AdS-plane wave and *pp*-wave solutions of generic gravity theories

Metin Gürses,[1,*] Tahsin Çağrı Şişman,[2,3,†] and Bayram Tekin[4,‡]

[1]*Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara, Turkey*
[2]*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*
[3]*Department of Astronautical Engineering, University of Turkish Aeronautical Association,
06790 Ankara, Turkey*
[4]*Department of Physics, Middle East Technical University, 06800 Ankara, Turkey*

We construct the anti–de Sitter-plane wave solutions of generic gravity theory built on the arbitrary powers of the Riemann tensor and its derivatives in analogy with the *pp*-wave solutions. In constructing the wave solutions of the generic theory, we show that the most general two-tensor built from the Riemann tensor and its derivatives can be written in terms of the traceless Ricci tensor. Quadratic gravity theory plays a major role; therefore, we revisit the wave solutions in this theory. As examples of our general formalism, we work out the six-dimensional conformal gravity and its nonconformal deformation as well as the tricritical gravity, the Lanczos-Lovelock theory, and string-generated cubic curvature theory.

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I. INTRODUCTION

At short distances, Einstein's gravity is expected to be replaced by a better-behaved effective theory with more powers of curvature and its derivatives which can be written in the most general form (with no matter fields) as

$$I = \int d^D x \sqrt{-g} f(g^{\alpha\beta}, R^{\mu}{}_{\nu\gamma\sigma}, \nabla_{\rho} R^{\mu}{}_{\nu\gamma\sigma}, \ldots,$$
$$(\nabla_{\rho_1} \nabla_{\rho_2} \ldots \nabla_{\rho_M}) R^{\mu}{}_{\nu\gamma\sigma}, \ldots). \tag{1}$$

Although we will give solutions to this theory, it is often more convenient to take the following power-series version:

$$I = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right.$$
$$+ \gamma (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2)$$
$$\left. + \sum_{n=3}^{\infty} C_n (\mathrm{Riem, Ric, R, \nabla Riem}, \ldots)^n \right\}, \tag{2}$$

where we have added a bare cosmological constant Λ_0 which—although not required at short distances—plays a major phenomenological role at long distances. We have separated the quadratic parts as they will play a role in the construction of solutions to the generic theory and we have also organized the third term in the quadratic curvature modifications into the Gauss-Bonnet form which is easier to handle as it gives second-order equations in the metric just like Einstein's theory. Note that the third line represents

*gurses@fen.bilkent.edu.tr
†tahsin.c.sisman@gmail.com
‡btekin@metu.edu.tr

all other possible contractions of the Riemann tensor and its derivatives which provide field equations that are beyond fourth order in the metric; for example, terms such as $R\Box R$ are also included in that summation. In a microscopic theory, such as string theory, the parameters $\alpha, \beta, \gamma, C_n, \Lambda_0, \kappa$ are expected to be computed and some of them obviously vanish due to constraints such as unitarity, supersymmetry, etc. Here, to stay as generic as possible and not focus too much on such constraints, we shall consider Eqs. (1) and (2) to be the *theory* and seek exact solutions for it. Of course we shall give some specific examples as noted in the Abstract. It should be mentioned that not all theories of the form (2) give healthy, stable theories when linearized about their vacua. For example, most theories yield higher time-derivative free theories that have the Ostrogradsky instability when small interactions are added. These considerations do not deter us from studying the most general action given by Eq. (1) or Eq. (2) since our theories include all possible viable theories as well as the instability-plagued ones. We know that the $f(R)$ gravity theories are free from the Ostrogradsky instability. In addition to this subclass of Eq. (2), whether one can obtain a theory that is free from the Ostrogradsky instability is still an open question.

Unlike the case of Einstein gravity (for which books containing exact solutions exist [1,2]), there are only a few solutions known for some variants or restricted versions of the theory (2); see for example Refs. [3–14]. In Ref. [15], we briefly sketched the proof that the anti–de Sitter (AdS) waves (both plane and spherical) that solve Einstein's gravity and the quadratic gravity also solve the generic theory (2), needless to say, with modified parameters. Here, we shall give a detailed proof for the AdS-plane wave case with a direct approach based on the proof that *pp*-wave solutions of Einstein's gravity and the quadratic gravity are solutions to the theory with $\Lambda_0 = 0$. As we shall see, having

a nonzero Λ_0 complicates the matter a great deal. (AdS-spherical waves require a separate approach which we shall come back to in another work.)

In this work, we will exclusively be interested in the exact solutions (not perturbative excitations) about the maximally symmetric vacua of the theory. Nevertheless, the fact that these exact solutions *linearize* the field equations just like the perturbative excitations, leads to the following remarkable consequence: these metrics can be used to test the unitarity of the underlying theory and to find the excitation masses and the degrees of freedom of the spin-2 sector. (There is an important caveat here: if the theory for these test metrics turns out to be nonunitary, then the theory is nonunitary. But, if the theory turns out to be unitary for these test metrics, then this does not mean that the theory is unitary; one still has to check the unitarity of the spin-0 sector.) In the examples that we shall study here, the procedure will be apparent.

AdS-plane waves [16,17] and AdS-spherical waves [18] of quadratic gravity theories played a central role in Ref. [15]. We shall study here the AdS-plane wave (sometimes called the Siklos metric [19]) given as

$$ds^2 = \frac{\ell^2}{z^2}(2dudv + d\vec{x} \cdot d\vec{x} + dz^2) + 2V(u, \vec{x}, z)du^2, \quad (3)$$

where u and v are null coordinates, $\vec{x} = (x^i)$ with $i = 1, \ldots, D - 3$, and ℓ is the AdS radius related to the effective cosmological constant as $\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}$. For this D-dimensional metric, the Ricci tensor can be computed to be

$$R_{\mu\nu} = -\frac{(D-1)}{\ell^2}g_{\mu\nu} + \rho\lambda_\mu\lambda_\nu, \quad (4)$$

where the vector is $\lambda_\mu = \delta_\mu^u$ and the scalar function is

$$\rho \equiv -\left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V, \quad (5)$$

with $\Box \equiv \nabla^\mu\nabla_\mu$ and ∇_μ is compatible with the full metric (3). For these spacetimes, as we showed in Ref. [15], the field equations of Eq. (2) reduce to

$$eg_{\mu\nu} + a_0 S_{\mu\nu} + a_1 \Box S_{\mu\nu} + \cdots + a_n \Box^n S_{\mu\nu} + \cdots = 0, \quad (6)$$

where $S_{\mu\nu}$ is the traceless Ricci tensor. Taking the trace gives $e = 0$, which determines the effective cosmological constant of the theory. $S_{\mu\nu} = 0$, which is the Einsteinian solution, naturally solves the full theory. In order for Eq. (3) to be a solution to the cosmological Einstein theory, V satisfies the $\rho = 0$ equation, namely

$$\left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V(u, \vec{x}, z) = 0, \quad (7)$$

whose solution is[1]

$$V(u, \vec{x}, z) = z^{\frac{D-5}{2}}[c_1 I_{\frac{D-1}{2}}(z\xi) + c_2 K_{\frac{D-1}{2}}(z\xi)]\sin(\vec{\xi} \cdot \vec{x} + c_3), \quad (8)$$

with I, K being the modified Bessel functions, $|\vec{\xi}| = \xi$ and the c_i's are arbitrary functions of the null coordinate u [16,20]. Further assuming $\xi = 0$, the solution becomes [21] (see also Ref. [20])

$$V(u, z) = c(u)z^{D-3}, \quad (9)$$

where we omit the other solution, that is $\frac{1}{z^2}$, since it can be added to the "background" AdS part which is the $V = 0$ case of the metric (3). This is all in the cosmological Einstein theory. But, observe that neither Eq. (8) nor Eq. (9) depend explicitly on the cosmological constant of the theory. The dependence of the metric on the cosmological constant is only in the "AdS background" part. This leads to the fact that these Einsteinian solutions remain intact in the most general theory (2) with the only adjustment being that the cosmological constant that appears in the AdS background part depends on the parameters of the full theory. As we shall show in Sec. VI, one can find the cosmological constant, which will be determined by nonderivative terms in the action (2), without going through the cumbersome task of finding the field equations.

Now, let us consider the same metric as a solution to quadratic gravity. AdS-plane wave solutions of quadratic gravity again solve the field equations of the full theory (6) which will be more apparent when the field equations are represented in the factorized form (84). In this case, the metric function V satisfies a more complicated fourth-order equation

$$\left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2} - M^2\right)$$
$$\times \left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V(u, \vec{x}, z) = 0, \quad (10)$$

where the "mass" parameter reads

$$M^2 \equiv -\frac{1}{\beta}\left(\frac{1}{\kappa} - \frac{2}{\ell^2}((D-1)(D\alpha + \beta) + (D-3)(D-4)\gamma)\right). \quad (11)$$

Assuming $M^2 \neq 0$, that is the nondegenerate case, the most general solution of Eq. (10) can be constructed from two second-order parts; one is the pure Einstein theory

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[1]Since the equation is linear in V, the most general solution will be a sum or an integral over the arbitrary parameter ξ if no further condition is given. As the most general solution is easy to write we do not depict it here.

$$\left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V_a(u,\vec{x},z) = 0, \qquad (12)$$

and the other is a "massive" version of the theory

$$\left(\Box + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2} - M^2\right)V_b(u,\vec{x},z) = 0, \quad (13)$$

with $V = V_a + V_b$. Since we already know V_a from Eq. (8), let us write V_b

$$V_b(u,\vec{x},z) = z^{\frac{D-5}{2}}[c_{b,1}I_{\nu_b}(z\xi_b) + c_{b,2}K_{\nu_b}(z\xi_b)]$$
$$\times \sin(\vec{\xi}_b \cdot \vec{x} + c_{b,3}), \qquad (14)$$

where $\nu_b = \frac{1}{2}\sqrt{(D-1)^2 + 4\ell^2 M^2}$ [16]. If, on the other hand, $M^2 = 0$, which also includes the critical gravity [22,23], the solution becomes highly complicated in the most general case $\xi \neq 0$ (this was given in the Appendix of Ref. [16] which we do not reproduce here). For the special case of $\xi = 0$, the solution is[2]

$$V(u,z) = c_{a,1}z^{D-3} + z^{\frac{D-5}{2}}(c_{b,1}z^{|\nu_b|} + c_{b,2}z^{-|\nu_b|}), \quad (15)$$

for $M^2 \neq 0$, and

$$V(u,z) = c_1 z^{D-3} + \frac{1}{D-1}\left(c_2 z^{D-3} - \frac{c_3}{z^2}\right)\ln\left(\frac{z}{\ell}\right), \quad (16)$$

for $M^2 = 0$. Note that all the $c_{a,i}$'s and $c_{b,i}$'s appearing in the solutions of the quadratic gravity are arbitrary functions of u.

