

Finsler-type modification of the Coulomb lawYakov Itin^{*}*Institute of Mathematics, The Hebrew University of Jerusalem and Jerusalem College of Technology,
Jerusalem 91904, Israel*Claus Lämmerzahl[†] and Volker Perlick[‡]*ZARM, University of Bremen, Am Fallturm, 28359 Bremen, Germany*

(Received 10 November 2014; published 19 December 2014)

Finsler geometry is a natural generalization of pseudo-Riemannian geometry. It can be motivated e.g. by a modified version of the Ehlers-Pirani-Schild axiomatic approach to space-time theory. Also, some scenarios of quantum gravity suggest a modified dispersion relation which could be phrased in terms of Finsler geometry. On a Finslerian space-time, the universality of free fall is still satisfied but local Lorentz invariance is violated in a way not covered by standard Lorentz invariance violation schemes. In this paper we consider a Finslerian modification of Maxwell's equations. The corrections to the Coulomb potential and to the hydrogen energy levels are computed. We find that the Finsler metric corrections yield a splitting of the energy levels. Experimental data provide bounds for the Finsler parameters.

DOI: 10.1103/PhysRevD.90.124057

PACS numbers: 04.50.Kd, 11.30.Cp

I. INTRODUCTION

A widely expected consequence of a (still-to-be-found) theory of quantum gravity is a small modification of general relativity. Such a modification may be encoded in a scalar-tensor theory as it comes out from the low energy limit of string theory leading e.g. to a violation of the universality of free fall [1,2]. Other consequences might be that, in addition to the metric, there could be a further geometric field like torsion leading to an effective Riemann-Cartan geometry.

Another modification of the usual pseudo-Riemannian geometry is Finsler geometry. It comes about naturally in some scenarios inspired from quantum gravity. For example, it was shown in [3] that a modified dispersion relation suggested by quantum gravity can be interpreted in terms of Finsler geometry. Further motivation comes from very special relativity [4]: As demonstrated in [5], some deformations of very special relativity lead in a natural way to Finsler geometry. Finsler geometry has also been considered in the context of analogue gravity [6].

Finsler geometry is a framework which still respects the universality of free fall but violates local Lorentz invariance. The way in which local Lorentz invariance is violated is beyond the usual Lorentz invariance violation schemes like the $\chi - g$ formalism [7], the $THE\mu$ framework [8] or the Standard Model extension [9]. Furthermore, though the universality of free fall is valid in a Finslerian setting, gravity cannot be transformed away locally [10]; that is, there is no Einstein elevator. On a more basic level, a

Finslerian geometry may result from a relaxed version of the Ehlers-Pirani-Schild axiomatics [11] by not requiring the world function to be twice differentiable.

Therefore, in view of considering all possible deviations from standard Riemannian geometry reflecting effects from quantum gravity, and in view of more fundamental issues, it might be of general interest to study further consequences of Finsler geometry. Since electromagnetic phenomena provide very precise tools for exploring the geometry of space-time, in this paper we will set up a generalization of Maxwell's equations in a Finslerian space-time and derive possible consequences for atomic physics which can be compared with experiments.

II. FINSLER GEOMETRY**A. Positive definite Finsler structures**

The central idea of Finsler geometry was already proposed by Riemann in his famous habilitation lecture devoted to the geometry of curved manifolds [12]. In parallel to the (Riemannian) geometry based on a second rank symmetric nondegenerate metrical tensor $g_{\alpha\beta}(x)$ with the line element $ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$, Riemann briefly discussed a geometry based on a fourth-rank totally symmetric tensor $g_{\alpha\beta\gamma\delta}(x)$ with the line element

$$ds^4 = g_{\alpha\beta\gamma\delta}(x)dx^\alpha dx^\beta dx^\gamma dx^\delta. \quad (2.1)$$

An intensive study and a further generalization of this type of geometry was given by Finsler [13] in 1918 in his dissertation. Finsler geometry is based on a *Finsler function* $F(x, y)$ that assigns a length

^{*}itin@math.huji.ac.il[†]claus.laemmerzahl@zarm.uni-bremen.de[‡]volker.perlick@zarm.uni-bremen.de

$$S = \int_{s_1}^{s_2} F(x(s), \dot{x}(s)) ds \quad (2.2)$$

to each curve. One requires that $F(x, y)$ is positively homogeneous of degree one,

$$F(x, \lambda y) = \lambda F(x, y) \quad \text{for } \lambda > 0, \quad (2.3)$$

to make sure that the length of a curve is independent of its parametrization, and that the *Finsler metric*

$$g_{\alpha\beta}(x, y) = \frac{\partial^2(F(x, y)^2)}{\partial y^\alpha \partial y^\beta} \quad (2.4)$$

is positive definite for all $y \neq 0$.

The unparametrized geodesics of a Finsler geometry are the extremals of the length functional (2.2) where the end points are kept fixed. The affinely parametrized geodesics are the extremals of the “energy functional”

$$E = \int_{s_1}^{s_2} F(x(s), \dot{x}(s))^2 ds, \quad (2.5)$$

where the end points and the parameter interval are kept fixed. Riemannian geometry is, of course, a special case of Finsler geometry, characterized by the additional property that the metric $g_{\alpha\beta}$ is independent of y .

The theory of positive definite Finsler metrics, which is detailed e.g. in [14] and [15], has several applications to physics, where the underlying manifold is to be interpreted as three-dimensional space, so the greek indices take values 1,2,3. For example, the Lagrangian of a charged particle in a magnetostatic field is given by a Finsler function of the Randers form

$$F(x, y) = \sqrt{h_{\mu\nu}(x)y^\mu y^\nu} + A_\mu(x)y^\mu, \quad (2.6)$$

where $h_{\mu\nu}(x)$ is a Riemannian metric (i.e., positive definite) and $A_\mu(x)$ is a one-form. It can be shown that the corresponding Finsler metric (2.4) is, indeed, positive definite for all $y \neq 0$ provided that $F(x, y) > 0$ for all $y \neq 0$; see [15], Sec. 11.1. To mention another example, light propagation in an anisotropic medium that is time independent is characterized by two positive definite spatial Finsler metrics [16,17]. If these two metrics coincide (i.e., if there is no birefringence), they are necessarily Riemannian [18,19]. Positive definite Finsler metrics have also been used for describing the propagation of seismic waves; see e.g. [20].

B. Finsler structures of Lorentzian signature

In applications to space-time physics, the Euclidean signature of the metric must be replaced by a Lorentzian signature. Following Beem [21], this can be done by considering, instead of the function $F(x, y)^2$, a

Lagrangian $L(x, y)$ that may take positive, zero and negative values. [Notice that it is the *square* of the Finsler function that enters into the definition of the metric tensor (2.4).]

