Exploring special relative locality with de Sitter momentum-space

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Relative locality is a recent approach to the quantum-gravity problem which allows the taming of nonlocality effects which may arise in some models which try to describe Planck-scale physics. I here explore the effect of relative locality on basic special-relativistic phenomena. In particular I study the deformations due to relative locality of special-relativistic transformation laws for momenta at all orders in the rapidity parameter ξ. I underline how those transformations also define the relative locality characteristic (momentum-dependent) invariant metric. I focus my analysis on the well studied de Sitter momentum-space framework, and I investigate the differences and similarities between this model and special relativity, from the definition of the boost parameter γ to a first discussion of transverse effects characteristic of relative locality on clocks observables.

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I. INTRODUCTION

Relative locality (RL) is a quite young approach to the quantum-gravity problem which formalizes nonlocalities and other characteristic features of deformed symmetries models, introducing some sort of momentum-space curvature [1–4] that influences the localization process, at a characteristic scale that we assume to be of the order of the Planck-scale $\ell \sim 1/M_P$.[1] A strong motivation to explore this feature has emerged both from the theoretical and the phenomenological sides, since the Planck-scale curvature of momentum space introduces corrections to the travel times of particles, opening also an opportunity for experimental tests [5].

So far many aspects of this theory have been examined: from the implications for interaction vertices conservation laws [2,6,7] to some attempts to generalize relative locality to curve space-time scenarios [8,9]. However, in the literature we are still lacking a clear explanation of the properties of the theory transformation laws as deformation of Lorentz ones, though the argument has been analyzed so from the algebraical point of view [10–12], as from the phenomenological one [13,14]. In this paper we will discuss analogies and divergences between special relativity and its relative locality version, using the well-studied de Sitter momentum-space formalism [3,4] at first order in the deformation parameter ℓ.

Relative locality was at first meant [15] as a realization of deformed special relativity (DSR; see for instance Ref. [16] and references therein) suggesting a way to introduce the coordinate space. The curved momentum-space geometric interpretation was later introduced [1,2] without explicitly including a relativity principle. It may be possible, then, to also formalize Lorentz symmetries breakdown models, such as models with a preferential spacelike direction [17], using a curved momentum-space framework. For our purposes, however, we need a ten-generators symmetry algebra, and therefore the choice to study a de Sitter-like curved momentum space is pretty straightforward.

To give a satisfying characterization to the relative locality features that we will encounter, we will need to work with boost transformations at all orders in the rapidity parameter ξ, in $2+1$ dimensions (the $3+1$-dimensional generalization is straightforward). We will then give a brief description of the RL-boost transverse effects [18,19] on momenta, showing also how those ℓ-deformed transformations naturally implement a Rainbow metric formalism [20] for the invariant line element.

A key element of our analysis is the nontrivial coordinate system, defined in analogy with de Sitter space-time conserved charges $\Pi_0 = p_0 - Hx^k p_k$, $\Pi_i = p_i$ [4,21]. These coordinates satisfy the following nontrivial Poisson brackets:

$$\{\chi^i, \chi^0\} = \ell \chi^i, \tag{1}$$

where in our case the index i can assume the values $i = L, T$ (longitudinal and transverse direction). The reason why this coordinatization is more suitable for this kind of discussion, as we will explain later in detail, is that the worldline expression in χ^α coordinates is momentum independent [4,22], and therefore we do not encounter any theoretical problem in fixing a reflexive, symmetric and transitive definition for a time interval. Relative locality effects on clocks observables in $2+1$ dimensions will then be discussed at the end of the paper.

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[1] In this paper lengths will have the dimensions of an inverse mass, since from now on we will adopt the natural units system $c = \hbar = 1$.

A. About de Sitter momentum space

Our mathematical formalism is based on deSitter momentum space in $2+1$ dimensions for which the metric is

$$\tilde{\eta}^{\alpha\beta}(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(1+2\ell p_0) & 0 \\ 0 & 0 & -(1+2\ell p_0) \end{pmatrix}. \quad (2)$$