It was announced in Ref. [15] that these AdS-plane wave solutions of Einstein gravity and the quadratic gravity also solve the most general theory defined by the action (2) with redefined parameters that are M^2 and ℓ^2. This work expounds upon the results of Ref. [15]. In doing this, we show that the pp-wave spacetimes in the Kerr-Schild form having the metric

$$ds^2 = 2dudv + d\vec{x}\cdot d\vec{x} + 2V(u,\vec{x})du^2, \qquad (17)$$

where $\vec{x} = (x^i)$ with $i = 1, ..., D-2$, and the AdS-plane wave spacetimes have analogous algebraic properties, and with these specific properties in both cases the highly complicated field equations of the generic gravity theory reduce to somewhat simpler equations that admit exact solutions as exemplified above. For the pp-wave

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[2]When $\nu_b = 0$, the Breitenlohner-Freedman (BF) bound [24] is saturated and the solution turns into a logarithmic one given in Ref. [16] as

$$V(u,z) = c_1 z^{D-3} + z^{\frac{D-5}{2}}\left[c_1 + c_2\ln\left(\frac{z}{\ell}\right)\right].$$

spacetimes (17), in complete analogy with Eq. (6), the field equations for the full theory (2) reduce to

$$a_0 R_{\mu\nu} + a_1\Box R_{\mu\nu} + \cdots + a_n\Box^n R_{\mu\nu} + \cdots = 0, \quad (18)$$

which is solved by the Einsteinian solution $R_{\mu\nu} = 0$. Once one considers plane waves, which are a subclass of pp-wave spacetimes with the metric

$$ds^2 = 2dudv + d\vec{x}\cdot d\vec{x} + h_{ij}(u)x^i x^j du^2, \qquad (19)$$

where $\vec{x} = (x^i)$ with $i = 1, ..., D-2$, and h_{ij} is symmetric and traceless, $R_{\mu\nu}$ vanishes and one has a solution of Eq. (18) for any h_{ij}. Thus, the plane-wave solutions of Einstein's gravity solve the generic theory [4]. The pp-wave metric (17) solves Einstein's gravity if the metric function V satisfies the Laplace equation for the $(D-2)$-dimensional space, and the fact that these solutions solve the generic gravity theory (18) was first shown in Ref. [6]. In addition, if vanishing scalar invariant spacetimes, of which Eq. (17) is a member, satisfy $\Box R_{\mu\nu} = 0$, the field equations of Eq. (18) again reduce to the Einsteinian ones [9]. In addition to these Einstein gravity-based considerations, as we shall show below by putting Eq. (18) in the factorized form (37), one can observe that the pp-wave solutions of quadratic curvature gravity which satisfy $(b_1\Box + b_0)R_{\mu\nu} = 0$ also solve the generic theory. Note that one can extend these solutions to theories with pure radiation sources, that is $T_{\mu\nu}dx^\mu dx^\nu = T_{uu}du^2$. With these kinds of sources and metrics satisfying $\Box R_{\mu\nu} = 0$, the field equations take the form $a_0 R_{uu} = T_{uu}$, and the case of $T_{uu} = T_{uu}(u)$ was considered in Refs. [4,6,9]. A solution to $a_0 R_{uu} = T_{uu}(u)$ can be found, for example, by relaxing the traceless condition on $h_{ij}(u)$ of Eq. (19); then one simply has the algebraic equation $a_0\sum_{i=1}^{D-2}h_{ii}(u) = -T_{uu}(u)$ [4].

The layout of the paper is as follows. In Sec. II, pp-wave spacetimes in generic gravity theory are discussed to set the stage for the AdS-plane waves discussed in Sec. III which also includes the proof of the theorem that a generic two-tensor can be reduced to a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher orders of $S_{\mu\nu}$ (such as, for example, $\Box^n S_{\mu\nu}$). Section IV is devoted to the field equations of quadratic gravity for pp-wave and AdS-wave *Ansätze* which play a major role in generic gravity theories. In Sec. V, we study the wave solutions of $f(R^{\mu\nu}_{\alpha\beta})$ theories where the action depends on the Riemann tensor but not on its derivatives. As two examples, we study the cubic gravity generated by string theory and the Lanczos-Lovelock theory. In Sec. VI, we show that Einsteinian wave solutions solve the generic gravity theory and as an example, we study the AdS-plane wave solutions of the six-dimensional conformal gravity and its nonconformal deformation as well as the tricritical gravity. In the Appendices, we expound upon some of the calculations given in the text.

II. pp-WAVE SPACETIMES IN GENERIC GRAVITY THEORY

As discussed above, analogies with the pp-wave solution will play a role in our proof so we first study the simpler pp-wave case. The pp-wave spacetime is a spacetime with plane-fronted parallel rays (for further properties of pp-waves see, for example, Refs. [25,26]). A subclass of these metrics can be put into the Kerr-Schild form as

$$g_{\mu\nu} = \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \qquad (20)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and the following relations hold:

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = 0, \qquad \lambda^\mu\partial_\mu V = 0. \qquad (21)$$

The pp-wave spacetimes have special algebraic properties. The Riemann and Ricci tensors of pp waves in the Kerr-Schild form are classified as Type N according to the "null alignment classification" [27,28]. When the Riemann and Ricci tensors are calculated by using Eq. (20), they, respectively, become

$$R_{\mu\alpha\nu\beta} = \lambda_\mu\lambda_\beta\partial_\alpha\partial_\nu V + \lambda_\alpha\lambda_\nu\partial_\mu\partial_\beta V - \lambda_\mu\lambda_\nu\partial_\alpha\partial_\beta V$$
$$- \lambda_\alpha\lambda_\beta\partial_\mu\partial_\nu V, \qquad (22)$$

and

$$R_{\mu\nu} = -\lambda_\mu\lambda_\nu\partial^2 V, \qquad (23)$$

which make the Type-N properties explicit. With these forms of the Riemann and Ricci tensors, notice that any contraction with the λ^μ vector yields zero. The scalar curvature is zero for the metric (20). Besides the scalar curvature, it has vanishing scalar invariants (VSIs). Since the Riemann and Ricci tensors are of Type N, and the scalar curvature is zero, the pp-wave spacetimes are also Type-N Weyl. Lastly, since the λ^μ vector is covariantly constant, it is nonexpanding, shear-free, and nontwisting; therefore, the pp-wave metrics belong to the Kundt class of metrics.

The two tensors of pp-wave spacetimes also have a special structure: any second-rank tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $R_{\mu\nu}$ and higher orders of $R_{\mu\nu}$ (such as, for example, $\Box^n R_{\mu\nu}$ with n a positive integer). This result follows from the corresponding property of Type-N Weyl and Type-N Ricci spacetimes given in Ref. [10] as the pp-wave spacetimes in the Kerr-Schild form share these properties. Although the pp-wave result was implied in Ref. [10], here we provide the proof along the lines of Ref. [6] since it gives some insight on the corresponding proof for the AdS-plane wave given below.

A. Two-tensors in a pp-wave spacetime

A generic two-tensor of the pp-wave spacetimes can be, symbolically, represented as

$$[R^{n_0}(\nabla^{n_1} R)(\nabla^{n_2} R)...(\nabla^{n_m} R)]_{\mu\nu}, \qquad (24)$$

where R denotes the Riemann tensor, and $\nabla^{n_i} R$ represents the $(0, n_i + 4)$-rank tensor constructed by n_i covariant derivatives acting on the Riemann tensor, so the term in $[...]_{\mu\nu}$ is a $(0, 4n_0 + 4m + \sum_{i=1}^m n_i)$-rank tensor whose indices are contracted until two indices, μ and ν, are left free. Here, the important point to notice is that each Riemann tensor has two λ's [Eq. (22)], so in total there are $2(n_0 + m)$ λ vectors. The remaining tensor structure involves just $\nabla^n V$'s.

Here is what we will prove: the generic two-tensors of the form (24) will boil down to a linear combination of $R_{\mu\nu}$ and $\Box^n R_{\mu\nu}$'s.

The first step of the proof is to show that the λ vector cannot make a nonzero contraction. It is easy to show this by using mathematical induction. With the identity $\lambda^\mu\partial_\mu V = 0$, the λ contraction of the term $\nabla^2 V$ is simply zero

$$\lambda^\mu\nabla_\nu\partial_\mu V = 0, \qquad (25)$$

after using the fact that λ is covariantly constant. Then, to show that the λ contraction of the term $\nabla^n V$ reduces to a lower-order term, we first observe that

$$\lambda^{\mu_j}\nabla_{\mu_1}...\nabla_{\mu_j}...\nabla_{\mu_n} V = \nabla_{\mu_1}(\lambda^{\mu_j}\nabla_{\mu_2}...\nabla_{\mu_j}...\nabla_{\mu_n} V). \quad (26)$$

Second, when λ is contracted with the first covariant derivative, by using $[\nabla_\alpha, \nabla_\beta]V^\rho = R_{\alpha\beta}{}^\rho{}_\sigma V^\sigma$ and $\lambda^\mu R_{\mu\alpha\nu\beta} = 0$, one has

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_n} V = \lambda^{\mu_1}[\nabla_{\mu_1}, \nabla_{\mu_2}]...\nabla_{\mu_n} V$$
$$+ \lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}...\nabla_{\mu_n} V$$
$$= \lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}...\nabla_{\mu_n} V, \qquad (27)$$

which completes the reduction of the nth-order term to the $(n-1)$th order. Thus, λ cannot make a nonzero contraction either with other λ's or with $\nabla^n V$'s.

Although we achieved our goal, let us discuss another proof of this step which gives some insight into the corresponding discussion in the AdS-plane wave case. For the pp-wave metrics in the Kerr-Schild form, one can choose the coordinates in such a way that the metric takes the form

$$ds^2 = 2dudv + d\vec{x}\cdot d\vec{x} + 2V(u, \vec{x})du^2, \qquad (28)$$

where $\vec{x} = (x^i)$ with $i = 1, ..., D - 2$, and u and v are null coordinates, so

$$\lambda_\mu dx^\mu = du \Rightarrow \lambda^\mu \partial_\mu = \partial_v \Rightarrow \lambda^\mu \partial_\mu V = \partial_v V = 0. \quad (29)$$

With this choice of the metric, $\nabla_\mu \lambda^\nu = 0$ leads to $\Gamma^\nu_{\mu\sigma}\lambda^\sigma = 0$. Now, let us look at the expansion of $\nabla^n V$ which has the form

$$
\begin{aligned}
\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_n}V &= \partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_n}V \\
&- (\partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_{n-2}}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n})\partial_{\sigma_1}V \\
&- \Gamma^{\sigma_1}_{\mu_{n-1}\mu_n}\partial_{\mu_1}\partial_{\mu_2}...\partial_{\mu_{n-2}}\partial_{\sigma_1}V - \cdots \\
&- (-1)^{n-1}\Gamma^{\sigma_1}_{\mu_1\mu_2}\Gamma^{\sigma_2}_{\sigma_1\mu_3}...\Gamma^{\sigma_{n-1}}_{\sigma_{n-2}\mu_n}\partial_{\sigma_{n-1}}V.
\end{aligned}
$$
$$(30)$$

The structures appearing in this expansion are the Christoffel connection, and partial derivatives of both V and the Christoffel connection. When one has a λ contraction, some terms involve a contraction of λ with one of the partial derivatives acting on V which yields an immediate zero since $\lambda^\mu = \delta^\mu_v$ and $\partial_v V = 0$. In addition, a λ contraction with a Christoffel connection also yields zero. On the other hand, if λ is contracted with one of the partial derivatives acting on a Christoffel connection, one needs to use the definition of the Riemann tensor, for example as

$$
\begin{aligned}
\lambda^{\mu_j}&\partial_{\mu_1}...\partial_{\mu_j}...\partial_{\mu_{n-2}}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n} \\
&= \partial_{\mu_1}...\partial_{\mu_{n-2}}(\lambda^{\mu_j}\partial_{\mu_j}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n}) \\
&= \partial_{\mu_1}...\partial_{\mu_{n-2}}[\lambda^{\mu_j}(R^{\sigma_1}{}_{\mu_n\mu_j\mu_{n-1}} + \partial_{\mu_{n-1}}\Gamma^{\sigma_1}_{\mu_j\mu_n} \\
&\quad -\Gamma^{\sigma_1}_{\mu_j\alpha}\Gamma^{\alpha}_{\mu_{n-1}\mu_n} + \Gamma^{\sigma_1}_{\mu_{n-1}\alpha}\Gamma^{\alpha}_{\mu_j\mu_n})],
\end{aligned}
$$
$$(31)$$

where the terms in the square brackets are just zero since $\lambda^\mu R_{\mu\alpha\nu\beta} = 0$ and $\Gamma^\nu_{\mu\sigma}\lambda^\sigma = 0$.

