PHYSICAL REVIEW D **90,** 124036 (2014)

Quantization and stability of bumblebee electrodynamics

C. A. Hernaski[*]

Indiana University Center for Spacetime Symmetries, Bloomington, Indiana 47405, USA
(Received 21 October 2014; published 10 December 2014)

The quantization of a vector model presenting spontaneous breaking of Lorentz symmetry in flat Minkowski spacetime is discussed. The Stueckelberg trick of introducing an auxiliary field along with a local symmetry in the initial Lagrangian is used to convert the second-class constraints present in the initial Lagrangian to first-class ones. An additional deformation is employed in the resulting Lagrangian to handle properly the first-class constraints, and the equivalence with the initial model is demonstrated using the BRST invariance of the deformed Lagrangian. The framework for performing perturbation theory is constructed, and the structure of the Fock space is discussed. Despite the presence of ghost and tachyon modes in the spectrum of the free theory, it is shown that one can implement consistent conditions to define a unitary and stable reduced Fock space. Within the restricted Fock space, the free model turns out to be equivalent to the Maxwell electrodynamics in the temporal gauge.

DOI: 10.1103/PhysRevD.90.124036 PACS numbers: 04.60.Ds, 11.30.Cp, 11.30.Er, 11.30.Qc

I. INTRODUCTION

A great amount of effort has been employed in the construction of a consistent quantum theory of gravity. This relies mainly on theoretical grounds, since direct experimental evidence from the Planck scale $\simeq 10^{19}$ GeV, where such a theory plays a major role, is presently unattainable. However, it may happen that these fundamental theories can provide some key signals that our currently low-energy-scale experiments are able to detect. Since CPT and Lorentz symmetry are among the tenets of our present understanding of nature at the fundamental level, minor deviations from these symmetries could be detected in low-energy experiments. In this way, CPT and Lorentz violation is one of these key signals and, interestingly, it may occur in many of the candidates for fundamental theories, like string theory [1,2], loop quantum gravity [3], noncommutative field theories [4], and nontrivial spacetime topology [5].

The effective field theory that accounts for the possible deviations of the known physical phenomena due to Lorentz and CPT violations is the Standard-Model Extension (SME) [6–8]. In this framework, the usual Lagrangians of the Standard Model (SM) of the elementary particles of physics and of Einstein's general theory of relativity are supplemented by Lorentz-violating (LV) operators. In the nongravitational sector of the SME, these LV operators are constructed by considering all SM operators contracted with LV tensorial coefficients in a coordinate-invariant way. The gravity sector, in turn, follows the same idea, but considering diffeomorphism tensors instead of the SM operators.

The LV coefficients can be generated in many different ways. One particularly elegant and generic one is through spontaneous Lorentz and CPT violation [1,9]. In this case, along with Lorentz and CPT violation, other important consequences can arise, like the appearance of Nambu-Goldstone (NG) and Higgs modes. Unlike the effective framework provided by the SME, the properties of these modes are, in general, model dependent and cannot be completely discussed without knowledge of the underlying fundamental theory. However, in many cases, some features of the propagation of these modes can be discovered in a model-independent way. In Ref. [10], for instance, the effects of the NG modes on the metric field are taken into account using the coordinate invariance requirement. In the work of Ref. [11], some general conclusions about the fate of the NG modes are also obtained without considering any particular theory. It is also worth mentioning that, unlike the nongravitational SME sector, the gravity sector needs to take into account the NG modes to keep its consistency as is shown in Ref. [8].

Being an effective model, the SME is expected to apply at low energies up to some characteristic energy scale frequently related to the Planck scale, and for this reason, it is unsurprising that some inconsistencies can arise in the analysis of some phenomena if the typical energy scale under consideration is pushed to arbitrarily large values. Concerning the photon sector of the SME, the works in Ref. [12] investigated the subtle issues of microcausality and unitarity. In Ref. [13], focusing in the fermion sector, the authors conclude that some problems can arise for energies of the order of the Planck scale. It is also suggested that contributions coming from the extra modes, due to the spontaneous symmetry-breaking mechanism in the fundamental theory, could help in the consistency of the models. To better understand the role of the NG modes in the problem of the stability and causality, it seems relevant to consider the quantization of models presenting spontaneous Lorentz and CPT violation and try to extract from them some general features.

[*]chernask@indiana.edu

The most studied LV models contemplating the role of the extra modes arising from the spontaneous symmetry-breaking mechanism involve the vacuum condensation of a vector field. These are called "bumblebee models" and were discussed in curved and flat spacetimes [1,10,11,14,15]. Besides the NG excitations, other modes can also appear, like massive ones called Higgs modes and Lagrange multiplier modes. The propagation properties of all the dynamical modes depend on the form of the kinetic and potential term considered in the Lagrangian. In Ref. [11], it was shown that even when the extra modes do not propagate, they can give interesting and measurable contributions to the Coulomb and Newton potentials.

Bumblebee models have been extensively investigated not only as toy models to probe the role of the several excitations originated from the Lorentz symmetry-breaking mechanism, but also as an alternative to the $U(1)$ gauge theory in the consistent description of the photon. In this case, the masslessness of the photon is unrelated to the invariance of the system under a local symmetry, but is instead related to its identification as a NG mode. Surprisingly, with some assumptions, these LV vector models turn out to be equivalent to the Maxwell electromagnetism in a special nonlinear gauge. Actually, a very interesting model considered by Nambu [16] already described the photon as a NG mode due to spontaneous Lorentz violation. In Nambu's model, Lorentz violation is introduced by choosing a nonlinear gauge condition and inserting it directly into the Maxwell Lagrangian coupled to a conserved current. However, unlike the bumblebee models, Lorentz violation in Nambu's model is unphysical, since it is a consequence of a special gauge choice.

This work establishes a suitable formalism for the canonical quantization of a particular bumblebee model with a Maxwell-type kinetic term and a smooth quartic potential responsible for triggering the spontaneous breaking of Lorentz symmetry. This model was introduced by Kostelecký and Samuel (KS) in Ref. [1] and was investigated in Refs. [11,14]. Besides the massless NG modes, it propagates a massive tachyonic excitation leading to instabilities. However, it will be shown that one can consistently choose a region of the phase space of the solutions where the tachyon does not propagate, and within this phase space slice the model is classically equivalent to the Maxwell theory in a nonlinear gauge. Since the model has second-class constraints, the most direct way to apply the methods of canonical quantization is through Dirac's method of the quantization of constrained systems. However, to avoid the difficulties of Dirac's method, this work makes use of the Stueckelberg method. It consists of the enlargement of the field content along with the introduction of a local symmetry in the Lagrangian to turn the second-class constraints into first-class ones. The first-class constraints, in turn, are handled with the usual procedure of quantization of gauge theories. First, a gauge-fixing term will be introduced in the gauge-invariant Lagrangian, and it will be shown that the new regular Lagrangian is BRST invariant, resulting in its equivalence to the KS model. Truncating the Lagrangian up to quadratic terms, the basic components to perform a systematic quantum analysis of the model are constructed. These include the derivation of the dispersion relations of the propagating modes, the subtle Fourier-mode expansion of the free fields, and the correct identification of the creation and annihilation operators as well as their algebra. The perturbative conditions for the absence of negative-norm states and tachyonic excitations are also discussed. The resulting free model is tachyon free with a positive-normed Fock space that coincides with that of the Maxwell electrodynamics in the temporal gauge. To test the consistency of the treatment, the analysis of the stability of the free model is made and compared with the classical discussion of Ref. [14].

This paper is organized as follows: Section II reviews the main classical properties of the KS model. The implementation of the perturbation analysis and application of the Stueckelberg method to the KS model is discussed in Sec. III. In Sec. IV, the Fourier-mode expansion of the fields and the construction of the extended Fock space of the deformed KS model are performed. The conditions for the absence of negative-norm states and the stability of the free model are discussed in Sec. V. Finally, in Sec. VI, the results are summarized. In Appendix A, the BRST invariance of the full proposed Lagrangian is demonstrated. Appendix B presents some technical calculations concerning the Fourier expansion of the fields.

II. SPONTANEOUS LORENTZ SYMMETRY VIOLATION AND CLASSICAL STABILITY OF THE KOSTELECKÝ-SAMUEL MODEL

The starting point is the specific KS model with a smooth quartic potential:

$$\mathcal{L}_{\mathrm{KS}} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\kappa}{4}(B_\mu B^\mu - b^2)^2 - B_\mu J^\mu, \qquad (1)$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \qquad (2)$$

κ is a dimensionless positive constant, and J^μ is an assumed conserved current composed of matter fields, and it is also the source for the B_μ field. In the present analysis, the dynamics for the matter fields that compose the current J^μ will be disregarded. b^2 is a positive constant with dimension of $(\mathrm{mass})^2$, and it will be convenient, for the coming discussions, to consider it as the quadratic scalar $b^2 = b_\mu b^\mu$, formed out of the constant timelike vector b_μ. The first term in the Lagrangian (1) is the usual Maxwell term, so it is invariant under $U(1)$ gauge transformations.

However, the potential term, $V = \frac{\kappa}{2}(B_\mu B^\mu - b^2)^2$, breaks this gauge invariance. One can also see that the minimum of the potential occurs for $B^2 = b^2$. Therefore, the field B_μ acquires a nonvanishing vacuum expectation value. This indicates the occurrence of spontaneous breaking of Lorentz symmetry. The vacuum is degenerate, and any choice between the possible vacuum states leads to equivalent physical scenarios. For definiteness, the vacuum state is chosen to be such that $\langle B_\mu \rangle = b_\mu$, where $\langle\cdot\rangle$ means vacuum expectation value. According to Ref. [17], one can classify the potential propagating modes in a spontaneously symmetry-broken model in five types. For the present purposes, due to the characteristics of the KS model, only two of them are relevant: the NG modes, which are massless excitations satisfying the condition $V'(X) = 0$, where the prime means derivative with respect to $X = B_\mu B^\mu - b^2$; and massive modes that satisfy $V' \neq 0$. Since $V' = \kappa(B_\mu B^\mu - b^2)$, the massive mode will be present whenever $(B_\mu B^\mu - b^2) \neq 0$.

As discussed in Ref. [14], the Hamiltonian associated with Lagrangian (1) is unbounded from below for general field configurations. Nonetheless, initial conditions can be chosen such that, for such field configurations, the Hamiltonian remains positive.