Using this metric we can define the invariant line element in momentum space as the geodesic distance from the momentum-space origin

$$dk^2(0,p) = \int_\lambda \tilde{\eta}^{\alpha\beta}(p)\dot{\lambda}_\alpha\dot{\lambda}_\beta ds = \mathcal{C}(p), \quad (3)$$

in which s is the variable with which we parametrize our geodesic $\lambda(s)$ connecting the point at p in which a particle lies at the point $p=0$; and in which

$$\mathcal{C}(p) = p_0^2 - p^2 - \ell p_0 p^2, \quad (4)$$

where, by definition, in $2+1$ dimensions $p^2 = p_L^2 + p_T^2$. With the $\mathcal{C}(p)$ being invariant, we can identify it as the Casimir operator of the de Sitter momentum-space transformation generators algebra,

$$\{p_0, p_i\} = 0, \quad \{p_i, p_j\} = 0, \quad \{\mathcal{N}_{(i)}, \mathcal{R}\} = \epsilon_{ij}\mathcal{N}_{(j)}, \quad (5)$$

$$\{\mathcal{N}_{(i)}, \mathcal{N}_{(j)}\} = \epsilon_{ij}\mathcal{R}, \quad \{\mathcal{N}_{(i)}, p_0\} = -p_i, \quad (6)$$

$$\{\mathcal{N}_{(i)}, p_j\} = -\delta^i_j\left(p_0 - \ell p_0^2 + \frac{\ell}{2}p^2\right) + \ell p_i p_j, \quad (7)$$

in which the boost $\mathcal{N}_{(i)}$ and the rotation $\mathcal{R}$ generators can be represented in terms of the χ^μ coordinates as

$$\mathcal{N}_{(i)} = \chi^0 p_i + \chi^i\left(p_0 - \ell p_0^2 + \frac{\ell}{2}p^2\right), \quad (8)$$

$$\mathcal{R} = \chi^L p_T - \chi^T p_L. \quad (9)$$

An important de Sitter momentum-space feature to take into account is the deformation of the symplectic structure between momenta and coordinates, given by the nontrivial relation between the coordinate components (1). Therefore,

$$\begin{aligned} &\{p_0, \chi^0\} = 1, \qquad \{p_0, \chi^j\} = 0 \\ &\{p_i, \chi^0\} = -\ell p_i, \qquad \{p_i, \chi^j\} = \delta^j_i. \end{aligned} \quad (10)$$

It is easy to check that (10) and (1) satisfy all Jacobi identities.

We can obtain the finite action of the boost transformation by means of the Poisson brackets of its generator $\mathcal{N}_{(i)}$ through the map

$$\mathcal{B}_{(i)} \triangleright f(x,p) = f(x,p) - \xi\{\mathcal{N}_{(i)}, f(x,p)\} + \frac{\xi^2}{2!}\{\mathcal{N}_{(i)}, \{\mathcal{N}_{(i)}, f(x,p)\}\} + \cdots \quad (11)$$

However, in the following sections, instead of summing all the ξ^n contributes, we will, for sake of simplicity, just integrate the first-order term of the series expansion.

II. BOOST PARAMETERS IN RELATIVE LOCALITY

In this first paragraph, we will review some basic concepts of relative locality in $1+1$ dimensions, and, at the end, we will show how the special-relativistic parameter β and γ find a rather simple interpretation even in a curve-momentum-space framework. In special relativity, in order to identify the physical meaning of β, we take advantage of the mathematical relation between hyperbolic sine and cosine,

$$\cosh^2(\xi) - \sinh^2(\xi) = 1, \quad (12)$$

and then we redefine the two functions as

$$\cosh(\xi) = \gamma \qquad \sinh(\xi) = \beta\gamma; \quad (13)$$

therefore, (12) determines the connection between the two parameters (which still have no physical interpretation for now),

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \quad (14)$$

Of course those relations find a useful application in describing the coordinate and momenta transformations between two observers boosted with respect to one another. We let the special-relativistic boost generator $\mathcal{N}_{SR} = x^1 p_0 + x^0 p_1$ act on momenta p_α through the Poisson-bracket formalism. Then we find the infinitesimal variation of momentum-space coordinates with respect the *rapidity* parameter ξ:

$$\begin{cases} \frac{dp_0}{d\xi} = -\{\mathcal{N}_{SR}, p_0\} = p_1 \\ \frac{dp_1}{d\xi} = -\{\mathcal{N}_{SR}, p_1\} = p_0. \end{cases} \quad (15)$$

System (15) can be easily solved for example using the *ab initio* conditions $p_0(0) = \mu$, $p_1(0) = 0$. With this choice we find the usual

$$p_0(\xi) = \mu\cosh(\xi) \qquad p_1(\xi) = \mu\sinh(\xi). \quad (16)$$

Now from equation (12) it is straightforward to obtain the special-relativistic invariant dispersion relation

$$p_0^2 - p_1^2 = \mu^2$$

and also the physical interpretation for the β parameter,

$$\beta = \frac{\sinh(\xi)}{\cosh(\xi)} = \frac{p_1}{p_0} \equiv v_1,$$

which is the particle's velocity we find in the expression of special-relativistic worldlines.