More precisely, a Finsler structure of Lorentzian signature is a function $L(x, y)$ that is positively homogeneous of degree two,

$$L(x, \lambda y) = \lambda^2 L(x, y) \quad \text{for } \lambda > 0, \quad (2.7)$$

and for which the Finsler metric

$$g_{ij}(x, y) = \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \quad (2.8)$$

is nondegenerate and of Lorentzian signature for all $y \neq 0$. (Actually, it is recommendable to relax the latter condition by requiring the conditions on the Finsler metric to hold only for *almost* all $y \neq 0$; see [22].) In applications to physics, the underlying manifold is to be interpreted as space-time, so the Latin indices take values 0,1,2,3.

The homogeneity condition (2.7) implies that

$$L(x, y) = \frac{1}{2} g_{ij}(x, y) y^i y^j. \quad (2.9)$$

The affinely parametrized geodesics of such a Finsler structure are, by definition, the extremals of the “energy functional”

$$E = \int_{s_1}^{s_2} L(x(s), \dot{x}(s)) ds. \quad (2.10)$$

The homogeneity condition assures that L is a constant of motion, so the geodesics can be classified as timelike ($L < 0$), lightlike ($L = 0$) and spacelike ($L > 0$).

III. MAXWELL'S EQUATIONS ON A FLAT FINSLER SPACE-TIME

In this section we discuss how Maxwell's equations must be modified if the underlying space-time is Finslerian. We mention that there are different views on this issue; see e.g. Pfeifer and Wohlfarth [23] for an alternative approach. We follow a line of thought that was sketched already in the appendix of [22]. Our guiding principles are that the electromagnetic field strength should be a field on space-time (and not on the tangent bundle, as in [23]), and that the lightlike Finsler geodesics should be the bicharacteristics (i.e., the “rays”) of Maxwell's equations.

A. Flat Finsler space-times

As in this paper we are interested in laboratory experiments, where space-time curvature plays no role, we assume that the underlying Finsler structure is flat. We prescribe this Finsler structure in terms of a Lagrangian,

following Beem's definition. The flatness assumption means that we can choose the coordinates such that the Lagrangian is independent of x ,

$$L(y) = \frac{1}{2} g_{ij}(y) y^i y^j. \quad (3.1)$$

This is analogous to the pseudo-Riemannian case where the flatness assumption means that the coordinates can be chosen such that the g_{ij} are independent of x . Here and in the following, Latin indices take values 0,1,2,3 and Greek indices take values 1,2,3.

As a consequence of (2.7) and (2.8), the Finsler metric is homogeneous of degree zero,

$$y^k \frac{\partial g_{ij}(y)}{\partial y^k} = 0, \quad (3.2)$$

and its derivative is totally symmetric,

$$\frac{\partial g_{ij}(y)}{\partial y^k} = \frac{\partial g_{ki}(y)}{\partial y^j} = \frac{\partial g_{jk}(y)}{\partial y^i}. \quad (3.3)$$

We will later assume that $g_{ij}(y)$ is a small perturbation of the Minkowski metric, but in this section we will not need this specification.

B. Hamiltonian vs Lagrangian formalism

Recall that the lightlike geodesics of our Finsler structure are the extremals of the functional (2.5) with $L(x, y) = 0$. In the case at hand, where L is assumed to be independent of x , the lightlike geodesics are the straight lines $x^i(s) = a^i + y^i s$ with $L(y) = 0$. To characterize these curves in terms of a Hamiltonian, rather than in terms of a Lagrangian, we introduce the canonical momenta

$$p_i = \frac{\partial L(y)}{\partial y^i} \quad (3.4)$$

and the Hamiltonian

$$H(p) = p_i y^i - L(y). \quad (3.5)$$

In (3.5), the y^i must be expressed in terms of the p_j with the help of (3.4). The nondegeneracy of the Finsler metric guarantees that this can be done for all $y \neq 0$.

With (3.1), (3.3) and (3.2) we see that (3.4) can be written more explicitly as

$$p_i = g_{in}(y) y^n + \frac{1}{2} \frac{\partial g_{mn}(y)}{\partial y^i} y^m y^n = g_{in}(y) y^n. \quad (3.6)$$

Thereupon, the Hamiltonian (3.5) reads

$$H(p) = \frac{1}{2} g^{jj}(p) p_i p_j, \quad (3.7)$$

where

$$g^{jj}(p) = \frac{\partial^2 H(p)}{\partial p_i \partial p_j} \quad (3.8)$$

is the inverse of $g_{jk}(y)$, with the y^i expressed in terms of the p_i by (3.4). In accordance with (3.2) and (3.3) we have

$$p_k \frac{\partial g^{jj}(p)}{\partial p_k} = 0, \quad (3.9)$$

$$\frac{\partial g^{jj}(p)}{\partial p_k} = \frac{\partial g^{ki}(p)}{\partial p_j} = \frac{\partial g^{jk}(p)}{\partial p_i}. \quad (3.10)$$

The Hamiltonian H is homogeneous of degree two with respect to p , i.e.

$$p_k H^k(p) = 2H(p), \quad (3.11)$$

where we have introduced, as an abbreviation,

$$H^k(p) = \frac{\partial H(p)}{\partial p_k} = g^{kj}(p) p_j. \quad (3.12)$$

The lightlike Finsler geodesics (i.e., the lightlike straight lines in the case at hand) are the solutions to Hamilton's equations with $H(p) = 0$.

C. Modified Maxwell's equations

If the space-time metric is the unperturbed Minkowski metric, $g^{jk} = \eta^{jk}$ where $(\eta^{jk}) = \text{diag}(-1, 1, 1, 1)$, Maxwell's equations read

$$\partial_l F_{jk} + \partial_j F_{kl} + \partial_k F_{lj} = 0. \quad (3.13)$$

$$\eta^{kl} \partial_l F_{kj} = -\mu_0 J_j. \quad (3.14)$$

Here the two-form F_{kj} is the electromagnetic field strength, J_j is the current density and μ_0 is the permeability of the vacuum. If the current is given, (3.13) and (3.14) give a system of first-order partial differential equations for the electromagnetic field strength.