Since λ cannot make a nonzero contraction, there should be at most two λ's—that is, one Riemann tensor—so the nonzero terms of the form (24) reduce to

$$R_{\mu\nu}, \quad \text{or} \quad [\nabla^{2n}R]_{\mu\nu}, \quad (32)$$

where an even number of covariant derivatives is required to have a two-tensor. After determining the nonzero terms required by the first step of the proof, in the second step, let us discuss the structure of these nonzero terms of the form $[\nabla^n R]_{\mu\nu}$. In obtaining a two-tensor by contracting the indices of $[\nabla^n R]_{\mu\nu}$, one should either have

$$g^{\alpha\beta}\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{2n}}R_{\mu\alpha\nu\beta} = \nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{2n}}R_{\mu\nu}, \quad (33)$$

or

$$\nabla_{\mu_1}\nabla_{\mu_2}...\nabla^\alpha...\nabla^\beta...\nabla_{\mu_{2n-2}}R_{\mu\alpha\nu\beta}. \quad (34)$$

In Eq. (34), one can rearrange the order of the derivatives. Each change of order introduces a Riemann tensor, and as we have just shown, a two-tensor contraction in the presence of this additional Riemann tensor gives zero. The only nonzero part is the original term which in the final form reads

$$\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{2n-2}}\nabla^\alpha\nabla^\beta R_{\mu\alpha\nu\beta} = \nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_{2n-2}}\Box R_{\mu\nu},$$
$$(35)$$

where we used the Bianchi identity on the Riemann tensor. Further contractions in Eqs. (33) and (35) should be between the indices of the derivatives and as we have shown we can change the order of the derivatives without introducing an additional term; then, one has

$$[\nabla^{2n}R]_{\mu\nu} = \Box^n R_{\mu\nu}.$$

As a result, the nonzero terms are in the form $R_{\mu\nu}$ and $\Box^n R_{\mu\nu}$, where n is a positive integer. Any two-tensor of the pp-wave spacetimes in the Kerr-Schild form is a linear combination of these terms. This completes the proof.

Before proceeding to the field equations, we note that with this result about the two-tensors, the VSI property of the pp waves in the Kerr-Schild form is explicit since $R_{\mu\nu}$ is traceless.[3]

B. Field equations of the generic theory for a pp-wave spacetime

Once the above result is used, the field equations of the most general theory (2) with $\Lambda_0 = 0$ reduce to

$$\sum_{n=0}^{N} a_n \Box^n R_{\mu\nu} = 0, \quad (36)$$

where the a_n's are constants depending on the parameters of the theory (namely on κ, α, β, γ, C_n), and N can be as large as possible. Note that a pp-wave metric [Eq. (20)] solving $R_{\mu\nu} = 0$ is a solution of Eq. (36). This fact was demonstrated in Ref. [6] without finding the explicit form of Eq. (36) by taking $R_{\mu\nu} = 0$ as an assumption from the beginning. The plane waves, which are special pp waves with $V(u, \vec{x}) = h_{ij}(u)x^i x^j$ where h_{ij} is symmetric and traceless, provide a solution to Eq. (36) for any h_{ij} by satisfying $R_{\mu\nu} = 0$ [4]. As discussed in Ref. [9], one can also follow the method of constraining pp-wave spacetimes such that $R_{\mu\nu}$ is the only nonzero two-tensor, which effectively means $\Box R_{\mu\nu} = 0$; then, the field equations of the generic gravity theory reduce to the Einstein gravity ones. On the other hand, obtaining Eq. (36) makes one realize that the pp-wave solutions of the quadratic gravity theory also solve the generic gravity theory (2). To show this, we first notice that one can factorize Eq. (36) as

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[3]For the proof of the VSI property of plane waves, see Ref. [29].

$$\prod_{n=1}^{N}(\Box + b_n)R_{\mu\nu} = 0, \qquad (37)$$

where the b_n's are constants depending on the original parameters of the theory. Here, the b_n's can be real or complex and once they are complex, they must appear in complex-conjugate pairs.

To further reduce Eq. (37), by first using the covariantly constant property of λ^μ, one has

$$\Box R_{\mu\nu} = \Box(-\lambda_\mu\lambda_\nu\partial^2 V) = -\lambda_\mu\lambda_\nu\Box\partial^2 V. \qquad (38)$$

Here, we note that for *any* scalar function ϕ (not necessarily V) satisfying $\lambda^\mu\partial_\mu\phi = \partial_v\phi = 0$, one has

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \eta^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma\phi, \qquad (39)$$

after also using $\Gamma^\nu_{\mu\sigma}\lambda^\sigma = 0$ which is valid in the coordinates we have chosen [Eq. (28)]. Here, $\eta_{\mu\nu}$ is the flat metric in null coordinates. In addition, for Eq. (28), the Christoffel connection has the form

$$\Gamma^\sigma_{\mu\nu} = \lambda^\sigma\lambda_\nu\partial_\mu V + \lambda^\sigma\lambda_\mu\partial_\nu V - \lambda_\mu\lambda_\nu\eta^{\sigma\beta}\partial_\beta V, \qquad (40)$$

which leads to $\eta^{\mu\nu}\Gamma^\sigma_{\mu\nu} = 0$; therefore, one has

$$\Box\phi = \partial^2\phi. \qquad (41)$$

Furthermore, since $\partial^2 = 2\frac{\partial^2}{\partial u\partial v} + \hat{\partial}^2,$ where $\hat{\partial}^2 \equiv \sum_{i=1}^{D-2}\frac{\partial^2}{\partial x^i\partial x^i}$, and $\partial_v\phi = 0$, we have

$$\Box\phi = \hat{\partial}^2\phi. \qquad (42)$$

With this property and $\partial_v\hat{\partial}...\hat{\partial}V = 0$, one has

$$\Box^n V = \hat{\partial}^{2n}V, \qquad (43)$$

which reduces Eq. (37) to

$$\lambda_\mu\lambda_\nu\hat{\partial}^2\prod_{n=1}^{N}(\hat{\partial}^2 + b_n)V = 0. \qquad (44)$$

Note that this equation is linear in V, so one can make an important observation for pp-wave metrics in the Kerr-Schild form. One can consider the pp-wave metric (20) as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \equiv 2V\lambda_\mu\lambda_\nu$, and with this definition the Ricci tensor becomes

$$R_{\mu\nu} = -\frac{1}{2}\partial^2 h_{\mu\nu}, \qquad (45)$$

after using the fact that λ_μ is covariantly constant. Then, once one considers this form of the Ricci tensor and $\Box^n R_{\mu\nu} = \partial^{2n}R_{\mu\nu}$ in either Eq. (36) or Eq. (37), it is clear

that the field equations of the generic theory (2) for pp waves are linear in $h_{\mu\nu}$ as in the case of a perturbative expansion of the field equations around a flat background for a small metric perturbation $\|h\| \equiv \|g - \eta\| \ll 1$.

This observation suggests that there are two possible ways to find the field equations of the generic gravity theory for pp waves, namely, (i) by deriving the field equations and directly putting the pp-wave metric *Ansatz* (20) into them, or (ii) by linearizing the derived field equations around the flat background and putting $h_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$ in these linearized equations. Although the second way involves an additional linearization step, the idea itself provides a shortcut in finding the field equations of pp waves for a gravity theory described with a Lagrangian density which is constructed from the Riemann tensor but not its derivatives. Namely, due to linearization in the field equations, only up to the quadratic curvature order of these theories contributes to the field equations. This idea is made explicit in the examples discussed in Sec. V. Lastly, since $h_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$ is transverse, $\partial_\mu h^{\mu\nu} = 0$, and traceless, $\eta^{\mu\nu}h_{\mu\nu} = 0$, to find the field equations by following the second way, one needs only the linearized field equations for the transverse-traceless metric perturbation.

Assuming nonvanishing and distinct b_n's, the most general solution of Eq. (44) is

$$V = V_E + \Re\left(\sum_{n=1}^{N}V_n\right), \qquad (46)$$

where V_E is the solution to Einstein's theory, namely $\hat{\partial}^2 V_E = 0$, $\Re$ represents the real part, and the V_n's solve the equation of the quadratic gravity theory, i.e. $(\hat{\partial}^2 + b_n)V_n = 0$ (in case the reader has any doubt that this equation is the quadratic gravity theory's equation for the pp wave, we shall show this explicitly below). Then, the pp-wave solution of Einstein gravity also solves a generic gravity theory which was already known in the literature [6]. Here, the novel result is that the pp-wave solutions of the quadratic gravity theory also solve the generic theory. These solutions are of the form

$$V_n(u,\vec{x}) = c_{1,n}(u)\sin(\vec{\xi}_n \cdot \vec{x} + c_{2,n}(u)), \qquad (47)$$

with $|\vec{\xi}_n|^2 \equiv b_n$. Here, we consider the case with real b_n since the b_n's are related to the masses of the perturbative excitations around a flat background as $M_{n,\text{flat}}^2 = -b_n$. What we have learned in the pp-wave case will be applied to the Λ_0 case below.

III. ADS-PLANE WAVE SPACETIMES IN GENERIC GRAVITY THEORY

AdS-plane waves are a member of the Kerr-Schild-Kundt metrics given as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \tag{48}$$

where $\bar{g}_{\mu\nu}$ is the AdS metric and the following relations hold:

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)},$$
$$\xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0. \tag{49}$$

The second identity serves as a definition of the ξ vector where the symmetrization convention is $\xi_{(\mu}\lambda_{\nu)} \equiv \frac{1}{2}(\xi_\mu\lambda_\nu + \lambda_\mu\xi_\nu)$.

As in the case of the pp-wave spacetimes, the AdS-plane wave also satisfies special algebraic properties. However, instead of the Riemann and Ricci tensors, the Weyl tensor and the traceless Ricci tensor, that is $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{D}g_{\mu\nu}$, are Type N. By using the results in Ref. [18], the traceless Ricci and Weyl tensors can be calculated as

$$S_{\mu\nu} = \rho\lambda_\mu\lambda_\nu, \tag{50}$$

and

$$C_{\mu\alpha\nu\beta} = 4\lambda_{[\mu}\Omega_{\alpha][\beta}\lambda_{\nu]}, \tag{51}$$

where the square brackets denote antisymmetrization, and ρ is defined as

$$\rho \equiv -\left(\Box + 2\xi^\mu\partial_\mu + \frac{1}{2}\xi^\mu\xi_\mu - \frac{2(D-2)}{\ell^2}\right)V, \tag{52}$$

and the symmetric tensor $\Omega_{\alpha\beta}$ is defined as

$$\Omega_{\alpha\beta} \equiv -\left[\nabla_\alpha\partial_\beta V + \xi_{(\alpha}\partial_{\beta)}V + \frac{1}{2}\xi_\alpha\xi_\beta V \right.$$
$$\left. + \frac{1}{D-2}g_{\alpha\beta}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right]. \tag{53}$$

In fact, these forms follow from Eqs. (48) and (49), and the derivations are given in Appendix A. In the given forms above, Type-N properties of the Weyl and traceless Ricci tensors are explicit. It can also be seen that the λ^μ contractions with the traceless Ricci tensor are zero. This is also the case for the Weyl tensor, since $\Omega_{\alpha\beta}$ satisfies

$$\lambda^\alpha\Omega_{\alpha\beta} = \frac{1}{2}\lambda_\beta\Omega^\alpha_\alpha, \tag{54}$$

where

$$\Omega^\alpha_\alpha = \xi^\alpha\partial_\alpha V - \frac{2}{D-2}\rho + \frac{4}{\ell^2}V. \tag{55}$$

The scalar curvature for the AdS-plane waves is constant $R = -D(D-1)/\ell^2$. In addition, these spacetimes have constant scalar invariants (CSI), for example

$$R_{\mu\alpha\beta\gamma}R^{\mu\alpha\beta\gamma} = \frac{2D(D-1)}{\ell^4}, \qquad R_{\mu\sigma}R^{\mu\sigma} = \frac{D(D-1)^2}{\ell^4}. \tag{56}$$

Finally, due to $\nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$ and $\lambda^\mu\xi_\mu = 0$, the λ^μ vector is nonexpanding, shear-free, and nontwisting; therefore, the AdS-plane wave metrics belong to the Kundt class of metrics.

