

Thermodynamic instability of topological black holes in Gauss-Bonnet gravity with a generalized electrodynamics

S. H. Hendi^{*}

*Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
and Research Institute for Astrophysics and Astronomy of Maragha (RIAAM),
P.O. Box 55134-441, Maragha 71454, Iran*

S. Panahiyan[†]

*Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
(Received 24 August 2014; published 3 December 2014)*

Motivated by the string corrections on the gravity and electrodynamics sides, we consider a quadratic Maxwell invariant term as a correction of the Maxwell Lagrangian to obtain exact solutions of higher dimensional topological black holes in Gauss-Bonnet gravity. We first investigate the asymptotically flat solutions and obtain conserved and thermodynamic quantities which satisfy the first law of thermodynamics. We also analyze thermodynamic stability of the solutions by calculating the heat capacity and the Hessian matrix. Then, we focus on horizon-flat solutions with an anti-de Sitter (AdS) asymptote and produce a rotating spacetime with a suitable transformation. In addition, we calculate the conserved and thermodynamic quantities for asymptotically AdS black branes which satisfy the first law of thermodynamics. Finally, we perform thermodynamic instability criterion to investigate the effects of nonlinear electrodynamics in canonical and grand canonical ensembles.

DOI: [10.1103/PhysRevD.90.124008](https://doi.org/10.1103/PhysRevD.90.124008)

PACS numbers: 04.40.Nr, 04.20.Jb, 04.70.Bw, 04.70.Dy

I. INTRODUCTION

Nonlinear theories have been studied extensively in the context of various physical phenomena. In other words, because of the nonlinear nature of most physical systems, linear theories could not precisely describe the experimental consequences and we must inevitably consider the nonlinear models.

Classical electrodynamics, whose field equations originate from the Maxwell Lagrangian, contains various problems which motivate one to consider nonlinear electrodynamics (NLED). One of the main problematic items of Maxwell theory is infinite self-energy of a pointlike charge. The first successful Lorentz invariance theory for solving this problem was introduced by Born and Infeld [1]. After Born-Infeld (BI) theory, although some different NLED have been introduced with various motivations, their weak field limits lead to Maxwell theory [2–5]. Amongst the NLED theories, the so-called BI types, whose first nonlinear correction is a quadratic function of Maxwell invariant, are completely special [4–6]. In addition to the interesting properties of BI-type NLED theories [7], it may be shown that these theories may arise as a low energy limit of heterotic string theory [8], which leads to increased interest. Taking into account the first leading correction term of BI-type theories which originated from the string theory, one can investigate the effect of NLED versus Maxwell theory [9,10].

In this paper, we consider Gauss-Bonnet (GB) gravity as a natural generalization of Einstein gravity with a quadratic power of Maxwell invariant, in addition to the Maxwell Lagrangian as a source. On the gravity point of view, string theories in their low energy limit give rise to effective models of gravity in higher dimensions which involve higher curvature terms, while from the electrodynamics viewpoint, a quadratic power of Maxwell invariant supplements the Maxwell Lagrangian with the development of superstring theory. In the weak field limit, GB gravity and the Lagrangian of the mentioned NLED reduce to Einstein gravity and Maxwell Lagrangian, respectively.

On the other hand, it has been shown that there is a close relationship between black hole thermodynamics and the nature of quantum gravity. For black holes to be physical objects, it is necessary for them to be stable under external perturbation. There are two kinds of stability: physical (dynamical) stability and thermodynamical stability, which is of interest in this paper. The most interesting parts of black hole thermodynamics are thermal instability and Hawking-Page phase transition [11]. It has been shown that there is no Hawking-Page phase transition for the Schwarzschild-AdS black hole whose horizons have vanishing or negative constant curvature [12].

Thermodynamic properties of black holes in GB gravity with NLED have been studied before [13]. In this paper, we will obtain the black hole solutions of GB-Maxwell corrected (GB-MC) gravity in arbitrary dimensions and discuss thermal instability conditions.

^{*}hendi@shirazu.ac.ir
[†]ziexify@gmail.com

One may question the motivation for considering GB-MC gravity. As we know, the Maxwell theory has acceptable consequences, to a large extent, in various physical areas. So, in transition from the Maxwell theory to NLED, it is allowable to consider the effects of small nonlinearity variations, not strong effects. In other words, in order to obtain physical results with experimental agreements, one should regard the nonlinearity as a correction to the Maxwell field. Eventually, we should note that, although various theories of NLED have been created with different primitive motivations, only their weak nonlinearities have physical and experimental importance. So the effects of nonlinearity should be considered as a perturbation to Maxwell theory. In addition, in a gravitational framework the GB gravity is a natural generalization of Einstein gravity (not a perturbation in general) in higher dimensions. One may regard GB and MC as the corrections of an Einstein-Maxwell black hole.

The layout of the paper will be in this order: In Sec. II, we will introduce the structure of action which contains GB gravity coupled with MC. Then, we will solve the field equations and discuss the geometric properties of the topological black holes. We will study thermodynamic properties and stability of the asymptotically flat spacetime in Sec. III. In the following section, we will generalize our solutions to rotating black branes with anti-de Sitter (AdS) asymptote and calculate thermodynamic and conserved quantities of rotating solutions. The final part is devoted to investigating the stability of the solutions in canonical and grand canonical ensembles. Conclusions are drawn in the last section.

II. TOPOLOGICAL BLACK HOLES

The Lagrangian of Einstein-GB gravity coupled to NLED can be written as

$$L_{\text{tot}} = L_E - 2\Lambda + \alpha L_{\text{GB}} + L(F), \quad (1)$$

$$d\Omega_k^2 = \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2\theta_j d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sinh^2\theta_1 d\theta_2^2 + \sinh^2\theta_1 \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2\theta_j d\theta_i^2 & k = -1. \\ \sum_{i=1}^{n-1} d\phi_i^2 & k = 0 \end{cases} \quad (7)$$

Since we are looking for the black hole solution with a radial electric field, we know that the nonzero components of the electromagnetic field are

$$F_{tr} = -F_{rt}. \quad (8)$$

where the Lagrangian of Einstein gravity is the Ricci scalar, $L_E = R$, and Λ is the cosmological constant. In the third term of Eq. (1), α is the GB coefficient with dimension $(\text{length})^2$ and L_{GB} is the Lagrangian of GB gravity,

$$L_{\text{GB}} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2. \quad (2)$$

The last term in Eq. (1) is the Lagrangian of NLED, which we choose a quadratic correction in addition to the Maxwell Lagrangian

$$L(\mathcal{F}) = -\mathcal{F} + \beta\mathcal{F}^2 + O(\beta^2), \quad (3)$$

where β is the nonlinearity parameter and the Maxwell invariant $\mathcal{F} = F_{ab}F^{ab}$, where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic field tensor and A_b is the gauge potential. For the vanishing nonlinearity parameter, this Lagrangian yields the standard Maxwell theory, as it should. Regarding the gauge-gravity Lagrangian (1) and using the variational method, one can obtain the following field equations:

$$G_{ab}^E + \Lambda g_{ab} + \alpha G_{ab}^{\text{GB}} = \frac{1}{2}g_{ab}L(\mathcal{F}) - 2L_{\mathcal{F}}F_{ac}F_b^c, \quad (4)$$

$$\partial_a(\sqrt{-g}L_{\mathcal{F}}F^{ab}) = 0, \quad (5)$$

where G_{ab}^E is the Einstein tensor, $G_{ab}^{\text{GB}} = 2(R_{acde}R_b^{cde} - 2R_{acbd}R^{cd} - 2R_{ac}R_b^c + RR_{ab}) - \frac{1}{2}L_{\text{GB}}g_{ab}$ and $L_{\mathcal{F}} = \frac{dL(\mathcal{F})}{d\mathcal{F}}$.

Now we are interested in topological static black hole solutions; therefore, we take into account the following static metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \quad (6)$$

where $d\Omega_k^2$ represents the line element of an $(n-1)$ -dimensional hypersurface with the constant curvature $(n-1)(n-2)k$ and volume V_{n-1} with the following explicit form:

One can use Eq. (5) to obtain the explicit form of F_{tr} with the following form:

$$F_{tr} = \frac{q}{r^{n-3}} - \frac{4q^3\beta}{r^{3n-3}} + O(\beta^2), \quad (9)$$

which for small values of β reduces to the Maxwell electromagnetic field tensor.