In relative locality we proceed in a quite similar way. The RL version of (15) was already found in DSR literature [13,23] and defined in a curved momentum-space framework in Ref. [3] at all orders in the deformation parameter ℓ. It is sufficient for our purposes to discuss the first-order expansion formalism which was already used to explore synchrotron radiation in deformed special relativity [14]. Indeed the transformations of our curve-momentum-space coordinates can be obtained from the deformed boost generator [see (8)] action:

$$\begin{cases} \frac{dp_0(\xi)}{d\xi} = -\{\mathcal{N}, p_0\} = p_1(\xi) \\ \frac{dp_1(\xi)}{d\xi} = -\{\mathcal{N}, p_1\} = p_0(\xi) - \ell p_0^2(\xi) - \frac{\ell}{2} p_1^2(\xi) \end{cases}. \quad (17)$$

This differential equation system can be solved by perturbing the solutions we found in the classical case (16) as

$$p_0(\xi) = \mu \cosh(\xi) + \ell a(\xi),$$
$$p_1(\xi) = \mu \sinh(\xi) + \ell b(\xi). \quad (18)$$

Thus, using (18), we reduce (17) to the relations

$$a(\xi) - \frac{d^2 a(\xi)}{d\xi^2} = \mu^2 \left(\cosh^2(\xi) + \frac{1}{2} \sinh^2(\xi) \right) \quad (19)$$

$$b(\xi) = \frac{da(\xi)}{d\xi}, \quad (20)$$

from which we finally obtain the solutions

$$p_0(\xi) = \mu \cosh(\xi) - \ell \frac{\mu^2}{2} \sinh^2(\xi) \quad (21)$$

$$p_1(\xi) = \mu \sinh(\xi) - \ell \mu^2 \sinh(\xi) \cosh(\xi). \quad (22)$$

We can now verify that if we assume the energy-momentum dispersion relation to be deformed according to (4) we still can obtain a coherent picture for the invariance of the particle mass; in fact since

$$\sinh(\xi) \simeq \frac{p_1}{\mu} \left(1 + \ell \frac{p_0}{\mu} \right), \qquad \cosh(\xi) \simeq \frac{p_0}{\mu} + \frac{\ell}{2} \frac{p_1^2}{\mu}, \quad (23)$$

we can again rely on (12) (which is purely a relation between hyperbolic functions and then not model dependent at all) to define our modified dispersion relation (MDR), invariant under deformed boost transformations

$$p_0^2 - p_1^2 - \ell p_1^2 p_0 = \mu^2.$$

We can now recover the generic definitions (13) for β, γ (which as stated above still is not model dependent). Therefore,

$$\beta = \tanh(\xi) = \frac{|p_1|}{\sqrt{p_1^2 + \mu^2}} + \ell |p_1| \left(1 - \frac{p_1^2}{p_1^2 + \mu^2} \right). \quad (24)$$

This result is very important, since also in the ℓ-deformed framework β can be interpreted as the velocity of a boosted particle in the laboratory reference frame. We can in fact notice that relation (24) is exactly the coordinate velocity found in previous relative locality works [4,22].

A. Velocity in relative locality

To better explain the liaison between the β parameter and the velocity expression in relative locality, we need to introduce some elements of its phase space formalisation. In this framework we can describe particles worldlines in terms of an auxiliary parameter τ. The dependence of coordinates χ^α on the worldline parameter τ can be found using Casimir (4) as a Hamiltonian: $\dot{\chi}^\beta = d\chi^\beta / d\tau = \{\mathcal{C}, \chi^\beta\}$. This leads to

$$\dot{\chi}^0 = \{\mathcal{C}, \chi^0\} = 2p_0 + \ell p_1^2, \quad (25)$$

$$\dot{\chi}^1 = \{\mathcal{C}, \chi^1\} = -2p_1 - \ell p_0 p_1. \quad (26)$$

The worldline expression can then be found pretty easily by integrating

$$\chi^1 - \bar{\chi}^1 = \int_{\bar{\chi}^0}^{\chi^0} \frac{\dot{\chi}^1}{\dot{\chi}^0} d\chi^0 = v_\chi(p)(\chi^0 - \bar{\chi}^0). \quad (27)$$

It is not hard to verify that $|v_\chi|(p) = \beta(p)$ and then, as stated before, that parameter β founds a simple physical interpretation in the relative locality framework. In Ref. [22] the quantity $v_\chi(p)$ was referred to as *coordinate velocity*, since, in order to obtain a satisfactory formalization of the entire particle emission-reception process, we also need to take into account how to relate different translated observers coordinatizations. In fact, taking into account the nontrivial symplectic structure defined in (10), we obtain deformed relations between the coordinates of an observer at the emission of a particle (Alice) and the receiver's (Bob) ones:

$$\chi^\mu_B = \chi^\mu_A + a^\nu\{p_\nu, \chi^\mu\} = \chi^\mu_A + a^\nu(\delta^\mu_\nu - \ell\delta^1_\nu\delta^\mu_0 p_1). \quad (28)$$

This means that an event, such as the emission of two particles with different energies, defined as "local" by Alice may not be local for Bob as well.