Like pp-wave spacetimes, the two tensors of AdS-plane wave spacetimes also have a special structure. In Ref. [15], while sketching a proof using the boost weight decomposition [27], we gave the following theorem:

Consider a Kundt spacetime for which the Weyl and the traceless Ricci tensors are of type N, and all scalar invariants are constant. Then, any second-rank symmetric tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher orders of $S_{\mu\nu}$ (such as, for example, $\Box^n S_{\mu\nu}$).

The AdS-plane wave spacetimes belong to this class. Below, we give a direct proof of this theorem that is specific to the AdS-plane waves.

A. Two-tensors in an AdS-plane wave spacetime

A generic two-tensor obtained by contracting any number of Riemann tensors and their covariant derivatives can be symbolically written as

$$[R^{r_0}(\nabla^{r_1}R)(\nabla^{r_2}R)\dots(\nabla^{r_s}R)]_{\mu\nu}, \tag{57}$$

where the same conventions as in the pp-wave case are used. Since the Riemann tensor is

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + \frac{2}{D-2}(g_{\mu[\nu}S_{\beta]\alpha} - g_{\alpha[\nu}S_{\beta]\mu})$$
$$+ \frac{2R}{D(D-1)}g_{\mu[\nu}g_{\beta]\alpha}, \tag{58}$$

equivalently, one can write Eq. (57) as a sum of terms in the form

$$[C^{m_0}(\nabla^{m_1}C)(\nabla^{m_2}C)\dots(\nabla^{m_k}C)S^{n_0}(\nabla^{n_1}S)$$
$$\times (\nabla^{n_2}S)\dots(\nabla^{n_l}S)]_{\mu\nu}, \tag{59}$$

where C and S represent the Weyl tensor and the traceless-Ricci tensor, respectively. Note that one may consider adding the metric to Eq. (59) to make the discussion more complete, but it would be a trivial addition which would boil down to Eq. (59) after carrying out contractions involving the $g_{\mu\nu}$'s.

Here is what we will prove: the generic two-tensors of the form (59) will boil down to a linear combination of $S_{\mu\nu}$ and $\Box^n S_{\mu\nu}$'s.

The proof is somewhat lengthy and lasts until the end of this section. The reader who is not interested in the proof, but rather just in the applications of the result can skip this section. Now, let us give the proof which involves two steps:

(1) First, we prove that $[C^{m_0}(\nabla^{m_1}C)(\nabla^{m_2}C) \dots(\nabla^{m_k}C)S^{n_0}(\nabla^{n_1}S)(\nabla^{n_2}S)\dots(\nabla^{n_l}S)]_{\mu\nu} = 0$ unless $(m_0, k, n_0, l) = (0, 1, 0, 0)$, or $(m_0, k, n_0, l) = (0, 0, 1, 0)$ or $(m_0, k, n_0, l) = (0, 0, 0, 1)$.

(2) For even[4] n, we then prove that $[\nabla^n S]_{\mu\nu}$ and $[\nabla^n C]_{\mu\nu}$ have a second-rank tensor contraction which is a linear combination of $\Box^{\frac{n}{2}}S$, $\Box^{\frac{n}{2}-1}S$, $\dots$, $\Box S$, and S.

1. $[C^{m_0}(\nabla^{m_1}C)(\nabla^{m_2}C)\dots(\nabla^{m_k}C)S^{n_0}(\nabla^{n_1}S)(\nabla^{n_2}S)\dots(\nabla^{n_l}S)]_{\mu\nu} = 0$ if $m_0 \neq 0$ and $n_0 + k + l > 1$

Before giving the precise proof, let us present the basic idea. If one considers the forms of the Weyl tensor and the traceless Ricci tensor together with the property $\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$, then one can see that the generic term (59) represents the sum of terms that are made up of $2(m_0 + k + n_0 + l)$ λ vectors and various combinations of the derivatives of V, the ξ vector and its derivatives. Without loss of generality, one can assume $m_1 < m_2 < \dots < m_k$ and $n_1 < n_2 < \dots < n_l$; then, the building blocks of Eq. (59) are

$$\lambda, \xi, \nabla^p V, \nabla^r \xi,$$

$$p = 1, \dots, \max(n_l, m_k) + 2; \quad r = 1, \dots, \max(n_l, m_k).$$

We proved that these building blocks (other than λ) generate a free-index λ vector when they are contracted with a λ vector. In addition, the remaining tensor structure just involves the same buildings blocks that have the same or lower derivative order than the order before contraction. Naturally, any tensor that is made up of these building blocks inherits this property. Due to this property, it is not possible to lower the number of λ vectors by contractions and these λ vectors sooner or later yield a zero contraction. Therefore, to get a nonzero term from Eq. (59), the unique possibility is to have at most two λ vectors, that is

$$2(m_0 + k + n_0 + l) = 2 \Rightarrow m_0 + k + n_0 + l = 1,$$

yielding either $(m_0, k, n_0, l) = (1, 0, 0, 0)$, $(m_0, k, n_0, l) = (0, 1, 0, 0)$, $(m_0, k, n_0, l) = (0, 0, 1, 0)$, or $(m_0, k, n_0, l) = (0, 0, 0, 1)$. But, $(m_0, k, n_0, l) = (1, 0, 0, 0)$ is $[C]_{\mu\nu}$ which is just zero. Thus, the possible nonzero terms coming from Eq. (59) are in the form $S_{\mu\nu}$, $[\nabla^n S]_{\mu\nu}$, and $[\nabla^m C]_{\mu\nu}$ which are studied in Sec. III A 2.

Now, let us start our rigorous proof and first show how a free-index λ vector is generated by any λ contraction.

To this end, we consider the behavior of the $(0, n)$-rank tensor $\nabla^{n-1}\xi$ under λ contractions. To analyze $\nabla^{n-1}\xi$, we work in the null frame in which the metric has the form

$$ds^2 = \frac{\ell^2}{z^2}(2dudv + d\vec{x}\cdot d\vec{x} + dz^2) + 2V(u, \vec{x}, z)du^2, \tag{60}$$

where u and v are null coordinates, and $\vec{x} = (x^i)$ with $i = 1, \dots, D-3$. Thus, λ_μ and λ^μ are of the form

$$\lambda_\mu dx^\mu = du, \qquad \lambda^\mu \partial_\mu = \frac{z^2}{\ell^2}\partial_v, \tag{61}$$

which shows why the metric function V does not depend on the coordinate v due to the relation $\lambda^\mu \partial_\mu V = 0$. In addition, ξ_μ and ξ^μ become [18]

$$\xi_\mu = \frac{2}{z}\delta^z_\mu, \qquad \xi^\mu = \frac{2z}{\ell^2}\delta^\mu_z. \tag{62}$$

The properties $\xi^\mu \lambda_\mu = 0$ and $[\nabla_\mu, \nabla_\alpha]\lambda^\mu = -\frac{(D-1)}{\ell^2}\lambda_\alpha$ yield the following identities for the ξ^μ vector:

$$\lambda^\mu \nabla_\alpha \xi_\mu = -\frac{2}{\ell^2}\lambda_\alpha, \tag{63}$$

and

$$\lambda^\mu \nabla_\mu \xi_\alpha = -\frac{2}{\ell^2}\lambda_\alpha, \tag{64}$$

where we also used $\nabla_\mu \xi^\mu = -\frac{2(D-1)}{\ell^2}$.

Now, let us look at $\nabla^{n-1}\xi$ in the explicit form:

$$\begin{aligned}
\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_{n-1}}\xi_{\mu_n} = {} & \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-1}}\xi_{\mu_n} \\
& - (\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-2}}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n})\xi_{\sigma_1} \\
& - \Gamma^{\sigma_1}_{\mu_{n-1}\mu_n}\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-2}}\xi_{\sigma_1} - \cdots \\
& - (-1)^{n-1}\Gamma^{\sigma_1}_{\mu_1\mu_2}\Gamma^{\sigma_2}_{\sigma_1\mu_3}\dots\Gamma^{\sigma_{n-1}}_{\sigma_{n-2}\mu_n}\xi_{\sigma_{n-1}}.
\end{aligned} \tag{65}$$

The structures appearing in this expression are the Christoffel connection, and partial derivatives of both ξ_μ and the Christoffel connection. In considering possible λ^μ contractions with the terms in this expansion, we first note that Eq. (62) yields

$$\lambda^{\mu_n}\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-1}}\xi_{\mu_n} = 0, \qquad \lambda^{\mu_j}\partial_{\mu_1}\dots\partial_{\mu_j}\dots\partial_{\mu_{n-1}}\xi_{\mu_n} = 0. \tag{66}$$

In addition, since $\partial_v g_{\alpha\beta} = 0$, a λ^μ contraction with the derivatives acting on a Christoffel connection also yields zero:

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[4]Note that for odd n, it is not possible to have a two-tensor contraction.

$$\lambda^{\mu_j}\partial_{\mu_1}\dots\partial_{\mu_j}\dots\partial_{\mu_{n-2}}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n}=0. \tag{67}$$

Moving to the λ^μ and λ_μ contractions of the Christoffel connection, the property $\nabla_\alpha\lambda_\beta = \xi_{(\alpha}\lambda_{\beta)}$ leads to

$$\Gamma^\sigma_{\alpha\beta}\lambda_\sigma = -\frac{1}{2}(\lambda_\alpha\xi_\beta + \xi_\alpha\lambda_\beta), \tag{68}$$

$$\Gamma^\alpha_{\beta\sigma}\lambda^\sigma = \frac{1}{2}(\xi^\alpha\lambda_\beta - \lambda^\alpha\xi_\beta), \tag{69}$$

in the null frame. In addition, when λ^μ or λ_μ contracts with a Christoffel connection under the action of the partial derivatives, one has

$$\lambda_\sigma\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\Gamma^\sigma_{\alpha\beta} = -\frac{1}{2}\lambda_\alpha\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\xi_\beta$$
$$-\frac{1}{2}\lambda_\beta\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\xi_\alpha, \tag{70}$$

$$\lambda^\sigma\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\Gamma^\alpha_{\beta\sigma} = \frac{1}{2}(\xi^\alpha\lambda_\beta - \lambda^\alpha\xi_\beta)z\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}\frac{1}{z}, \tag{71}$$

where a new structure (that is, partial derivatives acting on $1/z$) appears; however, it yields zero after a further λ^μ contraction.

Having discussed all possible λ contraction patterns [Eqs. (66)–(71)] with the structures involved in the expansion of $\nabla^{n-1}\xi$, we now show that a λ vector contraction with the $(0,n)$-rank tensor $\nabla^{n-1}\xi$ provides a free-index λ one-form (we mean λ_μ). To see this, we first notice that the possible nonzero contractions of the λ vector (we mean λ^μ), which are Eqs. (69) and (71), always consist of two terms such that one of them involves a λ one-form and the other involves a λ vector. If the reproduced λ one-form is noncontracting, then we have achieved the goal of having a free-index λ one-form. However, if it is contracting, then it must make a contraction in the form of either Eq. (68) or Eq. (70), so this contracted λ one-form generates new λ one-forms. The same procedure holds for these newly generated λ one-forms and when all the possible λ one-form contractions are carried out, one always ends up with a free-index λ one-form. On the other hand, returning to the λ vector reproduced after the first contraction, it necessarily makes a contraction and if this contraction is not zero, it should again be in the form of either Eq. (69) or Eq. (71). Thus, one should follow the same procedure until the newly generated λ vector makes a zero contraction and this is in fact the case since for a λ vector, there is a limited number of nonzero contraction possibilities generating a new λ vector in each term in the $\nabla^{n-1}\xi$ expansion (65).