We find that the nonzero independent components of Eq. (4) can be written by applying electromagnetic field tensor (9):

$$e_{tt} = r^3 \left(1 + \frac{2(1-f)\alpha'}{r^2} \right) f' - \alpha'(n-4)(1-f)^2 + \frac{2\Lambda r^4}{(n-1)} + \frac{2q^2}{(n-1)r^{2n-6}} - \frac{4q^4\beta}{(n-1)r^{4n-8}} + O(\beta^2), \quad (10)$$

$$e_{\theta\theta} = r^4 \left[1 + \frac{2(1-f)\alpha'}{r^2} \right] f'' - 2\alpha' r^2 f'^2 + 2r(n-2) \left[r^2 + 2\alpha' \frac{(n-4)}{(n-2)} (1-f) \right] f' - \alpha'(n-4)(n-5)(1-f)^2 - (n-2)(n-3)r^2(1-f) + 2\Lambda r^4 - \frac{2q^2}{r^{2n-6}} + \frac{12q^4\beta}{r^{4n-8}} + O(\beta^2), \quad (11)$$

where $\alpha' = (n-2)(n-3)\alpha$.

It is straightforward to show that the following metric function satisfies all of the field equations simultaneously:

$$f(r) = k + \frac{r^2}{2\alpha'} (1 - \sqrt{\Psi(r)}), \quad (12)$$

with

$$\Psi(r) = 1 + \frac{8\alpha'}{n(n-1)} \left(\Lambda + \frac{n(n-1)m}{2r^n} - \frac{nq^2}{(n-2)r^{2n-2}} + \frac{2nq^4\beta}{r^{4n-4}(3n-4)} \right) + O(\beta^2), \quad (13)$$

where m is an integration constant that is related to mass. One can see that for small values of β , the metric function reduces to usual GB-Maxwell gravity. Expansion of the metric function for small values of the GB parameter leads to

$$f(r) = f_{\text{EM}} - \frac{4q^4}{(n-1)(3n-4)r^{4n-6}}\beta + \frac{(k-f_{\text{EM}})^2}{r^2}\alpha' + \frac{8q^4(k-f_{\text{EM}})}{(n-1)(3n-4)r^{4n-4}}\alpha'\beta + O(\alpha^2, \beta^2), \quad (14)$$

where the metric function of Einstein-Maxwell gravity is

$$f_{\text{EM}} = k - \frac{2\Lambda r^2}{n(n-1)} - \frac{m}{r^{n-2}} + \frac{2q^2}{(n-1)(n-2)r^{2n-4}}. \quad (15)$$

In order to consider the asymptotic behavior of the solution, we put $m = q = 0$, where the metric function reduces to

$$f(r) = k + \frac{r^2}{2\alpha'} \left(1 - \sqrt{1 + \frac{8\alpha'\Lambda}{n(n-1)}} \right). \quad (16)$$

This result puts a restriction on α' which states that for $\alpha' \leq \frac{-n(n-1)}{8\Lambda}$, the metric function will be real asymptotically. On the other hand, for $\Lambda = 0$, asymptotically flat solutions are available only for $k = 1$.

The next step is to look for singularities. It is a matter of calculation to show that the Kretschmann scalar of metric (6) is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = f''^2 + 2(n-1)\frac{f'^2}{r^2} + 2(n-1)(n-2)\frac{f^2}{r^4}, \quad (17)$$

where its series expansion for small and large values of r will be

$$\lim_{r \rightarrow 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto r^{-4(n-1)}, \quad (18)$$

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{4(n+2)\Lambda}{n\alpha'} - \frac{\varsigma}{\alpha'} \left(1 + \frac{(n-1)(n+2)}{\alpha'} \right), \quad (19)$$

$$\varsigma = \sqrt{\frac{n-2}{n} + \frac{8\alpha'\Lambda}{n(n-1)}} - 1, \quad (20)$$

which, for small values of the GB parameter, will lead to

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{8(n+2)}{(n+2)n^2}\Lambda^2 - \frac{4}{n(n-1)}\Lambda + O(\alpha'). \quad (21)$$

Equations (18)–(21) show that there is an essential singularity located at $r = 0$ and that the asymptotic behavior of the solutions are AdS with an effective cosmological constant. If this solution contains the horizon, then our metric function is interpreted as a black hole. In order to investigate the existence of the horizon, one should find the root of $g^{rr} = f(r) = 0$. If we suppose that the metric function is positive for large and small r , by considering the possibility of the existence of only one extreme root $r_+ = r_{\text{ext}}$, we know that $f(r)$ has a minimum at $r = r_{\text{ext}}$. So we can investigate the roots of f' ,

$$k(n-1)(n-2)r_{\text{ext}}^{4n-6} \left[\frac{\alpha'(n-4)}{(n-2)r_{\text{ext}}^2} + 1 \right] + 4q^4\beta - 2r_{\text{ext}}^{2n-2}q^2 - 2r_{\text{ext}}^{4n-4}\Lambda + O(\beta^2) = 0, \quad (22)$$

where, for example, in the Ricci-flat case ($k = 0$), we obtain

$$r_{\text{ext}} = \frac{[q^2 \Lambda^{2n-3} (\pm \sqrt{(1 + 8\beta\Lambda) - 1})]^{\frac{1}{2n-2}}}{\Lambda} + O(\beta^2) \quad (23)$$

It is matter of calculation to show that the extremal mass for $k = 0$ is

$$m_{\text{ext}} = \frac{2q^2}{r_{\text{ext}}^{n-2}(n-1)(n-2)} - \frac{2r_{\text{ext}}^n \Lambda}{n(n-1)} - \frac{4q^4 \beta}{r_{\text{ext}}^{3n-4}(3n-4)(n-1)} + O(\beta^2). \quad (24)$$

We should note that the obtained solutions may be interpreted as black holes with inner and outer horizons provided $m > m_{\text{ext}}$, an extreme black hole for $m = m_{\text{ext}}$, and naked singularity otherwise. One can obtain the Hawking temperature by surface gravity definition with the following explicit form:

$$T = \frac{k(n-1)(n-2)r_+^{4n-6} \left(1 + \frac{(n-4)\alpha'}{(n-2)r_+^2}\right) - 2r_+^{4n-4}\Lambda - 2r_+^{2n-2}q^2 + 4q^4\beta}{4\pi(n-1)r_+^{4n-5} \left(1 + \frac{2k\alpha'}{r_+^2}\right)} + O(\beta^2). \quad (25)$$

It is worthwhile to mention that extremal black holes have zero temperature.

III. THERMODYNAMICS OF ASYMPTOTICALLY FLAT BLACK HOLES ($k = 1$ AND $\Lambda = 0$)

In this section we are interested in the thermodynamics of asymptotically flat solutions. Using the series expansion of the metric function (12) with $k = 1$ for large values of distance, we obtain

$$f(r) = 1 - \frac{m}{r^{n-2}} + \frac{2q^2}{(n-1)(n-2)r^{2n-4}} + O\left(\frac{1}{r^{2n-2}}\right), \quad (26)$$

which shows that the solutions are asymptotically flat.

According to area law, the entropy of black holes is one quarter of the horizon area. This relation is acceptable for Einstein gravity, whereas we are not allowed to use it for higher derivative gravity. For our GB case we can use the Wald formula for calculating the entropy of asymptotically flat black hole solutions:

$$S = \frac{1}{4} \int d^{n-1}x \sqrt{\gamma} (1 + 2\alpha\tilde{R}), \quad (27)$$

where \tilde{R} is the Ricci scalar for the induced metric γ_{ab} on the $(n-1)$ -dimensional boundary. Calculations show that one can obtain

$$S = \frac{V_{n-1}}{4} \left(1 + \frac{2(n-1)\alpha'}{(n-3)r_+^2}\right) r_+^{n-1}, \quad (28)$$

which confirms that asymptotically flat black holes violate the area law.

Considering the flux of the electric field at infinity, one can find the electric charge of black holes with the following form:

$$Q = \frac{q}{4\pi}. \quad (29)$$

Next, for the electric potential, Φ , we use the following definition:

$$\begin{aligned} \Phi &= A_{\mu}\chi^{\mu}|_{r \rightarrow \infty} - A_{\mu}\chi^{\mu}|_{r \rightarrow r_+} \\ &= \frac{q}{(n-2)r_+^{n-2}} \left(1 - \frac{4(n-2)q^2\beta}{(3n-4)r_+^{2n-2}}\right) + O(\beta^2), \end{aligned} \quad (30)$$

where χ^{μ} is the null generator of the horizon. It is notable that although the electric potential depends on the nonlinearity of electrodynamics, the electric charge does not.

In order to calculate the finite mass of the black hole, we use the ADM (Arnowitt-Deser-Misner) approach for large values of r , which will result in [14]

$$M = \frac{V_{n-1}}{16\pi} m(n-1). \quad (31)$$

We should note that one can obtain m from $f(r = r_+) = 0$, so the total mass depends on GB and NLED parameters.