The conjugate momenta associated with the B_μ fields are defined by

$$\Pi^\mu \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 B_\mu)}. \tag{3}$$

From this definition and from Eqs. (1) and (2), one has

$$\Pi^\mu = -B^{0\mu}. \tag{4}$$

This immediately shows that only three out of the four components of B_μ actually propagate. In fact, $\Pi^0 = 0$ is identified as the primary constraint on the phase space of the model. Following the usual Lagrangian approach, the equations of motion can be derived. They are given by

$$\partial_\mu B^{\mu\nu} - \kappa(B_\mu B^\mu - b^2)B^\nu = J^\nu.$$

Considering $\nu = 0$, one gets the consistency condition for the primary constraint. The two constraints

$$\phi = \Pi^0 \approx 0, \tag{5}$$

$$\chi = \partial_i \Pi^i - \kappa(B_\mu B^\mu - b^2)B^0 - J^0 \approx 0 \tag{6}$$

define the constrained phase space of the model. The symbol "≈" means weakly equal, which is used in equality relations only valid on the constraint surface. There are two kinds of constraints: first- and second-class ones. A first-class constraint is one whose Poisson brackets, when calculated in the extended phase space with any other constraint, vanish, whereas a second class possesses at least one nonvanishing Poisson bracket with another constraint. In the present case, the Poisson bracket between the two constraints in Eqs. (5) and (6) is nonvanishing, so they are second class.

An important fact about constrained systems is that the number of propagating degrees of freedom is different from the number that one begins with in the Lagrangian. Given a system described by N degrees of freedom and with n_1 and n_2 first- and second-class constraints, respectively, the number of propagating degrees of freedom is $N - n_1 - \frac{n_2}{2}$. From the above discussion, the model described by Lagrangian (1) has $N = 4$, $n_1 = 0$, and $n_2 = 2$. So, only three out of the four degrees of freedom actually propagate in the model. The dynamics of the fields is governed by the extended Hamiltonian, which is given by the canonical Hamiltonian $\mathcal{H}_c = \Pi^\mu B_\mu - \mathcal{L}$ up to additional multiples of the constraints. The coefficients multiplying the constraints can be determined by consistency requirements in the case of second-class constraints or remain arbitrary in the case of first-class ones.

Using the constraints in Eqs. (5) and (6) and integration by parts, the canonical Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2}(\Pi^i)^2 + \frac{1}{4}(B_{jk})^2 - \frac{1}{4}\kappa(3B_0^2 + B_j^2 + b^2)(B_0^2 - B_j^2 - b^2) + B_i J^i. \tag{7}$$

The situation when $\vec{J} = 0$ is the one in which the external matter fields do no work on the B_μ fields. A stable model should have a positive Hamiltonian in this limit. This is not the case for the Hamiltonian given in Eq. (7) when general field configurations are considered. However, positivity can be attained if the field configurations are restricted to satisfy the condition $(B_0^2 - B_j^2 - b^2) = 0$. It can be shown, using the extended Hamiltonian, that this choice of the initial conditions is preserved by the field dynamics [14]. This condition avoids the propagation of the massive mode, and the restricted phase space turns out to be equivalent to the phase space of the Maxwell electrodynamics in a nonlinear LV gauge. One of the main goals of this work is to develop a framework suitable for the quantum analysis of this stability issue. This will be the subject of the following sections.

III. PERTURBATION ANALYSIS AND STUECKELBERG METHOD

Since the KS model exhibits a phase where Lorentz symmetry is spontaneously broken, it is convenient to redefine the vector field B_μ as a perturbation, β_μ, around its expectation vacuum value b_μ. That is,

$$B_\mu = b_\mu + \beta_\mu. \tag{8}$$

In terms of this expansion, the Lagrangian (1) is written as

$$\mathcal{L}_{\rm KS} = -\frac{1}{4}\beta_{\mu\nu}\beta^{\mu\nu} - \frac{\kappa}{4}(4b_\mu\beta^\mu b_\nu\beta^\nu + \beta_\mu\beta^\mu\beta_\nu\beta^\nu + 4b_\mu\beta^\mu\beta_\nu\beta^\nu) - \beta_\mu J^\mu - b_\mu J^\mu, \tag{9}$$

where

$$\beta_{\mu\nu} = \partial_\mu\beta_\nu - \partial_\nu\beta_\mu. \tag{10}$$

The presence of the second-class constraints (5) and (6) hampers the direct application of the canonical quantization rules. There are many situations where it is desirable to look for alternatives to the standard Dirac method of quantization of constrained systems. For gauge-invariant models, which are examples of models presenting first-class constraints, there are powerful tools associated with this procedure, like the Gupta-Bleuler and BRST quantization. In this case, the original gauge-invariant Lagrangian is deformed by adding suitable gauge-noninvariant terms and the Fadeev-Popov ghosts. The extra degrees of freedom, inserted by the gauge-violating terms, are eliminated by imposing additional constraints on the final set of quantum states. The absence of unphysical degrees of freedom in the second-class constrained systems foils the direct application of this procedure.

To make use of the same framework described in the quantization of gauge theories, the model given by Lagrangian (9) will be considered as a gauge-fixing limit of some gauge-invariant one. This will be done by enlarging the field content of the KS model and introducing a suitable gauge symmetry, in such a way that the new degrees are of no physical consequence. This technique is known in the literature as the Stueckelberg method. Despite the lack of physical consequences, the presence of a new field and a new symmetry provide a greater flexibility in the mathematical treatment of some properties of the model. The successful application of the Stueckelberg procedure in the analysis of the unitarity and renormalizability of massive vector theories is an example of its convenience. In this case, the Proca Lagrangian describes a massive vector field, and there is an apparent incompatibility between power-counting renormalizability and the absence of negative-norm states in the spectrum of the theory. However, by implementing the Stueckelberg method one can define an equivalent Lagrangian where the renormalizability and unitarity are evident [18]. A review of the Stueckelberg method can be found, for example, in Ref. [19].

The Stueckelberg field is introduced in the Lagrangian (9) through the substitution

$$\beta_\mu \longrightarrow \beta_\mu - \frac{1}{\sqrt{\kappa}}\partial_\mu\phi. \tag{11}$$

With this substitution, the Kostelecký-Samuel-Stueckelberg (KSS) Lagrangian is defined as

$$\mathcal{L}_{\rm KSS} = \mathcal{L}_{\rm KS}\left(\beta_\mu \longrightarrow \beta_\mu - \frac{1}{\sqrt{\kappa}}\partial_\mu\phi\right). \tag{12}$$

This Lagrangian is invariant under the following gauge transformations:

$$\beta'_\mu = \beta_\mu + \partial_\mu\chi, \tag{13}$$

$$\phi' = \phi + \sqrt{\kappa}\chi, \tag{14}$$

where χ is some arbitrary smooth function of the spacetime coordinates. The invariance can easily be seen by noticing that the combination on the right-hand side of expression (11) is invariant under these transformations. The arbitrariness in the field content permits one to choose the function χ in Eqs. (13) and (14) such that the ϕ field vanishes, and the original Lagrangian (9) is recovered. In the standard terminology of gauge theories, this is known as the unitary gauge. Nevertheless, more interesting is to take advantage of the gauge freedom of the Lagrangian (12) and make use of the above mentioned machinery employed in the treatment of gauge-invariant models. In this vein, the Lagrangian (12) will be deformed by adding to it a gauge-violating term, promoting, in this way, the propagation of the gauge degrees of freedom. As a commonly used terminology in the literature, this term will be referred to as a gauge-fixing term.

It is convenient to choose a gauge-fixing term that provides dynamics for the 0 component of the β_μ field and cancels out the mixing between the β_μ and ϕ fields in the Lagrangian (12). The first criterion enables one to avoid the presence of the second-class constraints in Eqs. (5) and (6), whereas the second is only for making the future correspondence between the fields and the particle quantum states more transparent. One can easily verify that the following gauge-fixing Lagrangian meets these requirements:

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}(b^\nu b^\mu\partial_\nu\beta_\mu - 2\xi\sqrt{\kappa}\phi)^2, \tag{15}$$

where ξ is a parameter whose value can be chosen conveniently.

Gathering this gauge-fixing Lagrangian with $\mathcal{L}_{\rm KSS}$, one gets the total Lagrangian, $\mathcal{L}_T = \mathcal{L}_{\rm KSS} + \mathcal{L}_{\rm gf}$, to be discussed from now on. The introduction of a gauge-fixing term into the Lagrangian can sometimes be dangerous, since this term provides dynamics for the unphysical degrees of freedom and can lead to nontrivial consequences. The BRST quantization method is an interesting tool to analyze this issue. In Appendix A, it is shown that $\mathcal{L}_T = \mathcal{L}_{\rm KSS} + \mathcal{L}_{\rm gf}$ is BRST invariant if the Fadeev-Popov ghosts are added, but these decouple from the other fields and can be discarded without affecting physical results. Furthermore, $\mathcal{L}_T$ with the ghost fields included differs from $\mathcal{L}_{\rm KSS}$ by a term that is in the image of the BRST operator. If a physical state is defined as a state without ghosts, by imposing the Gupta-Bleuler condition, this term does not bring any contribution to the physical states. The net result is that the physical Hilbert space construct from $\mathcal{L}_T$ is the same as the one construct from $\mathcal{L}_{\rm KSS}$. For a review of the consequences of the BRST invariance, see for example Ref. [20].