For any number of λ^μ contractions with the $(0,n)$-rank tensor $\nabla^{n-1}\xi$, the case is the same and each λ^μ contraction generates a free-index λ_μ one-form in each term in the

$\nabla^{n-1}\xi$ expansion (65) if it makes a nonzero contraction. To see this, we just note that after each λ^μ contraction the remaining structures are the original ones (the ξ_μ one-form, the Christoffel connection, and partial derivatives of both the ξ_μ one-form and the Christoffel connection) in addition to the newly generated ξ^μ vectors and $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}(1/z)$-type forms which yield zero under a further λ^μ contraction. Therefore, the discussion of the further λ^μ contractions is not different from the single λ^μ contraction, and each λ^μ contraction generates a free-index λ one-form.

Moving to the other tensor structure appearing in Eq. (59), the $(0,n)$-rank tensor $\nabla^n V$ also shares the same properties as $\nabla^{n-1}\xi$ under λ^μ contractions. Expanding $\nabla^n V$ yields

$$\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_{n-1}}\partial_{\mu_n}V = \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-1}}\partial_{\mu_n}V$$
$$- (\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-2}}\Gamma^{\sigma_1}_{\mu_{n-1}\mu_n})\partial_{\sigma_1}V$$
$$- \Gamma^{\sigma_1}_{\mu_{n-1}\mu_n}\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-2}}\partial_{\sigma_1}V - \cdots$$
$$- (-1)^{n-1}\Gamma^{\sigma_1}_{\mu_1\mu_2}\Gamma^{\sigma_2}_{\sigma_1\mu_3}\dots\Gamma^{\sigma_{n-1}}_{\sigma_{n-2}\mu_n}\partial_{\sigma_{n-1}}V, \tag{72}$$

where $\partial_\mu V$ simply replaces ξ_μ in the above discussion. When this expansion is contracted with the λ^μ vectors, the possible contraction patterns are the same as for the $\nabla^{n-1}\xi$ case except that the $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_{n-1}}\xi_{\mu_n}$ term is replaced by $\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_n}V$ which also yields a zero under a λ^μ contraction as

$$\lambda^{\mu_j}\partial_{\mu_1}\dots\partial_{\mu_j}\dots\partial_{\mu_n}V = 0. \tag{73}$$

Therefore, after exactly the same discussion as in the case of $\nabla^{n-1}\xi$, one can show that each λ^μ contraction with $\nabla^n V$ generates a free-index λ_μ one-form.

We established that each λ vector contraction with the $(0,n)$-rank tensors $\nabla^{n-1}\xi$ and $\nabla^n V$ generates a free-index λ one-form; however, after a certain number of λ vector contractions, these tensors become necessarily zero, because the possible nonzero λ vector contractions are made with the indices of the Christoffel connections and the maximum number of Christoffel connections is just $(n-1)$ for both cases. These $(n-1)$ Christoffel connections involve n free down indices. Each λ vector contraction reduces the number of contractible down indices[5] by two since it also introduces a free-index λ one-form. Thus, if n is even, then $n/2$ is the maximum number of λ vector contractions before one necessarily gets a zero. On the other hand, for odd n, $(n-1)/2$ is the maximum number of nonzero λ vector contractions.

In obtaining a two-tensor from the rank $(0, 4(m_0 + k) + 2(n_0 + l) + \sum_{i=1}^k m_i + \sum_{i=1}^l n_i)$ tensor

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[5]The ones giving a nonzero result.

$$[C^{m_0}(\nabla^{m_1}C)(\nabla^{m_2}C)...(\nabla^{m_k}C)S^{n_0}(\nabla^{n_1}S)(\nabla^{n_2}S)...(\nabla^{n_l}S)], \tag{74}$$

one may prefer to make contractions involving λ one-forms first. To have a nonzero contraction, λ one-forms should be contracted with either $\nabla^{n-1}\xi$ tensors or $\nabla^n V$ tensors. However, since these contractions generate new λ one-forms, the number of λ one-forms cannot be reduced by contractions. In addition, there is a limit for getting a nonzero contraction from the tensors $\nabla^{n-1}\xi$ and $\nabla^n V$. As a result, in the presence of more than two λ one-forms, one cannot get rid off these λ one-forms by contraction and they, sooner or later, make zero contractions.

To get a nonzero two-tensor from Eq. (74), there should be at most two λ one-forms and they should provide the two-tensor structure. Then, the possibilities are

$$S_{\mu\nu}, \qquad [\nabla^n C]_{\mu\nu}, \qquad [\nabla^n S]_{\mu\nu}, \tag{75}$$

where we have not included $[C]_{\mu\nu}$ as the Weyl tensor is traceless. Note that to have a two-tensor, the number of covariant derivatives acting on the Weyl tensor and the traceless Ricci tensor should be even. Next, we will reduce the last two expressions to the desired form.

2. Reduction of $[\nabla^n S]_{\mu\nu}$ and $[\nabla^n C]_{\mu\nu}$ to $\sum_{i=0}^{\frac{n}{2}} d_i(D,R)\Box^i S$

First, let us analyze the $[\nabla^n S]_{\mu\nu}$ term where n is even because we want to get a two-tensor contraction from $\nabla^n S$.

The lowest-order term is $[\nabla^2 S]_{\mu\nu}$ which has two contraction possibilities: $\Box S_{\mu\nu}$ and $\nabla^\alpha\nabla_\mu S_{\alpha\nu}$. The first contraction possibility is already in the desired form. For the second possibility, changing the order of the covariant derivatives yields

$$\nabla^\alpha\nabla_\mu S_{\alpha\nu} = \nabla_\mu\nabla^\alpha S_{\alpha\nu} + [\nabla^\alpha, \nabla_\mu]S_{\alpha\nu}, \tag{76}$$

where the first term is zero due to the Bianchi identity and the constancy of the scalar curvature; finally, it takes the form

$$\nabla^\alpha\nabla_\mu S_{\alpha\nu} = \frac{R}{D-1}S_{\mu\nu}, \tag{77}$$

after using Eq. (58). Having discussed the lowest order, to use mathematical induction, let us analyze a generic nth-order derivative term $[\nabla^n S]_{\mu\nu}$. The contraction patterns for this term are as follows: (i) the two free indices can be on the S tensor; (ii) at least one of the free indices is on the covariant derivatives. In the first contraction pattern, the indices of the covariant derivatives are totally contracted among themselves and it is possible to rearrange the order of the covariant derivatives to put the term in the form $\Box^{\frac{n}{2}}S_{\mu\nu}$ by using

$$\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r-1}\mu_r} = \nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r-1}\mu_r} + [\nabla_{\mu_1}, \nabla_{\mu_2}]\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r-1}\mu_r}$$

$$= \nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r-1}\mu_r} + \sum_{s=3}^{r-2}R_{\mu_1\mu_2\mu_s}{}^{\mu_{r+1}}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_{r+1}}\left(\prod_{i_2=s+1}^{r-2}\nabla_{\mu_{i_2}}\right)S_{\mu_{r-1}\mu_r}$$

$$+ R_{\mu_1\mu_2\mu_{r-1}}{}^{\mu_{r+1}}\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r+1}\mu_r} + R_{\mu_1\mu_2\mu_r}{}^{\mu_{r+1}}\left(\prod_{i=3}^{r-2}\nabla_{\mu_i}\right)S_{\mu_{r-1}\mu_{r+1}}. \tag{78}$$

In addition, if one uses Eq. (58), then the parts of the Riemann tensor involving the Weyl and the traceless Ricci tensors just yield zeros as we proved above. The remaining nonzero part of the Riemann tensor in which the tensor structure is just two metrics [that is the third term in Eq. (58)] reduces the terms involving the Riemann tensor to $(n-2)$th-order terms as $[\nabla^{n-2}S]_{\mu\nu}$. Thus, the first contraction pattern of $[\nabla^n S]_{\mu\nu}$ yields a sum involving $\Box^{\frac{n}{2}}S_{\mu\nu}$ and $[\nabla^{n-2}S]_{\mu\nu}$ terms. On the other hand, for the second contraction pattern, at least one of the covariant derivatives is contracted with S and in order to use the Bianchi identity $\nabla^\rho S_{\mu\rho} = 0$, one needs to change the order of the covariant derivative contracting with S until it is next to S by using Eq. (78). Again, during this process terms involving the

Riemann tensor and $(n-2)$ covariant derivatives are introduced, and after the use of Eq. (58), these terms become $[\nabla^{n-2}S]_{\mu\nu}$. Thus, the second contraction pattern of $[\nabla^n S]_{\mu\nu}$ yields a sum involving $[\nabla^{n-2}S]_{\mu\nu}$ terms. Then, just as we showed that the lowest-order derivative term $[\nabla^2 S]_{\mu\nu}$ satisfies the desired pattern and that the nth-order term $[\nabla^n S]_{\mu\nu}$ reduces to a sum involving a desired term $\Box^{\frac{n}{2}}S_{\mu\nu}$ and $(n-2)$th-order terms $[\nabla^{n-2}S]_{\mu\nu}$, it is clear by mathematical induction that $[\nabla^n S]_{\mu\nu}$ can be represented as a sum in the form

$$[\nabla^n S]_{\mu\nu} = \sum_{i=0}^{\frac{n}{2}} d_i(D,R)\Box^i S_{\mu\nu}, \tag{79}$$

where $d_{n/2}$ is just one, and the dimension and scalar curvature dependence of the other d_i's are due to the Riemann tensors that are transformed via Eq. (58).

Now, let us move to the term $[\nabla^n C]_{\mu\nu}$ where n is again even because we want to get a two-tensor contraction from $\nabla^n C$. Since the Weyl tensor is traceless, at least two covariant derivatives should be contracted with the Weyl tensor when obtaining a nonzero two-tensor from $\nabla^n C$. Then, at the lowest order $[\nabla^2 C]_{\mu\nu} = \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta}$, one can use the following identity for the Weyl tensor assuming that the metric is Eq. (48):

$$\nabla^\mu \nabla^\nu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2}\left(\Box S_{\alpha\beta} - \frac{R}{D-1}S_{\alpha\beta}\right), \qquad (80)$$

which is derived in Appendix B, and then we immediately obtain the desired form. Now, we move to the nth-order term $[\nabla^n C]_{\mu\nu}$ for which again one can change the order of the covariant derivatives in such a way that two of the covariant derivatives contracting with the Weyl tensor are moved next to it in order to use the Bianchi identity (80). As before, during the order change of the covariant derivatives Riemann tensors are introduced. After the use of Eq. (58), only the part of the Riemann tensor involving two metrics yields a nonzero contribution, so the terms involving the Riemann tensor reduce to $(n-2)$th-order terms $[\nabla^{n-2}C]_{\mu\nu}$. Thus, the nth-order term $[\nabla^n C]_{\mu\nu}$ reduces to the sum of $[\nabla^{n-2}\Box S]_{\mu\nu}$, $[\nabla^{n-2}S]_{\mu\nu}$, and $[\nabla^{n-2}C]_{\mu\nu}$ terms. Then, just as we showed that the lowest-order derivative term $[\nabla^2 C]_{\mu\nu}$ can be converted to the $[\nabla^2 S]_{\mu\nu}$ case and that the nth-order term $[\nabla^n C]_{\mu\nu}$ reduces to a sum involving the $[\nabla^n S]_{\mu\nu}$ and $[\nabla^{n-2}S]_{\mu\nu}$ cases and $(n-2)$th-order terms $[\nabla^{n-2}C]_{\mu\nu}$, it is clear by mathematical induction that $[\nabla^n C]_{\mu\nu}$ can be represented as a sum involving just $[\nabla^m S]_{\mu\nu}$ terms where $n \geq m \geq 0$. Then, the $[\nabla^n C]_{\mu\nu}$ case reduces to the $[\nabla^n S]_{\mu\nu}$ case which is of the desired form [Eq. (79)].