If we replace the Minkowski metric η^{kl} with our flat Finsler metric $g^{kl}(p)$, we see that there is no reason to modify (3.13) because it does not involve the metric. As to (3.14), it is most natural to replace

$$\eta^{kl} \partial_l \mapsto g^{kl}(-i\partial) \partial_l, \quad (3.15)$$

where i is the imaginary unit and $g^{kl}(-i\partial)$ stands for the expression that results if in $g^{kl}(p)$ the p_j are replaced with $-i\partial_j = -i\partial/\partial x^j$. As $g^{kl}(p)$ is not in general a polynomial

in the momentum coordinates, $g^{kl}(-i\partial)\partial_l$ is not in general a differential operator but rather a pseudodifferential operator. (For background material on pseudodifferential operators see e.g. [24].) With the replacement (3.15), the Maxwell equation (3.14) becomes a pseudodifferential equation,

$$g^{kl}(-i\partial)\partial_l F_{kj} = -\mu_0 J_j. \quad (3.16)$$

By (3.12), this equation can be equivalently rewritten as

$$iH^k(-i\partial)F_{kj} = -\mu_0 J_j. \quad (3.17)$$

As the current and the field strength are both real, the operator $iH^k(-i\partial)$ should map real functions to real functions. This is the case if the Hamiltonian is even, $H(-p) = H(p)$, i.e., if the homogeneity property (2.7) is true also for negative λ . If this condition is satisfied, (3.13) and (3.17) determine a perfectly reasonable dynamical system for the field strength if the current is given. Note that if H satisfies the property

$$H(-ip) = -H(p), \quad (3.18)$$

we may write

$$iH^k(-i\partial) = H^k(\partial), \quad (3.19)$$

and (3.17) is manifestly real. The Hamiltonians (4.2) and (4.9) to be considered below both satisfy (3.18), where in the case of (4.2) the correct branch of the square root, $i^{4/2} = -1$, has to be chosen.

To support our claim that (3.13) and (3.17) are the correct Finsler versions of Maxwell's equations, we apply the operator ∂_m to (3.17) for the case that $J_j = 0$,

$$0 = \partial_m(H^k(-i\partial)F_{kj}) = H^k(-i\partial)(\partial_m F_{kj}). \quad (3.20)$$

By (3.13), this can be rewritten as

$$\begin{aligned} 0 &= H^k(-i\partial)(\partial_k F_{jm} + \partial_j F_{mk}) \\ &= \partial_k(H^k(-i\partial)F_{jm}) + \partial_j(H^k(-i\partial)F_{mk}). \end{aligned} \quad (3.21)$$

The second term vanishes because of $J_m = 0$. Using (3.11) we find that F_{jm} satisfies a generalized wave equation,

$$H(-i\partial)F_{jm} = 0. \quad (3.22)$$

If we solve this equation with a plane-wave ansatz for the electromagnetic field,

$$F_{jm}(x) = \text{Re}\{f_{jm} \exp(ik_l x^l)\}, \quad (3.23)$$

we find that the wave covector k_l has to satisfy the equation

$$H(k) = 0, \quad (3.24)$$

i.e., that in our flat Finsler space-time electromagnetic waves propagate along lightlike straight lines. This observation supports our claim that (3.13) and (3.17) are the correct Finsler versions of Maxwell's equations.

To give further support to this claim, we now demonstrate that (3.17) can be brought into a form which is adapted to the formalism of premetric electrodynamics; cf. [25]. To that end we have to show that (3.17) can be rewritten as

$$\partial_l \mathcal{H}^{ml} = -J^m, \quad (3.25)$$

where the excitation \mathcal{H}^{ml} is related to the field strength F_{kj} by a certain constitutive law. We write (3.17) in the equivalent form of (3.16), and we apply the pseudodifferential operator $g^{mj}(-i\partial)$. Then we obtain

$$g^{mj}(-i\partial)g^{kl}(-i\partial)\partial_l F_{kj} = -\mu_0 J^m \quad (3.26)$$

with $J^m = g^{mj}(-i\partial)J_j$. Since g^{kl} is independent of the x^i , this can be rewritten as

$$\partial_l(\kappa^{mlkj}(-i\partial)F_{kj}) = -J^m \quad (3.27)$$

with a constitutive operator

$$\kappa^{mlkj}(-i\partial) = \frac{1}{2\mu_0}(g^{mj}(-i\partial)g^{kl}(-i\partial) - g^{mk}(-i\partial)g^{jl}(-i\partial)). \quad (3.28)$$

This form is equivalent to the original equation (3.17). In particular, for $g^{ij} = \eta^{ij}$ we return to the standard Maxwell vacuum electrodynamics on Minkowski space-time. We have, thus, put our modified Maxwell equations in the premetric form, where the constitutive law

$$\mathcal{H}^{ml} = \kappa^{mlkj}(-i\partial)F_{kj} \quad (3.29)$$

involves the pseudodifferential operator (3.28). An important advantage of the premetric formulation is that, quite generally, (3.25) together with the antisymmetry of \mathcal{H}^{kl} immediately implies charge conservation, $\partial_m J^m = 0$.

The homogeneous part of Maxwell's equations (3.13) is automatically satisfied if we express the electromagnetic field in terms of a potential,

$$F_{ij} = \partial_i A_j - \partial_j A_i. \quad (3.30)$$

We mention in passing that then the inhomogeneous part (3.27) can be derived from the action

$$\mathcal{S} = \int \left(\frac{1}{4} \kappa^{kl ij} (-i\partial) F_{kl}(x) F_{ij}(x) - \mu_0 A_i(x) J^i(x) \right) d^4x, \quad (3.31)$$

where one has to take into account that the operator $\kappa^{kl ij}(-i\partial)$ commutes with the variational derivative.

In the following we will be interested in static fields. Then $\partial_0 A_i = 0$ and (3.17) implies

$$iH^k(-i\partial)\partial_k A_0 = -\mu_0 J_0. \quad (3.32)$$

We denote the four components of the potential by $(A_0 = -V/c, A_1, A_2, A_3)$ and the four components of the current density by $(J_0 = -c\rho, J_1, J_2, J_3)$. Then (3.32) can be rewritten, with the help of (3.11), as

$$2H(-i\partial)V = \frac{\rho}{\varepsilon_0}, \quad (3.33)$$

where ε_0 is the permittivity of the vacuum and we have used that $c^{-2} = \varepsilon_0 \mu_0$. If the metric is the unperturbed Minkowski metric, we have, of course, $2H(-i\partial)V = -\Delta V$ where Δ is the ordinary Laplacian. Equation (3.33) is the Finslerian modification of the Poisson equation that determines the electrostatic potential V of a static charge density ρ . This is the only equation from Finslerian electrodynamics that we will need in the following.

