

Modelling the evaporation of nonsingular black holes

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We present a model for studying the formation and evaporation of nonsingular (quantum corrected) black holes. The model is based on a generalized form of the dimensionally reduced, spherically symmetric Einstein-Hilbert action and includes a suitably generalized Polyakov action to provide a mechanism for radiation backreaction. The equations of motion describing self-gravitating scalar field collapse are derived in local form both in null co-ordinates and in Painlevé-Gullstrand (flat slice) co-ordinates. They provide the starting point for numerical studies of complete spacetimes containing dynamical horizons that bound a compact trapped region. Such spacetimes have been proposed in the past as solutions to the information loss problem because they possess neither an event horizon nor a singularity. Since the equations of motion in our model are derived from a diffeomorphism invariant action they preserve the constraint algebra and the resulting energy momentum tensor is manifestly conserved.

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I. INTRODUCTION AND BACKGROUND

General relativity predicts the existence of singularities at the center of all black holes, while Hawking's famous quantum calculation implies that event horizons shrink by emitting thermal radiation. These two properties of black holes, combined with the fact that special relativity forbids information from escaping from an event horizon, lead to one of the deepest puzzles of modern theoretical physics: the so-called information loss paradox. In its simplest form, the question is: What is the endpoint of black hole formation and evaporation and what happens to the information about the state of the matter that formed the black hole?

A variety of end states have been suggested over the years. Among them:

- (1) The black hole evaporates completely via thermal radiation so that the information is lost to the outside world (asymptotic region of the black hole spacetime) forever. This scenario entails a breakdown of the unitarity of quantum mechanics.
- (2) At late stages the radiation is no longer thermal and the information emerges via quantum gravity related, presumably causality violating, corrections.
- (3) The black hole stops emitting radiation when it nears the Planck scale, leaving behind a microscopic remnant that hides forever an enormous amount of information.

- (4) Most recently it has been suggested [1,2]¹ that a firewall of as yet unknown origin exists just outside the event horizon that destroys the entanglement between infalling particles and outgoing radiation, effectively eliminating the information loss problem, albeit at a significant cost.

The purpose of the present paper is to set up a semi-classical model that explicitly realizes an alternative proposal to resolve the information loss problem. This proposal is based on the principle that there are no true singularities in nature. This idea is not new. It was first put forward by Frolov and Vilkovisky [5,6] and discussed more recently by [7].² We start with the assumption that in the correct theory the classical singularity will be replaced by a semi-classical nonsingular region. Moreover, we assume that the dynamics of gravitational collapse and evaporation in this region can be described by effective semi-classical equations of motion. Our first goal is to construct a suitable "quantum corrected"³ Lagrangian, potentially relevant to four-dimensional black holes, from which to derive these semi-classical equations. Such an action must have three key attributes:

- (1) It must incorporate modifications to Einstein gravity at short distances that resolve the singularity.
- (2) It must have a radiation backreaction term.

¹See also [3,4] where this issue is addressed in a canonically quantized model of dust collapse.

²See also Varadarajan [8].

³These quantum corrections ultimately derive from the underlying, as yet unknown, quantum gravity theory. Later on we will discuss corrections to the action which mimic Hawking radiation. These corrections are derived from quantization of matter fields but, to avoid confusion, we will refer to them as radiation corrections or radiation terms.

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- (3) At large distance scales it must accurately reproduce Einstein's theory.

The expectation is that the resulting gravitational collapse and subsequent evaporation will produce neither an event horizon nor a singularity. Instead the result will be a completely nonsingular spacetime that contains a closed dynamical horizon bounding a compact trapped region. At late times there will therefore be no impediment to information about the collapsed matter escaping to null infinity. Such a scenario was discussed in [5,6] and first made explicit by Sean Hayward [9]. More recently, a compact, dynamical trapping horizon was realized via the numerical simulation of the spherical collapse of a massless scalar field in [10]. In this paper the singularity resolution and energy loss were modeled by introducing explicit modifications to the gravitational potential in the equations of motion. Since the equations were not derived from a diffeomorphism invariant action, they did not preserve the constraint algebra and were nonconservative. A more systematic energy conserving model was later constructed [11] and used to show that nonsingular black holes could indeed be formed via gravitational collapse⁴ provided that the effective quantum corrections to the gravitational potential were introduced within the framework of a (spherical symmetry preserving) diffeomorphism invariant action. The equations in this model were derived from a variational principle and hence are conservative, but they did not contain a mechanism for describing Hawking radiation.

In the following, we continue the above program by constructing a set of dynamical equations for four-dimensional (and higher) gravity that lead to nonsingular collapse, are energy conserving and include radiation backreaction. Modeling the effect of Hawking radiation is difficult and so far it is not well understood in more than two dimensions. In two dimensions, however, the conformal anomaly can be calculated at one-loop order and integrated to derive the nonlocal Polyakov action [15],

$$I_{\text{Poly}} \sim - \int d^2x \sqrt{-g} \mathcal{R} \frac{1}{D^2} \mathcal{R}, \quad (1)$$

where g is the determinant of the two-dimensional metric, $g_{\mu\nu}$, \mathcal{R} is the Ricci scalar calculated with $g_{\mu\nu}$ and D is the covariant derivative compatible with $g_{\mu\nu}$. This one-loop effective action is exact in the large N limit, where N is the number of conformally coupled scalar fields. It has been studied extensively in the context of two-dimensional models for black hole evaporation [16–19]. It has also been used in a toy model to simulate Hawking radiation in four-dimensional, spherically symmetric Einstein gravity [20,21]. This is the approach we will adopt. The new, and crucial features of our analysis are twofold: first we will

⁴See also [12] and [13] for closely related work and [14] for early work on singularity resolved black holes.

formulate the theory in terms of a local, diffeomorphism invariant effective action. Our equations are therefore energy conserving and allow for the use of slicings that extend into the horizon. Second, the action which is our starting point is a generalization of the spherically symmetric Einstein action that allows us to use quantum motivated corrections that resolve the singularity in the vacuum solution. Since the class of Lagrangians we consider obey a Birkhoff theorem, this guarantees that gravitational collapse will yield an exterior spacetime that asymptotes to the corresponding nonsingular solution. We therefore expect a complete absence of singularities in the dynamical evolution of the collapsing matter as found in [11]. This will in turn permit us to examine via rigorous calculations, albeit in a simplistic model, the question posed above: How does singularity resolution affect the end point of gravitational collapse and Hawking radiation?

Recently several other models have been proposed to investigate nonsingular, radiating black holes [22–25] using different methods to model the radiation and singularity resolution than those used in this paper. Using the Polyakov term to model the radiation backreaction has the advantage that it is rigorous, at least in two dimensions. It is also useful to have a variety of plausible models to hunt for generic features of the formation and evaporation process.

In the present paper, we describe the model and derive the equations of motion both in conformal gauge and in P-G-type coordinates. The organization is as follows: Sec. II discusses the classical action (without radiating terms). Section III discusses the addition of the radiation corrections via the generalized Polyakov action in local form. Section IV derives the equations of motion in conformal gauge and discusses singularity resolution in the energy momentum tensor while Sec. V presents the Hamiltonian analysis and the equations of motion in P-G-type coordinates. Section VI closes with conclusions and prospects. The numerical simulations of these equations are relegated to a subsequent paper.