As a result, the nonzero two-tensors of the AdS-plane wave spacetime can be written as a linear combination of the tensor $S_{\mu\nu}$, the $\Box^n S_{\mu\nu}$'s, and the metric $g_{\mu\nu}$. This completes the proof.

Note that with this result about the two-tensors, the CSI property of the AdS-plane wave spacetimes is explicit since $S_{\mu\nu}$ is traceless.

B. Field equations of the generic gravity theory for an AdS-plane wave spacetime

In Ref. [15], we studied the field equations of the generic gravity theory for the CSI Kundt spacetime of Type-N Weyl and Type-N traceless Ricci tensors. In addition, we also demonstrated how the field equations further reduce for Kerr-Schild-Kundt spacetimes to which AdS-plane waves belong. Let us recapitulate these results here. As an immediate result of our conclusions above, the field equations coming from Eq. (2) are

$$e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0. \qquad (81)$$

The trace of the field equation yields

$$e = 0, \qquad (82)$$

which determines the effective cosmological constant Λ or $1/\ell^2$ in terms of the parameters that appear in the Lagrangian. On the other hand, the traceless part of the field equation

$$\sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \qquad (83)$$

can be factorized as

$$\prod_{n=1}^{N} (\Box + b_n) S_{\mu\nu} = 0, \qquad (84)$$

where the b_n's are again functions of the parameters of the original theory, and in general they can be complex which appear in complex-conjugate pairs. To further reduce Eq. (84), we note that in Ref. [18], it was shown that for any ϕ satisfying $\lambda^\mu \partial_\mu \phi = \partial_v \phi = 0$, one has

$$\Box \phi = \bar{\Box} \phi, \qquad (85)$$

where $\bar{\Box} \equiv \bar{g}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu$. Therefore, $S_{\mu\nu} = \lambda_\mu \lambda_\nu \mathcal{O} V$ with

$$\begin{aligned}\mathcal{O} &\equiv -\left(\bar{\Box} + 2\xi^\mu \partial_\mu + \frac{1}{2}\xi^\mu \xi_\mu - \frac{2(D-2)}{\ell^2}\right) \\ &= -\left(\bar{\Box} + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right), \qquad (86)\end{aligned}$$

where the second equality is valid for AdS-plane waves in the coordinates (60). Using the results in Ref. [18], $\Box(\phi\lambda_\alpha\lambda_\beta)$ can be written as

$$\Box(\phi\lambda_\alpha\lambda_\beta) = \bar{\Box}(\phi\lambda_\alpha\lambda_\beta) = -\lambda_\alpha\lambda_\beta\left(\mathcal{O} + \frac{2}{\ell^2}\right)\phi, \qquad (87)$$

which is again valid for any ϕ satisfying $\lambda^\mu \partial_\mu \phi = \partial_v \phi = 0$; therefore, $\Box S_{\mu\nu}$ becomes

$$\Box S_{\mu\nu} = -\lambda_\mu \lambda_\nu \left(\mathcal{O} + \frac{2}{\ell^2}\right)\rho = -\lambda_\mu \lambda_\nu \left(\mathcal{O} + \frac{2}{\ell^2}\right)\mathcal{O} V. \qquad (88)$$

Then, Eq. (84) becomes

$$\lambda_\mu \lambda_\nu \mathcal{O} \prod_{n=1}^{N} \left(\mathcal{O} + \frac{2}{\ell^2} - b_n\right) V = 0, \qquad (89)$$

where we also used the fact that for any ϕ satisfying $\partial_v \phi = 0$, $\mathcal{O}\phi$ also satisfies the same property $\partial_v \mathcal{O}\phi = 0$.

Note that Eq. (89) is linear in the metric function V which suggests the linearization of the field equations of the generic theory for the AdS-plane waves. To make this more explicit, by using Eq. (87) $S_{\mu\nu}$ can be put in the form

$$S_{\mu\nu} = -\left(\bar{\Box} + \frac{2}{\ell^2}\right)(\lambda_\mu \lambda_\nu V) = -\frac{1}{2}\left(\bar{\Box} + \frac{2}{\ell^2}\right)h_{\mu\nu}, \quad (90)$$

after defining $h_{\mu\nu} \equiv 2V\lambda_\mu \lambda_\nu$ with which the AdS-plane wave metric becomes $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. In addition, using $\partial_v \phi = 0 \Rightarrow \partial_v \mathcal{O}\phi = 0$ and Eq. (87), $\Box^n S_{\mu\nu}$ becomes

$$\Box^n S_{\mu\nu} = (-1)^n \lambda_\mu \lambda_\nu \left(\mathcal{O} + \frac{2}{\ell^2}\right)^n \mathcal{O}V = \bar{\Box}^n S_{\mu\nu}. \quad (91)$$

Once Eqs. (90) and (91) are considered in either Eq. (83) or Eq. (84), it is obvious that the field equations of the generic theory (2) for AdS-plane waves are linear in $h_{\mu\nu}$ as in the case of a perturbative expansion of the field equations around an (A)dS background for a small metric perturbation $\|h\| \equiv \|g - \bar{g}\| \ll 1$.

As in the case of pp waves, this observation suggests that there are two possible ways to find the field equations of the generic gravity theory for AdS-plane waves, namely, (i) derive the field equations and directly plug the AdS-plane wave metric *Ansatz* (48) into them, or (ii) linearize the derived field equations around the (A)dS background and put $h_{\mu\nu} = 2V\lambda_\mu \lambda_\nu$ into these linearized equations. Again, as we discuss in Sec. V, the idea in the second way of finding the field equations for AdS-plane waves provides a shortcut to finding the field equations of a gravity theory described with a Lagrangian density which is constructed by the Riemann tensor but not its derivatives. Finally, $h_{\mu\nu} = 2V\lambda_\mu \lambda_\nu$ is transverse, $\bar{\nabla}_\mu h^{\mu\nu} = 0$, and traceless, $\bar{g}^{\mu\nu}h_{\mu\nu} = 0$, so one needs only the linearized field equations for the transverse-traceless metric perturbation.

Just like the discussion in the pp-wave case, assuming nonvanishing and distinct b_n's, the most general solution of Eq. (89) is

$$V = V_E + \Re\left(\sum_{n=1}^N V_n\right), \quad (92)$$

where $\Re$ represents the real part and V_E is the solution to the cosmological Einstein theory, namely

$$\mathcal{O}V_E = -\left(\bar{\Box} + \frac{4z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V_E$$
$$= -\left(\frac{z^2}{\ell^2}\hat{\partial}^2 + \frac{(6-D)z}{\ell^2}\partial_z - \frac{2(D-3)}{\ell^2}\right)V_E = 0, \quad (93)$$

where $\hat{\partial}^2 \equiv \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{D-3}\frac{\partial^2}{\partial x^i \partial x^i}$. Here, the second equality follows from the results in Ref. [16]. In addition, the V_n's solve the equation of the quadratic gravity theory, i.e. $(\mathcal{O} + \frac{2}{\ell^2} - b_n)V_n = 0$. As a result, the AdS-plane wave solutions of Einstein gravity and quadratic gravity, which were summarized in the Introduction (with $M_n^2 = -b_n + \frac{2}{\ell^2}$),[6] also solve a generic gravity theory [15].

When we let $b_n = -M_n^2 + \frac{2}{\ell^2}$ for all $n = 1, 2, ..., N$ and assume real M_n^2's as they represent the masses of the excitations, then we can express the exact solution of the generic gravity theory depending on u and z as a sum of the Einsteinian Kaigorodov solution

$$V_E(u, z) = c(u)z^{D-3}, \quad (94)$$

where we omitted the $1/z^2$ solution as it can be absorbed into the background AdS metric by the redefinition of the coordinate v, and the functions V_n defined in Eq. (92) are given as

$$V_n(u, z) = z^{\frac{D-5}{2}}(c_{n,1}(u)z^{|\nu_n|} + c_{n,2}(u)z^{-|\nu_n|}),$$
$$n = 1, 2, ..., N, \quad (95)$$

where $\nu_n = \frac{1}{2}\sqrt{(D-1)^2 + 4\ell^2 M_n^2}$ and all M_n^2's are assumed to be distinct. On the other hand, there can be many special cases in which some of M_n^2's are equal. In fact, these special cases do appear in the critical gravity theories [22,23,30–32] and the corresponding solutions always involve logarithms; for example, for the four-dimensional case see Refs. [16,17]. Here, let us mention the extreme case in which all N masses vanish. In this case, the field equation takes the form

$$\mathcal{O}^{N+1}V(u,z) = [z^2\partial_z^2 + (6-D)z\partial_z - 2(D-3)]^{N+1}V(u,z)$$
$$= 0, \quad (96)$$

which has the solution

$$V(u,z) = z^{D-3}\sum_{n=0}^N c_{n,1}(u)\ln^n z + \frac{1}{z^2}\sum_{n=1}^N c_{n,2}(u)\ln^n z, \quad (97)$$

where again we considered the $1/z^2$ solution as being absorbed into the AdS part.

---

[6]Note that M_n^2 represents the mass of a massive spin-2 excitation around the AdS background. To prevent any confusion in its definition observe that $\mathcal{O} \sim -\bar{\Box}$. In addition, remember that in the pp-wave case, we also defined $M_{n,\text{flat}}^2$, the mass around the flat background, and these two masses are related by $\lim_{\ell \to \infty} M_n^2 = M_{n,\text{flat}}^2$.

IV. pp WAVES AND AdS-PLANE WAVES IN QUADRATIC GRAVITY

Since quadratic gravity played a central role in constructing the solutions of the generic gravity theory, let us explicitly study the field equations of quadratic gravity in the context of these wave solutions. The field equations of quadratic gravity [33]

$$
\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right)
$$
$$
+ 2\alpha R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + (2\alpha + \beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R
$$
$$
+ \beta\Box\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + 2\beta\left(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}
$$
$$
+ 2\gamma\left[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} + R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau} - 2R_{\mu\sigma}R_\nu{}^\sigma\right.
$$
$$
\left. - \frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho} - 4R^2_{\sigma\rho} + R^2)\right] = 0, \tag{98}
$$

for AdS-plane waves (48) reduce to a trace part and an apparently nonlinear wave-type equation on the traceless Ricci tensor [34]

$$
\left(\frac{\Lambda_0}{\kappa} + \frac{(D-1)(D-2)}{2\kappa\ell^2} - f\frac{(D-1)^2(D-2)^2}{2\ell^4}\right)g_{\mu\nu}
$$
$$
+ \beta\left(\Box + \frac{2}{\ell^2} - M^2\right)S_{\mu\nu} = 0, \tag{99}
$$

where $S_{\mu\nu}$ and M^2 are given in Eqs. (50) and (11), respectively, and f is

$$
f \equiv (D\alpha + \beta)\frac{(D-4)}{(D-2)^2} + \gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}. \tag{100}
$$

The trace part of Eq. (99)

$$
\frac{\Lambda_0}{\kappa} + \frac{(D-1)(D-2)}{2\kappa\ell^2} - f\frac{(D-1)^2(D-2)^2}{2\ell^4} = 0, \tag{101}
$$

determines the effective cosmological constant, that is the AdS radius ℓ. Since $\Box S_{\mu\nu} = \bar{\Box}S_{\mu\nu}$, the traceless part of Eq. (99) [after using Eq. (87)] further reduces to

$$
\left(\bar{\Box} + \frac{2}{\ell^2} - M^2\right)\left(\bar{\Box} + \frac{2}{\ell^2}\right)(\lambda_\mu\lambda_\nu V) = 0. \tag{102}
$$

This is an exact equation for the AdS-plane waves, but it is also important to realize that (defining $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$) these are also the linearized field equations for transverse-traceless fluctuations, which represent the helicity ± 2 excitations, about the AdS

background whose radius is determined by the trace part of Eq. (99).