IV. THE FINSLERIAN MODIFICATION OF THE COULOMB FIELD

A. A Finsler perturbation of Minkowski space-time

We further specify our Finsler structure by assuming that the Hamiltonian (3.5) is a small perturbation of the standard Hamiltonian on Minkowski space-time. The latter reads

$$H_0(p) = \frac{1}{2} \eta^{ij} p_i p_j = \frac{1}{2} (-p_0^2 + \delta^{\mu\nu} p_\mu p_\nu). \quad (4.1)$$

We restrict to the case that the Finsler perturbation affects the spatial part only. The simplest nontrivial ansatz for such a perturbation is a square root of a fourth-order term,

$$H(p) = \frac{1}{2} (-p_0^2 + \sqrt{(\delta^{\mu\nu} \delta^{\rho\sigma} + 4\phi^{\mu\nu\rho\sigma}) p_\mu p_\nu p_\rho p_\sigma}), \quad (4.2)$$

where $\phi^{\mu\nu\rho\sigma}$ is totally symmetric. (A similar perturbation of Minkowski space-time was considered in [26].) We assume that the Finsler perturbation is so small that we can linearize all equations with respect to the $\phi^{\mu\nu\rho\sigma}$. Then the Hamiltonian simplifies to

$$H(p) = \frac{1}{2} \left(-p_0^2 + \delta^{\rho\sigma} p_\rho p_\sigma + \frac{2\phi^{\mu\nu\rho\sigma} p_\mu p_\nu p_\rho p_\sigma}{\delta^{\lambda\kappa} p_\lambda p_\kappa} \right). \quad (4.3)$$

We will now demonstrate that the trace part of $\phi^{\mu\nu\rho\sigma}$ can be eliminated with the help of a coordinate transformation. To that end, we decompose $\phi^{\mu\nu\rho\sigma}$ in the form

$$\phi^{\mu\nu\rho\sigma} p_\mu p_\nu p_\rho p_\sigma = (\tilde{\phi}^{\mu\nu} \delta^{\rho\sigma} + \tilde{\phi}^{\mu\nu\rho\sigma}) p_\mu p_\nu p_\rho p_\sigma, \quad (4.4)$$

where $\tilde{\phi}^{\mu\nu\rho\sigma}$ is totally symmetric and trace-free. Then (4.3) can be rewritten as

$$\begin{aligned} H(p) &= \frac{1}{2} \left(-p_0^2 + \delta^{\rho\sigma} p_\rho p_\sigma + 2\tilde{\phi}^{\rho\sigma} p_\rho p_\sigma + \frac{2\tilde{\phi}^{\mu\nu\rho\sigma} p_\mu p_\nu p_\rho p_\sigma}{\delta^{\lambda\kappa} p_\lambda p_\kappa} \right). \end{aligned} \quad (4.5)$$

After a linear coordinate transformation,

$$\tilde{x}^0 = x^0, \quad \tilde{x}^\sigma = (\delta^\sigma_\mu - \delta_{\mu\lambda} \tilde{\phi}^{\sigma\lambda}) x^\mu, \quad (4.6)$$

$$p_0 = \tilde{p}_0, \quad p_\mu = (\delta^\sigma_\mu - \delta_{\mu\lambda} \tilde{\phi}^{\sigma\lambda}) \tilde{p}_\sigma, \quad (4.7)$$

the Hamiltonian reads

$$H(\tilde{p}) = \frac{1}{2} \left(-\tilde{p}_0^2 + \delta^{\rho\sigma} \tilde{p}_\rho \tilde{p}_\sigma + \frac{2\tilde{\phi}^{\mu\nu\rho\sigma} \tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho \tilde{p}_\sigma}{\delta^{\lambda\kappa} \tilde{p}_\lambda \tilde{p}_\kappa} \right) \quad (4.8)$$

up to terms of quadratic order with respect to the Finsler perturbation. If we drop the tilde, we have found the final form of our Hamiltonian,

$$H(p) = \frac{1}{2} \left(\eta^{ij} p_i p_j + \frac{2\phi^{\mu\nu\rho\sigma} p_\mu p_\nu p_\rho p_\sigma}{\delta^{\lambda\kappa} p_\lambda p_\kappa} \right), \quad (4.9)$$

with $\phi^{\mu\nu\rho\sigma}$ totally symmetric and trace-free. A totally symmetric fourth-rank tensor in three dimensions has 15 independent components. The trace-free condition allows one to express 6 of them in terms of the other ones, e.g.

$$\begin{aligned} \phi^{1122} &= \frac{1}{2} (\phi^{3333} - \phi^{1111} - \phi^{2222}), \\ \phi^{1133} &= \frac{1}{2} (\phi^{2222} - \phi^{3333} - \phi^{1111}), \\ \phi^{2233} &= \frac{1}{2} (\phi^{1111} - \phi^{2222} - \phi^{3333}), \\ \phi^{1123} &= -\phi^{2223} - \phi^{2333}, \\ \phi^{1223} &= -\phi^{1113} - \phi^{1333}, \\ \phi^{1233} &= -\phi^{1112} - \phi^{1222}, \end{aligned} \quad (4.10)$$

so we are left with 9 independent Finsler perturbation coefficients.

B. The modified Coulomb field

With the Hamiltonian (4.9) inserted into (3.33), we want to find the solution where the source is a point charge at

rest. The equation we have to solve reads

$$\Delta V + 2 \frac{\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\Delta} V = -\frac{q}{\epsilon_0} \delta(\vec{r}). \quad (4.11)$$

Here and in the following we write

$$\vec{r} = (x^1, x^2, x^3), \quad r = \sqrt{\delta_{\alpha\beta} x^\alpha x^\beta}, \quad \Delta = \delta^{\alpha\beta} \partial_\alpha \partial_\beta. \quad (4.12)$$

We look for a solution to (4.11) in the form

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r} + \psi(\vec{r}), \quad (4.13)$$

where the first term on the right-hand side is the standard Coulomb solution of the unperturbed problem. As we agreed to linearize all equations with respect to the Finsler coefficients $\phi^{\alpha\beta\mu\nu}$, it is sufficient to determine ψ to within this approximation. Then ψ must satisfy the equation

$$\Delta\psi + 2 \frac{\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\Delta} \left(\frac{q}{4\pi\epsilon_0 r} \right) = 0. \quad (4.14)$$

Applying the Laplacian to this equation gives a linear fourth order partial differential equation,

$$\Delta^2 \psi = -2q \phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \left(\frac{q}{4\pi\epsilon_0 r} \right). \quad (4.15)$$

The right-hand side of this equation is easily calculated,

$$\Delta^2 \psi = -\frac{210}{4\pi\epsilon_0} \frac{q}{r^9} \phi^{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta, \quad (4.16)$$

where $x_\alpha = \delta_{\alpha\beta} x^\beta$. Here we have used that $\phi^{\alpha\beta\gamma\delta}$ is trace-free.