The intention of this work is to construct a framework to perform perturbative quantum calculations with the KS model. As a first effort in this direction, the attention will be mainly focused on the free part of the total Lagrangian $\mathcal{L}_{\rm KSS}+\mathcal{L}_{\rm gf}$; that is, the current J^μ will be switched off, and only quadratic terms in the β_μ and ϕ fields will be considered. By doing this, one assumes that the constant κ is sufficiently small to be considered as a perturbation parameter. Without the interaction terms, the KS Lagrangian (9) turns out to be of the same form as the Proca-like LV theories considered in Refs. [21,22]. In those works, an explicitly LV mass term of the form $m^2A_i^2$ is considered along with the Maxwell kinetic term for the vector field A^μ rendering the transverse modes to be massive. In the context of the electron-photon sector of the SME, which is $U(1)$ gauge invariant, there is the possibility that these gauge-violating mass terms can arise as a result of radiative corrections. In the work of Ref. [23], this issue is addressed, and the dispersion relations for a more general class of LV mass terms of the form $M_{\mu\nu}A^\mu A^\nu$ are discussed. The violation of the Lorentz symmetry in the mentioned works is explicit, since they do not take into account the NG and massive modes emerging from the spontaneous symmetry-breaking mechanism. In the present work, on the other hand, the considered vector field is assumed to describe these NG and massive modes. As a result, despite the form, the quadratic Lagrangian $-\frac{1}{4}\beta_{\mu\nu}\beta^{\mu\nu}-2\kappa b_\mu b_\nu\beta^\mu\beta^\nu$ still has a symmetry related to Lorentz invariance of the complete Lagrangian. To verify this, one can perform the infinitesimal transformation $\beta'_\mu=\beta_\mu+\omega_{\mu\nu}b^\nu$ with fixed b_μ in the quadratic Lagrangian and use the antisymmetry of the Lorentz group parameters $\omega_{\mu\nu}$. This nonlinear symmetry is a reminiscence of the Lorentz symmetry present in the full Lagrangian. The Lorentz group acts linearly on the field B^μ via $\Lambda^\mu{}_\nu B^\nu$. Since $B_\mu=b_\mu+\beta_\mu$, an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu=\delta^\mu_\nu+\omega^\mu{}_\nu$ acting on B_μ yields $b_\mu+\beta'_\mu=b_\mu+\beta_\mu+\omega^\mu{}_\nu b^\nu+\mathcal{O}(\omega^2)$, where $\omega^\mu{}_\nu$ is supposed to be of the same order of magnitude as the perturbation field β_μ. This gives the nonlinear transformation of the field β_μ. In Ref. [23] the mass matrix $M_{\mu\nu}$ cannot assume the form of a product of two vectors, and such a shift symmetry cannot be constructed. For the longitudinal mode, $b\cdot\beta$, this transformation has no effect, but for the transverse mode β^T, which satisfies $b\cdot\beta^T=0$, the shift symmetry is expected to avoid the appearance of a mass term generated by quantum corrections.

The stability of the treatment under the insertion of the self-interactions of the β_μ field and with the interactions with the auxiliary field ϕ, along with the external matter current, is of great importance, but is beyond the scope of the present work. Hence, taking into account only quadratic terms in the β_μ and ϕ fields, one gets the following free Lagrangian:

$$\mathcal{L}_{\rm free}=-\frac{1}{4}\beta_{\mu\nu}\beta^{\mu\nu}-\frac{b^\nu b^\mu b^\rho b^\sigma}{2\xi}\partial_\nu\beta_\mu\partial_\rho\beta_\sigma-\kappa b_\mu b_\nu\beta^\mu\beta^\nu \\ -b_\mu b_\nu\partial^\mu\phi\partial^\nu\phi-2\xi\kappa\phi^2. \tag{16}$$

The canonical conjugate momenta associated with the fields in this Lagrangian are given by

$$\Pi^\mu_\beta=-\tilde{\eta}^{\mu\sigma}\dot{\beta}_\sigma+\Gamma^{\mu\sigma 0i}\partial_i\beta_\sigma, \tag{17}$$

$$\Pi_\phi=-2b_0b^\mu\partial_\mu\phi, \tag{18}$$

with

$$\tilde{\eta}^{\mu\sigma}=\left(\eta^{00}\eta^{\mu\sigma}-\eta^{\mu 0}\eta^{\sigma 0}+\frac{b_0^2b^\mu b^\sigma}{\xi}\right), \tag{19}$$

$$\Gamma^{\mu\sigma 0i}=\left(\eta^{\mu i}\eta^{0\sigma}-\frac{b_0b^ib^\mu b^\sigma}{\xi}\right). \tag{20}$$

Since the constant vector b_μ is timelike, $\tilde{\eta}^{\mu\sigma}$ is an invertible matrix, and one can invert the relations in Eqs. (17) and (18) to write the time derivatives of the fields in terms of the canonical momenta and the fields themselves. So, as expected, this is a regular Lagrangian system, and its quantization follows the standard procedure of considering the observables as quantum operators acting on the Hilbert space of the particle states and the classical Poisson brackets being replaced by commutators. The equal-time canonical commutation relations (ETCR) are, therefore, given by

$$[\phi(t,\vec{x}),\Pi_\phi(t,\vec{y})]=i\delta^3(\vec{x}-\vec{y}), \tag{21}$$

$$[\beta_\nu(t,\vec{x}),\Pi^\mu_\beta(t,\vec{y})]=i\delta^\mu_\nu\delta^3(\vec{x}-\vec{y}), \tag{22}$$

and any other commutator vanishes.

From these commutators and the expressions for the conjugate momenta (17) and (18), some other useful commutation relations involving fields and time derivatives of them can be derived. Namely,

$$[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})]=-\frac{i}{2b_0^2}\delta^3(\vec{x}-\vec{y}), \tag{23}$$

$$[\dot{\phi}(x), \dot{\phi}(y)] = \frac{i}{b_0^3} b^i \partial_i^x \delta^3(\vec{x} - \vec{y}), \tag{24}$$

$$[\dot{\beta}_\sigma(t, \vec{y}), \beta_\nu(t, \vec{x})] = i\bar{\eta}_{\sigma\nu}\delta^3(\vec{x} - \vec{y}), \tag{25}$$

$$[\dot{\beta}_\mu(t, \vec{x}), \dot{\beta}_\nu(t, \vec{y})] = -i\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma}\lambda^{\rho\sigma 0i}\partial_i^x\delta^3(\vec{x} - \vec{y}), \tag{26}$$

with $\bar{\eta}$ being the inverse of the $\tilde{\eta}$ matrix (19), which is given explicitly by

$$\bar{\eta}_{\mu\nu} = \frac{\eta_{\mu\nu}}{\eta^{00}} + \left(\frac{\xi}{b_0^4} + \frac{b^2}{\eta^{00} b_0^2}\right)\eta_{0\mu}\eta_{0\nu} - \frac{1}{\eta^{00} b_0}(\eta_{0\mu} b_\nu + \eta_{0\nu} b_\mu), \tag{27}$$

and $\lambda^{\rho\sigma 0i}$ is the symmetric combination of the $\Gamma^{\mu\sigma 0i}$'s defined in Eq. (20):

$$\lambda^{\rho\sigma 0i} = \Gamma^{\rho\sigma 0i} + \Gamma^{\sigma\rho 0i}. \tag{28}$$

It can be noticed from the new Lagrangian (16) that the number of dynamical degrees of freedom has increased from three to five as compared with the initial Lagrangian (1). The two extra degrees of freedom are related to the 0 component of the β_μ field and to the Stueckelberg field ϕ, which came to the fore through the gauge-fixing Lagrangian (15). Evidently, if one desires to recover the properties of the initial KS model, it is necessary to deal properly with these extra degrees of freedom. This will be done in Sec. V by choosing a specific region of the full Hilbert space that accommodates the particle states of the model described by $\mathcal{L}_{\text{free}}$, thwarting the appearance of the extra degrees of freedom in the physical spectrum.

IV. FOURIER EXPANSION

In this section, the relations between the energy and momentum for the particle spectrum of the model described by Lagrangian (16) are obtained, and the expansion of the fields β_μ and ϕ in terms of Fourier modes is derived. Obtaining the dispersion relations for the propagating modes and the discussion of their physical properties is the subject of Sec. IV A. Section IV B introduces the concept of "pure-mode solutions," which are particular solutions of the equations of motion of the β_μ field that satisfy convenient orthogonality relations. Finally, the general solutions for the β_μ and ϕ fields are derived in Sec. IV C.

A. Dispersion relations

To proceed with the analysis of the model described by Lagrangian (16), the equations of motion are derived. They are

$$\partial_\mu\partial^\mu\beta^\nu - \partial_\mu\partial^\nu\beta^\mu + \frac{b^\nu b^\mu b^\rho b^\sigma}{\xi}\partial_\mu\partial_\rho\beta_\sigma - 2\kappa b_\mu b^\nu\beta^\mu = 0, \tag{29}$$

$$b^\mu b^\nu \partial_\mu\partial_\nu\phi - 2\xi\kappa\phi = 0. \tag{30}$$

Assuming that the fields can be expressed as Fourier integrals, one gets these equations in momentum space:

$$\left[-p^2\eta^{\mu\nu} + p^\nu p^\mu - \left(\frac{(p\cdot b)^2}{\xi} + 2\kappa\right)b^\mu b^\nu\right]\beta_\nu(p) = 0, \tag{31}$$

$$((p\cdot b)^2 + 2\xi\kappa)\phi(p) = 0. \tag{32}$$

The conditions for the existence of nontrivial solutions for these equations are given, respectively, by

$$\det\left[-p^2\eta^{\mu\nu} + p^\nu p^\mu - \left(\frac{(p\cdot b)^2}{\xi} + 2\kappa\right)b^\mu b^\nu\right] = 0, \tag{33}$$

$$(p\cdot b)^2 + 2\kappa\xi = 0. \tag{34}$$

The second equation provides the dispersion relation of the particle associated with the Stueckelberg field, whereas the roots of the first equation give the dispersion relations for the particles associated with β_μ. The latter are promptly obtained by solving Eq. (33). Since the expression inside the brackets is a 4×4 matrix, an eighth-order polynomial in the momentum p is expected from Eq. (33), and, in the most general scenario, eight distinct roots. Nevertheless, the polynomial only presents monomials with even powers in the four-momentum; therefore, it is invariant under the replacement $(p^0, \vec{p}) \longrightarrow (-p^0, -\vec{p})$. The solution with negative energy and negative three-momentum can be reinterpreted through a parity and time-reversal transformation (PT) as a solution with positive energy and positive three-momentum. This reflects the fact that CPT symmetry remains unbroken in this model, as can be directly seen from the Lagrangian (16). The symmetry under charge conjugation is trivial, since the field is real. However, parity and time reversal, considered in isolation, are not symmetries of the Lagrangian (16), which can also be verified by the noninvariance of Eq. (33) under the replacement $(p^0, \vec{p}) \longrightarrow (-p^0, \vec{p})$.

The dispersion relations obtained from Eq. (33) will be labeled by $\lambda = 0, 1, 2, 3$. They are given by

$$\lambda = 0\colon\ (p\cdot b)^2 + 2\kappa\xi = 0, \tag{35}$$

$$\lambda = 1, 2\colon\ p^2 = 0, \tag{36}$$

$$\lambda = 3\colon\ (p\cdot b) = 0. \tag{37}$$

The reason for the appearance of only three independent dispersion relations, instead of the four expected from CPT invariance, has to do with the remaining symmetry after the spontaneous symmetry breaking of Lorentz symmetry takes place. In the symmetric phase, the degrees of freedom of a four-vector field can be mapped to those of a spin-1 and a spin-0 particle. The presence of a timelike background vector in a CPT invariant field theory promotes a splitting in the dynamics of these degrees of freedom, and the particle states organize themselves into classes of opposite spin polarizations. For the four-vector case, its four degrees of freedom potentially describe two new spins 0: the original spin 0, and the 0 polarization of the original spin 1; and one new spin 1: the ± 1 polarizations of the original spin 1. So, these two polarizations of the spin 1 share the same dispersion relation, and this is the reason for the degeneracy of the massless pole in Eq. (36). In the following, it will be verified that the mode described by Eq. (36) is indeed a spin-1 particle, and it will be identified as the photon.