In this work, we have been interested in the exact solutions and not perturbative excitations, but as a side remark we can note that the fact that AdS-plane waves and pp waves lead to the linearized equations can be used to put constraints on the original parameters of the theory once unitarity of the linearized excitations is imposed. For example, since the excitations cannot be tachyonic or ghost-like, $M^2 \geq 0$ and this immediately says that b_n cannot be complex.

To obtain the field equations for pp waves in this theory with $\Lambda_0 = 0$, one simply takes the $\ell \to \infty$ limit. Note that in this limit $S_{\mu\nu}$ becomes equal to $R_{\mu\nu}$.

V. WAVE SOLUTIONS OF $f(\text{Riemann})$ THEORY

Let us now consider a subclass of the generic theory (1) whose action is built only on the contractions of the Riemann tensor and not its derivatives. Namely, the action is given as

$$
I = \int d^D x\sqrt{-g}f(R^{\mu\nu}_{\alpha\beta}), \tag{103}
$$

where we specifically choose $R^{\mu\nu}_{\alpha\beta} \equiv R^{\mu\nu}{}_{\alpha\beta}$ as the argument to remove the functional dependence on the inverse metric $g^{\mu\nu}$ without losing any generality, because any higher-curvature combination can be written in terms of $R^{\mu\nu}_{\alpha\beta}$ without use of either metric or its inverse.

This class of theories constitutes an important subclass for two reasons. First, as we discussed above, pp waves and AdS-plane waves (actually, AdS waves in general) linearize the field equations of a generic gravity theory, that is both plugging the pp-wave (AdS-plane wave) metric $g_{\mu\nu} = \eta_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$ ($g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$) into the field equations and plugging $h_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$ into the linearized field equations around a flat (AdS) background yield the same field equations. Second, for the $f(R^{\mu\nu}_{\alpha\beta})$ theory, one can construct a quadratic curvature gravity theory which has the same vacua and the same linearized field equations as the original $f(R^{\mu\nu}_{\alpha\beta})$ theory (see Refs. [35–41]). Once one constructs the equivalent quadratic curvature action (EQCA) corresponding to Eq. (103), by using the effective parameters of EQCA in the results obtained for the quadratic gravity case in Sec. IV, one can obtain the field equations of Eq. (103) for AdS-plane waves and pp waves without deriving the field equations of Eq. (103). The use of the EQCA procedure in finding the field equations for AdS-plane waves and pp waves provides a fair amount of simplification over the standard method of finding the field equations which can be quite complicated depending on the function f. With a known f, one can use the procedure given in Ref. [40] to find the corresponding EQCA:

(1) Calculate $f(\bar{R}^{\mu\nu}_{\rho\sigma})$, that is the value of the Lagrangian density for the maximally symmetric background

$$\bar{R}^{\mu\nu}_{\rho\sigma} = -\frac{1}{\ell^2}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho). \tag{104}$$

In addition, one takes the first- and second-order derivatives of $f(R^{\mu\nu}_{\rho\sigma})$ with respect to the Riemann tensor, and calculates them again for the background (104) to find

$$\left[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R, \tag{105}$$

$$\frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\alpha\beta}_{\lambda\gamma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma}R^{\alpha\beta}_{\lambda\gamma} \equiv \tilde{\alpha}R^2 + \tilde{\beta}R^\mu_\nu R^\nu_\mu + \tilde{\gamma}(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2), \tag{106}$$

where ζ, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ are to be determined from these equations.

(2) Construct the action

$$I_{\text{EQCA}} = \int d^D x\sqrt{-g}\left[\frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0) + \tilde{\alpha}R^2 + \tilde{\beta}R^\mu_\nu R^\nu_\mu + \tilde{\gamma}(R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2)\right], \tag{107}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are defined in step 1, while the remaining two parameters are given as

$$\frac{1}{\tilde{\kappa}} = \zeta + \frac{2}{\ell^2}[(D-1)(D\tilde{\alpha} + \tilde{\beta}) + (D-2)(D-3)\tilde{\gamma}], \tag{108}$$

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} = -\frac{1}{2}f(\bar{R}^{\mu\nu}_{\rho\sigma}) - \frac{D(D-1)}{2\ell^2}\left(\zeta + \frac{1}{\ell^2}[(D-1)(D\tilde{\alpha} + \tilde{\beta}) + (D-2)(D-3)\tilde{\gamma}]\right). \tag{109}$$

After constructing the EQCA corresponding to Eq. (103), the field equations of Eq. (103) for AdS-plane waves can be found by substituting the effective parameters of Eq. (107) into Eq. (99). Since these field equations are solved by the AdS-plane wave solutions of Einstein gravity and quadratic gravity listed in the Introduction, the AdS-plane wave solutions of Eq. (103) simply follow from these solutions by using the effective cosmological constant of Eq. (103) and M^2 of Eq. (103), which is calculated by putting the effective parameters of the EQCA into Eq. (11). The effective cosmological constant of Eq. (103) can be found from Eq. (101) after putting the effective parameters of the EQCA into it. Note that although Eq. (101) is, apparently, a quadratic equation in $1/\ell^2$, after putting the effective parameters into this equation it yields a different dependence on $1/\ell^2$ since these effective parameters also depend on ℓ^2.

As in the case of the quadratic curvature gravity, the $\ell \to \infty$ limit in the AdS-plane wave field equations gives the field equations for the pp waves for the theory with $\Lambda_0 = 0$. Equivalently, one may find the curvature expansion of $f(R^{\mu\nu}_{\alpha\beta})$ up to the quadratic order, and this part of the action determines the field equations for the pp-wave metric.

As an application with a given $f(R^{\mu\nu}_{\alpha\beta})$, we consider the cubic curvature gravity generated by the bosonic string theory at the second order in the inverse string tension α' [42] and the Lanczos-Lovelock theory [43,44].

A. Cubic gravity generated by string theory

The effective action for the bosonic string at $O[(\alpha')^2]$ is

$$I = \frac{1}{\kappa}\int d^D x\sqrt{-g}\left[R + \frac{\alpha'}{4}(R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2)\right.$$
$$\left. + \frac{(\alpha')^2}{24}(-2R^{\mu\alpha}_{\nu\beta}R^{\nu\gamma}_{\mu\lambda}R^{\beta\lambda}_{\alpha\gamma} + R^{\mu\nu}_{\alpha\beta}R^{\gamma\lambda}_{\mu\nu}R^{\alpha\beta}_{\gamma\lambda})\right], \tag{110}$$

where the bare cosmological constant is not introduced, so the theory admits a flat background in addition to the (A)dS ones. In Ref. [38], the EQCA of Eq. (110) was calculated as

$$I_{\text{EQCA}} = \frac{1}{\kappa}\int d^D x\sqrt{-g}\left[\left(1 + \frac{\alpha'^2(D-5)}{4\ell^4}\right)\right.$$
$$\times\left(R + 2\frac{\alpha'^2 D(D-1)(D-5)}{6(D-5)\alpha'^2\ell^2 + 24\ell^6}\right)$$
$$+ \frac{\alpha'^2}{2\ell^2}R^2 - \frac{7\alpha'^2}{4\ell^2}R^\mu_\nu R^\nu_\mu + \frac{\alpha'}{4}\left(1 - \frac{2\alpha'}{\ell^2}\right)$$
$$\left.\times(R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2)\right], \tag{111}$$

where the effective parameters depend on the yet-to-be-determined effective cosmological constant represented through the AdS radius ℓ. The field equation for ℓ^2 can

be found by using the effective parameters of Eq. (111) in Eq. (101) as

$$\frac{1}{\ell^2}\left(1 - \frac{(D-3)(D-4)\alpha'}{4\ell^2} - \frac{(D-5)(D-6)\alpha'^2}{12(D-2)\ell^4}\right) = 0. \tag{112}$$

Note that although we started with a quadratic equation in $1/\ell^2$, that is Eq. (101), we obtained a cubic equation as expected for a cubic curvature theory. For $D \geq 3$, there is always an AdS solution because $1/\ell^2 \sim \alpha'$ in addition to the flat solution, so that the theory admits an AdS-plane wave solution.

In addition to effective cosmological constant, we need the mass parameter M^2 to write the AdS-plane wave solutions. Using EQCA parameters in Eq. (11), M^2 can be found as [15]

$$M^2 = \frac{4\ell^2}{7\alpha'^2} - \frac{2(D-3)(D-4)}{7\alpha'} + \frac{29 - 9D}{7\ell^2}. \tag{113}$$

The AdS-plane wave solutions given in the Introduction are the solutions of Eq. (110) with this M^2. For example, for the $\xi = 0$ case one has

$$V(u, z) = c_1(u)z^{D-3} + z^{\frac{D-5}{2}}\left(c_2(u)z^{\frac{1}{2}\sqrt{(D-1)^2+4\ell^2M^2}}\right.$$
$$\left. + c_3(u)z^{-\frac{1}{2}\sqrt{(D-1)^2+4\ell^2M^2}}\right). \tag{114}$$

Here, we note that when using the solutions of Eq. (112) in Eq. (113), M^2 has the form $1/\alpha'$ (which is also suggested by dimensional analysis) and becomes negative for $D > 3$. On the other hand, the BF bound, that is $M^2 \geq -\frac{(D-1)^2}{4\ell^2}$, is satisfied for $D \leq 6$.

To discuss pp-wave solutions, one should take the $\ell \to \infty$ limit in the AdS-plane wave field equations. Taking this limit in Eq. (113) yields $M^2 \to \infty$ which suggests the absence of the massive operator part in the pp-wave field equations. This is in fact the case which becomes more clear by taking the $\ell \to \infty$ limit at the EQCA level. In this limit, Eq. (111) reduces to Einstein-Gauss-Bonnet theory which is the quadratic curvature order of the original action (110).[7] Therefore, as we discussed above, the quadratic curvature order of the original action determines the pp-wave field equations, and here it is the Einstein-Gauss-Bonnet theory whose equations reduce to the field equations of Einstein gravity at the linearized level. Therefore, the massive operator is absent and the pp-wave solutions of Eq. (110) are only the Einsteinian solutions.

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[7]This is expected since the EQCA is just the Taylor series expansion in curvature around the maximally symmetric background. Then, once the flat limit is taken in the EQCA, the action reduces to the quadratic curvature order of the original action.