The solution ψ of the biharmonic equation (4.16) must

- (a) be asymptotically zero for $r \rightarrow \infty$,
- (b) be linear with respect to $\phi^{\alpha\beta\gamma\delta}$,
- (c) have only one singular point located at the origin,
- (d) be constructed from the $\phi^{\alpha\beta\gamma\delta}$ and the x_α .

Under these circumstances we can guess the solution of (4.16) to be of the form

$$\psi = C \frac{\phi^{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta}{r^5}. \quad (4.17)$$

Note that we cannot add terms proportional to $\phi^{\alpha\beta\gamma\delta} \delta_{\alpha\beta} x_\gamma x_\delta$ or $\phi^{\alpha\beta\gamma\delta} \delta_{\alpha\beta} \delta_{\gamma\delta}$ because these terms vanish.

The biharmonic operator applied to (4.17) gives

$$\Delta^2 \psi = \frac{280C}{r^9} \phi^{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta. \quad (4.18)$$

By comparing (4.18) with (4.16) we obtain $C = -3q(16\pi\epsilon_0)^{-1}$. Thus the solution of (4.15) is

$$\psi = -\frac{3q}{16\pi\epsilon_0 r^5} \phi^{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta. \quad (4.19)$$

Consequently, we have the scalar potential of the point source in the form

$$V = \frac{q}{4\pi\epsilon_0 r} \left(1 - \frac{3}{4r^4} \phi^{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta \right). \quad (4.20)$$

In spherical coordinates this expression reads

$$V = \frac{q}{4\pi\epsilon_0 r} \left(1 - \frac{3}{4} \phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) \right), \quad (4.21)$$

where

$$\begin{aligned} \phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) = & \phi^{1111} \sin^4 \theta \cos^4 \varphi + \phi^{1112} \sin^4 \theta \cos^3 \varphi \sin \varphi + \phi^{1113} \sin^3 \theta \cos \theta \cos^3 \varphi \\ & + \phi^{1122} \sin^4 \theta \cos^2 \varphi \sin^2 \varphi + \phi^{1123} \sin^3 \theta \cos \theta \cos^2 \varphi \sin \varphi \\ & + \phi^{1133} \sin^2 \theta \cos^2 \theta \cos^2 \varphi + \phi^{1222} \sin^4 \theta \cos \varphi \sin^3 \varphi + \phi^{1223} \sin^3 \theta \cos \theta \cos \varphi \sin^2 \varphi \\ & + \phi^{1233} \sin^2 \theta \cos^2 \theta \cos \varphi \sin \varphi + \phi^{1333} \sin \theta \cos^3 \theta \cos \varphi + \phi^{2222} \sin^4 \theta \sin^4 \varphi + \phi^{2223} \sin^3 \theta \cos \theta \sin^3 \varphi \\ & + \phi^{2233} \sin^2 \theta \cos^2 \theta \sin^2 \varphi + \phi^{2333} \sin \theta \cos^3 \theta \sin \varphi + \phi^{3333} \cos^4 \theta. \end{aligned} \quad (4.22)$$

V. FINSLER CORRECTIONS OF THE HYDROGEN ENERGY LEVELS

A. Finsler modified Schrödinger equation

For an electron (mass = m and charge = $-e$) in the Coulomb field (4.21) of a proton (charge $q = e$), the Schrödinger equation reads

$$\begin{aligned} & -\frac{\hbar^2}{2m} \left(\Delta + 2 \frac{\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\Delta} \right) \Psi(\vec{r}) \\ & - \frac{e^2}{4\pi\epsilon_0 r} \left(1 - \frac{3}{4} \phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) \right) \Psi(\vec{r}) = E \Psi(\vec{r}). \end{aligned} \quad (5.1)$$

Here we have added to the potential term a Finsler correction according to our results from the preceding

section, and we have added to the Laplacian the same correction as in the electrodynamic equations; cf. (4.11). The latter assumption is based on the idea that the Finsler perturbation modifies the underlying geometry such that particles and light are affected in the same way. As an alternative, one might speculate that there are two different Finsler modifications of the space-time structure, one for particles and one for light. This would come up to a Finslerian bimetric theory. We will not investigate such a more complicated theory here but rather stick with (5.1). However, we mention that the order-of-magnitude estimates of the following calculations remain true for the more general (bimetric) theories as long as the perturbation of the Laplacian term does not exceed the corresponding term in (5.1) by several orders of magnitude.

To give further support to our Schrödinger equation (5.1), we demonstrate that it comes about as the nonrelativistic limit of a modified Klein-Gordon equation. The free Klein-Gordon equation in a Finsler space-time is naturally given by

$$2H(-i\hbar\partial)\Phi + m^2c^2\Phi = 0, \quad (5.2)$$

where H is the four-dimensional Hamiltonian. This can also be derived from an action principle. In our model,

$$2H(-i\hbar\partial) = \hbar^2 \left(\frac{1}{c^2} \partial_t^2 - \delta^{\mu\nu} \partial_\mu \partial_\nu - \frac{2\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\delta^{\tau\lambda} \partial_\tau \partial_\lambda} \right). \quad (5.3)$$

We want to derive the nonrelativistic limit of this Finslerian Klein-Gordon equation. For that we use the formalism described in [27]. We make an ansatz where the wave function is given by an exponential function of a sum of terms of different orders of c^{-2} ,

$$\Phi(x) = \exp \left(\frac{i}{\hbar} (c^2 S_0(x) + S_1(x) + c^{-2} S_2(x) + \dots) \right). \quad (5.4)$$

Here the functions $S_N(x)$ may take complex values. As we are looking for solutions to (5.2) that are small perturbations of plane harmonic waves $\sim e^{ik_\mu x^\mu}$ and, hence, have no zeros, the ansatz (5.4) is no restriction of generality. We insert this ansatz into the Klein-Gordon equation and equate equal powers of c . The equation of leading order, c^4 , is

$$(\delta^{\mu\nu} \delta^{\rho\sigma} + 2\phi^{\mu\nu\rho\sigma}) \partial_\mu S_0 \partial_\nu S_0 \partial_\rho S_0 \partial_\sigma S_0 = 0. \quad (5.5)$$

