)\cos^2\theta\tag{B3}$$

where

$$\Sigma=\tilde{r}^2+b^2\sin^2\theta+a^2\cos^2\theta,\tag{B4}$$

$$\Delta(\tilde{r})=(\tilde{r}^2+a^2)(\tilde{r}^2+b^2)-\mu\tilde{r}^2.\tag{B5}$$

The metric on a constant-time slice will be

$$h=\frac{\Sigma}{\Delta(\tilde{r})}\mathrm{d}r^2+\Sigma\mathrm{d}\theta^2+\lambda_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j.\tag{B6}$$

This metric is singular at two roots $\tilde{r}_\pm$ of $\Delta(\tilde{r})$ which correspond to spacetime inner and outer horizons. One can define a quasi-isotropic coordinate as

$$\tilde{r}^2=r^2+\frac{1}{2}(\mu-a^2-b^2)+\frac{\mu(\mu-2a^2-2b^2)+(a^2-b^2)^2}{16r^2}.\tag{B7}$$

Note that the outer horizon at $\tilde{r}_+$ is shifted to $r=0$ and the slice metric will be

$$h=\frac{\Sigma}{r^2}(\mathrm{d}r^2+r^2\mathrm{d}\theta^2)+\lambda_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j\tag{B8}$$

where $0<r<\infty$, $0<\theta<\pi/2$, and $0<\phi_1,\phi_2<2\pi$. The point $r=0$ is another asymptotic infinity (see Fig. 1) and one can show this by computing the distance to $r=0$ along a curve of constant (θ,ϕ_1,ϕ_2) from $r=r_0$, i.e.

$$\text{Distance}=\int_r^{r_0}\frac{\sqrt{\Sigma}}{r}dr\to\infty\quad\text{as}\quad r\to 0.\tag{B9}$$

In the extreme limit $\mu=(a+b)^2$ the quasi-isotropic radius simplifies to [10]

$$\tilde{r}^2=r^2+ab.\tag{B10}$$

The conformal metric $\tilde{h}$ can be determined by the relations

$$\Phi^2=\frac{\sqrt{\det\lambda}}{\rho},\qquad e^{2U}=\frac{\rho\Sigma}{r^4\sqrt{\det\lambda}},\qquad \lambda'_{ij}=\Phi^{-2}\lambda_{ij}\tag{B11}$$

where $\rho=\frac{1}{2}r^2\sin 2\theta$ and $z=\frac{1}{2}r^2\cos 2\theta$. The potentials in the general case are cumbersome, but in the extreme case they simplify to

$$\begin{aligned}Y^1&=\frac{a(a^2-b^2)(r^2+ab+b^2)\cos^2\theta-r^2a(2a^2+2ab+r^2)}{(a-b)^2}+\frac{a(r^2+ab+a^2)^2(r^2+ab+b^2)}{\Sigma(a-b)^2},\\ Y^2&=\frac{br^2((a+b)^2+r^2)-b(a^2-b^2)(r^2+ab+a^2)\cos^2\theta}{(a-b)^2}-\frac{b(r^2+ab+a^2)(r^2+ab+b^2)^2}{\Sigma(a-b)^2}.\end{aligned}\tag{B12}$$

The expansion at infinity is

$$Y^1 = \frac{a^3(a+b)^2}{(a-b)^2} - \frac{4J_1}{\pi}\cos^2\theta(2-\cos^2\theta) + \mathcal{O}(r^{-2}), \quad \text{(B13)}$$

$$Y^2 = -\frac{ab^2(a+b)^2}{(a-b)^2} - \frac{4J_2}{\pi}\cos^4\theta + \mathcal{O}(r^{-2}). \quad \text{(B14)}$$

The asymptotic behavior of the conformal factor at infinity is given by

$$\Phi = 1 + \frac{\mu}{4r^2} + \mathcal{O}(r^{-4}) \quad r \to \infty. \quad \text{(B15)}$$

The region $r \to 0$ corresponds to another asymptotic region. In the nonextreme case, we have

$$\Phi = \frac{\sqrt[4]{(\mu-(a+b)^2)^2(\mu-(a-b)^2)^2}}{4r^2} + \mathcal{O}(1),$$
$$\Phi_{,r} = \mathcal{O}(r^{-3}), \qquad r \to 0 \quad \text{(B16)}$$

and it is easy to verify that $\tilde{h}$ approaches the flat metric on $\mathbb{R}^4$. Hence this region is an asymptotically flat end. In the extreme case, however, one can check that

$$\Phi = \frac{(ab(a+b)^3)^{1/4}}{(a\cos^2\theta + b\sin^2\theta)r} + O(r) \quad r \to 0. \quad \text{(B17)}$$