Next, by using Eqs. (12), (28), and (29) and considering M as a function of the extensive parameters S and Q , we have

$$\begin{aligned} M(S, Q) &= \frac{r_+^{n-2}}{16\pi} \left[(n-1) \left(1 + \frac{\alpha'}{r_+^2}\right) + \frac{32\pi^2 Q^2}{(n-2)r_+^{2n-4}} \right. \\ &\quad \left. - \frac{1024\pi^4 Q^4 \beta}{(3n-4)r_+^{4n-6}} \right] + O(\beta^2). \end{aligned} \quad (32)$$

Now we calculate the temperature and electric potential as the intensive parameter with the following relation:

$$T = \left(\frac{\partial M}{\partial S} \right)_Q = \left(\frac{\partial M}{\partial r_+} \right)_Q / \left(\frac{\partial S}{\partial r_+} \right)_Q, \quad \Phi = \left(\frac{\partial M}{\partial Q} \right)_S, \quad (33)$$

which are the same as the ones that were calculated in Eqs. (25) and (30). Thus, the conserved and thermodynamic quantities satisfy the first law of thermodynamics,

$$dM = TdS + \Phi dQ. \quad (34)$$

$$\begin{aligned} \left(\frac{\partial^2 M}{\partial S^2} \right)_Q = & - \left\{ r_+^{4n-8} (n-1)(n-2) \left[\frac{2(n-4)\alpha'^2}{(n-2)} + \frac{(n-8)\alpha' r_+^2}{(n-2)} + r_+^4 \right] - 2r_+^{2n-2} q^2 [2\alpha'(2n-5) + (2n-3)r_+^2] \right. \\ & \left. + 2\beta q^4 [4\alpha'(4n-7) + (8n-10)r_+^2] \right\} / [\pi(n-1)^2 r_+^{5n-10} (2\alpha' + r_+^2)^3] + O(\beta^2). \end{aligned} \quad (35)$$

One may study the behavior of $\left(\frac{\partial^2 M}{\partial S^2} \right)_Q$ for small values of the GB parameter with the following form:

$$\begin{aligned} \left(\frac{\partial^2 M}{\partial S^2} \right)_Q = & - \frac{(n-1)(n-2)r_+^{4n-4} - 2(2n-3)r_+^{2n} q^2}{(n-1)^2 r_+^{5n-4} \pi} - \frac{4(4n-5)q^4}{(n-1)^2 r_+^{5n-6} \pi} \beta \\ & + \frac{[(5n-4)r_+^{4n} - 16r_+^{2n+4} q^2]}{(n-1)r_+^{5n+2} \pi} \alpha' + \frac{64q^4}{(n-1)r_+^{5n-4} \pi} \beta \alpha' + O(\alpha'^2, \beta^2). \end{aligned} \quad (36)$$

In addition, it is worthwhile to mention that for small and large values of r_+ , we obtain

$$\left(\frac{\partial^2 M}{\partial S^2} \right)_Q \Big|_{\text{Small } r_+} = \begin{cases} -\frac{(4n-7)\beta q^4}{\pi(n-1)^2 \alpha'^2 r_+^{5n-10}} < 0, & \alpha' \neq 0, \beta \neq 0 \\ -\frac{4(4n-5)\beta q^4}{\pi(n-1)^2 r_+^{5n-6}} < 0, & \alpha' = 0, \beta \neq 0 \\ \frac{(2n-5)q^2}{2\pi(n-1)^2 \alpha'^2 r_+^{n-8}} > 0, & \alpha' \neq 0, \beta = 0 \\ \frac{2(2n-3)q^2}{\pi(n-1)^2 r_+^{3n-4}} > 0, & \alpha' = 0, \beta = 0 \end{cases}, \quad (37)$$

$$\left(\frac{\partial^2 M}{\partial S^2} \right)_Q \Big|_{\text{Large } r_+} = -\frac{n-2}{\pi(n-1)r_+^n} < 0, \quad (38)$$

which indicate that for nonzero α' and β , asymptotically flat black holes with a large or small horizon radius are not stable. In other words, there is an upper limit, $r_{+\text{max}}$, as well as a lower limit, $r_{+\text{min}}$, for the asymptotically flat stable black holes ($r_{+\text{min}} < r_+ < r_{+\text{max}}$). Although this result does not depend on the value of the GB parameter, asymptotically flat black holes with a small horizon radius may be stable in the presence of a pure Maxwell field (with

A. Stability of the solutions

Here, we investigate the thermodynamic stability of charged black hole solutions of GB gravity with NLED. In the grand canonical ensemble, the thermodynamic stability may be carried out by calculating the determinant of the Hessian matrix of $M(S, Q)$ with respect to its extensive variables X_i , $\mathbf{H}_{X_i X_j}^M = \left(\frac{\partial^2 M}{\partial X_i \partial X_j} \right)$. In the canonical ensemble, the electric charge is a fixed parameter, and therefore the positivity of the heat capacity $C_Q = T_+ / \left(\frac{\partial^2 M}{\partial S^2} \right)_Q$ is sufficient to ensure the local stability. Since the physical black hole solutions have positive temperature, it is sufficient to check the positivity of $\left(\frac{\partial^2 M}{\partial S^2} \right)_Q = \left(\frac{\partial T}{\partial r_+} \right)_Q / \left(\frac{\partial S}{\partial r_+} \right)_Q$,

an absence of nonlinearity of electrodynamic correction). In other words, taking into account thermal stability in the canonical ensemble, the nonlinearity of electrostatics has a reasonable influence on a small horizon radius.

In order to investigate the effects of GB gravity and NLED on the local stability for arbitrary r_+ , we plot Figs. 1–6. Here, we consider left panels which are appropriate for the canonical ensemble analysis.

In Figs. 1 and 2 we investigate the effects of considering GB gravity. One can find that when the GB parameter is less than a critical value ($\alpha' < \alpha'_c$), T_+ has a real positive root at r_0 . In this case, $\left(\frac{\partial^2 M}{\partial S^2} \right)_Q > 0$ for $r_0 < r_+ < r_{+\text{max}}$. In other words, small black holes are not physical and large black holes are not stable. Moreover, we find that r_0 ($r_{+\text{max}}$) decrease (increase) when we increase α' . In Fig. 2, we consider $\alpha' > \alpha'_c$, in which T_+ is a positive definite function of r_+ . In this case one can obtain stable solutions for $r_{+\text{min}} < r_+ < r_{+\text{max}}$, which confirms that small and large black holes are unstable. We should note that increasing α' leads to increasing (decreasing) the value of $r_{+\text{min}}$ ($r_{+\text{max}}$) and, therefore, decreases the range of stability.

Besides, we plot Figs. 3 and 4 to analyze the effects of NLED. We find that for the fixed values of q , n , and α' , there

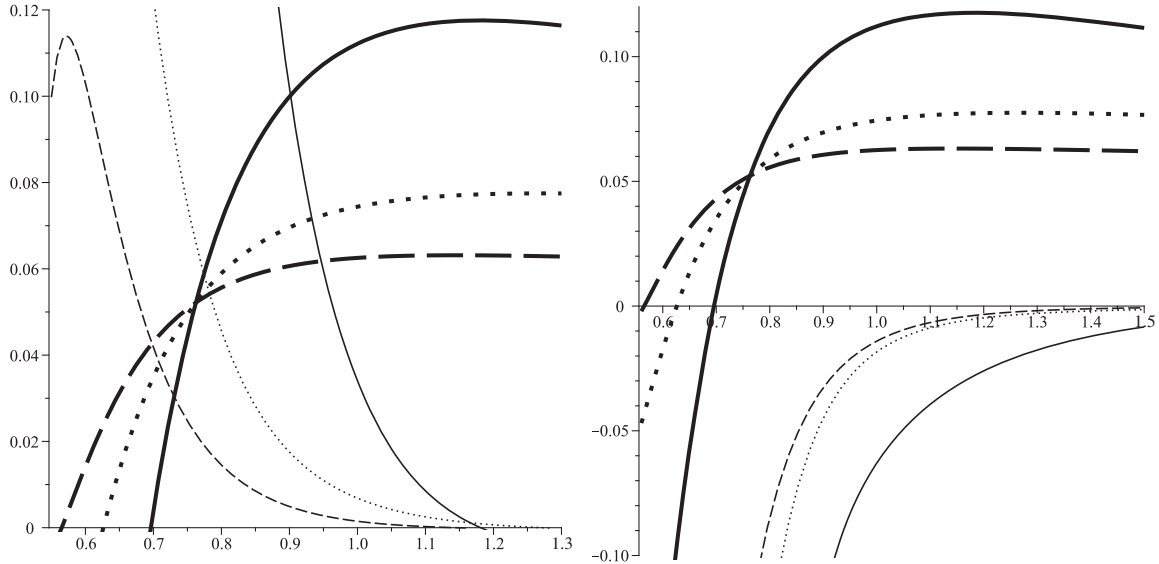


FIG. 1. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial s^2})_Q$ (left panel) and $|H_{S,Q}^M|$ (right panel) versus r_+ for $n = 5, q = 1, \beta = 0.001$, and $\alpha = 0.1$ (solid line), $\alpha = 0.3$ (dotted line), and $\alpha = 0.5$ (dashed line). Bold lines represent the temperature.

is a critical value for the nonlinearity parameter, β_c , in which for $\beta < \beta_c$, the temperature is positive for $r_+ > r_0$. This case is the same as that in Fig. 1 with $\alpha' < \alpha'_c$, and we may obtain asymptotically flat stable black holes for $r_0 < r_+ < r_{+max}$. In Fig. 4 we set $\beta > \beta_c$ to investigate the stability condition for positive definite temperature. One finds there are lower and upper limits for the horizon radius of stable black holes. It means that for $\beta > \beta_c$, asymptotically flat black holes are stable when $r_{+min} < r_+ < r_{+max}$.