One can also note the occurrence of the same dispersion relation in Eqs. (34) and (35), as well as their dependence on the gauge-fixing parameter ξ. In fact, these two modes are the ones brought about by the gauge-fixing Lagrangian (15) and are, in this sense, unphysical. Therefore, no concern needs to be dedicated to their dependence on the gauge-fixing parameter or possible issues with the appearance of negative energies. However, since the components of the four-momentum are the reciprocal coordinates of the spacetime coordinates in the Fourier expansion, they need to be real. This restricts the gauge parameter to assume only negative values and highlights a remarkable difference in the role played by the gauge-fixing term in this model as compared with the SM gauge theories, where no such dependence of the dispersion relations on the gauge parameter appears, and no restriction in their values is present.

The four-momentum in the dispersion relation (37) is spacelike and gauge independent. So, one cannot advocate that the associated excitation will not appear in the physical spectrum. Indeed, its presence is really an indication of an instability in the model. This kind of instability should already be expected, since as seen in Sec. II, besides the NG modes, the KS model propagates a massive mode that renders the Hamiltonian to be unbounded from below. In that classical discussion, it was argued that the instability could be avoided by choosing a suitable slice of the full phase space of the field solutions. The framework to address this question in the quantized picture will be discussed in Sec. V.

B. Suitable particular solutions for the β_μ field

The decomposition of the vector field β_μ in terms of Fourier modes is subtle. To this end, it is convenient to define first what will be called "pure-mode solutions," $\beta_\mu^{(\lambda)}(\vec{p})$, that satisfy

$$\left[-p^2\eta^{\mu\nu}+p^\nu p^\mu-\left(\frac{(p\cdot b)^2}{\xi}+2\kappa\right)b^\mu b^\nu\right]\Big|_{p_0=p_0^\lambda(\vec{p})}\beta_\nu^{(\lambda)}(\vec{p})=0, \tag{38}$$

where $p_0^{(\lambda)}$ are the solutions for the dispersion relations (35)–(37). For $\lambda=0,1,2$, there are actually two solutions of the general type $p_{0\pm}^{(\lambda)}=f(\vec{p})\pm\sqrt{h(\vec{p})}$, but they are not independent due to the invariance of the expression inside the brackets under the substitution $p_\mu\to -p_\mu$. So, only one of the two needs to be considered. Conventionally, it is assumed that $p_0^{(\lambda)}$ corresponds to $p_{0+}^{(\lambda)}$. Up to normalization constants, one can show that these particular solutions are given by

$$\beta_\mu^{(0)}(\vec{p})=\left(\frac{\vec{b}\cdot\vec{p}}{b_0}+\frac{\sqrt{-2\xi\kappa}}{b_0},\vec{p}\right)\equiv p_\mu^{(0)}(\vec{p}), \tag{39}$$

$$\beta_\mu^{(i)}(\vec{p})=\epsilon_\mu^{(i)}(\vec{p}),\quad i=1,2, \tag{40}$$

$$\beta_\mu^{(3)}(\vec{p})=\left(\frac{\vec{b}\cdot\vec{p}}{b_0},\vec{p}\right)\equiv p_\mu^{(3)}(\vec{p}), \tag{41}$$

where $\epsilon_\mu^{(i)}(\vec{p})$ are two independent spacelike four-vectors that are simultaneously orthogonal to $p_\mu^{(i)}=(|\vec{p}|,\vec{p})$ and to the background vector b_μ. From their properties, one can derive the projector on this orthogonal subspace:

$$\sum_{i=1}^{2}\epsilon_\mu^{(i)}(\vec{p})\epsilon_\nu^{(i)}(\vec{p})=-\eta_{\mu\nu}+\frac{1}{\bar{p}\cdot b}(b_\mu\bar{p}_\nu+b_\nu\bar{p}_\mu)-\frac{1}{(\bar{p}\cdot b)^2}\bar{p}_\mu\bar{p}_\nu, \tag{42}$$

where $\bar{p}_\mu\equiv(|\vec{p}|,\vec{p})$.

It can be shown that this projector also appears in the propagator for the β_μ field, and, after the exclusion of the unphysical modes from the Fock space, it is identified as the propagator for the transverse physical excitations. Moreover, it also coincides with the propagator for the Maxwell theory in the temporal gauge. There, the background vector, b_μ, is assumed to have no physical consequences, since it is introduced only for choosing a particular gauge. However, in the present case, this vector could give rise to measurable effects through the coupling with the matter current, as can be seen from the Lagrangian (9).

Although the solutions (39)–(41) are particular ones for the equation of motion in momentum space (31), they are interesting because they satisfy suitable orthogonal relations. To derive such relations, Eq. (38) is rewritten as

$$[(p_0^{(\lambda)})^2\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i} p_0^{(\lambda)} p_i]\beta_\nu^{(\lambda)}(\vec{p}) = \left[\vec{p}^2\eta^{\mu\nu} + \left(\delta_i^\nu\delta_j^\mu - \frac{b_i b_j b^\mu b^\nu}{\xi}\right)p^i p^j - 2\kappa b^\mu b^\nu\right]\beta_\nu^{(\lambda)}(\vec{p}), \tag{43}$$

where $\tilde{\eta}$ and $\lambda^{\mu\nu 0i}$, defined respectively in Eqs. (19) and (28), are used. Multiplying both sides of this equation by $\beta^{\lambda'}(\vec{p})$, with $\lambda \neq \lambda'$, and subtracting from the analogous relation with λ and λ' interchanged, yields

$$\beta_\mu^{(\lambda')}(\vec{p})[(p_0^{(\lambda)} + p_0^{(\lambda')})\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i} p_i]\beta_\nu^{(\lambda)}(\vec{p}) = 0. \tag{44}$$

For general λ and λ', one can write

$$\beta_\mu^{(\lambda')}(\vec{p})[(p_0^{(\lambda)} + p_0^{(\lambda')})\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i} p_i]\beta_\nu^{(\lambda)}(\vec{p}) = \eta^{\lambda\lambda'} N^{(\lambda)}(\vec{p}). \tag{45}$$

When $\lambda' = \lambda$, the results need to be calculated explicitly. Using the explicit results for the pure-mode solutions (39)–(41), one has

$$N^{(0)} = -4b_0\kappa\sqrt{-2\kappa\xi}, \tag{46}$$

$$N^{(i)} = 2|\vec{p}|, \quad i = 1, 2, \tag{47}$$

$$N^{(3)} = 0. \tag{48}$$

Another useful orthogonality relation can be obtained by multiplying Eq. (43) by $\beta_\mu^{(\lambda')}(-\vec{p})$, switching λ for λ' and $\vec{p}$ for $-\vec{p}$, and subtracting the obtained expression by the original one multiplied by $\beta_\mu^{(\lambda')}(-\vec{p})$. This gives

$$\beta_\mu^{(\lambda)}(\vec{p})[(p_0^{(\lambda')}(-\vec{p}) - p_0^{(\lambda)}(\vec{p}))\tilde{\eta}^{\mu\nu} + \lambda^{\mu\nu 0i} p_i]\beta_\nu^{(\lambda')}(-\vec{p}) = 0. \tag{49}$$

In Ref. [24], the quantization of the photon sector within the framework of the SME is considered. For the Lagrangian considered in that work, the general solution of the equations of motion can be written as a combination of the pure-mode solutions. The analogous orthogonality relations can be used for writing the creation and annihilation operators in terms of the fields and canonical momenta and for getting the algebra of these operators. In the present case, a similar expansion would run into trouble, since the normalization factor for the massive mode in Eq. (48) vanishes, and one cannot invert the expansion for this mode. Furthermore, as was already emphasized, the expansion in terms of the pure-mode solutions (39)–(41) fails to provide the most general solution of the equation of motion (29).

C. General solutions

To construct the more general solution for β_μ following from the equation of motion (29), it is convenient to try to decouple the dynamics for the longitudinal modes. Multiplying the equation of motion (29) by ∂_μ and b_μ yields the two coupled equations for these longitudinal modes:

$$(b\cdot\partial)\partial\cdot\beta - \left(\Box + \frac{b^2}{\xi}((b\cdot\partial)^2 - 2\kappa\xi)\right)b\cdot\beta = 0, \tag{50}$$

$$((b\cdot\partial)^2 - 2\kappa\xi)(b\cdot\partial)b\cdot\beta = 0. \tag{51}$$

The solution for the last equation can be promptly obtained, since it is completely decoupled from the other modes. In possession of this solution, one can use it in Eq. (50) to obtain the solution for $\partial\cdot\beta$. Finally, both solutions can be used in Eq. (29) to get the solution for the transverse modes.

From Eq. (51), the solution for $b\cdot\beta$ can be conveniently expressed as

$$b\cdot\beta(x) = \frac{\xi}{(2\pi)^3}\int d^4p\,\delta(((p\cdot b)^2 + 2\xi\kappa)p\cdot b)\bar{c}(p)e^{-ip\cdot x}. \tag{52}$$

$\bar{c}(p)$ is a complex function of the four independent variables p_0 and $\vec{p}$. From the reality of the field $b\cdot\beta$, this function satisfies the condition $\bar{c}^\dagger(-p_0, -\vec{p}) = \bar{c}(p_0, \vec{p})$. Using this condition and the properties of the delta function, one obtains

$$b\cdot\beta(x) = \frac{1}{(2\pi)^3|b_0|}\int d^3p\left(d(\vec{p})e^{-ip^{(3)}\cdot x} - \frac{1}{4\kappa}(c(\vec{p})e^{-ip^{(0)}\cdot x} + c^\dagger(\vec{p})e^{ip^{(0)}\cdot x})\right), \tag{53}$$

where

$$c(\vec{p}) \equiv \bar{c}\left(\frac{\vec{b}\cdot\vec{p}}{b_0} + \frac{\sqrt{-2\xi\kappa}}{b_0}, \vec{p}\right) \tag{54}$$

and

$$d(\vec{p}) \equiv -\frac{1}{2\kappa}\bar{c}\left(\frac{\vec{b}\cdot\vec{p}}{b_0}, \vec{p}\right) = d^\dagger(-\vec{p}). \tag{55}$$

The solution for $\partial\cdot\beta$ in Eq. (50) can be constructed as the sum of the solution of the homogeneous equation $(b\cdot\partial)\partial\cdot\beta = 0$ plus a particular solution for the inhomogeneous one, since the inhomogeneous part is explicitly known from Eq. (53). The solution for the homogeneous equation can be derived straightforwardly following the previous reasoning to reach the solution for the $b\cdot\beta$ field in Eq. (53). Denoting $\partial\cdot\beta(x)$ as $S(x)$, one has

$$S^H(x) = \frac{1}{(2\pi)^3|b_0|}\int d^3p s(\vec{p})e^{-ip^{(3)}\cdot x}, \tag{56}$$

where $s^\dagger(-\vec{p}) = s(\vec{p})$, and the superscript H on the left-hand side of this equation stands for homogeneous.