This nontrivial translation framework may result in being a little bit confusing. The same process can be, however, also expressed using coordinates x^α with trivial symplectic sector $\{p_\mu, x^\nu\} = \delta^\nu_\mu$ as explained in Ref. [4]. In this framework the worldlines expression is

$$x^1 - \bar{x}^1 = \int_{\bar{x}^0}^{x^0} \frac{\dot{x}^1}{\dot{x}^0} dx^0 = v_x(p)(x^0 - \bar{x}^0), \quad (29)$$

where

$$|v_x|(p) = \frac{\partial p_0}{\partial p_1} = \frac{|p_1|}{\sqrt{p_1^2 + \mu^2}} + \ell|p_1|$$

is the particle *physical velocity*, since in this formalism the relative locality effect is only expressed through velocity momentum dependence [15].

Those two formalisms, the nontrivial translation one, formalized in terms of χ^α coordinates, and the physical velocity one (which relates on a standard-symplectic sector), coordinatized through x^α, are not in contradiction and predict the same physical effect as shown in Fig. 1.

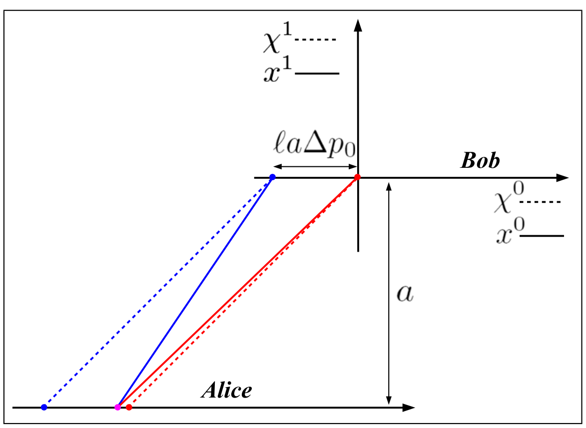

FIG. 1 (color online). Two photons are emitted in Alice's origin: a low-energetic one (in red on the right) and an ultraviolet one (in blue on the left). The relative locality effect of this model is represented by the time delay between the arrival of the two photons in Bob's spatial origin (at a certain distance a from the emission). The effect is the same no matter what formalism Bob uses to describe phenomena: both the nontrivial translation formalism (dashed worldlines) and the physical velocity one (straight worldlines) predict the same time delay $\Delta t = \ell a \Delta p_0$ between the arrival of the high-energetic photon and the low-energetic one.

It is important for phenomenological purposes to notice that the identification between the β parameter and the coordinate velocity v_χ cannot be done in a symmetry-breakdown scenario, since in this case we do not modify (15), and then the relation between $\beta_{\rm LIV}$ and worldline velocity is unavoidably nontrivial: $\beta_{\rm LIV} = v_1(1 - \ell p_0)$. The possibility of having a departure from the identification between the β parameter and velocity v is usually not taken into account in Lorentz-invariance violation literature (see *exempli gratia* Refs. [24,25]) and maybe should be better deepened.

B. Deformed Lorentz momenta transformations in 2 + 1 dimensions

It is maybe important to deepen our exploration on relative locality with de Sitter momentum space in more than one spatial dimension, since it shows a peculiar feature which in the literature is called *transverse relative locality* [15,18,19,26]. This feature is an important aspect of theories with relativity of locality since it provides interesting phenomenological effects as we will see further. In $2+1$ dimensions the system of differential equations (17) is enriched by a transverse-component equation:

$$\begin{cases} \frac{dp_0(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, p_0\} = p_L(\xi) \\ \frac{dp_L(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, p_L\} = p_0(\xi) - \ell p_0^2(\xi) + \frac{\ell}{2}|p|^2(\xi) - \ell p_L^2. \\ \frac{dp_T(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, p_T\} = -\ell p_L p_T \end{cases} \quad (30)$$