We study the effects of electric charge in Figs. 5 and 6. These figures show that for fixed values of α' and β , there is a critical value for the electric charge, q_c , in which the temperature is positive definite for $q < q_c$. When $q > q_c$,

there is a real positive root (r_0) for the temperature which is an increasing function of q . We should note that the general behaviors of Figs. 5 and 6 are, respectively, the same as those in Figs. 1 and 2. In other words, asymptotically flat black holes are stable for $r_{+min} < r_+ < r_{+max}$ and one should replace r_{+min} with r_0 for $q > q_c$. It is worthwhile to mention that both r_{+min} and r_{+max} are increasing functions of q .

Finally, considering Figs. 1–6, we should note that, in canonical ensemble, asymptotically flat black holes are stable for $r_{+min} < r_+ < r_{+max}$, in which one must replace r_{+min} with r_0 when T_+ has a real positive root ($q > q_c$ or $\beta > \beta_c$ or $\alpha' < \alpha'_c$).

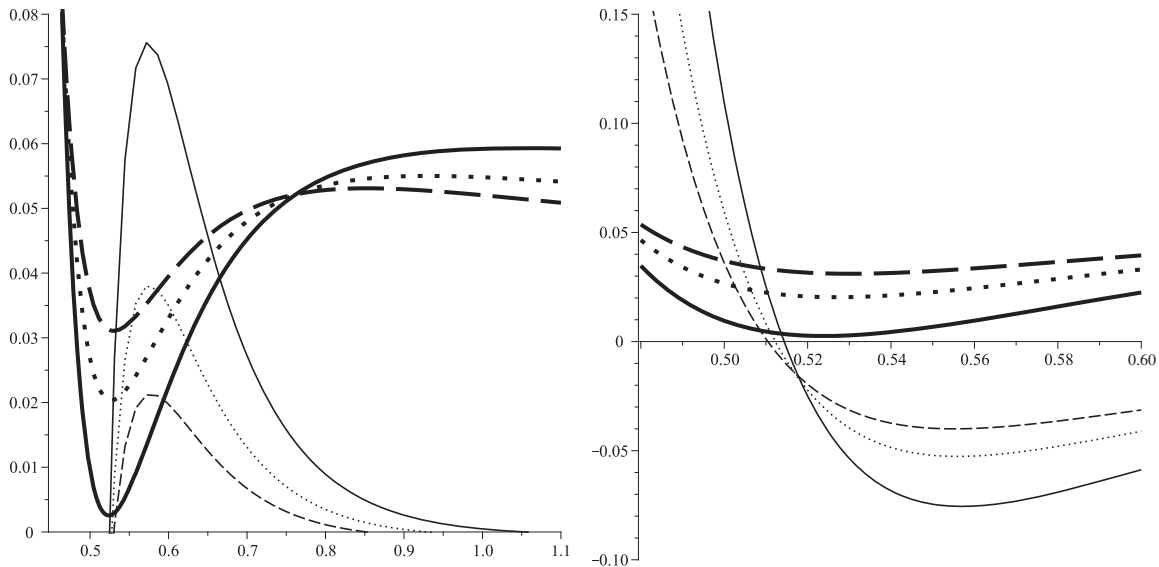


FIG. 2. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial s^2})_Q$ (left panel) and $|\frac{H_{S,Q}^M}{10}|$ (right panel) versus r_+ for $n = 5, q = 1, \beta = 0.001$, and $\alpha = 0.6$ (solid line), $\alpha = 0.8$ (dotted line), and $\alpha = 0.1$ (dashed line). Bold lines represent the temperature.

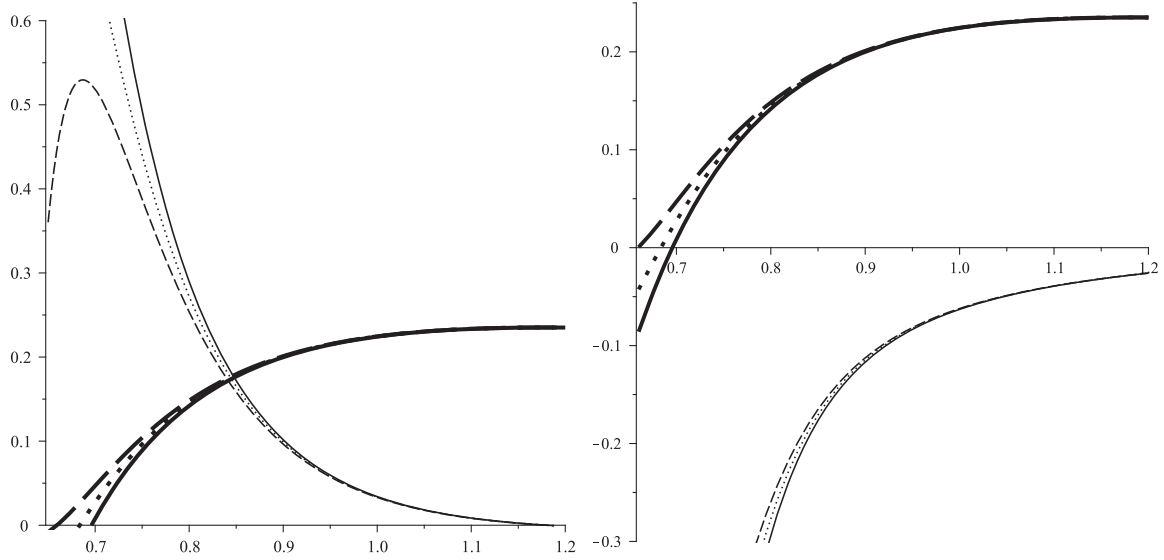


FIG. 3. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial s^2})_Q$ (left panel) and $|\mathbf{H}_{S,Q}^M|$ (right panel) versus r_+ for $n = 5$, $q = 1$, $\alpha = 0.1$, and $\beta = 0.001$ (solid line), $\beta = 0.003$ (dotted line), and $\beta = 0.005$ (dashed line). Bold lines represent the temperature.

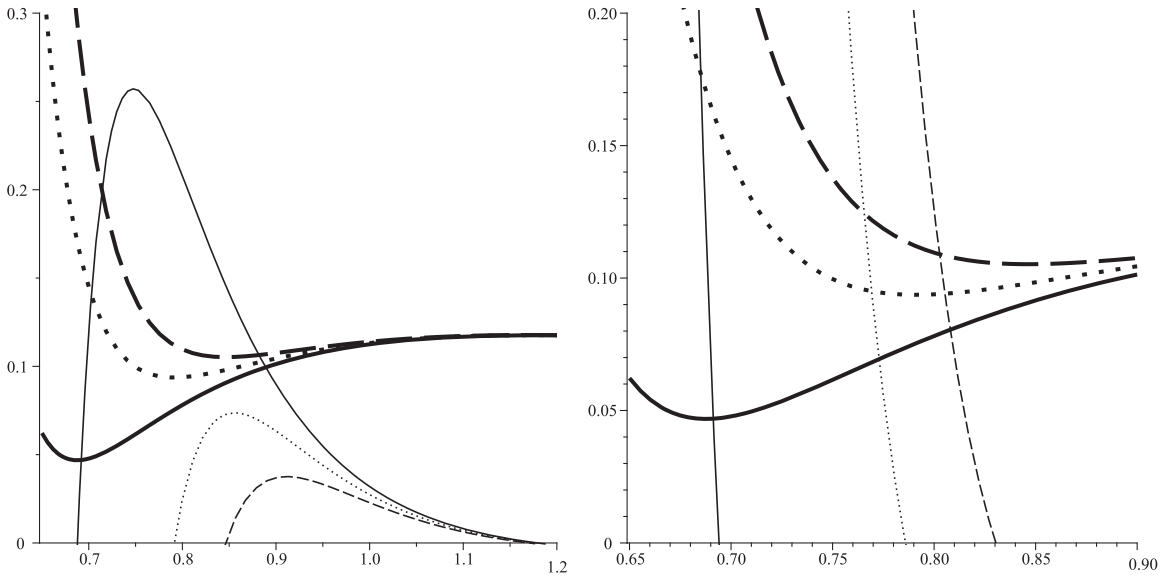


FIG. 4. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial s^2})_Q$ (left panel) and $|\mathbf{H}_{S,Q}^M|$ (right panel) versus r_+ for $n = 5$, $q = 1$, $\alpha = 0.1$, and $\beta = 0.01$ (solid line), $\beta = 0.03$ (dotted line), and $\beta = 0.05$ (dashed line). Bold lines represent the temperature.