As the Finsler coefficients are small, this implies $\partial_\mu S_0 = 0$, i.e., S_0 can only be a function of time, $S_0(x) = S_0(t)$. The next order, c^2 , yields the equation

$$\left(\frac{dS_0(t)}{dt} \right)^2 - m^2 = 0, \quad (5.6)$$

which possesses the solutions

$$S_0(t) = \pm mt + \text{const} \quad (5.7)$$

where, for physical reasons, we do not consider the plus sign. The equation of next order, c^0 , gives for the function $\Phi_1(x) = e^{\frac{i}{\hbar} S_1(x)}$ the equation of motion

$$i\hbar \frac{\partial \Phi_1(x)}{\partial t} = -\frac{\hbar^2}{2m} \left(\Delta + \frac{2\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\Delta} \right) \Phi_1(x). \quad (5.8)$$

This represents the free Schrödinger equation in our Finsler space-time. Coupling to an electrostatic potential V will be performed through

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} - \frac{i}{\hbar} eV(x), \quad (5.9)$$

which gives us the time-dependent Schrödinger equation with coupling to an electrostatic potential,

$$i\hbar \frac{\partial \Phi_1(x)}{\partial t} = -\frac{\hbar^2}{2m} \left(\Delta + \frac{2\phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{\Delta} \right) \Phi_1(x) - eV(x) \Phi_1(x). \quad (5.10)$$

Upon inserting for V our expression for the perturbed Coulomb potential, the time-independent Schrödinger equation (5.1) results from a separation ansatz $\Phi_1(x) = \Psi(\vec{r}) e^{-iEt/\hbar}$. Note that in (5.1) the radial variable r can be separated from the angular variables θ and φ exactly as in the ordinary theory. The two angular variables, however, cannot be separated from each other.

B. Finsler modified energy levels

We want to determine the bound states and the energy levels by the perturbation method to within linear order in the Finsler coefficients $\phi^{\alpha\beta\gamma\delta}$. This will give us the splitting of the hydrogen spectral lines as produced by the Finsler perturbation. Of course, as we are considering the simple Kepler problem as the unperturbed situation, this splitting is to be viewed on top of all the other (fine-structure and hyperfine-structure) splittings of the hydrogen spectral lines which are well understood.

We denote the unperturbed bound states of the Coulomb potential by

$$\begin{aligned} \Psi_{nlm}(\vec{r}) &= \sqrt{\frac{2^3(n-l-1)!}{n^3 a_0^3 2n(n+l)!}} e^{-\frac{r}{na_0}} \left(\frac{2r}{na_0} \right)^l \\ &\times L_{n-l-1}^{2l+1} \left(\frac{2r}{na_0} \right) Y_l^m(\theta, \varphi), \end{aligned} \quad (5.11)$$

where

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \quad (5.12)$$

is the Bohr radius, the L_p^q are the generalized Laguerre polynomials and the Y_l^m are the spherical harmonics. The quantum numbers n, l and m take the values

$$n = 1, 2, \dots; \quad l = 0, \dots, n-1; \quad m = 0, \dots, \pm l. \quad (5.13)$$

The corresponding unperturbed eigenvalues are

$$E_n = -\frac{\text{Ry}}{n^2}, \quad \text{Ry} = \frac{e^2}{8\pi\epsilon_0 a_0}. \quad (5.14)$$

The first-order corrections to the eigenvalues are determined by the matrix elements

$$M_{nlm, n'l'm'} = -\left\langle \Psi_{nlm} \left| \frac{\hbar^2 \phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{m\Delta} \Psi_{n'l'm'} \right. \right\rangle + \left\langle \Psi_{nlm} \left| \frac{3e^2}{16\pi\epsilon_0 r} \phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) \Psi_{n'l'm'} \right. \right\rangle. \quad (5.15)$$

The first scalar product on the right-hand side can be calculated more easily in the momentum representation,

$$-\left\langle \Psi_{nlm} \left| \frac{\hbar^2 \phi^{\alpha\beta\gamma\delta} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta}{m\Delta} \Psi_{n'l'm'} \right. \right\rangle = \frac{1}{m} \langle \hat{\Psi}_{nlm} | \phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) p^2 \hat{\Psi}_{n'l'm'} \rangle, \quad (5.16)$$

where $\hat{\Psi}_{nlm}(\vec{p})$ is the Fourier transform of $\Psi_{nlm}(\vec{r})$ which is given by [28]

$$\hat{\Psi}_{nlm}(\vec{p}) = \sqrt{\frac{2a_0^3 \hbar n(n-l-1)!}{\pi(n+l)!}} \frac{2^{2l+2} (\hbar a_0 p)^l}{(a_0^2 p^2 + \hbar^2)^{l+2}} \times C_{n-l-1}^{l+1} \left(\frac{a_0^2 p^2 - \hbar^2}{a_0^2 p^2 + \hbar^2} \right) Y_l^m(\theta, \varphi), \quad (5.17)$$

where the C_s^k are the Gegenbauer polynomials.

We now calculate the necessary matrix elements one by one to determine the perturbations of the lowest energy levels.

The ground state, $n = 1$, is nondegenerate. Under the Finsler perturbation, its energy value is shifted in first-order perturbation theory according to

$$E_1 \rightarrow E_1 + \Delta E_1, \quad (5.18)$$

where

$$\Delta E_1 = M_{100,100}. \quad (5.19)$$

Calculation of this matrix element yields

$$\Delta E_1 = \frac{7\text{Ry}}{12} (\phi^{1111} + \phi^{2222} + \phi^{3333}), \quad (5.20)$$

where we have used the trace-free condition.

The next level, $n = 2$, is fourfold degenerate in the unperturbed situation. Under the Finsler perturbation, it will in general split into four levels,

$$E_2 \rightarrow E_2 + \Delta E_2^A, \quad A = 1, 2, 3, 4, \quad (5.21)$$

where, in first-order perturbation theory, the ΔE_2^A are the eigenvalues of the perturbation matrix $(M_{2lm, 2l'm'})$. The entries of this (4×4) matrix can be calculated. Using again the trace-free condition, we find

$$M_{200,200} = \frac{19\text{Ry}}{48} (\phi^{1111} + \phi^{2222} + \phi^{3333}), \quad (5.22)$$

$$M_{210,210} = \frac{19\text{Ry}}{112} (\phi^{1111} + \phi^{2222} + 5\phi^{3333}), \quad (5.23)$$

$$M_{211,211} = M_{21(-1),21(-1)} = \frac{19\text{Ry}}{112} (3\phi^{1111} + 3\phi^{2222} + \phi^{3333}), \quad (5.24)$$

$$M_{200,210} = M_{200,211} = M_{200,21(-1)} = 0, \quad (5.25)$$

$$M_{210,211} = -\overline{M_{210,21(-1)}} = -\frac{19\text{Ry}}{\sqrt{2}140} (\phi^{1113} + \phi^{1333} + i(\phi^{2223} + \phi^{2333})), \quad (5.26)$$

$$M_{211,21(-1)} = \frac{19\text{Ry}}{56} \left(-\phi^{1111} + \phi^{2222} + \frac{2i}{5} (\phi^{1112} + \phi^{1222}) \right), \quad (5.27)$$

where overlining means complex conjugation.