We can solve the system perturbatively as done in the previous section with system (17), fixing the generic *ab initio* conditions $p_0(0) = \bar{p}_0$, $p_L(0) = \bar{p}_L$ and $p_T(0) = \bar{p}_T$; given those we find the generic solutions

$$p_0(\xi) = \bar{p}_0\cosh(\xi) + \bar{p}_L\sinh(\xi) - \frac{\ell}{2}(\cosh(\xi) - 1) \times(-\bar{p}_T^2 + \bar{p}_0^2(\cosh(\xi)+1) + \bar{p}_L^2\cosh(\xi) + 2\bar{p}_0\bar{p}_L\sinh(\xi)), \quad (31)$$

$$p_L(\xi) = \bar{p}_L\cosh(\xi) + \bar{p}_0\sinh(\xi) + \ell\left(\bar{p}_0\bar{p}_L(1-\cosh(\xi))^2 + \left(\frac{\bar{p}^2}{2} - (\bar{p}_0^2 + \bar{p}_L^2)\cosh(\xi)\right)\sinh(\xi)\right), \quad (32)$$

$$p_T(\xi) = \bar{p}_T + \ell\bar{p}_T(\bar{p}_0(1-\cosh(\xi)) - \bar{p}_L\sinh(\xi)). \quad (33)$$

It is very easy to verify that those solutions reduces to (21) and (22) if we fix the initial conditions as $\bar{p}_0 = \mu$, $\bar{p}_L = 0$ and $\bar{p}_T = 0$. Another important property of solutions (31), (32) and (33) is that they verify the invariance of the deformed dispersion relation defined by the Casimir (4) at all orders in ξ; in fact we observe that

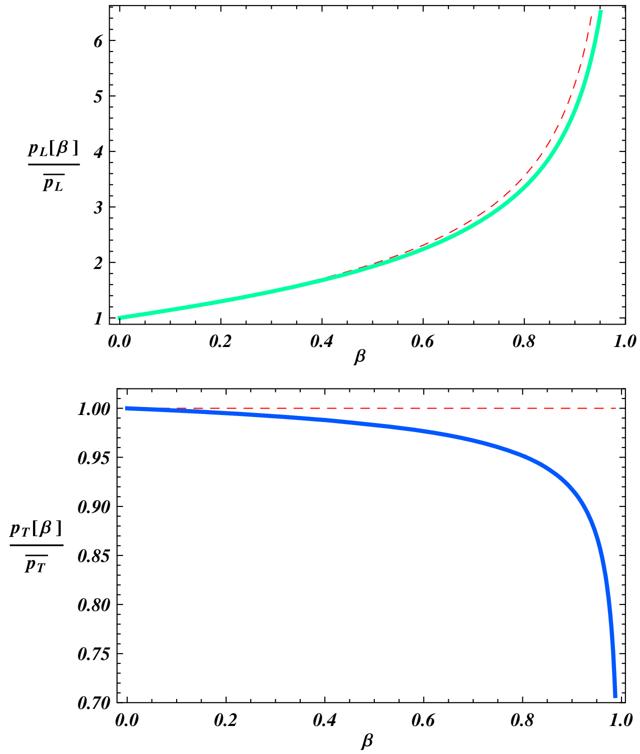

FIG. 2 (color online). In those pictures we represent the behavior respectively of the p_L and the p_T components of momenta, for different values of the β parameter. The straight lines obey (32) and (33) transformation laws, while the dashed ones represent the special relativistic case. Of course in order to show explicitly the differences between those two theories, the momenta absolute value has been fixed at some consistent fraction of our deformation scale ($|p| \sim 0.03\ell^{-1}$).

$$p_0^2(\xi) - (p_L^2(\xi) + p_T^2(\xi)) - \ell p_0(\xi)(p_L^2(\xi) + p_T^2(\xi)) = \bar{p}_0^2 - (\bar{p}_L^2 + \bar{p}_T^2) - \ell \bar{p}_0(\bar{p}_L^2 + \bar{p}_T^2) = \mu^2, \tag{34}$$

as we could expect, given relation $\{\mathcal{N}_{(i)}, \mathcal{C}\} = 0$. One thing we can notice from Eq. (34) is that the invariance of the dispersion relation is strictly related to the transformations of all the components of momenta. While in special relativity (SR) the transformation of the p_0 and the p_L components leads to them compensating each other (where L is chosen as the boost direction), in RL we need to take into account also the transverse one to ensure the invariance of the MDR. Since (32) and (33) balance each other harmoniously, there is no point in studying the evolution of the angle $\theta = \arctan(p_T(\beta)/p_L(\beta))$ between the two momenta (we would obtain a practically indistinguishable behavior from the SR one). On the other hand, it may be of some interest to analyze the behavior of the single momentum components (see Fig. 2).

While $p_L(\beta)$ basically follows the special relativistic curve, $p_T(\beta)$ shows a sensitively different behavior than the SR case, at some orders of magnitude below our deformation scale ℓ. This could be an important feature for further phenomenological investigations of relative locality, for example for what concerns the study of deformed particle vertices. We will not deepen those aspects in this paper for they might deserve dedicated studies, and instead we are here more interested in characterizing deformed Lorentz-transformations effects also for space-time.