Next, in the grand canonical ensemble, we calculate the determinant of the Hessian matrix of the asymptotically flat black holes. After some algebraic manipulation, one obtains

$$\begin{aligned}
 |\mathbf{H}_{S,Q}^M| = & 4 \left\{ 2q^2(3n-4)r_+^{4n-4}[r_+^2 - 2\alpha'] + 4q^4r_+^{2n-2}[2\alpha'(11n-24) - (7n-8)r_+^2] \right. \\
 & + 12q^2(n-1)(n-2)^2r_+^{4n-8} \left[\frac{(n-8)\alpha'r_+^2}{(n-2)} + \frac{2(n-4)\alpha'^2}{(n-2)} + r_+^4 \right] \beta \\
 & \left. - (n-1)(n-2)(3n-4)r_+^{6n-10} \left[\frac{(n-8)\alpha'r_+^2}{(n-2)} + \frac{2(n-4)\alpha'^2}{(n-2)} + r_+^4 \right] \right\} / [r_+^{8n-14}] \\
 & \times (r_+^2 + 2\alpha')^3(n-1)^2(3n-4)(n-2) + O(\beta^2). \tag{39}
 \end{aligned}$$

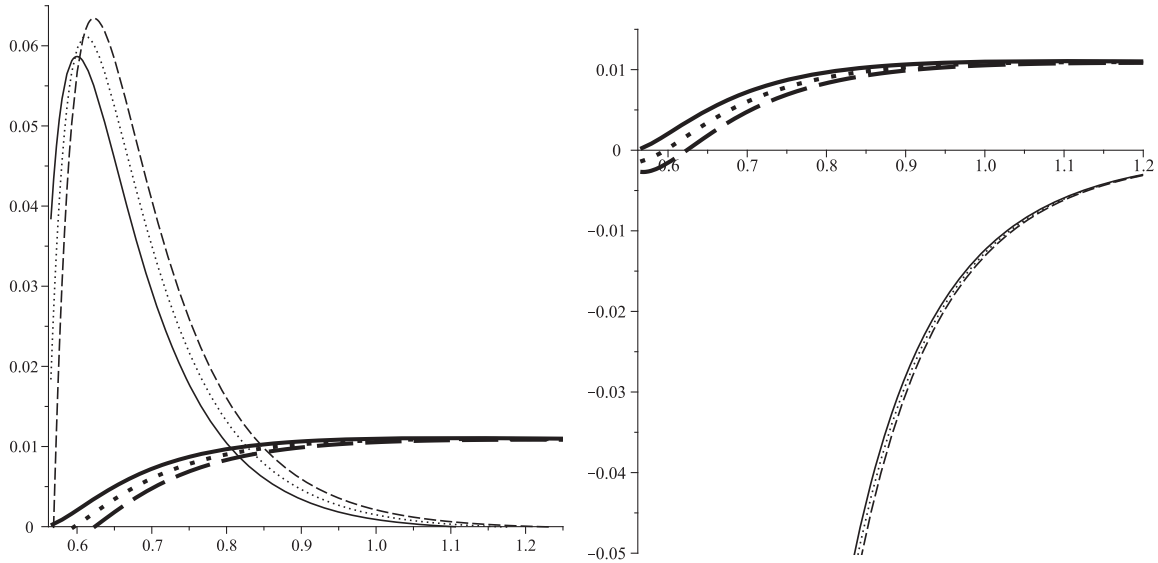


FIG. 5. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial S^2})_Q$ (left panel) and $|\mathbf{H}_{S,Q}^M|$ (right panel) versus r_+ for $n = 5, \beta = 0.001, \alpha = 0.7$, and $q = 1.2$ (solid line), $q = 1.3$ (dotted line), and $q = 1.4$ (dashed line). Bold lines represent the temperature.

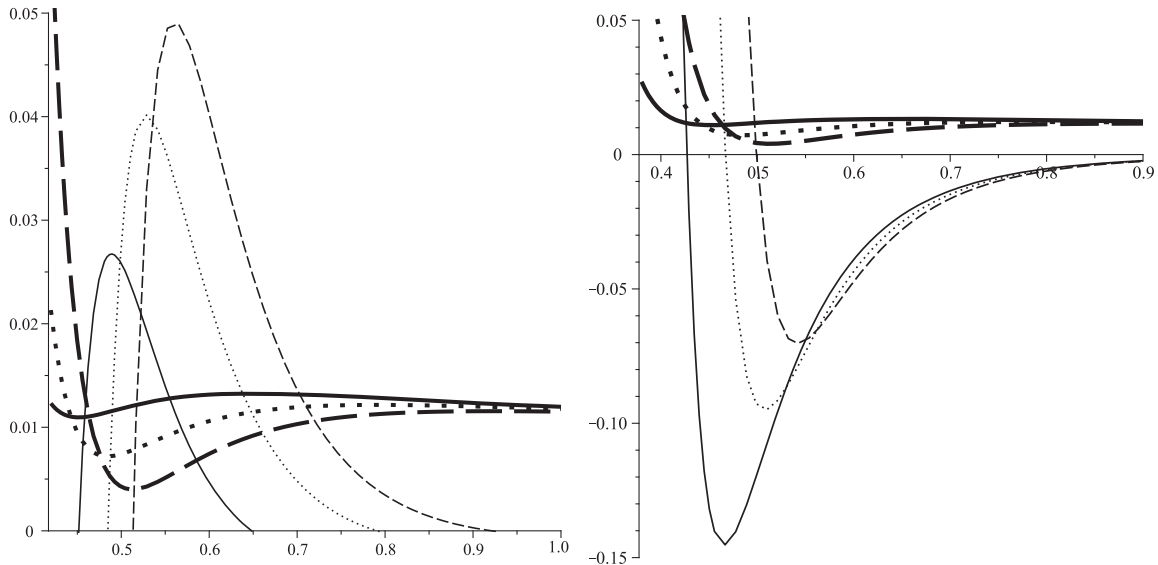


FIG. 6. Asymptotically flat solutions $(\frac{\partial^2 M}{\partial S^2})_Q$ (left panel) and $|\mathbf{H}_{S,Q}^M|$ (right panel) versus r_+ for $n = 5, \beta = 0.001, \alpha = 0.7$, and $q = 0.5$ (solid line), $q = 0.7$ (dotted line), and $q = 0.9$ (dashed line). Bold lines represent the temperature.

In addition, it is worthwhile to mention that for small and large values of r_+ , we obtain

$$|\mathbf{H}_{S,Q}^M|_{\text{Small } r_+} = \begin{cases} \frac{4(11n-24)\beta q^4}{\pi(n-1)^2(3n-4)\alpha^2 r_+^{6n-12}} > 0, & \alpha' \neq 0, \beta \neq 0 \\ -\frac{16(7n-8)\beta q^4}{r_+^{6n-8}(n-1)^2(n-2)(3n-4)} < 0, & \alpha' = 0, \beta \neq 0 \\ -\frac{2q^2}{(n-1)^2(n-2)\alpha^2 r_+^{4n-10}} < 0, & \alpha' \neq 0, \beta = 0 \\ \frac{8q^2}{(n-1)^2(n-2)r_+^{4n-6}} > 0, & \alpha' = 0, \beta = 0 \end{cases}, \quad (40)$$

$$|\mathbf{H}_{S,Q}^M|_{\text{Large } r_+} = -\frac{4}{\pi(n-1)r_+^n} < 0, \quad (41)$$

which indicate that, for nonvanishing α' and β , asymptotically flat black holes with a large horizon radius are not stable. In other words, there is an upper limit, $r_{+\text{max}}$, for the asymptotically flat stable black holes ($r_+ < r_{+\text{max}}$). In the absence of the GB gravity case ($\alpha' = 0, \beta \neq 0$), there is a lower limit for stable solutions. This means that for $\alpha' = 0$, the results of the stability conditions for the canonical and grand canonical ensembles are identical and both

ensembles show that small and large black holes are unstable. We should note that, in the presence of GB gravity, thermal stability is ensemble dependent. These results are shown in Figs. 1–6.