II. EFFECTIVE GRAVITATIONAL ACTION

We start from Einstein gravity in n spacetime dimensions minimally coupled to a massless scalar field, whose action is:

$$I_{(n)} = \frac{1}{16\pi G^{(n)}} \int d^n x \sqrt{-\bar{g}} (\mathcal{R}(\bar{g}) - 8\pi G^{(n)} (\bar{D}\psi)^2) \quad (2)$$

where $G^{(n)}$ is the higher-dimensional gravitational constant, $\mathcal{R}(\bar{g})$ is the Ricci scalar calculated using the n -dimensional metric, \bar{g} , \bar{D} is the derivative compatible with \bar{g} and ψ is a scalar field. After imposing spherical symmetry, integrating out the angular variables and absorbing a factor of $\sqrt{8\pi G^{(n)}}$ into the scalar field to make it dimensionless, this action takes the form [26,27]:

$$\begin{aligned}
 I_{(2)} = & \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} [R^{n-2} \mathcal{R} \\
 & + (n-2)(n-3)R^{(n-4)}(DR)^2 \\
 & + (n-2)(n-3)R^{(n-4)} - R^{n-2}(D\psi)^2], \quad (3)
 \end{aligned}$$

where R is the areal radius, \mathcal{R} is the Ricci scalar of g which is the (t, x) part of the higher-dimensional metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + R^2 d\Omega^{(n-2)}, \quad (4)$$

and we have defined the length parameter

$$l^{(n-2)} := \frac{16\pi G^{(n)}}{\mathcal{A}_{(n-2)}}, \quad (5)$$

with $\mathcal{A}_{(n-2)}$ being the invariant volume of a unit $n-2$ -sphere. In terms of these parameters the well known Schwarzschild-Tangherlini solution is:

$$\begin{aligned}
 ds^2 = & - \left(1 - \frac{l^{(n-2)} M}{(n-2)R^{n-3}} \right) dt^2 \\
 & + \left(1 - \frac{l^{(n-2)} M}{(n-2)R^{n-3}} \right)^{-1} dR^2 + R^2 d\Omega^{(n-2)} \quad (6)
 \end{aligned}$$

where M is the ADM mass in the vacuum case.

We now generalize the above allowing the coefficients of each of the three terms in (3) to be arbitrary functions of the areal radius $R(x)$, as

$$\begin{aligned}
 I_{\text{general}} = & \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} \\
 & \times \{ \phi(R) \mathcal{R} + h(R)(DR)^2 + V(R) + B(R)(D\psi)^2 \}. \quad (7)
 \end{aligned}$$

The above is the most general action that contains at most two derivatives of the metric and areal radius and yields a Hamiltonian that is quadratic in their conjugate momenta. It can in principle be further generalized to take the general form of a dimensionally reduced higher curvature Lovelock gravity, whose equations of motion are second order, but higher order in momenta. See [28–30] for a Hamiltonian analysis of spherically symmetric Lovelock gravity. We relegate the investigation of such theories to future work.

For (7) to reduce to the GR case, two conditions must be satisfied,

$$\phi = -B = R^{n-2} \quad (8)$$

$$h = V = \phi'', \quad (9)$$

where a prime means differentiation with respect to the areal radius. In this paper we are concerned with the case where (9) is obeyed reducing the number of free functions

by two (although, for the purpose of generality we do not substitute (9) into our equations until we discuss important physical results). As we will show, (9) will be necessary to remove the singularity in the vacuum solution while simultaneously removing another one in the radiating term. It will also become important when defining a mass function (see Appendix B) and when finding boundary terms at infinity which make the variational principal well defined (see Appendix C).

The action (7) belongs to the class of theories called generic two-dimensional dilaton gravity (see [31] and [32] for reviews). These theories obey a Birkhoff theorem [33]. The most general vacuum solution can be found by defining a new metric $\tilde{g}_{\mu\nu} := \omega^2 g_{\mu\nu}$ which puts the action in the form

$$I = l^{-(n-2)} \int d^2x \sqrt{-\tilde{g}} (\phi \mathcal{R}(\tilde{g}) + V/\omega^2) \quad (10)$$

whose equations of motion can readily be solved [33]. The corresponding metric in the current parametrization is

$$\begin{aligned}
 ds^2 = & \frac{j}{\omega^2} \left[- \left(1 - \frac{l^{(n-2)} M}{j} \right) dt^2 \right. \\
 & \left. + \left(1 - \frac{l^{(n-2)} M}{j} \right)^{-1} \left(\frac{\phi'}{j} \right)^2 dR^2 \right] + R^2 d\Omega^2, \quad (11)
 \end{aligned}$$

where

$$\ln(\omega^2) := \int \frac{h}{\phi'} dR \quad (12)$$

$$j := \int (\phi' V / \omega^2) dR. \quad (13)$$

The solution contains a single parameter, M , and has at least one Killing vector ($\partial/\partial t$). There are horizons whenever $j = l^{(n-2)} M$, and the Killing vector is timelike in the asymptotic region, $j > l^{(n-2)} M$. Most importantly, the arbitrary functions of the areal radius $\phi(R)$, $h(R)$ and $V(R)$ can, as we shall see, be chosen so that the resulting vacuum solutions are nonsingular.

As an example of a nonsingular metric that can be obtained as a solution to the equations derived from an action of the form (7) consider the following metric that was originally proposed by Poisson and Israel [34]:

$$\begin{aligned}
 ds^2 = & - \left(1 - \frac{l^2 M R^2}{2(R^3 + \nu^3)} \right) dt^2 \\
 & + \left(1 - \frac{l^2 M R^2}{2(R^3 + \nu^3)} \right)^{-1} dR^2 + R^2 d\Omega^2. \quad (14)
 \end{aligned}$$

This is of the same form as (11) with the following identifications:

$$\phi' = j = 2 \frac{R^3 + \nu^3}{R^2} \quad (15)$$

$$\phi = \frac{R^4 - 2\nu^3 R}{R^2} \quad (16)$$

$$\omega^2 = j = 2 \frac{R^3 + \nu^3}{R^2} \quad (17)$$

$$h = \ln(\omega^2)' \phi' = j' = 2 \frac{R^3 - 2\nu^3}{R^3} \quad (18)$$

$$V = \omega^2 j' / \phi' = 2 \frac{R^3 - 2\nu^3}{R^3}. \quad (19)$$

In the above ν is the parameter that determines the scale of the quantum corrections. The above metric approaches a de Sitter metric as $R \rightarrow 0$ and hence is manifestly nonsingular. The resulting spacetime has two horizons and the effective stress energy tensor violates the weak energy condition at short length scales (of the order of ν). The properties of this metric are discussed more fully in [11]. They can easily be generalized to higher dimensions with

$$j(R) = (n-2) \frac{R^{n-1} + \nu^{n-1}}{R^2}. \quad (20)$$

A qualitatively different example of a nonsingular static spacetime was derived from polymer quantum gravity in [35] and [36]. This spacetime contains a single bifurcative horizon that surrounds an infinite Kasner type universe on the interior. It describes, in effect, a wormhole whose minimum throat radius shrinks to the polymerization scale before reexpanding to infinity. Details of how it fits into the current formalism are given in Appendix A.

III. RADIATION TERMS: THE GENERALIZED POLYAKOV ACTION

The radiation term (1) assumes that the matter under consideration is minimally coupled to gravity, i.e. B is a constant in (7). We will use a more general version that allows for more general matter couplings [37–40],

$$I_{\text{Poly}} \sim - \int d^2x \sqrt{-g} \left[\mathcal{R} \frac{1}{D^2} \mathcal{R} + b(R) \left(\frac{1}{D^2} \mathcal{R} - \ln \mu^2 \right) (DR)^2 + c(R) \mathcal{R} \right], \quad (21)$$

where μ is a constant related to the renormalization procedure used to obtain the Polyakov action. The forms of b and c have been suggested by [40] to be $b = -3(B')^2/B^2$ and $c = -6 \ln B$. Since it may be interesting to also investigate the case where $b = c = 0$ (as done by [20]) we perform the algebra without making any

assumptions about b and c until we discuss important, physical results (see Appendix C).