For the particular solution S^P, one could make use of the Green function method. The caveat here is that the convolution of the Green function for the operator $b\cdot\partial$ with the distribution $\delta(((p\cdot b)^2 + 2\xi\kappa)p\cdot b)$ is ill defined. Here, only the final solution for S^P is presented, leaving the details for Appendix VI:

$$S^P(x) = -i\Box\int\frac{d^3p}{4|b_0|\sqrt{-2\kappa\xi}}(c(\vec{p})e^{-ip^{(0)}\cdot x} - c^\dagger(\vec{p})e^{ip^{(0)}\cdot x}) + (\Box - 2\kappa b^2)\int\frac{d^3p}{b_0^2}x^0 d(\vec{p})e^{-ip^{(3)}\cdot x}. \tag{57}$$

Finally, this solution and the homogeneous part from Eq. (56) can be used, along with the solution for $b\cdot\beta$ in Eq. (53), to obtain the inhomogeneous part of the differential equation (29). The solution for the entire field β_μ can again be expressed as a sum of a homogeneous plus an inhomogeneous part. The homogeneous part can be written as

$$\beta^H_\mu(x) = \frac{1}{(2\pi)^3}\int d^4p a_\mu(p)\delta(p^2)e^{-ip\cdot x}, \tag{58}$$

where the set of the four vectors $a_\mu(p)\delta(p^2)$, defined for each momentum $\vec{p}$, can be expanded in terms of some convenient complete basis of vectors for each point $\vec{p}$. For the present purposes, a suitable basis can be built using the background vector b_μ, the lightlike four-momentum $\bar{p}_\mu \equiv (|\vec{p}|, \vec{p})$, and the two spacelike vectors in Eq. (40). Since b_μ is timelike and the $\epsilon^{(i)}_\mu(\vec{p})$'s are simultaneously orthogonal to p_μ and b_μ, this set of vectors forms a complete basis for each momentum $\vec{p}$. In terms of this basis, the set of vectors $a_\mu(\vec{p}) \equiv a_\mu(|\vec{p}|, \vec{p})$ can be expressed as

$$a_\mu(\vec{p}) = \sum_{i=1}^{2} a^{(i)}(\vec{p})\epsilon^{(i)}_\mu(\vec{p}) + a^{(3)}(\vec{p})\bar{p}_\mu + a^{(4)}(\vec{p})b_\mu. \tag{59}$$

Concerning the particular solution, there is no obstruction for the convolution of the Green function of the d'Alembertian operator with the expressions in Eqs. (53), (56), and (57). Referring to the Green function of the $\Box$ operator as $G^{(1)}$, one can write formally

$$\beta^P_\mu(x) = \partial_\mu(G^{(1)} * S^H) + \Box\partial_\mu(G^{(1)} * G^{(3)} * C) + 2\kappa b_\mu(G^{(1)} * D) - (\Box - 2\kappa b^2)\partial_\mu\frac{d}{d\tau}(G^{(1)} * D(x;\tau))|_{\tau=0}, \tag{60}$$

where the symbol "$*$" means convolution, and the functions $C(x)$ and $D(x)$ are defined in Eqs. (B3) and (B5), respectively. $G^{(3)}$ is the Green function for the operator $b\cdot\partial$, given in Eq. (B9), and the Green function $G^{(1)}$ can be chosen to be

$$G^{(1)} = -\frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{p^2 + i\epsilon}. \tag{61}$$

Since the equation of motion (29) for the β_μ field is second order in time, one should expect the presence of four pairs of arbitrary functions of the three-momenta to be fixed by the initial conditions. However, it can be seen from Eqs. (53), (56), (57), and (59) that there are six pairs at disposal instead: $(a^{(l)}(\vec{p}), a^{\dagger(l)}(\vec{p}))$ with $l = 1, \ldots, 4$; $(c(\vec{p}), c^\dagger(\vec{p}))$; and $(s(\vec{p}), d(\vec{p}))$. This apparent overcounting problem is solved when it is imposed that $b\cdot\beta$ and $\partial\cdot\beta$, calculated from the Eqs. (58) and (60), match the expressions in Eqs. (53), (56), and (57). These two conditions imply in the vanishing of the functions $a^{(3)}$ and $a^{(4)}$ in the expansion (59). Despite the length, for convenience, the final result for the expansion of the β_μ field is presented here:

$$\begin{aligned}\beta_\mu(x) = {} & \frac{1}{(2\pi)^3}\int\frac{d^3p}{2|\vec{p}|}\sum_{i=1}^{2}(a^{(i)}(\vec{p})\epsilon^{(i)}_\mu(\vec{p})e^{-ip^{(1)}\cdot x} + a^{\dagger(i)}(\vec{p})\epsilon^{(i)}_\mu(\vec{p})e^{ip^{(1)}\cdot x}) \\ & - \frac{i}{4\kappa|b_0|\sqrt{-2\kappa\xi}}\partial_\mu\int\frac{d^3p}{(2\pi)^3}(c(\vec{p})e^{-ip^{(0)}\cdot x} - c^\dagger(\vec{p})e^{ip^{(0)}\cdot x}) \\ & + \frac{2\kappa b^2}{(2\pi)^3 b_0^2}\int d^3p\left(\frac{\delta^0_\mu - ix^0p^{(3)}_\mu}{(p^{(3)})^2} - \frac{2p^{(3)}_0p^{(3)}_\mu}{(p^{(3)})^4}\right)d(\vec{p})e^{-ip^{(3)}(\vec{p})\cdot x} \\ & - \frac{2\kappa b_\mu}{(2\pi)^3|b_0|}\int d^3p\frac{e^{-ip^{(3)}\cdot x}}{(p^{(3)})^2}d(\vec{p}) - \frac{1}{(2\pi)^3|b_0|}\partial_\mu\int d^3p\frac{e^{-ip^{(3)}\cdot x}}{(p^{(3)})^2}s(\vec{p}) + \partial_\mu\int\frac{d^3p}{(2\pi)^3b_0^2}x^0d(\vec{p})e^{-ip^{(3)}\cdot x}.\end{aligned} \tag{62}$$

Concerning the general solution for the Stueckelberg field $\phi(x)$, one can notice from Eq. (32) that its solution can be expressed as

$$\phi(x) = \frac{1}{(2\pi)^3 2\sqrt{\kappa}} \int d^4 p \delta((p\cdot b)^2 + 2\xi\kappa)\bar{g}(p)e^{-ip\cdot x}. \tag{63}$$

Using the properties of the delta function, this expansion yields

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{4\sqrt{\kappa}|b_0|\sqrt{-2\xi\kappa}} \times (g(\vec{p})e^{-ip^{(0)}\cdot x} + g^\dagger(\vec{p})e^{ip^{(0)}\cdot x}), \tag{64}$$

since ϕ is a real field, and therefore $\bar{g}^\dagger(-p_0, -\vec{p}) = \bar{g}(p_0, \vec{p})$. The function $g(\vec{p})$ was also defined by $\bar{g}(\frac{\vec{b}\cdot\vec{p}}{b_0} + \frac{\sqrt{-2\xi\kappa}}{b_0}, \vec{p})$.

The operators $g(\vec{p})$ and $g^\dagger(\vec{p})$ will soon be identified as the annihilation and creation operators of the Stueckelberg field. The momentum-space operators for the transverse field and for the gauge modes in expansion (62) arrange themselves in the standard way, and it will be verified in the next section that they can be indeed identified as creation and annihilation operators for these modes. Nevertheless, the expansion for the longitudinal sector is not so enlightening, and the role of the operators $s(\vec{p})$ and $d(\vec{p})$ in the structure of the Fock space is unclear. This issue will now be addressed.

V. STABILITY

In this section, the conditions for the suppression of the unphysical modes are discussed and implemented. In Sec. V A, the commutation relations for the Fourier modes that appear in field expansions (62) and (64) are obtained, and their main properties are analyzed. The Gupta-Bleuler condition for the absence of the gauge modes is also discussed in this subsection. In Sec. V B, the two Fourier-mode operators for the longitudinal sector, s and d, are mapped into new ones with a simpler action on the Fock space, and a condition for the absence of tachyons is obtained from them.

A. Fourier-mode algebra

The Fourier expansions for the free fields ϕ and β_μ in Eqs. (62) and (64) are very convenient to make perturbative calculations. However, one still needs to address the question of the fate of the gauge and Stueckelberg modes, which are unphysical, and the more subtle question of the presence of tachyonic excitation in the spectrum of the model. For this intent, the Fock space of the present model will be constructed, and the implementation of the conditions to handle properly the existence of the unphysical excitations will be discussed in this section.