Now we need to discuss ensemble dependency [15]. As we know, in the canonical ensemble the internal energy is allowed to fluctuate with fixed electric charge and, therefore, the black hole and the heat reservoir remain in thermal equilibrium with a certain temperature. While in the grand canonical ensemble, the black hole is in both thermal and electrical equilibrium, with its reservoir held at a constant temperature and a constant potential. In other words, different boundary conditions lead to different ensembles.

In the usual discussions of the stability criterion of black holes, one expects ensemble independence of the system. Indeed, different ensembles which imply different boundary conditions should lead to similar stability conditions in the usual thermodynamical systems.

Ensemble dependency of a system may come from two subjects. One of them is real ensemble dependency, which may occasionally occur in some thermodynamical systems. The existence of ensemble dependency was seen in the usual thermodynamical systems [16]. Another one, which is not real, can be removed by improving our thermodynamic viewpoint. In other words, our lack of knowledge may lead to ensemble dependency and we should improve our thermodynamical understanding to obtain ensemble independency. In our black hole model (with a spherical horizon), we find that the nonlinearity of electrodynamics results in the same changes for the behavior of both canonical and grand canonical ensembles. One can see that the ensemble dependency comes from the contribution of Gauss-Bonnet gravity. This means that, in the absence of the GB parameter, both ensembles have the same stability conditions. This shows that the GB parameter makes more of a contribution to the thermodynamical behavior of the black hole systems. In other words, one may consider the GB parameter not only as a fixed parameter, but also as a thermodynamical parameter [17]. By doing so, the Hessian matrix for this system will be modified and, therefore, we may expect to see that this modification solves the ensemble dependency of the black hole system.

Another way to solve ensemble dependency is through considering the fact that thermodynamical systems described by a thermodynamical potential must be invariant under the Legendre transformation. To do so, one can use the method that was introduced in [18] and can use geometrothermodynamics to solve ensemble dependency of the system [18,19].

IV. ASYMPTOTICALLY AdS SPINNING BLACK BRANES ($k = 0$ & $\Lambda \neq 0$)

In this section, we will investigate rotating AdS spacetime. In order to investigate the asymptotic behavior of the solutions for $k = 0$ and $\Lambda \neq 0$, one can use the series

expansion of the metric function for a large value of r . We obtain

$$f(r) = -\frac{2\Lambda_{\text{eff}}}{n(n-1)}r^2 - \frac{m}{\left(1 + \frac{4\alpha'\Lambda_{\text{eff}}}{n(n-1)}\right)r^{n-2}} + \frac{2q^2}{(n-1)(n-2)\left(1 + \frac{4\alpha'\Lambda_{\text{eff}}}{n(n-1)}\right)r^{2n-4}} + O\left(\frac{1}{r^{2n-2}}\right), \quad (42)$$

where

$$\Lambda_{\text{eff}} = \frac{n(n-1)\left(\sqrt{1 + \frac{8\alpha'\Lambda}{n(n-1)}} - 1\right)}{4\alpha'}. \quad (43)$$

Equation (42) shows that for $k = 0$ the solutions are asymptotically AdS with an effective cosmological constant Λ_{eff} . When constructing a rotating spacetime, one can apply the following transformation in the static spacetime with $k = 0$:

$$\begin{aligned} t &\rightarrow \Xi t - a_i \phi_i, \\ \phi &\rightarrow \Xi \phi_i - \frac{a_i}{l} t. \end{aligned} \quad (44)$$

Using this transformation, the metric of $(n+1)$ -dimensional asymptotically AdS rotating spacetime with p rotation parameters is

$$\begin{aligned} ds^2 &= -f(r)\left(\Xi dt - \sum_{i=1}^p a_i d\phi_i\right)^2 + \frac{dr^2}{f(r)} \\ &+ \frac{r^2}{l^4} \sum_{i=1}^p (a_i dt - \Xi l^2 d\phi_i)^2 \\ &- \frac{r^2}{l^2} \sum_{i<j}^p (a_i d\phi_i - a_j d\phi_j)^2 + r^2 dX^2, \end{aligned} \quad (45)$$

where $\Xi = \sqrt{1 + \sum_{i=1}^p \frac{a_i^2}{l^2}}$ and dX^2 is the Euclidean metric on the $(n-1-p)$ -dimensional submanifold. The fourth term in Eq. (45) comes from the fact that, for a rotating spacetime with more than one rotating parameter, we should consider cross terms associated with rotating coordinates. The rotation group in $(n+1)$ dimensions is $SO(n)$ and, therefore, $p \leq [n/2]$. Considering the aforementioned transformation, for the gauge potential, we should write

$$A_\mu = h(r)(\Xi \delta_\mu^0 - a_i \delta_\mu^i) \quad (\text{no sum on } i). \quad (46)$$

Calculations show that metric function (12), with $k = 0$ and $h(r) = \int F_{tr} dr$ [using the F_{tr} calculated in Eq. (9)], satisfies all of the field equations for the aforementioned rotating AdS spacetime. Straightforward calculations

confirm that there is a curvature singularity located at $r = 0$ which is covered with the horizon(s).

Using the analytic continuation of the metric by setting $t \rightarrow i\tau$ and $a_i \rightarrow ia_i$ and regularity at the event horizon, r_+ , helps us to obtain the Hawking temperature and the angular velocities of the black branes:

$$T_+ = \frac{f'(r_+)}{4\pi\Xi} = -\frac{r_+^{4n-4}\Lambda + r_+^{2n-2}q^2 - 2q^4\beta}{2\pi\Xi(n-1)r_+^{4n-5}} + O(\beta^2), \quad (47)$$

$$\Omega_i = \frac{a_i}{\Xi l^2}. \quad (48)$$

Equation (47) shows that, unlike black hole solutions with a spherical horizon, the temperature of the aforementioned black brane with a flat horizon does not depend on GB gravity.

Next, we calculate the electric charge and potential of the solutions. The electric charge per unit volume V_{n-1} can be found by calculating the flux of the electric field at infinity, yielding

$$Q = \frac{q\Xi}{4\pi}. \quad (49)$$

On the other hand, because of the applied transformation and changes in the metric, we now have Killing vectors in the form of $\chi = \partial_t + \sum_{i=1}^p \Omega_i \partial_{\phi_i}$ which is the null generator of the horizon. The electric potential is obtained as

$$\begin{aligned} \Phi &= A_{\mu}\chi^{\mu}|_{r \rightarrow \infty} - A_{\mu}\chi^{\mu}|_{r \rightarrow r_+} \\ &= \frac{q}{\Xi(n-2)r_+^{n-2}} \left(1 - \frac{4(n-2)q^2\beta}{(3n-4)r_+^{2n-2}} \right) + O(\beta^2). \end{aligned} \quad (50)$$

Now we desire to calculate the entropy, the angular momentum, and the finite mass to check the first law of thermodynamics. In general, the action and the conserved quantities of the spacetime diverge when evaluated on the solutions. In order to overcome this problem and due to the fact that our spacetime is asymptotically AdS, we can use the counterterm method to calculate the finite action and the conserved quantities. One can show that for the obtained solutions with flat boundary $R_{abcd}(\gamma) = 0$, the finite action is

$$I_{\text{finite}} = I_G + I_b + I_{ct}, \quad (51)$$

where

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} L_{\text{tot}} \quad (52)$$

$$I_b = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \{K + 2\alpha(J - 2\hat{G}_{ab}K^{ab})\} \quad (53)$$

$$I_{ct} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \left(\frac{n-1}{l_{\text{eff}}} \right), \quad (54)$$

where γ and K are, respectively, the trace of the induced metric, γ_{ab} , and the extrinsic curvature, K^{ab} , on the boundary $\partial\mathcal{M}$, \hat{G}_{ab} is the Einstein tensor calculated on the boundary, J is the trace of

$$J_{ab} = \frac{1}{3} (K_{cd}K^{cd}K_{ab} + 2KK_{ac}K_b^c - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}), \quad (55)$$

and l_{eff} is a scale length factor that depends on l and α , which must reduce to l as α vanishes. It is worthwhile to mention that, for a spacetime with zero curvature boundary, I_{ct} has exactly the same value as that of the Einstein gravity in which l is replaced by l_{eff} . Using the Brown-York method of a quasilocal definition with Eq. (51)–(54), one can introduce a divergence-free stress-energy tensor as follows:

$$T^{ab} = \frac{1}{8\pi} \left[(K^{ab} - K\gamma^{ab}) + 2\alpha(3J^{ab} - J\gamma^{ab}) + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right]. \quad (56)$$

Then the quasilocal conserved quantities associated with the stress tensors of Eq. (56) can be written as

$$Q(\xi) = \int_B d^{n-1}\varphi \sqrt{\gamma} T_{ab} n^a \xi^b, \quad (57)$$

where n^a is the timelike unit normal vector to the boundary B . Taking into account Eq. (57), with $\xi = \partial/\partial t$ as a Killing vector, one can calculate the mass per unit volume V_{n-1} as

$$M = \frac{1}{16\pi} m(n\Xi^2 - 1), \quad (58)$$

where

$$m = m(r = r_+). \quad (59)$$

Equations (58) and (59) indicate that, unlike the asymptotic flat solutions with a spherical boundary, the finite mass does not depend on the GB parameter for the boundary flat rotating solutions.