The nonlocal character of the actions (1) and (21) is often dealt with by working in double-null coordinates,

$$ds^2 = e^{2f} dudv, \quad (22)$$

where f is a scalar function and the Ricci scalar takes the form

$$\mathcal{R} = -8e^{-2f} f_{,uv} = 2D^2 f. \quad (23)$$

It is sometimes advantageous, however, to work in other coordinate systems, particularly when performing numerical simulations that go past horizon formation (see numerics section of [41]). In this case auxiliary fields, z_1 and z_2 can be used to write the action as [40,42,43]

$$I_{\text{Poly}} \sim - \int \sqrt{-g} [\mathcal{R}(z_1 + z_2) + D_A z_2 D^A z_1 + b(DR)^2 (z_1 - \ln \mu^2) + c\mathcal{R}], \quad (24)$$

where $A = 0, 1$. The equations of motion for z_1 and z_2 are given by

$$D^2 z_1 - \mathcal{R} = 0, \quad (25)$$

$$D^2 z_2 - \mathcal{R} - b(DR)^2 = 0. \quad (26)$$

Inserting (25) and (26) into (24) gives (21).

Inspired by (7) and (24) we define the action that we will consider for the rest of this paper as

$$I = \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} \{ \phi(R) \mathcal{R} + h(R) (DR)^2 + V(R) + W [\mathcal{R}(z_1 + z_2) + D_A z_2 D^A z_1 + b(R) (DR)^2 (z_1 - \ln \mu^2) + c(R) \mathcal{R}] + B(R) (D\psi)^2 \}, \quad (27)$$

where W is a coupling constant.

It is worth noting here that Ayal and Piran [20] (see also [21]) considered a model similar to this one. In fact they considered the equations of motion, in null gauge, that would come from varying (27) with the fields given as those for the four-dimensional GR case with $b = c = 0$ and $z_1 = z_2$. They eventually modified those equations of motion to remove a singularity at $R = 0$ in the effective stress energy tensor. The subsequent equations, however, did not obey the Bianchi identities, since they altered the equations of motion directly and not their corresponding action.

IV. DOUBLE NULL COORDINATES

In this section we derive the covariant equations of motion, constraints and effective energy momentum tensor from the action (27) in null gauge.

A. Action

We start with the metric

$$g_{\mu\nu} = \begin{bmatrix} 0 & e^{2f}/2 \\ e^{2f}/2 & 0 \end{bmatrix} \quad (28)$$

from which the Ricci scalar is given by (23). Using (28) the action, (27) becomes

$$I = \int d^2x \left\{ -4\phi(R)f_{,uv} + 2h(R)R_{,u}R_{,v} + \frac{1}{2}e^{2f}V(R) \right. \\ \left. + W[-4f_{,uv}(z_1 + z_2) + (z_{1,u}z_{2,v} + z_{2,u}z_{1,v}) \right. \\ \left. + 2b(R)R_{,u}R_{,v}(z_1 - \ln \mu^2) - 4c(R)f_{,uv}] \right. \\ \left. + 2B(R)\psi_{,u}\psi_{,v} \right\}, \quad (29)$$

where we have absorbed the factor of l^{n-2} into the action.

B. Equations of motion

Varying (29) gives the equations of motion

$$\frac{\delta I}{\delta R} = 0 = -4\phi'f_{,uv} - 2h'R_{,u}R_{,v} - 4hR_{,uv} + (e^{2f}/2)V' \\ + 2B'\psi_{,u}\psi_{,v} + W[-2(b'R_{,u}R_{,v} + 2bR_{,uv})(z_1 - \ln \mu^2) \\ - 2b(R_{,u}z_{1,v} + R_{,v}z_{1,u}) - 4c'f_{,uv}], \quad (30)$$

$$\frac{\delta I}{\delta f} = 0 = -4R_{,uv}\phi' - 4R_{,u}R_{,v}\phi'' + e^{2f}V \\ + W[-4z_{1,uv} - 4z_{2,uv} - 4R_{,uv}c' - 4R_{,u}R_{,v}c''], \quad (31)$$

$$\frac{\delta I}{\delta \psi} = 0 = -2B'R_{,u}\psi_{,v} - 4B\psi_{,uv} - 2B'R_{,v}\psi_{,u}, \quad (32)$$

$$\frac{\delta I}{\delta z_2} = 0 = -4Wf_{,uv} - 2Wz_{1,uv} \quad (33)$$

and

$$\frac{\delta I}{\delta z_1} = 0 = -4Wf_{,uv} - 2Wz_{2,uv} + 2WbR_{,u}R_{,v}. \quad (34)$$

Equations (33) and (34) can be used to write (30) and (31) with out any dependence on z_1 or z_2 as

$$-4\phi'f_{,uv} - 2h'R_{,u}R_{,v} - 4hR_{,uv} + (e^{2f}/2)V' + 2B'\psi_{,u}\psi_{,v} \\ + W[-2(b'R_{,u}R_{,v} + 2bR_{,uv})(-2f - \ln \mu^2) \\ + 4b(R_{,u}f_{,v} + R_{,v}f_{,u}) - 4c'f_{,uv}] = 0 \quad (35)$$

and

$$-4R_{,uv}\phi' - 4R_{,u}R_{,v}\phi'' + e^{2f}V \\ + W[16f_{,uv} - 4bR_{,u}R_{,v} - 4R_{,uv}c' - 4R_{,u}R_{,v}c''] = 0 \quad (36)$$

Using the GR values for ϕ , h , V and B as well as $W = -\alpha/8\pi$, $b = c = 0$, $n = 4$ and $u \rightarrow -u$ (due to different sign conventions) we find that (35) and (36) reduce to (11a) and (11b) of [20] (where α is the coupling constant used in that paper).

C. Constraint equations

To find the constraint equations we use the most general metric,

$$g_{\mu\nu} = \begin{bmatrix} A & e^{2f}/2 \\ e^{2f}/2 & C \end{bmatrix}, \quad (37)$$

which corresponds to the Ricci scalar

$$\mathcal{R} = (-g)^{-2} \{ A[(C_{,u})^2/2 + C_{,v}A_{,v}/2 - e^{-2f}C_{,v}f_{,u}] \\ + C[(A_{,v})^2/2 + C_{,u}A_{,u}/2 - e^{-2f}A_{,u}f_{,v}] \\ + (e^{2f}/2)[-e^{2f}A_{,v}f_{,v} - e^{2f}C_{,u}f_{,u} \\ + A_{,u}C_{,v}/2 - C_{,u}A_{,v}/2 + 8ACf_{,u}f_{,v}] \} \\ + (-g)^{-1} \{ A_{,vv} - 2e^{2f}f_{,uv} + C_{,uu} \}. \quad (38)$$

From (38) we can see the following useful relationships (up to boundary terms)

$$\left(\frac{\delta}{\delta A} \int d^2x \sqrt{-g} \phi \mathcal{R} \right) \Big|_{A=C=0} \\ = (2\phi_{,vv} - 4f_{,v}\phi_{,v})e^{-2f} \\ = (2\phi''(R_{,v})^2 + 2\phi'R_{,vv} - 4f_{,v}\phi'R_{,v})e^{-2f} \quad (39)$$

$$\left(\frac{\delta}{\delta A} \int d^2x \sqrt{-g} h(DR)^2 \right) \Big|_{A=C=0} \\ = -2he^{-2f}(R_{,v})^2. \quad (40)$$

This gives the constraint

$$\begin{aligned} \left. \frac{\delta I}{\delta A} \right|_{A=C=0} = 0 &= 2e^{-2f}[\phi''(R,v)^2 + \phi' R_{,vv} - 2\phi' f_{,v} R_{,v}] \\ &\quad - 2e^{-2f} h(R,v)^2 - 2e^{-2f} B(\psi,v)^2 \\ &\quad + W\{2e^{-2f}[z_{1,vv} + z_{2,vv} - 2f_{,v} z_{1,v} - 2f_{,v} z_{2,v}] \\ &\quad - 2e^{-2f} z_{1,v} z_{2,v} - 2e^{-2f} b(z_1 - \ln \mu^2)(R,v)^2 \\ &\quad + 2e^{-2f}[c''(R,v)^2 + c' R_{,vv} - 2c' f_{,v} R_{,v}]\}. \end{aligned} \quad (41)$$

Note from (33) that $z_1(u, v) = -2f(u, v) + p_u(u) + p_v(v)$, where p_u and p_v are some functions. If we assume that p_u and p_v are constants and rearrange (41) and use (33) and (34) we get

$$\begin{aligned} &-R_{,vv} + 2f_{,v} R_{,v} \\ &= \frac{\tilde{\phi}'' + 2Wb(f + \ln |\mu|) - h}{\tilde{\phi}'} (R,v)^2 \\ &\quad - \frac{B}{\tilde{\phi}'} (\psi,v)^2 - \frac{4W}{\tilde{\phi}'^2} \{f_{,vv} - (f_{,v})^2 - \bar{b}_{,v}/4\}, \end{aligned} \quad (42)$$

where we define

$$\tilde{\phi} := \phi + Wc \quad (43)$$

$$\bar{b}(v) := \int du (bR_{,u} R_{,v}), \quad (44)$$

and we have used (34) to write

$$z_{2,vv} = -2f_{,vv} + \bar{b}_{,v}. \quad (45)$$

There is also a second constraint equation which can be found by swapping u and v in (42).