To begin with, the identification of the right operators that create and annihilate all the propagating modes in this theory is useful. So, one proceeds with the inversion of the expansions (62) and (64). It can be directly shown that the modes g and $g^\dagger$ can be expressed in terms of ϕ and $\dot{\phi}$ as

$$g(\vec{p}) = \int d^3x e^{ip^{(0)}\cdot x}(i\overleftrightarrow{\partial}_0^S 2\sqrt{\kappa}b_0^2 + 4\sqrt{\kappa}b_0\sqrt{-2\kappa\xi})\phi(x), \tag{65}$$

$$g^\dagger(\vec{p}) = \int d^3x e^{-ip^{(0)}\cdot x}(-i\overleftrightarrow{\partial}_0^S 2\sqrt{\kappa}b_0^2 + 4\sqrt{\kappa}b_0\sqrt{-2\kappa\xi})\phi(x), \tag{66}$$

with x^0 arbitrary, and for two arbitrary functions, f_1 and f_2, $\overleftrightarrow{\partial}_0^S$ is defined by

$$f_1(x)\overleftrightarrow{\partial}_0^S f_2(x) \equiv f_1(x)\partial_0 f_2(x) + \partial_0 f_1(x) f_2(x), \tag{67}$$

where "S" stands for symmetric to differ from the antisymmetric, $\overleftrightarrow{\partial}_0^A$, defined by

$$f_1(x)\overleftrightarrow{\partial}_0^A f_2(x) \equiv f_1(x)\partial_0 f_2(x) - \partial_0 f_1(x) f_2(x). \tag{68}$$

If the only vectors that appeared in the expansion (62) for the β_μ field were the pure-mode solutions (39)–(41), one could obtain the inverse of that expansion using the orthogonality relations (45) and (49). This would give

$$a^{(i)}(\vec{p}) = -\int d^3x e^{ip^{(1)}\cdot x}(i\overleftrightarrow{\partial}_0^A\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i}p_i)\epsilon_\mu^{(i)}(\vec{p})\beta_\nu(x), \tag{69}$$

$$a^{\dagger(i)}(\vec{p}) = -\int d^3x e^{-ip^{(1)}\cdot x}(-i\overleftrightarrow{\partial}_0^A\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i}p_i)\epsilon_\mu^{(i)}(\vec{p})\beta_\nu(x), \tag{70}$$

$$c(\vec{p}) = \int d^3x e^{ip^{(0)}\cdot x}(i\overleftrightarrow{\partial}_0^A\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i}p_i)p_\mu^{(0)}(\vec{p})\beta_\nu(x), \tag{71}$$

$$c^\dagger(\vec{p}) = \int d^3x e^{-ip^{(0)}\cdot x}(-i\overleftrightarrow{\partial}_0^A\tilde{\eta}^{\mu\nu} - \lambda^{\mu\nu 0i}p_i)p_\mu^{(0)}(\vec{p})\beta_\nu(x). \tag{72}$$

However, besides the pure-mode solutions, there are other four-vectors composing the full expansion (62). Thence, extra algebraic relations between the vector quantities that appear in this expansion would be needed. It turns out that these extra contributions, coming from the substitution of the entire field β_μ in the expressions above, cancel out, and Eqs. (69)–(72) are in fact the right relations for these modes.

The last two operators, $s(\vec{p})$ and $d(\vec{p})$, can be obtained by considering $\partial\cdot\beta$ and $b\cdot\beta$, respectively, directly in the expansion (62) for β_μ and performing suitable Fourier transformations on the result. The outcome of this procedure is given by

$$d(\vec{p}) = |b_0| \int d^3x e^{ip^{(3)}\cdot x} b\cdot\beta(x) + \frac{1}{4\kappa}\left(c(\vec{p})e^{-i\frac{\sqrt{-2\kappa\xi}}{b_0}x^0} + c^\dagger(-\vec{p})e^{i\frac{\sqrt{-2\kappa\xi}}{b_0}x^0}\right), \quad (73)$$

$$s(\vec{p}) = -i\frac{1}{4\kappa\sqrt{-2\kappa\xi}}\left((p^{(0)}(\vec{p}))^2 c(\vec{p})e^{-i\frac{\sqrt{-2\kappa\xi}}{b_0}x^0} - (p^{(0)}(-\vec{p}))^2 c^\dagger(-\vec{p})e^{i\frac{\sqrt{-2\kappa\xi}}{b_0}x^0}\right) + \frac{1}{|b_0|}d(\vec{q})(2ip_0^{(3)}(\vec{p}) + x^0((p^{(3)}(\vec{p}))^2 + 2\kappa b^2)) + |b_0|\int d^3x e^{ip^{(3)}\cdot x}\partial\cdot\beta(x). \quad (74)$$

If the expressions (71) and (72) for $c(\vec{p})$ and $c^\dagger(\vec{p})$, respectively, are used in the equations for $d(\vec{p})$ and $s(\vec{p})$ above, this completes the task of writing the Fourier-mode operators in terms of the fields and their time derivatives.

From the expressions (65), (66), (71), and (72) for g, $g^\dagger$, c, and $c^\dagger$, respectively, and using the ETCR in Eqs. (21)–(26) along with the orthogonality relations in Eqs. (45) and (49), one can get the algebra of the Fourier modes

$$[g(\vec{p}), g^\dagger(\vec{q})] = -(2\pi)^3 4|b_0|\kappa\sqrt{-2\xi\kappa}\delta(\vec{p}-\vec{q}), \quad (75)$$

$$[c(\vec{p}), c^\dagger(\vec{q})] = (2\pi)^3 4|b_0|\kappa\sqrt{-2\kappa\xi}\delta(\vec{p}-\vec{q}), \quad (76)$$

$$[a^{(i)}(\vec{p}), a^{\dagger(j)}(\vec{q})] = -(2\pi)^3\delta^{ij}2|\vec{p}|\delta(\vec{p}-\vec{q}), \quad (77)$$

$$[s(\vec{p}), s(\vec{q})] = -(2\pi)^3\delta(\vec{p}+\vec{q})2b^2 p_0^{(3)}(\vec{p}), \quad (78)$$

$$[s(\vec{p}), d(\vec{q})] = \frac{i}{2\kappa}(2\pi)^3|b_0|(q^{(3)})^2\delta(\vec{p}+\vec{q}). \quad (79)$$

All the other possible commutators vanish.

The algebraic properties of the operators $(g(\vec{p}), g^\dagger(\vec{q}))$, $(c(\vec{p}), c^\dagger(\vec{q}))$, and $(a^{(i)}(\vec{p}), a^{\dagger(i)}(\vec{q}))$ are identical to the standard algebra of annihilation and creation operators. In this way, the vacuum of the theory can be defined as the state annihilated by the operators $g(\vec{p})$, $c(\vec{p})$, and $a(\vec{p})$. However, due to the plus sign in the commutator between $c(\vec{p})$ and $c^\dagger(\vec{q})$, the Fock space generated from the vacuum state by the successive operation of the creation operators $g^\dagger(\vec{q})$, $c^\dagger(\vec{q})$, and $a^\dagger(\vec{q})$ can present negative-norm states and cannot correspond to the physical Hilbert space. Despite the right sign in the commutator between $g(\vec{p})$ and $g^\dagger(\vec{q})$, it depends on the gauge parameter as well as $[c(\vec{p}), c^\dagger(\vec{q})]$. These modes were both introduced through the gauge-fixing term $(b^\nu b^\mu\partial_\nu\beta_\mu - 2\xi\sqrt{\kappa}\phi)$, and for their elimination, one imposes that the expectation value of this term between physical states must vanish. Moreover, to preserve the linear structure of the Hilbert space, one follows the Gupta-Bleuler procedure by imposing that the physical states must belong to the kernel of the annihilation part of the gauge-fixing term. That is,

$$(b\cdot\partial b\cdot\beta - 2\xi\sqrt{\kappa}\phi)^+|\text{Phys}\rangle = 0. \quad (80)$$

The expansion (53) for the $b\cdot\beta$ field contains, besides the modes $c(\vec{p})$ and $c^\dagger(\vec{p})$, the Fourier mode $d(\vec{p})$, and the condition (80) would also contain gauge-independent modes. It turns out that the operator $b\cdot\partial$ acting on $b\cdot\beta$ kills exactly the contribution for the $d(\vec{p})$ mode, and the condition (80) only contains the annihilation operators $c(\vec{p})$ and $g(\vec{p})$ indeed. Using the expansions (53) and (64) for $b\cdot\beta$ and ϕ, respectively, the condition (80) is equivalent to

$$(ic(\vec{p}) + g(\vec{p}))|\text{Phys}\rangle = 0. \quad (81)$$

The algebraic properties of the $s(\vec{p})$ and $d(\vec{p})$ operators in Eqs. (78) and (79) require further analysis. The appearance of the unusual i factor on the right-hand side of Eq. (79) is consistent with the constraint conditions obeyed by these operators: $s^\dagger(-\vec{p}) = s(\vec{p})$ and $d^\dagger(-\vec{p}) = d(\vec{p})$. One can verify this fact by taking the adjoint of Eq. (79) and making the replacements $\vec{p}\longrightarrow -\vec{p}$ and $\vec{q}\longrightarrow -\vec{q}$. Furthermore, these operators cannot be interpreted directly as creation and annihilation operators without contradicting their algebraic relations. As a single example, suppose that $s(\vec{p})$ is an annihilation operator. Therefore, it annihilates the vacuum, and $\langle 0|[s(\vec{p}), s(\vec{q})]|0\rangle$ should also give a null result, but this contradicts the nonvanishing right-hand side of Eq. (78). Many other examples can be given whenever $s(\vec{p})$ or $d(\vec{p})$ are supposed to be creation or annihilation operators. The correct interpretation of the role of these operators in the structure of the Fock space plays a fundamental role in the present analysis, since this sector houses the massive tachyonic mode, and the stability of the model demands its suppression. To accomplish this purpose, it is convenient to construct other operators from s and d such that they have a simpler action in the Fock space.

B. Condition for the absence of tachyons

Due to the constraints obeyed by $s(\vec{p})$ and $d(\vec{p})$, they carry enough information to be mapped in a one-to-one way into two adjointed-related complex operators. Let the latter be denoted by $\tau(\vec{p})$ and $\tau^\dagger(\vec{p})$. Consider the complex linear transformation

$$\begin{pmatrix} \tau(\vec{p}) \\ \tau^\dagger(-\vec{p}) \end{pmatrix} = \begin{pmatrix} \rho(\vec{p}) & \sigma(\vec{p}) \\ \rho^*(-\vec{p}) & \sigma^*(-\vec{p}) \end{pmatrix} \begin{pmatrix} s(\vec{p}) \\ d(\vec{p}) \end{pmatrix}, \tag{82}$$

where ρ and σ are two arbitrary complex functions of $\vec{p}$, such that τ and $\tau^\dagger$ satisfy the following commutation relations:

$$[\tau(\vec{p}), \tau(\vec{q})] = 0, \tag{83}$$

$$[\tau(\vec{p}), \tau^\dagger(\vec{q})] \neq 0. \tag{84}$$

These conditions, together with the requirement that the transformation (82) be one to one, yield

$$\left(\rho(\vec{p})\rho(-\vec{p})(-2b^2 p_0^{(3)}(\vec{p})) + (\rho(\vec{p})\sigma(-\vec{p}) - \rho(-\vec{p})\sigma(\vec{p}))\frac{i}{2\kappa} b_0 (p^{(3)})^2\right) = 0, \tag{85}$$

$$\left(|\rho(\vec{p})|^2(-2b^2 p_0^{(3)}(\vec{p})) + (\rho(\vec{p})\sigma^*(\vec{p}) - \rho^*(\vec{p})\sigma(\vec{p}))\frac{i}{2\kappa} b_0 (p^{(3)})^2\right) \neq 0, \tag{86}$$

$$\rho(\vec{p})\sigma^*(-\vec{p}) - \rho^*(-\vec{p})\sigma(\vec{p}) \neq 0. \tag{87}$$

It is straightforward to show that there are numerous possibilities for the choices of ρ and σ that satisfy these conditions. It seems that no specific choice is preferable to any other. For the present purposes, it is just assumed that some choice of ρ and σ is made such that it satisfies the requirements (85)–(87). Now, since the algebra in Eqs. (83) and (84) is the standard one for creation and annihilation operators, $\tau^\dagger$ and τ can be identified as creation and annihilation operators for the tachyonic mode, respectively. Thus, as a last condition in the definition of a physical state, one imposes

$$\tau(\vec{p})|\text{Phys}\rangle = 0. \tag{88}$$

This condition, together with the condition (81) for the suppression of the gauge modes, is sufficient to show that the only contribution for the expansion of the β_μ and ϕ fields between physical states comes from the transverse modes. These, in turn, are creation and annihilation operators for a massless spin-1 particle, which can be seen from the expression for the transverse projector (42) and the algebraic relation (77). As already mentioned before, the projection operator (42) is the effective physical propagator at the tree level for the β_μ field, and it coincides with the photon propagator of the Maxwell theory in the temporal gauge. In this way, the KS model provides an alternative to the gauge-invariant description of the photon by considering it an as NG mode arising from the spontaneous breaking of Lorentz symmetry.