Considering another Killing vector $\zeta = \partial/\partial\phi_i$ of rotating spacetime which is related to angular momentum, one can calculate the angular momentum per unit volume,

$$J_i = \frac{1}{16\pi} nm\Xi a_i. \quad (60)$$

We should note that for static solutions ($a_i = 0$), the angular momentum vanishes, which confirms that the a_i 's are the rotational parameters of the spacetime.

As we mentioned before, the Wald formula can be applied to asymptotically flat spacetime. Here, we encounter asymptotically AdS solutions, so we calculate the entropy through the use of the Gibbs-Duhem relation,

$$S = \frac{1}{T} \left(M - Q\Phi - \sum_{i=1}^k \Omega_i J_i \right) - I_{\text{finite}}. \quad (61)$$

First we calculate the finite action I_{finite} for the rotating metric. We find that the finite action per unit volume may be written as

$$I_{\text{finite}} = \frac{1}{8\pi n(n-1)T_+} \left(-\Lambda r_+^n + \frac{nq^2}{(n-2)r_+^{n-2}} - \frac{2nq^4\beta}{(3n-4)r_+^{3n-4}} + O(\beta^2) \right). \quad (62)$$

Using the finite conserved and thermodynamic quantities with the finite action, we obtain

$$S = \frac{\Xi}{4} r_+^{n-1}, \quad (63)$$

which confirms that the entropy obeys the area law for asymptotically AdS black branes with zero curvature horizon.

Now we want to check the first law of thermodynamics. To do so, we obtain the mass as a function of the extensive quantities S , J , and Q . Using the expression for the electric charge, the mass, the angular momentum, and the entropy given, respectively, in Eqs. (49), (58), (60), and (63) and the fact that $f(r = r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S, J, Q) = \frac{(nZ-1)J}{nl\sqrt{Z(Z-1)}}, \quad (64)$$

where $J = |\mathbf{J}| = \sqrt{\sum_{i=1}^p J_i^2}$ and $Z = \Xi^2$ is the positive real root of the following equation

$$2^{\frac{2}{n-1}} nl \sqrt{Z(Z-1)} \left(\pi^4 Q^4 \beta - \frac{(3n-4)S^2 \pi^2 Q^2}{2(n-2)} + \frac{(3n-4)\Lambda S^4}{2n} \right) + \left(\frac{S}{\sqrt{Z}} \right)^{\frac{n-2}{n-1}} S^2 Z \pi (n-1)(3n-4)J = 0. \quad (65)$$

Taking into account S , J , and Q as the extensive parameters of M , we can define the intensive parameters conjugate to them as

$$T = \left(\frac{\partial M}{\partial S} \right)_{J, Q}, \quad \Omega_i = \left(\frac{\partial M}{\partial J_i} \right)_{S, Q}, \quad \Phi = \left(\frac{\partial M}{\partial Q} \right)_{S, J}. \quad (66)$$

It is a matter of calculation to show that the intensive quantities calculated by Eq. (66) coincide with Eqs. (47), (48), and (50). Therefore, the first law of thermodynamics is satisfied:

$$dM = TdS + \sum_i^p \Omega_i dJ_i + \Phi dQ. \quad (67)$$

A. Stability of the solutions

Now we are in a position to calculate the heat capacity and the Hessian matrix to check the local stability of these solutions in context of canonical and grand canonical ensembles. For canonical ensembles, where electric charge and angular momenta are fixed parameters, the positivity of $(\frac{\partial^2 M}{\partial S^2})_{J, Q}$ is sufficient to ensure the local stability. Therefore, we obtain

$$\left(\frac{\partial^2 M}{\partial S^2} \right)_{J, Q} = \frac{2r_+^{-2}A_1}{\pi\Xi^2(n-1)\psi\sigma} - \frac{4(n-2)^2q^4r_+^{-4}A_2\beta}{\pi\Xi^2(n-1)(3n-4)\sigma\psi^2} + O(\beta^2), \quad (68)$$

$$A_1 = \frac{n(3\sigma-n)q^4}{r_+^{3n-6}} + \frac{2(3\sigma-n^2)\Lambda q^2}{r_+^{n-4}} + \frac{[(n+2)\sigma-n^2]\Lambda^2}{r_+^{2-n}}, \quad (69)$$

$$A_2 = \frac{[(5n^2-42n+40)\Xi^2 + (7n^2+11n-20)]\Lambda^2}{r_+^{n-6}} - \frac{2n[(13n^2-50n+40)\Xi^2 - (n^2-19n+20)]\Lambda q^2}{(n-2)r_+^{3n-8}} + \frac{n^2[(17n-20)\sigma - n(5n-6)]q^4}{(n-2)^2r_+^{5n-10}}, \quad (70)$$

$$\psi = [nq^2 - \Lambda(n-2)r_+^{2n-2}], \quad (71)$$

$$\sigma = [(n-2)\Xi^2 + 1]. \quad (72)$$

As we mentioned before, in the grand canonical ensemble, the positivity of the determinant of the Hessian matrix of $M(S, Q, J)$ with respect to its extensive variables X_i , $\mathbf{H}_{X_i X_j}^M = (\frac{\partial^2 M}{\partial X_i \partial X_j})$ is sufficient to ensure the local stability. It is a matter of calculation to show that the determinant of $\mathbf{H}_{S, Q, J}^M$ is

$$|\mathbf{H}_{A, Q, J}^M| = \kappa(q^2 - \Lambda r_+^{2n-2}) - \frac{4\kappa q^2 [3(n-2)^2 \Lambda^2 r_+^{4n-4} + 3n(n-1)q^4 - 2(n-2)(3n-2)\Lambda q^2 r_+^{2n-2}] \beta}{(3n-4)[nq^2 - (n-2)\Lambda r_+^{2n-2}] r_+^{2n-2}} + O(\beta^2), \quad (73)$$

where

$$x = \frac{64\pi r_+^{4-3n} [nq^2 - (n-2)\Lambda r_+^{2n-2}]^{-1}}{[(n-2)\Xi^2 + 1]\Xi^6 l^2}. \quad (74)$$

Regarding Eq. (47) with the mentioned $(\frac{\partial^2 M}{\partial S^2})_{J,Q}$ and also $|\mathbf{H}_{A,Q,J}^M|$, we find that, unlike the asymptotically flat case

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_{J,Q} \Big|_{\text{Small } r_+} = \begin{cases} -\frac{4[(5n^2-23n+20)(\Xi^2-1)+(12n^2-31n+20)\Xi^2]q^4\beta}{\pi\Xi^2(n-1)(3n-4)\sigma r_+^{5n-6}} < 0, & \beta \neq 0 \\ \frac{2\{3[(n-2)\Xi^2+1]-n\}q^2}{\pi\Xi^2(n-1)[(n-2)\Xi^2+1]r_+^{3n-4}} > 0, & \beta = 0 \end{cases}, \quad (75)$$

$$\left(\frac{\partial^2 M}{\partial S^2}\right)_{J,Q} \Big|_{\text{Large } r_+} = -\frac{2[(n^2-4)(\Xi^2-1)+n-2]\Lambda}{\pi\Xi^2(n-1)(n-2)[(n-2)\Xi^2+1]r_+^{n+2}} > 0, \quad (76)$$

and

$$H_{S,Q,J}^M \Big|_{\text{Small } r_+} = \begin{cases} -\frac{768\pi(n-1)q^2\beta}{n(3n-4)[(n-2)\Xi^2+1]\Xi^6 l^2 r_+^{5n-6}} < 0, & \beta \neq 0 \\ \frac{64\pi}{n[(n-2)\Xi^2+1]\Xi^6 l^2 r_+^{3n-4}} > 0, & \beta = 0 \end{cases}, \quad (77)$$

$$H_{S,Q,J}^M \Big|_{\text{Large } r_+} = \frac{64\pi}{(n-2)l^2\Xi^6[(n-2)\Xi^2+1]r_+^{3n-4}} > 0. \quad (78)$$

Both ensembles confirm that, in the presence of NLED ($\beta \neq 0$), although the black branes with small r_+ are unstable, large black branes are stable. It is notable that the instability of the small black branes is due to the presence of the NLED and, in the absence of the non-linearity effect, small black branes are stable.

V. CLOSING REMARKS

In this paper, we regarded both the gravity and the electrodynamic string corrections of Einstein-Maxwell gravity to obtain black hole solutions with spherical, hyperbolic, and flat horizon topology.