D. Effective stress energy tensor

We now use the equations of motion and Einstein's equations to calculate the energy momentum tensor. Using (2.18) of [27] we find that the nonangular components of the Einstein tensor in n dimensions are given by

$$G_{uu}^{(n)} = \frac{(n-2)}{R} (-R_{,uu} + 2f_{,u} R_{,u}) \quad (46)$$

$$G_{vv}^{(n)} = \frac{(n-2)}{R} (-R_{,vv} + 2f_{,v} R_{,v}) \quad (47)$$

$$G_{uv}^{(n)} = \frac{(n-2)}{R} \left[R_{,uv} - \frac{(n-3)}{R} \left(\frac{e^{2f}}{4} - R_{,u} R_{,v} \right) \right]. \quad (48)$$

Concentrating on the $u-v$ component, using the equation of motion (36) to solve for $R_{,uv}$ and the expression for the mass (B8),

$$\begin{aligned} (DR)^2 &= \frac{j\omega^2}{\phi'^2} \left(1 - \frac{M}{j} \right) \\ &= 4e^{-2f} R_{,u} R_{,v}, \end{aligned} \quad (49)$$

(where a factor of l^{n-2} has been absorbed into M) gives the following expression for the effective stress tensor:

$$\begin{aligned} G_{uv}^{(n)} &= \frac{(n-2)}{\tilde{\phi}' R} \left\{ \frac{e^{2f} M}{4j} \left(\tilde{\phi}'' - \frac{(n-3)\tilde{\phi}'}{R} \right) \right. \\ &\quad \left. + W \left[4f_{,uv} - \frac{e^{2f}}{4} \left(c'' + b \left(1 - \frac{M}{j} \right) \right) \right] \right\}. \end{aligned} \quad (51)$$

In the above we have assumed only that (9) is satisfied, which also implies that $j = \phi' = \omega^2$. This is necessary in order to eliminate terms in the stress energy tensor that are singular at $R = 0$. These conditions are satisfied for the two horizon quantum corrected black hole described by (15)–(19). Assuming reasonable regularity conditions on the mass function and c and b at the origin, the singularity observed by Piran and Ayal in the effective stress tensor is resolved as long as the singularity in $1/j$ is resolved. Similar statements apply to the diagonal components of the energy momentum tensor as we can see by comparing (47) to (42).

V. HAMILTON'S EQUATIONS IN NON-NULL COORDINATES

To find the equations of motion in non-null coordinates. We start with the general, ADM metric [44],

$$ds^2 = -N^2 dt^2 + \Lambda^2 (N_r dt + dx)^2. \quad (52)$$

For later use, we compute the following quantities:

$$(DR)^2 = -R_y^2 + \Lambda^{-2} R_{,x}^2, \quad (53)$$

$$\sqrt{-g_{(2)}} D^2 R = -\partial_t (\Lambda R_{,y}) + \partial_x (\Lambda N_r R_{,y} + \Lambda^{-1} N R_{,x}), \quad (54)$$

where we define the operator y acting on some field β by

$$\beta_{,y} := N^{-1} (\beta_{,t} - N_r \beta_{,x}). \quad (55)$$

Using the metric (52) the Ricci scalar can be written as

$$\begin{aligned} \sqrt{-g} \mathcal{R} &= 2\{N_{r,x} N_{,y} \Lambda N^{-1} - 2N_{r,x} \Lambda_{,y} - \Lambda (N_{r,x})_{,y} \\ &\quad + N^{-1} \Lambda (N_{r,x})^2 + N \Lambda_{,yy} - (N_{,x} \Lambda^{-1})_{,x}\}. \end{aligned} \quad (56)$$

To write the action in a form that lends itself to Hamiltonian analysis we define

$$\begin{aligned} \Phi(R(t, x), t, x) &:= \phi(R) + W(z_1 + z_2) + Wc(R) \\ &= \tilde{\phi}(R) + W(z_1 + z_2) \end{aligned} \quad (57)$$

and

$$h_z(R(t, x), z_1(t, x)) := h(R) + Wb(R)(z_1 - \ln \mu^2). \quad (58)$$

With these definitions the action looks like

$$I = \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} \{ \Phi \mathcal{R} + h_z(DR)^2 + V + W D_A z_2 D^A z_1 + B(D\psi)^2 \}. \quad (59)$$

We first work with the term containing the Ricci scalar. From (56) we can calculate this term up to total derivatives (t.d.) to be

$$\sqrt{-g} \mathcal{R} \Phi = 2\Phi_{,y} (N_{r,x} \Lambda - N \Lambda_{,y}) - 2N (\Phi_{,x} \Lambda^{-1})_{,x} + \text{t.d.} \quad (60)$$

Using this the action (59) becomes

$$\begin{aligned} I = \frac{1}{l^{n-2}} \int d^2x \{ & 2N^{-1} (\Phi_{,t} - N_r \Phi_{,x}) (N_{r,x} \Lambda - (\Lambda_{,t} - N_r \Lambda_{,x})) - 2N (\Phi_{,x} \Lambda^{-1})_{,x} + N \Lambda V \\ & + h_z [-N^{-1} \Lambda R_{,t}^2 + 2N_r N^{-1} \Lambda R_{,t} R_{,x} + (N \Lambda^{-1} - N_r^2 N^{-1} \Lambda) R_{,x}^2] \\ & + W [-N^{-1} \Lambda z_{1,t} z_{2,t} + N_r N^{-1} \Lambda (z_{1,x} z_{2,t} + z_{1,t} z_{2,x}) + (N \Lambda^{-1} - N_r^2 N^{-1} \Lambda) z_{1,x} z_{2,x}] \\ & + B [-N^{-1} \Lambda \psi_{,t}^2 + 2N_r N^{-1} \Lambda \psi_{,t} \psi_{,x} + (N \Lambda^{-1} - N_r^2 N^{-1} \Lambda) \psi_{,x}^2] \}. \end{aligned} \quad (61)$$