Another way to get the condition (88) is by imposing the positiveness of the expectation value between physical states of the Hamiltonian associated with the Lagrangian (16). Using the expressions for the conjugate momenta (17) and (18), the Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2}\Pi_i^\beta \Pi_\beta^i - \frac{\xi}{2b_0^4}\Pi_0^2 - \frac{b_i b^i}{2b_0^2}(\Pi_0^\beta)^2 + \frac{b^i}{b_0}\Pi_0^\beta \Pi_i^\beta + \Pi_\beta^i \partial_i \beta_0 - \frac{2b^i}{b_0}\Pi_0^\beta \partial_i \beta_0 - \frac{b^i b^j}{b_0^2}\Pi_0^\beta \partial_i \beta_j - \frac{1}{4b_0^2}\Pi_\phi^2 + \frac{1}{4}\beta_{ij}\beta^{ij} + \kappa b_\mu b_\nu \beta^\mu \beta^\nu - \frac{b^i}{b_0}\Pi_\phi \partial_i \phi + 2\xi\kappa\phi^2. \tag{89}$$

As discussed before, there are five propagating degrees of freedom, gauge dependence, and this Hamiltonian is unbounded from below. For this reason, one needs to implement conditions on the states of the Fock space so that, when restricted to these states, all the mentioned problems can be avoided. The extra degrees of freedom brought by the propagation of the gauge modes are kept outside of the physical region by the imposition of the Gupta-Bleuler condition (80), which can be easily restated using the expression for the conjugate momenta (17) as

$$\langle \Pi_\beta^0 + 2b_0^2\sqrt{\kappa}\phi \rangle_{\text{Phys}} = 0, \tag{90}$$

where the subscript "Phys" means that the expectation value is being calculated between physical states.

Considering the $\nu = 0$ component of the Hamiltonian version of the equation of motion (29) for β_μ and using the condition above, one obtains

$$\langle \Pi_\phi \rangle_{\text{Phys}} = \left\langle \frac{1}{\sqrt{\kappa}}\partial_i \Pi_\beta^i + 2b^i b_0 \partial_i \phi - 2\sqrt{\kappa} b_\mu b_0 \beta^\mu \right\rangle_{\text{Phys}}. \tag{91}$$

These two constraints on the physical states suppress the dynamics of two out of the five degrees of freedom, and $\langle \mathcal{H} \rangle_{\text{Phys}}$ can be conveniently written as

$$\langle \mathcal{H} \rangle_{\text{Phys}} = \left\langle -\frac{1}{2}(\Pi_\beta^i + 2\sqrt{\kappa} b_0 b^i \phi)(\Pi_i^\beta + 2\sqrt{\kappa} b_0 b^i \phi) + \frac{1}{4}\beta_{ij}\beta^{ij} \right\rangle_{\text{Phys}} - \left\langle \left(\beta_0 + \frac{1}{2\kappa b_0^2}\partial_i \Pi_\beta^i + \frac{2}{\sqrt{\kappa} b_0} b^i \partial_i \phi - \frac{1}{b_0} b_\mu \beta^\mu\right) \times \left(\frac{1}{2}\partial_i \Pi_\beta^i + \sqrt{\kappa} b^0 b^i \partial_i \phi\right) \right\rangle_{\text{Phys}}. \tag{92}$$

First, one notices the gauge independence of this expression as should be expected. To ensure the positiveness of this Hamiltonian is enough to require that

$$\left\langle \frac{1}{2}\partial_i \Pi^i_\beta + \sqrt{\kappa} b^0 b^i \partial_i \phi \right\rangle_{\text{Phys}} = 0. \tag{93}$$

Interestingly, this expression could be obtained by introducing the Stueckelberg field, through substitution (11), into the classical stability condition $b \cdot \beta = 0$ and considering it as a quantum expectation value between physical states. In this sense, the classical stability condition obtained in Ref. [14] is recovered.

From the expression (17) for the Π^μ_β field and using the constraint (90), equation (93) can be written as $\langle \partial_0 \partial_i \beta^i \rangle_{\text{Phys}} = \langle \partial_i \partial^i \beta^0 \rangle_{\text{Phys}}$. Using the field expansion (62) and after some algebraic manipulation, this amounts to the condition $\langle d(\vec{q}) \rangle_{\text{Phys}} = 0$, which is ensured by the condition (88). To show the dynamic consistency of this condition, consider the $\nu = i$ component of the Hamiltonian version of the equation of motion (29) for β_μ:

$$-\partial_0 \Pi^i_\beta + \partial_j \beta^{ji} - \frac{b^i b^j}{b_0^2} \partial_j \Pi^0_\beta - 2\kappa b_\mu b^i \beta^\mu = 0. \tag{94}$$

Multiplying this equation by ∂_i, taking the expectation value between physical states, and using the Gupta-Bleuler condition (90), one obtains

$$\left\langle \partial_i \dot{\Pi}^i_\beta + \frac{b^i b^j}{b_0^2} \partial_i \partial_j (-2 b_0^2 \sqrt{\kappa} \phi) + 2\kappa b_\mu b^i \partial_i \beta^\mu \right\rangle_{\text{Phys}} = 0. \tag{95}$$

The consistency of the condition (93) with the field dynamics is verified if the time derivative of the combination on the left-hand side of Eq. (93), $\langle \frac{1}{2}\partial_i \dot{\Pi}^i_\beta + \sqrt{\kappa} b^0 b^i \partial_i \dot{\phi} \rangle_{\text{Phys}}$, vanishes. From Eq. (18), $\dot{\phi} = -\frac{1}{2b_0^2}\Pi_\phi - \frac{b_i}{b_0}\partial^i \phi$. Using this relation and rewriting Π_ϕ and $\partial_i \dot{\Pi}^i_\beta$ in terms of the expressions obtained from Eqs. (91) and (95), one finally gets

$$\left\langle \frac{1}{2b_0\sqrt{\kappa}} \partial_i \dot{\Pi}^i_\beta + b^i \partial_i \dot{\phi} \right\rangle_{\text{Phys}} = \left\langle -b^j \partial_j \left(\frac{1}{2b_0^2\sqrt{\kappa}} \partial_i \Pi^i_\beta + \frac{b^i}{b_0^2} \partial_i \phi \right) \right\rangle_{\text{Phys}}, \tag{96}$$

which vanishes due to condition (93). Therefore, condition (88) is a stable one, and it ensures the absence of tachyons in the physical spectrum of the free theory.

With the analysis of this section, one concludes that the components to develop a systematic quantum analysis of bumblebee electrodynamics described by Lagrangian (1) can be consistently constructed. Moreover, in spite of the fact that the present approach to the canonical quantization needs the introduction of unphysical ghost modes, a physical Fock space free from pathologies can be defined.

VI. SUMMARY

In this work, the problems of the quantization and stability of a particular vector theory with a potential term that triggers the spontaneous symmetry breaking of Lorentz symmetry were addressed. In Sec. II, the main classical properties of the model described by the Lagrangian (1) were reviewed. Performing a Hamiltonian analysis, it was verified that the model exhibits two second-class constraints and only three out of the four degrees of freedom available in the vector field can be dynamical. Two of them correspond to the massless NG modes and form a massless spin-1 particle that, with the choice of the kinetic term, can be potentially identified as the photon. The other propagating mode corresponds to a field excitation that does not remain on the bottom of the potential, and for this reason is characterized as a massive particle. In fact, the mass of this particle was shown to be negative, leading to an instability in the model. However, at the end of that section it was shown that the instability can be avoided if suitable initial conditions are chosen, and the reduced phase space of this model is equivalent to that of the Maxwell electrodynamics in a nonlinear gauge.

The construction of a framework for discussing the quantum picture of the previous scenario was pursued in the subsequent sections. To avoid some of the complications of the Dirac method of quantization of constrained systems, the known Stueckelberg trick was used. This consisted of promoting an enlargement of the field content of the model with the simultaneous introduction of a local symmetry. By doing this, the second-class constraints were converted to first-class ones, and the widely known and successful methods for the quantization of gauge theories could be applied. In this vein, a convenient gauge-fixing term was added to the KS-Stueckelberg Lagrangian, and physical equivalence with the KS model was claimed based on the fact that the two Lagrangians differed by a BRST-invariant term.

To discuss the perturbation theory analysis, the free Lagrangian (16) was considered, and the Fourier decomposition of the free-field solutions was performed in Sec. IV. The dispersion relations of the propagating modes were obtained, and the appearance of two new unphysical degrees of freedom brought by the gauge-fixing term was observed. The other three degrees of freedom were the expected massless spin-1 and the tachyonic mode. In the sequence, the concept of pure-mode solutions was introduced in Eq. (38), which helped us to understand the structure of the Fourier decomposition. These are particular solutions for the equations of motion, where the 0 component of the four-momentum that appears in the matrix operator between brackets in Eq. (38) is one of the solutions, $p_0^{(\lambda)}$, for the dispersion relations (35)–(37). In some cases, like in the quantization of the Maxwell electrodynamics modified by the introduction of the gauge-fixing and finite-mass terms and in the photon sector of the SME, these pure-mode solutions form a basis of vectors for each value of the three-momentum $\vec{p}$, and they provide a natural set of polarization vectors appearing in the Fourier decomposition. However, in the present discussion, the pure-mode solution associated with the $\lambda = 0$ dispersion relation changes its spacelike, timelike or lightlike behavior for different choices of the three-momentum. It turns out that the four pure-mode solutions do not provide a basis of vectors for every choice of the three-momentum. This makes the Fourier decomposition of the vector field much more involved, as can be seen from the final result (62).