The perturbation matrix consists of a 1×1 block and a 3×3 block. Therefore, calculating the eigenvalues requires solving a third-order equation. This can be done explicitly, but the resulting expressions are rather awkward and will not be given here.

The transition from the E_2 level to the E_1 level is known as the Lyman- α line. Our Finsler perturbation causes a splitting of this line into four lines in general, a singlet ($l = 0$) and a triplet ($l = 1$). The Lyman- α line does *not* split if and only if the perturbation matrix $M_{2lm, 2l'm'}$ is a multiple of the unit matrix. This is the case if and only if

$$\phi^{1111} = \phi^{2222} = \phi^{3333} = 0, \quad \phi^{1112} + \phi^{1222} = \phi^{1113} + \phi^{1333} = \phi^{2223} + \phi^{2333} = 0. \quad (5.28)$$

[The six Finsler coefficients on the left-hand sides of (4.10) are then all zero.] This demonstrates that observations of the Lyman- α line alone cannot give us bounds on *all* Finsler coefficients. Even if we observe, with a certain measuring accuracy, that the Lyman- α line does not split, we could have arbitrary Finsler coefficients $\phi^{1112} = -\phi^{1222}$, $\phi^{1113} = -\phi^{1333}$ and $\phi^{2223} = -\phi^{2333}$.

One may consider the transition from the E_3 level to the E_1 level in addition which, in the unperturbed situation, gives rise to the Lyman- β line. The Lyman- β line splits, in general, into nine lines, a singlet ($l = 0$), a triplet ($l = 1$) and a quintuplet ($l = 2$). The energy shifts are determined by the eigenvalues of the matrix ($M_{3lm,3'l'm'}$). We calculate only two of these matrix elements,

$$M_{322,32(-2)} = \frac{13\text{Ry}}{252} (3\phi^{1111} + 3\phi^{2222} - \phi^{3333} + 2i(\phi^{1222} - \phi^{1112})) \quad (5.29)$$

and

$$M_{320,321} = -\overline{M_{320,32(-1)}} = -\frac{13\text{Ry}}{\sqrt{642}} (\phi^{1333} + i\phi^{2333}). \quad (5.30)$$

The Lyman- β line does not split if and only if the matrix ($M_{3lm,3'l'm'}$) is a multiple of the unit matrix. This requires, in particular, vanishing of the two off-diagonal matrix elements we have calculated; hence,

$$\phi^{1222} - \phi^{1112} = \phi^{1333} = \phi^{2333} = 0. \quad (5.31)$$

If neither the Lyman- α nor the Lyman- β line splits, both (5.28) and (5.31) have to hold, so in this case all Finsler coefficients must be zero. This demonstrates that we get bounds on *all* Finsler coefficients if we observe, with a certain measuring accuracy, that neither the Lyman- α line nor the Lyman- β line splits.

As a special case, we consider a Finsler perturbation that respects the symmetry about the z axis. This simplifying assumption seems reasonable in a laboratory on Earth if one believes that the Finsler anisotropy has a gravitational origin. Then the expression $\phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi)$ in (4.21) must be independent of φ . In combination with the trace-free condition, this symmetry assumption requires that (4.22) simplifies to

$$\phi^{\alpha\beta\gamma\delta} f_{\alpha\beta\gamma\delta}(\theta, \varphi) = \phi^{1111} (1 - 5\cos^2\theta + 10\cos^4\theta); \quad (5.32)$$

i.e., there is only one independent Finsler coefficient left.

The perturbation of the E_1 level (5.20) simplifies to

$$\Delta E_1 = \frac{14\text{Ry}}{3} \phi^{1111}. \quad (5.33)$$

The perturbation matrix ($M_{2lm,2'l'm'}$) becomes diagonal, so that the eigenvalues can easily be calculated. For the singlet we find

$$\Delta E_2^1 = \frac{19\text{Ry}}{6} \phi^{1111}, \quad (5.34)$$

whereas the triplet degenerates into two lines,

$$\Delta E_2^2 = \frac{38\text{Ry}}{7} \phi^{1111}, \quad (5.35)$$

$$\Delta E_2^3 = \Delta E_2^4 = \frac{57\text{Ry}}{28} \phi^{1111}. \quad (5.36)$$

This demonstrates that, in this case, the Lyman- α line splits into three lines. The spacing between the outermost lines is

$$\Delta E_2^2 - \Delta E_2^3 = \frac{95\text{Ry}}{28} \phi^{1111}. \quad (5.37)$$

If we observe, with a certain measuring accuracy $\delta\omega$ of the frequency, that the Lyman- α line does not split, we can deduce that

$$|\phi^{1111}| \leq \frac{28\hbar\delta\omega}{95\text{Ry}} \approx 1.4 \times 10^{-17} \delta\omega/\text{Hz}. \quad (5.38)$$

In the general case, without the special symmetry assumption, we get similar bounds for all Finsler coefficients from the observation that neither the Lyman- α nor the Lyman- β line splits. (Instead of 28/95, we get, of course, other numerical factors.)

VI. CONCLUSIONS

We have calculated the Finsler perturbation of atomic spectra for the simplest possible case, using the Schrödinger equation with the standard Coulomb potential for the unperturbed situation and a linearized metric perturbation that derives from the square root of a fourth-order term. We emphasize again that, if the results are to be compared with measurements of the hydrogen spectrum, the Finslerian splitting of the spectral lines has, of course, to be viewed as coming on top of all the other fine-structure and hyperfine-structure splittings that are well understood. Also, more complicated atomic spectra and more complicated Finslerian metric perturbations can be considered. What we wanted to estimate was the order of magnitude for the bounds on the Finsler perturbations that can be achieved by atomic spectroscopy. We see from (5.38) that these bounds are quite tight. Given the fact that, nowadays, frequencies can be measured in the optical and in the ultraviolet with an accuracy of up to $\delta\omega \approx 10^{-7}$ Hz, with this kind of measurements it should be possible to get an upper bound on the dimensionless Finsler coefficients of

about 10^{-24} . This bound is by several orders of magnitude smaller than the bounds from Solar System tests; cf. [22].