We can see from (61) that the conjugate momenta corresponding to N and N_r are zero, $P_N = P_{N_r} = 0$. The remaining conjugate momenta, corresponding to Λ , R , z_1 , z_2 and ψ are given by

$$\begin{aligned} P_\Lambda &= -l^{-(n-2)} 2N^{-1} (\Phi_{,t} - N_r \Phi_{,x}) \\ &= -l^{-(n-2)} 2(\tilde{\phi}' R_{,y} + W(z_1 + z_2)_{,y}), \end{aligned} \quad (62)$$

$$\begin{aligned} P_R &= l^{-(n-2)} [2N^{-1} \tilde{\phi}' (N_{r,x} \Lambda - (\Lambda_{,t} - N_r \Lambda_{,x})) \\ &\quad - 2\Lambda h_z N^{-1} (R_{,t} - N_r R_{,x})] \\ &= l^{-(n-2)} [2N^{-1} \tilde{\phi}' N_{r,x} \Lambda - 2\tilde{\phi}' \Lambda_{,y} - 2\Lambda h_z R_{,y}], \end{aligned} \quad (63)$$

$$\begin{aligned} P_{z_1} &= l^{-(n-2)} [2N^{-1} W (N_{r,x} \Lambda - (\Lambda_{,t} - N_r \Lambda_{,x})) \\ &\quad + W (-N^{-1} \Lambda z_{2,t} + N_r N^{-1} \Lambda z_{2,x})] \\ &= l^{-(n-2)} W [2N^{-1} N_{r,x} \Lambda - 2\Lambda_{,y} - \Lambda z_{2,y}], \end{aligned} \quad (64)$$

$$\begin{aligned} P_{z_2} &= l^{-(n-2)} [2N^{-1} W (N_{r,x} \Lambda - (\Lambda_{,t} - N_r \Lambda_{,x})) \\ &\quad + W (-N^{-1} \Lambda z_{1,t} + N_r N^{-1} \Lambda z_{1,x})] \\ &= l^{-(n-2)} W [2N^{-1} N_{r,x} \Lambda - 2\Lambda_{,y} - \Lambda z_{1,y}], \end{aligned} \quad (65)$$

and

$$\begin{aligned} P_\psi &= l^{-(n-2)} B [-2N^{-1} \Lambda \psi_{,t} + 2N_r N^{-1} \Lambda \psi_{,x}] \\ &= -2l^{-(n-2)} B \Lambda \psi_{,y}, \end{aligned} \quad (66)$$

respectively. By combining (62)–(65) we get

$$\begin{aligned} z_{1,y} &= \frac{-\Lambda P_\Lambda / 2 - \Lambda \tilde{\phi}' R_{,y} + P_{z_1} - P_{z_2}}{2W\Lambda} \\ &= -S_3 - \frac{P_{z_2}}{W\Lambda}, \end{aligned} \quad (67)$$

$$\begin{aligned} z_{2,y} &= \frac{-\Lambda P_\Lambda / 2 - \Lambda \tilde{\phi}' R_{,y} - P_{z_1} + P_{z_2}}{2W\Lambda} \\ &= -S_3 - \frac{P_{z_1}}{W\Lambda} \end{aligned} \quad (68)$$

and

$$\begin{aligned} R_{,y} &= \frac{\tilde{\phi}' (-\Lambda P_\Lambda / 2 + P_{z_1} + P_{z_2}) - 2W P_R}{\Lambda (\tilde{\phi}'^2 + 4W h_z)} \\ &= \frac{-\tilde{\phi}' S_1 - 2W P_R}{S_2}, \end{aligned} \quad (69)$$

where, for ease of notation we have defined

$$S_1 := \Lambda P_\Lambda / 2 - P_{z_1} - P_{z_2}, \quad (70)$$

$$S_2 := \Lambda (\tilde{\phi}'^2 + 4W h_z) \quad (71)$$

and

$$S_3 := \frac{2h_z S_1 - \tilde{\phi}' P_R}{S_2}, \quad (72)$$

and we have absorbed a factor of l^{n-2} into all of the conjugate momenta. Plugging (67)–(69) as well as (63) and (66) into the action (61), it can be shown that the Hamiltonian density, \mathcal{H} , can be written as a sum of two constraints,

$$\mathcal{H} = NH + N_r H_r + \text{t.d.}, \quad (73)$$

where the constraints are given by

$$\begin{aligned} H = & 2 \left(\frac{\tilde{\phi}' R_{,x}}{\Lambda} \right)_{,x} - \frac{h_z R_{,x}^2}{\Lambda} - \Lambda V \\ & + W \left[-\frac{z_{1,x} z_{2,x}}{\Lambda} + 2 \left(\frac{z_{1,x} + z_{2,x}}{\Lambda} \right)_{,x} \right] \\ & + \frac{h_z S_1^2 - \tilde{\phi}' P_R S_1 - W P_R^2}{S_2} - \frac{P_{z1} P_{z2}}{W\Lambda} \\ & - \frac{P_\psi^2}{4\Lambda B} - \frac{B\psi_{,x}^2}{\Lambda} \end{aligned} \quad (74)$$

and

$$H_r = -P_{\Lambda,x} \Lambda + P_R R_{,x} + P_{z1} z_{1,x} + P_{z2} z_{2,x} + P_\psi \psi_{,x}. \quad (75)$$

Note that we absorbed a factor of l^{n-2} into $H^{(M)}$, $H_r^{(M)}$, $H^{(G)}$ and $H_r^{(G)}$ (in addition to all of the conjugate momenta).

From (73) we can write down the equations of motion as

$$R_{,t} = N R_{,y} + N_r R_{,x} \quad (76)$$

$$\begin{aligned} P_{R,t} = & - \left(\frac{2N_{,x}}{\Lambda} \right)_{,x} \tilde{\phi}' - \left(\frac{2Nh_z R_{,x}}{\Lambda} \right)_{,x} \\ & + \frac{Nh'_z R_{,x}^2}{\Lambda} + N\Lambda V' \\ & + N \frac{-2\tilde{\phi}'' S_1 R_{,y} - \Lambda [2\tilde{\phi}' \tilde{\phi}'' + 4Wh'_z] R_{,y}^2}{4W} \\ & - \frac{NP_\psi^2 B'}{4\Lambda B^2} + \frac{NB'\psi_{,x}^2}{\Lambda} + (N_r P_R)_{,x} \end{aligned} \quad (77)$$

$$\Lambda_{,t} = N\Lambda \frac{S_3}{2} + (N_r \Lambda)_{,x} \quad (78)$$

$$\begin{aligned} P_{\Lambda,t} = & - \frac{2N}{\Lambda} \left(\frac{\tilde{\phi}' R_{,x}}{\Lambda} \right)_{,x} - \frac{2N_{,x} \tilde{\phi}' R_{,x}}{\Lambda^2} + 2NV \\ & - 2W \left[\frac{N}{\Lambda} \left(\frac{z_{1,x} + z_{2,x}}{\Lambda} \right)_{,x} + N_{,x} \left(\frac{z_{1,x} + z_{2,x}}{\Lambda^2} \right) \right] \\ & - \frac{NP_\Lambda S_3}{2} + \frac{NH}{\Lambda} + P_{\Lambda,x} N_r \end{aligned} \quad (79)$$

$$\psi_{,t} = -\frac{NP_\psi}{2\Lambda B} + N_r \psi_{,x} \quad (80)$$

$$P_{\psi,t} = \left(-\frac{2NB\psi_{,x}}{\Lambda} + N_r P_\psi \right)_{,x} \quad (81)$$

$$z_{1,t} = -NS_3 - \frac{NP_{z2}}{W\Lambda} + N_r z_{1,x} \quad (82)$$

$$\begin{aligned} P_{z1,t} = & \left[-\frac{W(Nz_{2,x} + 2N_{,x})}{\Lambda} + N_r P_{z1} \right]_{,x} \\ & - N\Lambda W b \left[R_{,y}^2 - \frac{R_{,x}^2}{\Lambda^2} \right] \end{aligned} \quad (83)$$

$$z_{2,t} = -NS_3 - \frac{NP_{z1}}{W\Lambda} + N_r z_{2,x} \quad (84)$$

$$P_{z2,t} = \left[-\frac{W(Nz_{1,x} + 2N_{,x})}{\Lambda} + N_r P_{z2} \right]_{,x}, \quad (85)$$

where we used the following useful identity,

$$\frac{S_1^2/\Lambda - S_2 R_{,y}^2}{4W} = \frac{h_z S_1^2 - \tilde{\phi}' P_R S_1 - W P_R^2}{S_2}. \quad (86)$$

Since our Hamiltonian is the sum of two first class constraints we must now pick two gauge choices. We will pick

$$\chi := R - x = 0, \quad (87)$$

which makes the spatial coordinate the areal radius.