III. COORDINATE TRANSFORMATIONS AND RAINBOW METRICS

In the literature many studies try to define the behavior of relative locality in the presence of space-time curvature [8,9]. To support those efforts, it may be of some interest to develop a phenomenology of RL effects, for example at a cosmological scale. An important mathematical tool which could be very useful in this kind of analysis is the Rainbow metrics formalism [20]. In this paper we have now the possibility to suggest how those momentum-dependent metrics should naturally arise in the Minkowskian limit of relative locality. In fact, as done for momenta (30), we can define the different space-time coordinatizations, which two boosted observers would use to describe physical phenomena, by solving the system

$$\begin{cases} \frac{d\chi^0(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, \chi^0\} = -\chi^L(\xi) + \ell(\chi^L(\xi)p_0(\xi) + \chi^0(\xi)p_L(\xi)) \\ \frac{d\chi^L(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, \chi^L\} = -\chi^0(\xi) \\ \frac{d\chi^T(\xi)}{d\xi} = -\{\mathcal{N}_{(L)}, \chi^T\} = -\ell(\chi^L(\xi)p_T(\xi) - \chi^T(\xi)p_L(\xi)). \end{cases} \tag{35}$$

As usual we opt for solving (35) perturbatively at first order in ℓ, using the solutions we found in the last section (31), (32) and (33), to write the explicit expressions for momenta $p_\mu(\xi)$. The solutions of system (35) for generic *ab initio* conditions $\chi^\mu(0) = \bar{\chi}^\mu$ are

$$\chi^0(\xi) = \bar{\chi}^0 \cosh(\xi) - \bar{\chi}^L \sinh(\xi) + \ell \sinh(\xi)(\bar{\chi}^L \bar{p}_0 + \bar{\chi}^0 \bar{p}_L), \tag{36}$$

$$\chi^L(\xi) = \bar{\chi}^L\cosh(\xi) - \bar{\chi}^0\sinh(\xi) + \ell(1-\cosh(\xi))(\bar{\chi}^L\bar{p}_0 + \bar{\chi}^0\bar{p}_L), \quad (37)$$

$$\chi^T(\xi) = \bar{\chi}^T + \ell((\cosh(\xi)-1)(\bar{\chi}^T\bar{p}_0 + \bar{\chi}^0\bar{p}_T) - \sinh(\xi)(\bar{\chi}^L\bar{p}_T - \bar{\chi}^T\bar{p}_L)). \quad (38)$$

Those solutions can help us to define the relative-locality-invariant line element ds^2 at all orders in ξ, and in the same exact way, we show the invariance of the dispersion relation in (34). We can therefore observe that two boosted observers will agree on

$$\chi^0(\xi)^2 - (\chi^L(\xi)^2 + \chi^T(\xi)^2)(1-2\ell p_0(\xi)) + 2\ell\chi^0(\xi)\chi^i(\xi)p_i(\xi) = (\bar{\chi}^0)^2 - ((\bar{\chi}^L)^2 + (\bar{\chi}^T)^2)(1-2\ell\bar{p}_0) + 2\ell\bar{\chi}^0\bar{\chi}^i\bar{p}_i. \quad (39)$$

Relation (39) can be more synthetically expressed through a metric formalism as

$$\Delta s^2 = \tilde{\eta}^{(\chi)}_{\mu\nu}(p)\chi^\mu\chi^\nu, \quad (40)$$

where the momentum-dependent $2+1$-dimensional Minkowskian metric $\tilde{\eta}$ is defined as

$$\tilde{\eta}^{(\chi)}_{\mu\nu}(p) = \begin{pmatrix} 1 & \ell p_L & \ell p_T \\ \ell p_L & -(1-2\ell p_0) & 0 \\ \ell p_T & 0 & -(1-2\ell p_0) \end{pmatrix}. \quad (41)$$

This example shows explicitly how the Rainbow metrics formalism is naturally implemented in the relative locality theory. The main difference between the Rainbow formalism used in Ref. [20] and the one we show in this paper is that in relative locality the definition of metric $\tilde{\eta}$ is not obtained through the modified dispersion relation as $m^2 = g^{\alpha\beta}_{(R)}(p)p_\alpha p_\beta$. Vice versa in RL both MDR and the space-time Rainbow metric are shaped on the curve momentum-space metric (2).

It may seem that metric (41) may not be dual to the momentum-space metric (2), because of the off-diagonal elements. That is not a problem because we do not expect metric $\tilde{\eta}^{(\chi)}_{\alpha\beta}$ to be dual to the momentum-space one, since noncommutative coordinates χ^μ have a nontrivial symplectic sector (10). Duality is instead required for commutative coordinates x^β which satisfy $\{p_\alpha, x^\beta\} = \delta^\beta_\alpha$. The liaison between χ^α and x^β coordinates is very well known in relative locality literature [1,3,22] and is

$$\chi^\alpha = \tau^\alpha_\beta(p)x^\beta = (\delta^\alpha_\beta - \ell\delta^\alpha_0\delta^j_\beta p_j)x^\beta, \quad (42)$$

where the $\tau^\alpha_\beta(p)$ are the translation de Sitter momentum-space killing vectors (see Ref. [4] for a clear discussion of the physical implications of this feature).