Using nuclear spectroscopy, rather than atomic spectroscopy, it might be possible to get even better bounds. The Hughes-Drever experiment (see e.g. Will [29]) comes to mind which gives the best bounds on anisotropic mass terms to date. It is based on magnetic resonance measurements of a Li-7 nucleus whose ground state of spin $3/2$ splits into four levels when a magnetic field is applied. Anisotropic mass terms would lead to an unequal spacing between these levels. It was also shown that the Hughes-Drever experiment gives very restrictive bounds on torsion; see [30]. The Finsler perturbations discussed in this paper are not exactly of the same mathematical form as anisotropic mass terms or torsion terms, but they also introduce some kind of spatial anisotropy. For this reason, it seems likely that a careful reanalysis of the Hughes-Drever experiment would also give some strong bounds on possible Finsler perturbations, probably even stronger than the bounds from atomic spectroscopy. However, there are two difficulties with the Hughes-Drever experiment, one from the theoretical side and one from the experimental side. Theoretically, the analysis of the experiment would have to be based on a wave equation for a particle with spin, i.e., on a Finsler generalization of a Dirac-type equation or on a nonrelativistic approximation thereof. The basic idea of how such a Dirac-type equation could be found in a Finsler setting is rather straightforward: One would have to linearize the corresponding Klein-Gordon equation with respect to the derivative operators; see e.g. [31]. However, the procedure is considerably more complicated than in the spinless case, and the details have not yet been worked out for the kind of Finsler perturbation discussed in this paper. Experimentally, a Hughes-Drever experiment in its

standard setting is performed by keeping the magnetic field fixed in the laboratory and waiting for 24 hours so that the Earth makes a full rotation with respect to the space-time background geometry. In this way, one can detect “cosmological” anisotropies, i.e. anisotropies in the background geometry, but not “gravitational” anisotropies which would rotate with the Earth. If one thinks of a Finsler perturbation as having a gravitational origin, it would be of a type that could not be detected with a Hughes-Drever experiment in its usual setting. One would have to rotate the magnetic field with respect to the laboratory which is technically more difficult.

For these two reasons, we have restricted in this paper to a test with atomic spectroscopy, rather than with nuclear spectroscopy of the Hughes-Drever type. It should be noted that such an atomic spectroscopy test applies not only to laboratory experiments on Earth, but to any situation where (hydrogen) spectral lines are observed. So it can be used also for estimating Finsler perturbations in the neighborhood of distant stars or gas clouds.

ACKNOWLEDGMENTS

We wish to thank the German-Israeli Foundation (GIF) for supporting this work within the project “Exploring the full range of electrodynamics.” During part of this work, V.P. was financially supported by Deutsche Forschungsgemeinschaft Grant No. LA905/10-1. Also, C.L. and V.P. gratefully acknowledge support from the Deutsche Forschungsgemeinschaft within the Research Training Group 1620 “Models of Gravity.” Moreover, we thank Nico Giulini for an important comment.

-
- [1] T. Damour, F. Piazza, and G. Veneziano, *Phys. Rev. Lett.* **89**, 081601 (2002).
 - [2] T. Damour, F. Piazza, and G. Veneziano, *Phys. Rev. D* **66**, 046007 (2002).
 - [3] F. Girelli, S. Liberati, and L. Sindoni, *Phys. Rev. D* **75**, 064015 (2007).
 - [4] A. G. Cohen and S. L. Glashow, *Phys. Rev. Lett.* **97**, 021601 (2006).
 - [5] G. W. Gibbons, J. G. Gomis, and C. N. Pope, *Phys. Rev. D* **76**, 081701 (2007).
 - [6] C. Barceló, S. Liberati, and M. Visser, *Living Rev. Relativity* **14**, 3 (2011), <http://www.livingreviews.org/lrr-2011-3>.
 - [7] W.-T. Ni, *Phys. Rev. Lett.* **38**, 301 (1977).
 - [8] K. S. Thorne, D. L. Lee, and A. P. Lightman, *Phys. Rev. D* **7**, 3563 (1973).
 - [9] V. A. Kostelecký and M. Mewes, *Phys. Rev. D* **66**, 056005 (2002).
 - [10] C. Lämmerzahl, in *Mass and Motion in General Relativity*, edited by L. Blanchet, A. Spallicci, and B. Whiting (Springer, Heidelberg, 2011), p. 25.
 - [11] J. Ehlers, F. A. E. Pirani, and A. Schild, in *General Relativity: Papers in Honour of J. L. Synge*, edited by L. O’Raifeartaigh (Clarendon Press, Oxford 1972), p. 63.
 - [12] B. Riemann, Habilitationsvorlesung, University of Göttingen (published 1892, Leipzig 1854).
 - [13] P. Finsler, Doctoral Dissertation, University of Göttingen (printed 1951, Birkhäuser, Basel, 1918).
 - [14] H. Rund, *The Differential Geometry of Finsler Spaces* (Springer, Berlin 1959).
 - [15] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry* (Springer, New York, 2000).
 - [16] V. Perlick, *Ray Optics, Fermat’s Principle, and Applications to General Relativity* (Springer, Berlin, 2000).
 - [17] V. Perlick, *Gen. Relativ. Gravit.* **38**, 365 (2006).

- [18] C. Lämmerzahl and F. W. Hehl, *Phys. Rev. D* **70**, 105022 (2004).
- [19] Y. Itin, *Phys. Rev. D* **72**, 087502 (2005).
- [20] V. Červený, *Studia geophysica et geodaetica* **46**, 567 (2002).
- [21] J. Beem, *Can. J. Math.* **22**, 1035 (1970).
- [22] C. Lämmerzahl, V. Perlick, and W. Hasse, *Phys. Rev. D* **86**, 104042 (2012).
- [23] C. Pfeifer and M. N. R. Wohlfarth, *Phys. Rev. D* **84**, 044039 (2011).
- [24] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE* (Birkhäuser, Boston, MA, 1991).
- [25] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics* (Birkhäuser, Basel, 2003).
- [26] C. Lämmerzahl, D. Lorek, and H. Dittus, *Gen. Relativ. Gravit.* **41**, 1345 (2009).
- [27] C. Kiefer and T. P. Singh, *Phys. Rev. D* **44**, 1067 (1991).
- [28] B. Podolsky and L. Pauling, *Phys. Rev.* **34**, 109 (1929).
- [29] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, UK, 1993).
- [30] C. Lämmerzahl, *Phys. Lett. A* **228**, 223 (1997).
- [31] J. Audretsch and C. Lämmerzahl, in *Semantical Aspects of Space-Time Geometry*, edited by U. Majer and H.-J. Schmidt (BI Verlag, Mannheim, Germany, 1993).