Before choosing the second gauge notice that the mass function (see Appendix B) is given by

$$\begin{aligned} M = & j \left(1 - \frac{(\phi')^2}{j\omega^2} (DR)^2 \right) \\ = & j \left(1 - \frac{(\phi')^2}{j\omega^2} \left[\left(\frac{R_{,x}}{\Lambda} \right)^2 - \left(\frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2} \right)^2 \right] \right), \end{aligned} \quad (88)$$

where we used (69). This inspires the choice of gauge

$$\Lambda - \sqrt{\frac{(\phi' R_{,x})^2}{j\omega^2}} = 0. \quad (89)$$

We are interested in the case where (9) is satisfied (which implies $(\tilde{\phi}' R_{,x})^2 j^{-1} \omega^{-2} = 1$) For this reason we choose

$$\xi := \Lambda - 1 = 0. \quad (90)$$

For both the GR case and the case of (15)–(19) the mass function is well defined and is the boundary term which

must be added to the Hamiltonian to make the variational principle well defined (see Appendix C). For our gauge choices [and assuming (9)] the mass function is given by

$$M = j \left(\frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2} \right)^2. \quad (91)$$

By comparing (87) and (90) to (52) we can also see that this gauge choice is regular and spatially flat at horizon formation, i.e. when

$$(DR)^2 = 0. \quad (92)$$

The consistency conditions on these two gauge choices can be obtained by setting $\dot{R} = 0$ and $\dot{\Lambda} = 0$ in (76) and (78). Note that we now use a dot to represent differentiation with respect to our time, T , in this choice of gauge and that a prime still represents differentiation with respect to the areal radius, R , which is now our spatial coordinate. The consistency conditions are given by

$$\frac{N_r}{N} = -R_{,y} \quad (93)$$

and

$$(-NR_{,y})' + N \frac{S_3}{2} = 0, \quad (94)$$

from which we can write N_r as

$$N_r = -\exp \left[\int dR \frac{S_3}{2R_{,y}} \right], \quad (95)$$

where we used (87) and (90) to redefine

$$S_1 := P_\Lambda/2 - P_{z_1} - P_{z_2}, \quad (96)$$

$$S_2 := \tilde{\phi}'^2 + 4Wh_z, \quad (97)$$

$$S_3 := \frac{2h_z S_1 - \tilde{\phi}' P_R}{S_2} \quad (98)$$

and

$$R_{,y} = \frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2}. \quad (99)$$

R and Λ are no longer phase space variables and neither are their conjugate momenta. P_R and P_Λ must be written in terms of the remaining phase space variables by setting the Hamiltonian and momentum constraints to zero; i.e., we must solve

$$H = 0 = 2\tilde{\phi}'' - h_z - V + W[-z_1' z_2' + 2(z_1'' + z_2'')] + \frac{h_z S_1^2 - \tilde{\phi}' P_R S_1 - WP_R^2}{S_2} - \frac{P_{z_1} P_{z_2}}{W} - \frac{P_\psi^2}{4B} - B\psi'^2 \quad (100)$$

and

$$H_r = 0 = -P'_\Lambda + P_R + P_{z_1} z_1' + P_{z_2} z_2' + P_\psi \psi' \quad (101)$$

for P_R and P_Λ .

At this point we can consider singularity resolution at the origin in both the metric, (52), (95), (93) and the higher-dimensional generalization of the energy momentum tensor, (46)–(48). Consider first the metric: we can set N to be a constant at the origin and so (93) tells us that if $R_{,y} = \sqrt{l^{n-2} M/j}$ is not singular at $R = 0$ then neither is the metric. So, we can resolve both the singularity in the energy momentum tensor and the one in the metric (in the case where (9) is satisfied) by choosing the functions $V(R)$, $\phi(R)$ and $h(R)$ in the action so that the factors containing $j (= \phi')$ are not singular at the origin. Note that this choice will not violate the Bianchi identities since we derived our equations of motion from a variational principal.

With the gauge choices, (87) and (90), the equations of motion for the remaining phase space variables can now be written as

$$\dot{\psi} = -N \frac{P_\psi}{2B} + N_r \psi', \quad (102)$$

$$\dot{P}_\psi = (-2NB\psi' + N_r P_\psi)', \quad (103)$$

$$\dot{z}_1 = -NS_3 - \frac{NP_{z_2}}{W} + N_r z_1', \quad (104)$$

$$\dot{P}_{z_1} = (-W(Nz_2' + 2N') + N_r P_{z_1})' - N W b(R_{,y}^2 - 1), \quad (105)$$

$$\dot{z}_2 = -NS_3 - \frac{NP_{z_1}}{W} + N_r z_2' \quad (106)$$

and

$$\dot{P}_{z_2} = (-W(Nz_1' + 2N') + N_r P_{z_2})'. \quad (107)$$

These equations of motion, (102)–(107) along with the consistency conditions, (95), (93), the constraints, (100), (101) and the definitions, (96)–(99) can be used to evolve appropriate initial conditions forward in time to show the formation of a black hole with a radiating term.

VI. CONCLUSIONS

In this paper we defined an action which mimics spherically symmetric, dimensionally reduced gravity with a radiating term. Although the radiating term was borrowed from the conformal anomaly and is only valid in two-dimensional physics we incorporated terms which account for the nonminimal coupling (in two dimensions) of the matter field. We derived Lagrange's equations in null gauge and found (in agreement with [20]) that the energy momentum tensor is singular at the origin. We found, however, that we could remove this singularity (as well as the singularity in the vacuum solution) by appropriate choice of coefficients in the action without violating the Bianchi identities since our equations of motion were derived from a variational principal. We then performed a detailed Hamiltonian analysis, including a prescription of the boundary conditions and corresponding boundary terms needed to make the variational principle well defined. From this Hamiltonian we imposed suitable gauge fixing conditions and derived the equations of motion in a family of non-null coordinates. The fact that these are first order in time derivatives makes them well suited to numerical simulations of black hole formation. We then chose a gauge such that the metric is well defined at horizon formation, which is well suited to the investigation of the dynamics past horizon formation.

At first glance the Lagrangian (27) that is our starting point contains many arbitrary functions. However, as we saw, the requirement that singularities be removed is quite restrictive. As shown in Sec. IV.D in order for the effective stress energy tensor to be regular it is necessary that $j = \omega^2 = (\phi')^2$ at $R = 0$. In addition, ϕ' must go to zero at the origin at least as fast as R^2 , as can be seen from (51). Finally, the remaining freedom in j , ϕ and V can be fixed so that the vacuum solution approaches the form derived via general arguments in [34]. This leaves the freedom in the matter coupling $B(R)$. The simplest choice is $B(R) = R^{n-2}$, i.e. the classical form obtained by dimensional reduction which, as shown in [11] yields nonsingular collapse. $b(R)$ and $c(R)$ are normally taken to be zero in two-dimensional models, but, when the coupling is not conformal these functions are determined by our choice of $B(R)$. It would be interesting to see what effect these terms have on the structure of the evaporating, nonsingular black hole spacetime. In an upcoming paper we will present the results of numerical simulations using these equations.