Using relation (42) we can find that

$$\Delta s^2 = \tilde{\eta}^{(\chi)}_{\mu\nu}(p)\chi^\mu\chi^\nu = \tilde{\eta}_{\mu\nu}(p)x^\mu x^\nu, \quad (43)$$

where

$$\tilde{\eta}_{\mu\nu}(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(1-2\ell p_0) & 0 \\ 0 & 0 & -(1-2\ell p_0) \end{pmatrix}. \quad (44)$$

Then, confronting (44) with (2), it is now clear how duality between space-time and momentum-space metrics is manifest, since $\tilde{\eta}^{\alpha\gamma}\tilde{\eta}_{\gamma\beta} = \delta^\alpha_\beta$.

A. Clocks and transverse effects

To explore special relative locality phenomenology, we should now define the procedure we use to identify what we call time intervals. As in usual special relativity, in RL we can rely on the absoluteness of the speed of light, using an Einstein clock of length a (see Fig. 3) to define time units. The only problem we should be careful about is the nontrivial relation between lengths and time intervals.

The procedure we are going to describe will make use of the same formalism already introduced in Sec. II. A, paying particular attention to the sign of photon velocities and momenta *pre-* and *postreflection* (and other small formal features that we will deepen throughout this section).

We can begin with noticing that according to (27) in χ^α coordinates our photons have trivial worldlines

$$\chi^T - \bar{\chi}^T = -(\chi^0 - \bar{\chi}^0); \quad (45)$$

on the other hand, we have deformed translations [4,22] due to the nontrivial symplectic sector (10). Then the ideal interaction point between a photon emitted in A and the mirror in B has coordinates

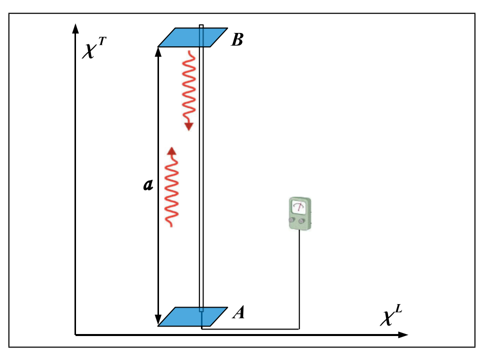

FIG. 3 (color online). Einstein synchronization convention: a photon is sent from the emission point A, set as the spatial origin of the coordinate frame in picture, at time $\chi^0 = 0$ toward the mirror in B. The following detection of the reflected photon in A gives us the definition of time.

$$\chi^{\nu}_{(B)} = \bar{\chi}^{\nu}_{(A)} - a^{\mu}\{p_{\mu},\chi^{\nu}_{(A)}\} = \bar{\chi}^{\nu}_{(A)} - a^{\mu}(\delta^{\nu}_{\mu} - \ell\delta^{\nu}_{0}\delta^{i}_{\mu}\bar{p}_{i}), \tag{46}$$

where with $(\bar{\chi}^{L}_{(A)} = 0, \bar{\chi}^{T}_{(A)} = 0)$ we indicate the emission point coordinates in the A frame. Then, using (46) with (45), we obtain that, according to the translated observer [whose spatial origin is in $(\chi^{L}_{(B)} = 0,\ \chi^{T}_{(B)} = 0)$], the emission point has coordinates

$$\bar{\chi}^{0}_{(B)} = -a^{0} + \ell a\bar{p}_{T}, \qquad \bar{\chi}^{L}_{(B)} = 0, \qquad \bar{\chi}^{T}_{(B)} = -a. \tag{47}$$

Then the observer in B infers different emission times for different photon energies. Moreover, using the worldline expression (45), we can verify that also the photon time of arrival at mirror in B is momentum dependent:

$$\chi^{0}_{(B)}(\chi^{L}_{(B)} = 0, \qquad \chi^{T}_{(B)} = 0) = a^{0} - a - \ell a\bar{p}_{T}. \tag{48}$$