With the Fourier decomposition of the Stueckelberg and vector fields, the construction of the physical Fock space of the model was discussed in Sec. V. The Fourier expansion of the fields was inverted and by using the ETCR (21)–(26), the algebra of the Fourier-mode operators was obtained. In the transverse sector, these operators could be directly interpreted as creation and annihilation operators of massless spin-1 particles. Due to the sign in the right-hand side of Eq. (76), one could also verify that the full Fock space is plagued by negative-norm states. These ghost states are commonly introduced in the quantization of gauge theories, and they are excluded from the physical Fock space by demanding that the extra gauge modes, brought by the gauge-fixing terms, be kept out of the physical Fock space. This was attained, in the present case, by imposing that a physical state should satisfy the condition (80). The information about the tachyon mode is contained in the s and d operators, but a condition for the absence of tachyons in the physical Fock space cannot be obtained directly from them, since they cannot be interpreted as creation or annihilation operators without contradicting their algebraic relations. The proposed solution for this problem was to redefine them in terms of the operators τ and $\tau^\dagger$ in such a way that these two new operators carried the same information as the previous ones and satisfied a usual creation and annihilation operator algebra. So, it was finally proposed that the condition (88) is the one that guarantees the absence of tachyons in the physical spectrum and, therefore, the stability of the free model. The same condition was regained by demanding the positiveness of the physical free Hamiltonian. Then, the stability condition was restated in terms of the fields which, in turn, was verified to be the quantum analogous to the classical stability condition discussed in Sec. II if the redefinition of the fields by the Stueckelberg method is taken into account.

The main attainment of this work was to construct the building blocks to develop a systematic quantum analysis of the KS model. This was achieved by showing that one can define a region of the full Hilbert space where the free model is stable and is, indeed, equivalent to the Maxwell electrodynamics in the temporal gauge. The present framework paves the road for further discussions of the quantum properties of the KS model that are presumably of great importance but lie beyond the scope of the present work, like the stability of the physical Fock space under radiative corrections, the coupling to the matter sector, and microcausality-related issues.

ACKNOWLEDGMENTS

The author gratefully acknowledges V. Alan Kostelecký for the useful discussions and Robert Bluhm and Robertus Potting for carefully reading the manuscript of the present work and for sending useful comments. This work has been supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior-Brazil).

APPENDIX A: BRST INVARIANCE

The BRST invariance of the full Lagrangians $\mathcal{L}_{\mathrm{KSS}}$ and $\mathcal{L}_{\mathrm{KSS}} + \mathcal{L}_{gf}$ will be shown in this appendix.

We introduce the scalar anticommutating fields $\omega(x)$ and $\omega^*(x)$. These are called ghost fields, since they satisfy a wrong spin statistics relation. From the invariance of the free Lagrangian $\mathcal{L}_{\mathrm{KSS}}$ under the gauge transformations (13) and (14), it can be seen that it is also invariant under the particular infinitesimal gauge transformations

$$\mathbf{s}\beta_\mu(x) = \partial_\mu \omega(x), \tag{A1}$$

$$\mathbf{s}\phi(x) = \sqrt{\kappa}\omega(x), \tag{A2}$$

$$\mathbf{s}\omega(x) = 0, \tag{A3}$$

where $\mathbf{s}$ is the operator that performs the infinitesimal BRST transformations. Since $\omega(x)$ and $\omega^*(x)$ are anticommutating fields, $\mathbf{s}$ is nilpotent, $\mathbf{s}^2 = 0$.

Instead of the gauge-fixing Lagrangian (15), consider the more general one,

$$\mathcal{L}_{gf} = \mathbf{s}\left[\omega^*\left(\mathcal{G}(\beta_\mu, \phi) + \frac{\xi}{2}h\right)\right], \tag{A4}$$

where $\mathcal{G}$ is some general functional of the fields β_μ and ϕ, and h is an auxiliary field called the Nakanishi-Lautrup field. Whatever the particular form of the functional $\mathcal{G}$, this Lagrangian is BRST invariant if the following transformations for the ω^* and h fields are assumed:

$$\mathbf{s}\omega^*(x) = h, \tag{A5}$$

$$\mathbf{s}h = 0. \tag{A6}$$

Notice that $\mathbf{s}$ obeys the Leibniz product rule, and it anticommutes with the ghost fields. Deriving the algebraic equation of motion for the auxiliary h field and using it to eliminate this field from the Lagrangian (A4), one obtains

$$\mathcal{L}_{gf} = -\omega^*(\mathbf{s}\mathcal{G}) - \frac{1}{2\xi}\mathcal{G}^2. \tag{A7}$$

Choosing $\mathcal{G} = (b^\nu b^\mu \partial_\nu \beta_\mu - 2\xi\sqrt{\kappa}\phi)^2$, one can write

$$\mathcal{L}_{gf} = -\omega^*(b^\nu b^\mu \partial_\nu \partial_\mu - 2\xi\sqrt{\kappa})\omega - \frac{1}{2\xi}(b^\nu b^\mu \partial_\nu \beta_\mu - 2\xi\sqrt{\kappa}\phi)^2. \tag{A8}$$

Therefore, with this choice for the functional $\mathcal{G}$, the ghost fields decouple and can be discarded from the Lagrangian without affecting physical results. In this way, the BRST invariance of the total Lagrangian $\mathcal{L}_{\rm KSS} + \mathcal{L}_{gf}$ is established.

APPENDIX B: PARTICULAR SOLUTION OF EQUATION (50)

In this appendix, the particular solution for the inhomogeneous differential equation (50) is derived. The problem is essentially that the naive convolution of the Green function with the inhomogeneous piece coming from (53) will lead to the product $\frac{1}{p\cdot b}\delta(p\cdot b)$, which is ill defined. However, the issue only abides in the convolution of the Green function with the d term in Eq. (53) that is where $\delta(p\cdot b)$ plays a role. There is no prevention in forming the convolution of the Green function with the c and $c^\dagger$ terms.

The particular solution can be derived by splitting it into two pieces:

$$S^P = S_1^P + S_2^P, \tag{B1}$$

where S_1^P is the particular solution for the equation

$$(b\cdot\partial)S_1^P(x) = \Box C(x) \tag{B2}$$

and $C(x)$ is the function defined by the first integral on the right-hand side of Eq. (57). That is,

$$C(x) = -\int \frac{d^3p}{4\kappa|b_0|(2\pi)^3}(c(\vec{p})e^{-ip^{(0)}\cdot x} + c^\dagger(\vec{p})e^{ip^{(0)}\cdot x}). \tag{B3}$$

The other piece of the particular solution in Eq. (B1) is a particular solution for the equation

$$(b\cdot\partial)S_2^P(x) = (\Box - 2\kappa b^2)D(x), \tag{B4}$$

where $D(x)$ is the function defined by the second integral on the right-hand side of Eq. (57). That is,

$$D(x) = \int \frac{d^3p}{(2\pi)^3|b_0|}d(\vec{p})e^{-ip^{(3)}\cdot x}. \tag{B5}$$

It can be easily verified that the two functions defined as particular solutions in Eqs. (B2) and (B4), when added, form a particular solution to Eq. (50), since

$$((b\cdot\partial)^2 - 2\kappa\xi)C(x) = 0 \tag{B6}$$

and

$$(b\cdot\partial)D(x) = 0. \tag{B7}$$

To obtain S_1^P, one considers the Green function for the operator $b\cdot\partial$, which must satisfy

$$(b\cdot\partial)G(x) = \delta(x). \tag{B8}$$

Since it is only needed for a particular solution, the following Green function that satisfies this equation is chosen:

$$G^{(3)}(x) = \frac{i}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot x}}{b\cdot p + i\epsilon}. \tag{B9}$$

Now, the particular solution for Eq. (B2) can be written as

$$S_1^P(x) = \Box(G^{(3)} * C)(x), \tag{B10}$$

where the symbol "$*$" means convolution, which for two arbitrary functions $f_1(x)$ and $f_2(x)$ is defined by

$$f_1 * f_2(x) = \int d^4y f_1(x-y)f_2(y). \tag{B11}$$

The property of the convolution operation

$$\mathcal{O}(f_1 * f_2) = (\mathcal{O}f_1 * f_2) = (f_1 * \mathcal{O}f_2), \tag{B12}$$

where $\mathcal{O}$ is a derivative operator, was also used in Eq. (B10).

As was observed before, one cannot apply the same technique for the obtainment of S_2^P, since the convolution $G^{(3)} * D$ is ill defined. To proceed with this calculation, one considers the continuous deformation, $D(x;\tau)$, of the function $D(x)$ that analogously to Eq. (B7) satisfies the differential equation

$$(b\cdot\partial + \tau)D(x;\tau) = 0, \tag{B13}$$

with the subsidiary condition that $D(x;\tau) \longrightarrow D(x)$ when $\tau \longrightarrow 0$. This gives

$$D(x;\tau) = \int \frac{d^3p}{(2\pi)^3|b_0|}d(\vec{p})e^{-i(\frac{\vec{b}\cdot\vec{p}}{b_0} - i\frac{\tau}{b_0})x^0 + i\vec{p}\cdot\vec{x}}. \tag{B14}$$

Similarly, $S(x;\tau)$ is defined in such a way that it satisfies the following differential equations:

$$(b\cdot\partial)S_2^P(x;\tau) = (\Box - 2\kappa b^2)D(x;\tau), \quad (B15)$$

$$(b\cdot\partial + \tau)S_2^P(x;\tau) = (\Box - 2\kappa b^2)D(x). \quad (B16)$$

Subtracting the first of these equations from the second, dividing both sides by τ, and taking the limit $\tau \longrightarrow 0$, one gets

$$S_2^P(x) = (\Box - 2\kappa b^2)\left.\frac{dD(x;\tau)}{d\tau}\right|_{\tau=0}. \quad (B17)$$

Gathering S_1^P and S_2^P along with the homogenous solution (56), one has the desired general solution for Eq. (B2). Using Eqs. (B9) and (B14), Eq. (57) is obtained.

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