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APPENDIX A: LOOP QUANTUM GRAVITY BLACK HOLE

An example of a nonsingular, single horizon black hole in four dimensions that can be derived as a solution to (7) is that of [36]. See also [35] for an earlier treatment.

$$\begin{aligned}
 ds^2 &= -\left(\sqrt{1 - \frac{k^2}{R^2} - \frac{l^2 M}{2R}}\right) dt^2 \\
 &\quad + \left(\sqrt{1 - \frac{k^2}{R^2} - \frac{l^2 M}{2R}}\right)^{-1} \frac{dR^2}{1 - \frac{k^2}{R^2}} + R^2 d\Omega^2 \\
 &= \sqrt{1 - \frac{k^2}{R^2}} \left[\left(1 - \frac{2GM}{R\sqrt{1 - \frac{k^2}{R^2}}}\right) dt^2 \right. \\
 &\quad \left. + \left(1 - \frac{l^2 M}{2R\sqrt{1 - \frac{k^2}{R^2}}}\right)^{-1} \left(\frac{dR}{1 - \frac{k^2}{R^2}}\right)^2 \right] \\
 &\quad + R^2 d\Omega^2, \tag{A1}
 \end{aligned}$$

where k is the polymerization (quantum gravity) length scale. This is of the same form as (11) with the following identifications:

$$j(R) = 2R\sqrt{1 - \frac{k^2}{R^2}} \tag{A2}$$

$$\phi'(R) = \frac{j(R)}{1 - \frac{k^2}{R^2}} = \frac{2R}{\sqrt{1 - \frac{k^2}{R^2}}} \tag{A3}$$

$$\phi(R) = \left(R\sqrt{R^2 - k^2} + k^2 \ln\left(R + \sqrt{R^2 - k^2}\right)\right) \tag{A4}$$

$$\omega^2 = \frac{j(R)}{\sqrt{1 - \frac{k^2}{R^2}}} = 2R \tag{A5}$$

$$h(R) = \ln(\omega^2)' \phi'(R) = \frac{\phi'(R)}{R} = \frac{1}{\sqrt{1 - k^2/R^2}} \tag{A6}$$

$$V(R) = \omega^2 \frac{j'(R)}{\phi'} = 2. \tag{A7}$$

As shown in [36] the above metric can be analytically continued to describe a complete nonsingular spacetime containing a single bifurcative horizon. The areal radius is bounded below by k and reexpands to infinity in the interior of the horizon. Note that for this black hole the conditions (9) are not satisfied, so that there are in principle terms in the effective stress tensor that are singular at $R = 0$. In the present case this is not an issue because, as previously mentioned, $R = 0$ is excluded from the complete, regular spacetime.

APPENDIX B: MISNER-SHARPE MASS FUNCTION

The following analysis describes the mass function used in this paper.

1. General relativity

In general relativity with no radiation terms the Misner-Sharp mass function is defined by [30]:

$$\begin{aligned} l^{n-2}M &= (n-2)R^{n-3}(1-(DR)^2) \\ &= (n-2)R^{n-3}(1-[\Lambda^{-2}R_{,x}^2 - R_{,y}^2]) \end{aligned} \quad (\text{B1})$$

where, from now on we absorb a factor of l^{n-2} into the mass function. In the case of four-dimensional GR, i.e., $\tilde{\phi} = \phi = R^2$, $h_z = h = 2 = V$, the mass function is given by

$$M = 2R \left(1 - \left[\left(\frac{R_{,x}}{\Lambda} \right)^2 - \left(\frac{P_\Lambda}{4R} \right)^2 \right] \right). \quad (\text{B2})$$

It is then easy to verify that:

$$-\tilde{H} := -\frac{R_{,x}}{\Lambda}H - \frac{P_\Lambda}{4\Lambda R}H_r = M_{,x} \quad (\text{B3})$$

It therefore makes sense to write the Hamiltonian in terms of the new Hamiltonian constraint:

$$\mathcal{H} = \tilde{N}\tilde{H} + \tilde{N}_r H_r = -\tilde{N}M_{,x} + \tilde{N}_r H_r \quad (\text{B4})$$

with suitably redefined lagrange multipliers:

$$\tilde{N} := \frac{\Lambda}{R_{,x}}N \quad (\text{B5})$$

$$\tilde{N}_r := N_r - \frac{P_\Lambda}{4RR_{,x}}N \quad (\text{B6})$$

Note that with asymptotically flat boundary conditions the only boundary terms that arise in the variation are of the form $\tilde{N}(\delta M)_{,x}$. The boundary term that needs to be added to the Hamiltonian in order to make the variational principle well defined is therefore:

$$H_B = \tilde{N}_\infty M_\infty \quad (\text{B7})$$

assuming that the mass function vanishes on the inner boundary: $M_{R=0} = 0$. This is discussed for the radiating case in Appendix C.

2. General ϕ , h and V with no radiation

The mass function for general ϕ , h and V with no radiating terms is given by

$$\begin{aligned} M &= j \left(1 - \frac{(\phi')^2}{j\omega^2} (DR)^2 \right) \\ &= j \left(1 - \frac{(\phi')^2}{j\omega^2} \left[\left(\frac{R_{,x}}{\Lambda} \right)^2 - \left(\frac{P_\Lambda}{2\phi'} \right)^2 \right] \right). \end{aligned} \quad (\text{B8})$$

This is a mass function in the sense that it can be written as a combination of constraints,

$$-\tilde{H} := -\frac{\phi' R_{,x}}{\omega^2 \Lambda} H - \frac{P_\Lambda}{2\omega^2 \Lambda} H_r = M_{,x}. \quad (\text{B9})$$

and it reduces to the Misner-Sharp mass function for the appropriate values of ϕ , h and V . As in the GR case we define

$$\mathcal{H} = \tilde{N}\tilde{H} + \tilde{N}_r H_r = -\tilde{N}M_{,x} + \tilde{N}_r H_r \quad (\text{B10})$$

with suitably redefined Lagrangian multipliers:

$$\tilde{N} := \frac{\omega^2 \Lambda}{\phi' R_{,x}} N \quad (\text{B11})$$

$$\tilde{N}_r := N_r - \frac{P_\Lambda}{2\phi' R_{,x}} N. \quad (\text{B12})$$

3. General ϕ , h and V with radiation

In the most general radiating case that we consider, the derivative of the mass function (88) cannot be written as a linear combination of the constraints. In the GR case where $\phi'' = h = V$ (and therefore $\phi'/j\omega^2 = 1$) it is shown in Appendix C that the boundary term which must be added to the Hamiltonian is of the form (B7) with M given by (B1). In terms of phase space variables, (69) can be used to give,

$$\begin{aligned} M &= (n-2)R^{n-3} \\ &\times \left(1 - \left[\left(\frac{R_{,x}}{\Lambda} \right)^2 - \left(\frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2} \right)^2 \right] \right) \end{aligned} \quad (\text{B13})$$

The derivative of this mass function also cannot in general be written as a combination of the constraints in the full radiating case. Nonetheless, the analysis of Appendix C applies to the non-GR case as long as the conditions $\phi'' = h = V$ are satisfied in the asymptotic region. In this case, as $R \rightarrow \infty$,

$$M \rightarrow j \left(1 - \left[\left(\frac{R_{,x}}{\Lambda} \right)^2 - \left(\frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2} \right)^2 \right] \right) \quad (\text{B14})$$

and this provides the boundary term needed to make the variational principle well defined, as well as the corresponding conserved energy.

APPENDIX C: BOUNDARY CONDITIONS AND BOUNDARY TERM

Here we derive the boundary conditions at $x \rightarrow \infty$ which give a finite Hamiltonian. For this to happen the Hamiltonian density (73) must go to zero faster than x^{-1} . For the nonradiation fields we adopt the boundary conditions used in [30] and [29] for slices which approach flat slice coordinates as $x \rightarrow \infty$,

$$N \simeq N_\infty(t) + \mathcal{O}(x^{-\epsilon_N}), \quad (\text{C1})$$

$$N_r \simeq N_r^\infty(t)x^{-(n-3)/2}, \quad (\text{C2})$$

$$\Lambda \simeq 1, \quad (\text{C3})$$

$$R \simeq x + R_1(t)x^{-\epsilon_R}, \quad (\text{C4})$$

where $\epsilon_N > 0$, $\epsilon_R > 1$ for $n = 4$ and $\epsilon_R > n - 4$ for $n > 4$. In this derivation we ignore the matter terms which are treated in [45]. For the radiation fields we assume the form

$$z_1 \simeq z_{10}^\infty(t) + z_1^\infty(t)x^{-\epsilon_{z1}} \quad (\text{C5})$$

$$z_2 \simeq z_{20}^\infty(t) + z_2^\infty(t)x^{-\epsilon_{z2}}. \quad (\text{C6})$$

We will use the fact that the Hamiltonian should be finite to find the conditions on z_{10}^∞ , z_{20}^∞ , ϵ_{z1} and ϵ_{z2} .