By examining the behavior of the metric h, one can see that the asymptotic region $r \to 0$ is a cylindrical end. In fact, an explicit computation of U and λ'_i shows that the conformal metric $\tilde{h}$ approaches the metric of a cone over an S^3 equipped with an inhomogeneous metric,

$$\tilde{h} = \Omega^2(dr^2 + r^2\gamma) \quad \text{(B18)}$$

where $\Omega = \Omega(\theta) \neq 0$ and γ is conformal to the inhomogeneous metric on cross sections of the horizon of the extreme Myers-Perry black hole.

The conformal factor in either case satisfies the conditions of Eqs. (30) and (41), and we have at the asymptotic ends

$$\frac{3}{8\pi}\lim_{r\to 0}\int_{S_r}\frac{\tilde{\nabla}_c\Phi}{\Phi}n^c ds_{\tilde{h}} = 0 \quad \text{(B19)}$$

where $ds_{\tilde{h}} = r^3\sin\theta\cos\theta + \mathcal{O}(r^6)$. Now, one can expand the function V at infinity and at the origin. As we discussed before we only consider the behavior of V near $\rho = 0$. We find

$$V = \frac{(a^2-b^2)\cos 2\theta}{4r^2} + \mathcal{O}(r^{-4}),$$
$$r \to \infty, \quad \text{(B20)}$$

$$V_+ = \frac{2z + a^2 + ab}{\sqrt{4z^2 + 3a^2b^2 + 2a^2z + b^4 + 2b^2z + 4abz + a^3b + 3ab^3}}, \qquad z \in I_+, \quad \text{(B21)}$$

$$V_- = \frac{-2z + b^2 + ab}{\sqrt{a^4 + 3a^3b + 3a^2b^2 - 2a^2z + ab^3 - 4abz - 2b^2z + 4z^2}}, \qquad z \in I_-. \quad \text{(B22)}$$

Thus V satisfies Eqs. (26) and (32). In particular, we read off $\bar{V} = \frac{1}{4}(a^2 - b^2)x$ and hence from Eq. (A3) we see that $\tilde{h}$ [see Eq. (B11)] has vanishing ADM mass. In addition, when $z \to \pm\infty$ we have that $V_\pm \to 1$ and $V_\pm$ are bounded continuous functions on rods $I_\pm$. Therefore, they are integrable. Let us consider the boundedness of other terms in the mass functional (51). We will consider explicitly the nonextreme case so the end is asymptotically flat. First we have the following expansion for v at origin and infinity:

$$(\mathrm{d}v)^2 = -\frac{\mu}{2r^5} + \mathcal{O}(r^{-7}), \qquad r \to \infty,$$
$$(\mathrm{d}v)^2 = -\frac{2}{r^3} + \mathcal{O}(r^{-1}), \qquad r \to 0 \quad \text{(B23)}$$

since the volume element is $\rho\mathrm{d}\rho\mathrm{d}z = r^5\sin\theta\cos\theta\mathrm{d}r\mathrm{d}\theta$, and $(dv)^2$ is bounded at the origin and infinity. Now we consider the term which is related to the scalar curvature in mass functional (51). We use the identity (47) and we have

$$\frac{\det \mathrm{d}\lambda'}{\det \lambda'} = \mathcal{O}(r^{-8}), \qquad r \to \infty,$$
$$\frac{\det \mathrm{d}\lambda'}{\det \lambda'} = \mathcal{O}(1), \qquad r \to 0. \quad \text{(B24)}$$

This is clearly bounded. One can check numerically over a range of (a, b) that $\det \mathrm{d}\lambda' < 0$ everywhere. The only term remaining is related to the full contraction of the extrinsic curvature and we have

$$\frac{\mathrm{Tr}(\lambda'^{-1}dYdY^t)}{2\det\lambda'} = \mathcal{O}(r^{-10}), \qquad r \to \infty,$$
$$e^{-6v}\frac{\mathrm{Tr}(\lambda'^{-1}dYdY^t)}{2\det\lambda'} = \mathcal{O}(r^2), \qquad r \to 0. \quad \text{(B25)}$$

Therefore, nonextreme Myers-Perry lies in the domain on which the mass functional (51) is defined. By similar steps the same result holds for the extreme case.

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