At the first step, we focused on asymptotically flat solutions. We used the Wald formula, the Gauss law, and the ADM approach to calculate entropy, electric charge, and finite mass, respectively. We checked that the conserved and thermodynamic quantities satisfied the first law of thermodynamics. Then we investigated the thermodynamic stability of the solutions in both canonical and grand canonical ensembles. Taking into account the canonical ensemble, we found that for nonzero α' and β , asymptotically flat black holes with a large or small horizon radius are unstable. This means that, in canonical ensemble, asymptotically flat black holes are stable for $r_{+\min} < r_+ < r_{+\max}$, in which one must replace $r_{+\min}$ with r_0 (the largest root of T_+) when T_+ has a real positive root

with the spherical horizon, neither the heat capacity nor the determinant of the Hessian matrix depend on the GB parameter for asymptotically AdS rotating solutions with zero curvature horizon. Therefore, we expect to obtain the same results in both canonical and grand canonical ensembles. Following the method of the previous section and regardless of the values of n , q , Λ , Ξ , and α , one finds

($q > q_c$ or $\beta < \beta_c$ or $\alpha' < \alpha'_c$). Moreover, we found that different values of α , β , and q can change the values of $r_{+\min}$ and $r_{+\max}$.

Then, we investigated the stability conditions in the grand canonical ensemble. We showed that there is an upper limit, $r_{+\max}$, for the asymptotically flat stable black holes ($r_+ < r_{+\max}$). We found that, although for $\alpha' = 0$ the results of the stability conditions for canonical and grand canonical ensembles are identical, these ensembles have different consequences in the presence of GB gravity. In other words, we noted that, in the presence of GB gravity, thermal stability is ensemble dependent.

In the next section, we considered the Ricci flat solutions with an AdS asymptote and produced a rotating spacetime by using an improper local transformation. In addition, we calculated the conserved and thermodynamic quantities for asymptotically AdS black branes which satisfy the first law of thermodynamics. Considering the thermodynamic instability criterion in canonical and grand canonical ensembles, we found that neither the heat capacity nor the Hessian matrix depend on the GB parameter. Therefore, both ensembles have identical stability conditions. We found that, although the black branes with small r_+ are unstable, large black branes are stable (unlike asymptotically flat static large black holes, which are unstable). It is notable that the instability of small black branes (holes) is due to the

presence of NLED and, in the absence of the nonlinearity effect, small black branes (holes) are stable.

To conclude, we found that for the Ricci-flat solutions, we obtained the same conditions for stable black holes in both canonical and grand canonical ensembles. In other words, in this case, we took into account S , Q and J as the thermodynamical extensive parameters, correctly, and obtained the same results for both ensembles. It is easy to show that, for rotating Ricci-flat solutions, one may obtain ensemble dependency if one, imprecisely, considers S and Q as the set of extensive parameters (regards J as a dynamical parameter, not a thermodynamic one). As one can confirm, unlike Ricci-flat solutions, the GB parameter in the spherical horizon of GB black holes contributes to finite mass, temperature, and, consequently, heat capacity. In other words, the GB parameter contributions lead to ensemble dependency in spherical horizon black holes. This means that a GB generalization of Einstein gravity not

only affects gravitational properties, but also thermodynamic aspects of the spherically symmetric black holes.

Finally, we should note that the modifications of the thermodynamic instability criterion in the presence of GB gravity depend on the choice of the ensemble. One may consider the GB parameter as a thermodynamic variable to remove the ensemble dependency. In addition, it is worthwhile to mention that we can regard asymptotically AdS black holes with spherical topology to investigate $P - V$ criticality in the extended phase space of the solutions by calculating the Gibbs free energy. These works are under examination.

ACKNOWLEDGMENTS

We thank the Shiraz University Research Council. This work has been supported financially by the Research Institute for Astronomy and Astrophysics of Maragha, Iran.

-
- [1] M. Born and L. Infeld, *Proc. R. Soc. A* **143**, 410 (1934); **144**, 425 (1934).
- [2] D. L. Wiltshire, *Phys. Rev. D* **38**, 2445 (1988); M. Cataldo and A. Garcia, *Phys. Lett. B* **456**, 28 (1999); S. Fernando and D. Krug, *Gen. Relativ. Gravit.* **35**, 129 (2003); T. Tamaki, *J. Cosmol. Astropart. Phys.* **05** (2004) 004; M. Aiello, R. Ferraro, and G. Giribet, *Phys. Rev. D* **70**, 104014 (2004); T. K. Dey, *Phys. Lett. B* **595**, 484 (2004); R. G. Cai, D. W. Pang, and A. Wang, *Phys. Rev. D* **70**, 124034 (2004); M. H. Dehghani and H. R. Rastegar-Sedehi, *Phys. Rev. D* **74**, 124018 (2006); M. H. Dehghani, N. Alinejadi, and S. H. Hendi, *Phys. Rev. D* **77**, 104025 (2008); S. H. Hendi, *J. Math. Phys. (N.Y.)* **49**, 082501 (2008).
- [3] M. Hassaine and C. Martinez, *Phys. Rev. D* **75**, 027502 (2007); *Classical Quantum Gravity* **25**, 195023 (2008); M. Maeda, M. Hassaine, and C. Martinez, *Phys. Rev. D* **79**, 044012 (2009).
- [4] H. H. Soleng, *Phys. Rev. D* **52**, 6178 (1995).
- [5] S. H. Hendi, *J. High Energy Phys.* **03** (2012) 065.
- [6] S. H. Hendi, *Ann. Phys. (Amsterdam)* **333**, 282 (2013); S. H. Hendi and A. Sheykhi, *Phys. Rev. D* **88**, 044044 (2013).
- [7] G. Boillat, *J. Math. Phys. (N.Y.)* **11**, 941 (1970); **11**, 1482 (1970); G. W. Gibbons and D. A. Rasheed, *Nucl. Phys.* **B454**, 185 (1995); Z. Zhao, Q. Pan, S. Chen, and J. Jing, *Nucl. Phys.* **B871**, 98 (2013).
- [8] E. S. Fradkin and A. A. Tseytlin, *Phys. Lett.* **163B**, 123 (1985); J. Ambjorn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo, *Phys. Lett. B* **480**, 399 (2000); N. Seiberg and E. Witten, *J. High Energy Phys.* **09** (1999) 032; R. Matsaev, M. Rahmanov, and A. Tseytlin, *Phys. Lett. B* **193**, 207 (1987); E. Bergshoff, E. Sezgin, C. Pope, and P. Townsend, *Phys. Lett. B* **188**, 70 (1987); Y. Kats, L. Motl, and M. Padi, *J. High Energy Phys.* **12** (2007) 068; R. G. Cai, Z. Y. Nie, and Y. W. Sun, *Phys. Rev. D* **78**, 126007 (2008); D. Anninos and G. Pastras, *J. High Energy Phys.* **07** (2009) 030.
- [9] S. H. Hendi and M. Momennia (to be published).
- [10] S. H. Hendi, *Eur. Phys. J. C* **73**, 2634 (2013).
- [11] S. W. Hawking and D. N. Page, *Commun. Math. Phys.* **87**, 577 (1983).
- [12] D. Birmingham, *Classical Quantum Gravity* **16**, 1197 (1999).
- [13] M. H. Dehghani and S. H. Hendi, *Int. J. Mod. Phys. D* **16**, 1829 (2007); O. Miskovic and R. Olea, *Phys. Rev. D* **83**, 024011 (2011); **83**, 064017 (2011); S. H. Hendi and B. Eslam Panah, *Phys. Lett. B* **684**, 77 (2010); S. H. Hendi, S. Panahiyan, and E. Mahmoudi, *Eur. Phys. J. C* **74**, 3079 (2014).
- [14] L. Brewin, *Gen. Relativ. Gravit.* **39**, 521 (2007).
- [15] G. L. Comer, *Classical Quantum Gravity* **9**, 947 (1992).
- [16] D. Chaudhuri, *Phys. Rev. E* **75**, 021803 (2007); D. J. Searles and D. J. Evans, *J. Chem. Phys.* **113**, 3503 (2000).
- [17] D. Kastor, S. Ray, and J. Traschen, *Classical Quantum Gravity* **27**, 235014 (2010); R. G. Cai, L. M. Cao, L. Li, and R. Q. Yang, *J. High Energy Phys.* **09** (2013) 005; S. W. Wei and Y. X. Liu, *Phys. Rev. D* **90**, 044057 (2014).
- [18] H. Quevedo, M. N. Quevedo, A. Sanchez, and S. Taj, *Phys. Scr.* **89**, 084007 (2014).
- [19] J. X. Mo, X. X. Zeng, G. Q. Li, X. Jiang, and W. B. Liu, *J. High Energy Phys.* **10** (2013) 056.