In this appendix we assume that the fields ϕ , h , V and B are those of GR case and that b and c go to zero at least as fast as (grow at least as slowly as)

$$b = -3(B'/B)^2 \quad (\text{C7})$$

$$c = -6 \ln B, \quad (\text{C8})$$

although we will discuss the applicability of this analysis to the singularity resolved case of (16), (18) and (19).

We can write down the asymptotic form of the conjugate momenta, up to dominant terms, using (62)–(64) and (65),

$$P_\Lambda \simeq 2(n-2)N_\infty^{-1}(-R_{1,t}x^{n-3-\epsilon_R} + N_r^\infty x^{(n-3)/2}) - 2WN_\infty^{-1}(z_{10,t}^\infty + z_{20,t}^\infty + z_{1,t}^\infty x^{-\epsilon_{z1}} + z_{2,t}^\infty x^{-\epsilon_{z2}}), \quad (\text{C9})$$

$$P_{\Lambda,x} \simeq 2(n-2)N_\infty^{-1}(-(n-3-\epsilon_R)R_{1,t}x^{n-4-\epsilon_R} + N_r^\infty((n-3)/2)x^{(n-5)/2}) + 2WN_\infty^{-1}(\epsilon_{z1}z_{1,t}^\infty x^{-\epsilon_{z1}-1} + \epsilon_{z2}z_{2,t}^\infty x^{-\epsilon_{z2}-1}), \quad (\text{C10})$$

$$P_R \simeq 2(n-2)N_\infty^{-1}(-(n-3)R_{1,t}x^{n-4-\epsilon_R} + N_r^\infty((n-3)/2)x^{(n-5)/2}) + 6W(n-2)^2N_\infty^{-1}(z_{10}^\infty + z_1^\infty x^{-\epsilon_{z1}} - \ln \mu^2) \times (R_{1,t}x^{-\epsilon_R-2} - N_r^\infty x^{-(n+1)/2}), \quad (\text{C11})$$

$$P_{z1} \simeq -WN_\infty^{-1}((n-3)N_r^\infty x^{-(n-1)/2} + z_{20,t}^\infty + z_{2,t}^\infty x^{-\epsilon_{z2}}), \quad (\text{C12})$$

$$P_{z2} \simeq -WN_\infty^{-1}((n-3)N_r^\infty x^{-(n-1)/2} + z_{10,t}^\infty + z_{1,t}^\infty x^{-\epsilon_{z1}}). \quad (\text{C13})$$

At this point it is advantageous to consider the $NP_{z1}P_{z2}/W\Lambda$ term in (73). The relevant, radiation terms in this term go as

$$z_{10,t}^\infty z_{20,t}^\infty + z_{1,t}^\infty z_{20,t}^\infty x^{-\epsilon_{z1}} + z_{2,t}^\infty z_{10,t}^\infty x^{-\epsilon_{z2}} + z_{1,t}^\infty z_{2,t}^\infty x^{-(\epsilon_{z1}+\epsilon_{z2})}, \quad (\text{C14})$$

which are not canceled by any of the other terms in (73). The form of (C14) means that $\epsilon_{z1} + \epsilon_{z2} > 1$ and $z_{10,t}^\infty z_{20,t}^\infty = 0$. In the case where $b = c = 0 \rightarrow z_1 = z_2 := z/2$, $P_{z1} = P_{z2} := P_z$ we can show that $z \sim x^{\epsilon_z}$ where $\epsilon_z > 1/2$. In the spirit of this limit we take

$$\epsilon_{z1} > 1/2, \quad \epsilon_{z2} > 1/2, \quad z_{10}^\infty = 0, \quad z_{20}^\infty = 0. \quad (\text{C15})$$

Using these conditions and keeping only the biggest terms we then find

$$P_\Lambda \simeq 2(n-2)N_\infty^{-1}(-R_{1,t}x^{n-3-\epsilon_R} + N_r^\infty x^{(n-3)/2}) \quad (\text{C16})$$

$$P_{\Lambda,x} \simeq 2(n-2)N_\infty^{-1}(-(n-3-\epsilon_R)R_{1,t}x^{n-4-\epsilon_R} + N_r^\infty((n-3)/2)x^{(n-5)/2}) \quad (\text{C17})$$

$$P_R \simeq 2(n-2)N_\infty^{-1}(-(n-3)R_{1,t}x^{n-4-\epsilon_R} + N_r^\infty((n-3)/2)x^{(n-5)/2}) \quad (\text{C18})$$

$$P_{z1} \simeq -WN_\infty^{-1}z_{2,t}^\infty x^{-\epsilon_{z2}} \quad (\text{C19})$$

$$P_{z2} \simeq -WN_\infty^{-1}z_{1,t}^\infty x^{-\epsilon_{z1}}. \quad (\text{C20})$$

Plugging these and (C15) into (74) and (75) and dropping all nondominant terms we can see that the Hamiltonian density reduces to the nonradiating case and the rest of the proof that the Hamiltonian is finite can be found in Appendix A.2 of [30].

It is important to note that there are some cancelations in the rest of the derivation which require $2\phi'' - h_z - V \rightarrow 0$ as $x \rightarrow \infty$. This occurs in the GR case as well as for (15), (18) and (19) assuming that b and c go to zero fast enough. These must be satisfied in order to use the boundary conditions given by (C1)–(C4).

We now take the variation of the action to find which boundary terms do not approach zero as $x \rightarrow \infty$. The variation of the action is given by

$$\begin{aligned}
 \delta I = & \int dt \int dx (\partial_t (\delta \Lambda P_\Lambda) - \delta \Lambda \dot{P}_\Lambda + \dot{\Lambda} \delta P_\Lambda \\
 & + \partial_t (\delta R P_R) - \delta R \dot{P}_R + \dot{R} \delta P_R \\
 & + \partial_t (\delta z_1 P_{z1}) - \delta z_1 \dot{P}_{z1} + \dot{z}_1 \delta P_{z1} \\
 & + \partial_t (\delta z_2 P_{z2}) - \delta z_2 \dot{P}_{z2} + \dot{z}_2 \delta P_{z2} \\
 & - \delta N H - N \delta H - \delta N_r H_r - N_r \delta H_r). \quad (C21)
 \end{aligned}$$

Assuming that all variations vanish at the time end points the only contributions to the boundary term come from the last line of (C21). Using the boundary conditions (C1)–(C6) and (C15) we calculate the boundary term. This calculation is tedious but straight forward and can be found in Appendix A.2 of [30] for the nonradiating case. The only boundary term which does not approach zero as $x \rightarrow \infty$ is

$$\begin{aligned}
 N_r \Lambda \delta P_\Lambda & \approx N_r^\infty x^{-(n-3)/2} [2(n-2) \delta (N_r^\infty / N_\infty) x^{(n-3)/2}] \\
 & = (n-2) N_\infty \delta (N_r^{\infty 2} / N_\infty^2). \quad (C22)
 \end{aligned}$$

The Misner-Sharp mass in this case (see (1.35) of [30]), with l set to 1, is given by

$$M = (n-2) R^{n-3} (N_r / N)^2, \quad (C23)$$

which goes to $(n-2)(N_r^\infty / N_\infty)^2$ as $x \rightarrow \infty$ and so the variation of the action is

$$\delta I = \int dt dx (\text{dynamical terms}) + \int dt [N \delta M]_{x=-\infty}^{x=+\infty}. \quad (C24)$$

This requires the addition of $NM|_{R=\infty}$ to the Hamiltonian.

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