All this may result in being a little bit weird to a reader facing relative-locality-related effects for the first time, but we should keep in mind that all those features are merely a coordinate artifact, due to the curvature of momentum space. This concept is even clearer when one clarifies what to expect from the entire emission-reflection-detection process. In fact since the detector is placed in A, we can check if such a momentum dependency is still present in the time interval measured by our device, by calculating where the observer in A would infer the emission point. First of all, we have to fix, using the inverse of transformations (46), how A would express the photon reflection point (48):

$$\chi^{\nu}_{(A)} = \bar{\chi}^{\nu}_{(B)} + a^{\mu}\{p_{\mu},\chi^{\nu}_{(B)}\} = \bar{\chi}^{\nu}_{(B)} + a^{\mu}(\delta^{\nu}_{\mu} - \ell\delta^{\nu}_{0}\delta^{i}_{\mu}\bar{p}_{i}). \tag{49}$$

Then, setting the result of (49) as a starting point for the worldlines (45), and considering that momentum p_T now points in the opposite direction, we obtain that the observer in A infers the emission time to be

$$\bar{\chi}^{0}_{(A)} = -2a - 2\ell a\bar{p}_{T}. \tag{50}$$

Therefore, as we expected, since the momentum dependence of the photon time of flight that B observes is a physical effect (46), we obtain that the time interval definition in relative locality depends explicitly on momentum-space curvature:

$$\Delta\chi^{0} \simeq 2a(1 - \ell\bar{p}_{0}). \tag{51}$$

The reason why we have formalized our theory using coordinates with apparently complicated relations between each other (1) and a nontrivial symplectic sector (10) is that we have been able to express the physical effect just as a feature of the deformed translations. If instead of using the χ^{α} coordinates we had used the commutative x^{α} ones, we would have payed the simplification of the mathematical formalism with a more complex description of the whole synchronization mechanism (though the physical result would have been the same).

Using this coordinatization, it is now easy to obtain the time-interval expression for a boosted observer. In fact, if we imagine observing the device in Fig. 3 from a reference frame boosted along the χ^{L} direction, since any transverse effect on momentum is suppressed by a factor $\mathcal{O}(\ell^{2})$, according to (36) we would define the time interval just as

$$\Delta\chi^{0}(\beta) = 2a\gamma(1 - \ell\bar{p}_{0}), \tag{52}$$

where a is defined in the rest frame. However, a boosted observer would not express the clock length in terms of a. If instead we wish to express our time interval in terms of the boosted reference frame observables, we should take into account also the relatively local transverse effect. Then, with $\mathcal{L}_{(a)}$ being the clock length measured by the boosted observer, using (38), Eq. (52) becomes

$$\Delta\chi^{0}(\beta) = 2\mathcal{L}_{(a)}\gamma(1 - \ell(2\gamma - 1 + \beta\gamma)\bar{p}_{0}). \tag{53}$$

To imagine a way to detect this effect, we can borrow a common idea in quantum-gravity literature, considering the time delay of two simultaneously emitted photons carrying different energies [5] in two different boosted reference frames. While in the clock's reference frame we expect a momentum-dependent time delay only amplified from the size of length a, on the other hand, according to a boosted observer, the two photons should reach the detector at different times, for which the difference for $\gamma \gg 1$ is $\delta T \sim \ell\gamma^{2}\mathcal{L}_{a}\delta E$. For this effect to have any significance, an ideal *gedanken experiment* based on it should then compare the observations of two boosted observers with high boost parameter γ, for the ricochet of two photons with big energy difference δE, in a clock with large $\mathcal{L}_{a}$, to compensate the tiny value of ℓ.

IV. CLOSING REMARKS

In quantum-gravity phenomenology, it is always complex to define observables and consequently to fix upper bounds to the parameters we use to formalize the effects. It is then, in my experience, useful to express those effects as corrections to the classical models. This is precisely the spirit of this whole article in which the manifestations of momentum-space curvature are expressed as a deformation of Lorentz transformations, modeled in terms of the usual β and γ parameters. With this formalization it is pretty simple to characterize the deformation effects, even the most unexpected ones, like the boost-related transverse relative locality. About this rather unexplored scenario of transverse effects in de Sitter momentum space, it may be interesting to verify if such features can be of some help in identifying an upper limit for phenomenological parameters, for example for analysis such as the one reported in Ref. [27], for which the identification of the origin point of detected particles is crucial.

Also interesting for phenomenological purposes is the discussion about the deformed (momentum-dependent) law for time-intervals dilatation (the boost parameter γ appears to act like a magnifier for RL effects), and it might require a dedicated research program to identify the most promising applications that might allow us to unveil such effects. But the payout that could be expected appears to be worth the effort, since such a novel window on the Planck-scale realm could have particularly significant impact on our ability to investigate the quantum-gravity problem.

Also for what concerns the more academic/conceptual side of the issues here discussed, these studies should motivate further investigation, particularly for what concerns the identification of a characteristic metric formalism for relative locality which could also be extremely important from the phenomenological side, as discussed in Sec. III.

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