B. Lanczos-Lovelock theory

The Lanczos-Lovelock theory is a special $f(R^{\mu\nu}_{\alpha\beta})$ theory which has at most second-order derivatives of the metric in its field equations just like Einstein gravity. Therefore, one expects a second-order differential equation for the metric function V as the (traceless) field equations for pp waves and AdS-plane waves. To find the explicit form of the field equations, one needs to construct the EQCA for the Lanczos-Lovelock theory given by the Lagrangian density

$$f_{\text{L-L}} = \sum_{n=0}^{[\frac{D}{2}]} C_n \delta^{\mu_1\cdots\mu_{2n}}_{\nu_1\cdots\nu_{2n}} \prod_{p=1}^{n} R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{115}$$

where the C_n's are dimensionful constants, $\delta^{\mu_1\cdots\mu_{2n}}_{\nu_1\cdots\nu_{2n}}$ is the generalized Kronecker delta, and $[\frac{D}{2}]$ denotes the integer part of its argument. In Ref. [39], the EQCA of Eq. (115) was calculated as

$$I_{\text{EQCA}} = \int d^D x \sqrt{-g}\left[\frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0)\right.$$
$$\left. + \tilde{\gamma}(R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2)\right], \tag{116}$$

where the effective parameters are the effective Newton's constant,

$$\frac{1}{\tilde{\kappa}} \equiv 2(D-2)!\sum_{n=0}^{[\frac{D}{2}]}(-1)^n C_n \frac{n(n-2)}{(D-2n)!}\left(\frac{2}{\ell^2}\right)^{n-1}, \tag{117}$$

the effective cosmological constant,

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} \equiv -\frac{D!}{4}\sum_{n=0}^{[\frac{D}{2}]}(-1)^n C_n \frac{(n-1)(n-2)}{(D-2n)!}\left(\frac{2}{\ell^2}\right)^n, \tag{118}$$

and the effective Gauss-Bonnet coefficient

$$\tilde{\gamma} \equiv 2(D-4)!\sum_{n=0}^{[\frac{D}{2}]}(-1)^n C_n \frac{n(n-1)}{(D-2n)!}\left(\frac{2}{\ell^2}\right)^{n-2}. \tag{119}$$

The AdS radius appearing in these effective parameters satisfies the equation

$$0 = \sum_{n=0}^{[\frac{D}{2}]}(-2)^n C_n \frac{(D-2n)}{(D-2n)!}\left(\frac{1}{\ell^2}\right)^n, \tag{120}$$

which can be found by plugging Eqs. (117)–(119) into Eq. (101). Again, notice that the quadratic equation (101) yields a $[\frac{D}{2}]$th-order equation in $1/\ell^2$ after the use of the ℓ-dependent parameters of the EQCA. Note that for even dimensions, the $D = 2n$ term does not contribute to the field equation (120). Since the EQCA is in the

Einstein-Gauss-Bonnet form, the traceless part of the field equations reduces to

$$\mathcal{O}V = 0, \qquad (121)$$

where $\mathcal{O}$ is defined in Eq. (93), if $1/\tilde{\kappa} \neq 0$, that is

$$\sum_{n=0}^{[\frac{D}{2}]} (-1)^n C_n \frac{n(D-2n)}{(D-2n)!} \left(\frac{2}{\ell^2}\right)^{n-1} \neq 0, \qquad (122)$$

holds. Thus, the Einsteinian solutions, such as the Kaigorodov solution

$$V(u, z) = c(u) z^{D-3}, \qquad (123)$$

solve Eq. (121). Note that even though V_E apparently does not show a dependence on the parameters of the theory, the metric depends on all of the parameters via the AdS radius ℓ. Hence, the above exercise is nontrivial.

Lastly, for the Chern-Simons Lovelock theory in odd dimensions [45], the constraint $1/\tilde{\kappa} = 0$ is satisfied, so the field equation becomes trivial.

VI. EINSTEINIAN WAVE SOLUTIONS OF THE GENERIC THEORY

A natural generalization of the above exercises is that the AdS-plane waves of the cosmological Einstein theory solve the generic gravity theory (2). The metric function V does not depend on the parameters of the theory; therefore, it is intact for all theories. But, the nontrivial part of the computation is to find the AdS radii for each theory. Fortunately, with the equivalent linear action (ELA) procedure that we used in Refs. [36,37,41], all one needs to do is (i) calculate the Lagrangian density in the maximally symmetric background (104) (let us call it $\bar{f}$), and (ii) compute the derivative of the Lagrangian density with respect to the Riemann tensor and evaluate it again in Eq. (104). With this result one finds

$$\left[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R, \qquad (124)$$

which is in fact the definition of ζ. Using these results, the ELA, which has the same vacua as the original theory, can be constructed as

$$I_{\text{ELA}} = \frac{1}{\tilde{\kappa}} \int d^D x \sqrt{-g}(R - 2\tilde{\Lambda}_0), \qquad (125)$$

where the effective Newton's constant and the effective bare cosmological constant are

$$\frac{1}{\tilde{\kappa}} = \zeta, \qquad (126)$$

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} = -\frac{1}{2}\bar{f} - \frac{D(D-1)}{2\ell^2}\zeta. \qquad (127)$$

Then, the AdS radii can be calculated from

$$\ell^2 = -\frac{(D-1)(D-2)}{2\tilde{\Lambda}_0}. \qquad (128)$$

Note that in the Lagrangian, the terms involving the derivatives of the Riemann tensor do not contribute to the maximally symmetric vacua at all because the field equations derived from these derivative terms always involve the derivatives of the Riemann tensor which vanish for the maximally symmetric metric.

As an example of this procedure, let us consider the conformal gravity with derivative terms in $D = 6$ dimensions.

A. Conformal gravity in $D = 6$

Conformal gravity in six dimensions[8] has the Lagrangian density [46,48]

$$\mathcal{L}_{\text{Conf}} = \beta\left(RR^\mu_\nu R^\nu_\mu - \frac{3}{25}R^3 - 2R^\mu_\nu R^\rho_\sigma R^{\nu\sigma}_{\mu\rho}\right.$$
$$\left. -R^\mu_\nu \Box R^\nu_\mu + \frac{3}{10}R\Box R\right). \qquad (129)$$

With the AdS-plane wave *Ansatz*, the field equations coming from this Lagrangian, which are given in Ref. [46], reduce to

$$\left(\Box + \frac{8}{\ell^2}\right)\left(\Box + \frac{6}{\ell^2}\right)S_{\mu\nu} = 0. \qquad (130)$$

Using the ELA procedure above (see Appendix C), it is easy to show that for this purely cubic theory the AdS radius is not fixed; therefore, any maximally symmetric space is a solution which is to be expected since the theory is conformal with no internal scale. But, once one imposes the existence of an AdS vacuum, one necessarily breaks the symmetry in the vacuum and picks up a unique cosmological constant (in Ref. [46], $\Lambda = -10$ was chosen in the $\ell = 1$ units).

To further reduce Eq. (130), using Eqs. (87) and (90) yields

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[8]Note that to define a conformal gravity in six dimensions, one can also use the two independent scalars constructed from three Weyl tensors; see for example Refs. [46,47]. For this purely cubic Weyl theory, the EQCA and the linearized field equations will be identically zero, so the AdS-plane wave field equations become trivial. The version of six-dimensional conformal gravity we have chosen here is for discussing the presence of derivatives of the Riemann tensor in the action.

$$\left(\Box+\frac{8}{\ell^2}\right)\left(\Box+\frac{6}{\ell^2}\right)\left(\Box+\frac{2}{\ell^2}\right)(\lambda_\mu\lambda_\nu V)=0, \quad (131)$$

which still looks like a nonlinear differential equation since the d'Alembertian operators are with respect to the full metric involving the V part. But, this apparent nonlinearity is a red herring since $\Box^n(\lambda_\alpha\lambda_\beta V) = \bar{\Box}^n(\lambda_\alpha\lambda_\beta V)$. In addition, one can move λ vectors to the left with the help of Eqs. (87) and (93) implying

$$\left(\Box+\frac{2}{\ell^2}\right)(\lambda_\alpha\lambda_\beta V) = \left(\bar{\Box}+\frac{2}{\ell^2}\right)(\lambda_\alpha\lambda_\beta V)$$

$$= \lambda_\alpha\lambda_\beta\left(\frac{z^2}{\ell^2}\hat{\partial}^2-\frac{6}{\ell^2}\right)V, \quad (132)$$

and one gets a linear differential equation

$$\lambda_\mu\lambda_\nu\left(\frac{z^2}{\ell^2}\hat{\partial}^2\right)\left(\frac{z^2}{\ell^2}\hat{\partial}^2-\frac{2}{\ell^2}\right)\left(\frac{z^2}{\ell^2}\hat{\partial}^2-\frac{6}{\ell^2}\right)V=0. \quad (133)$$

Assuming $V=V(u,z)$, the general solution reads

$$V(u,z)=\frac{c_1}{z^2}+\frac{c_2}{z}+c_3+c_4z+c_5z^2+c_6z^3, \quad (134)$$

where the first term can be added to the "background" AdS part and $c_i=c_i(u)$. Note that this is also the general solution to the linearized equations with $h_{\mu\nu}=2V\lambda_\mu\lambda_\nu$.

To this conformal $D=6$ action, one can add the cosmological Einstein and Weyl square theories as [32]

$$\mathcal{L}_{6D}=R+\frac{20}{\ell^2}+\frac{\alpha}{2}C^{\mu\nu}_{\alpha\beta}C^{\alpha\beta}_{\mu\nu}-\mathcal{L}_{\text{Conf}}, \quad (135)$$

whose field equations (by using the AdS-plane wave *Ansatz*) reduce to

$$\left[\beta\left(\Box+\frac{8}{\ell^2}\right)\left(\Box+\frac{6}{\ell^2}\right)+\frac{3}{2}\alpha\left(\Box+\frac{6}{\ell^2}\right)+1\right]S_{\mu\nu}=0. \quad (136)$$

Unlike the purely cubic theory above, this theory has a unique vacuum with $\Lambda=-10/\ell^2$ which is fixed by the cosmological Einstein part: neither the quadratic Weyl piece nor the cubic part contributes to the effective cosmological constant. Again assuming $V=V(u,z)$, the general AdS-plane wave solution to Eq. (136) with generic α and β consists of six power terms

$$V(u,z)=\sum_{i=1}^{6}c_i(u)z^{n_i}, \quad (137)$$

with powers

$$n_1=-2, \quad n_2=3,$$

$$n_{3,4,5,6}=\frac{1}{2}\pm\sqrt{\frac{5}{4}-\frac{3\alpha\ell^2}{4\beta}\pm\sqrt{\left(1+\frac{3\alpha\ell^2}{4\beta}\right)^2-\frac{\ell^4}{\beta}}}, \quad (138)$$

where again the first term can be added to the "background" AdS part with no consequence. The second term is the Kaigorodov solution, which can be expected without doing any calculation, and the rest are the nontrivial pieces.

Let us consider the specific case of the "tricritical gravity," that is $\alpha=-5\ell^2/12$ and $\beta=\ell^4/16$ [32], for which $n_{3,4,5,6}$ becomes -2 and 3, so that the differential equation (136) degenerates into the form

$$\lambda_\mu\lambda_\nu\left(\frac{z^2}{\ell^2}\hat{\partial}^2-\frac{6}{\ell^2}\right)^3V=0, \quad (139)$$

with nontrivial logarithmic solutions in addition to the expected Einsteinian parts, which are AdS and Kaigorodov parts,

$$V(u,z)=\frac{1}{z^2}\left[c_1+c_2\ln\left(\frac{z}{\ell}\right)+c_3\ln^2\left(\frac{z}{\ell}\right)\right]$$

$$+z^3\left[c_4+c_5\ln\left(\frac{z}{\ell}\right)+c_6\ln^2\left(\frac{z}{\ell}\right)\right], \quad (140)$$

where again $c_i=c_i(u)$. Note that both Eq. (137) and Eq. (140) are also the general solutions to the corresponding linearized equations for transverse-traceless perturbations.

VII. CONCLUSION

We have shown that the AdS-plane wave metric solves the most general gravity theory whose Lagrangian is an arbitrary function of the metric, the Riemann tensor and the covariant derivatives of the Riemann tensor. In doing so, we have also given the explicit proof of the theorem, briefly proved in Ref. [15], that two-tensors in these spacetimes can be written as a sum of $\Box^n S_{\mu\nu}$ with $n=0,1,2,\ldots$. In our proof, the pp-wave solution played a role, so we have revisited this spacetime and also constructed novel solutions for quadratic gravity that also extend to the generic gravity theories. We have devoted several sections to example theories such as the cubic curvature gravity generated by string theory, Lanczos-Lovelock gravity, and the recent $D=6$ conformal gravity, its nonconformal modifications, and tricritical gravity.

Our exact solutions linearize the field equations, and hence they also match the perturbative solutions for transverse-traceless perturbations. Therefore, the particle spectrum of these theories can be read from these metrics, save the spin-0 mode. Once the unitarity constraints on the particle spectrum are considered, this result can be used to choose the viable theories. For example, requiring

nontachyonic physical excitations, that is $M_n^2 = -b_n + \frac{2}{\ell^2} > 0$ and $\lim_{\ell \to \infty} M_n^2 = M_{n,\text{flat}}^2 > 0$, puts tight constraints on the theory parameters such that they should yield real, upper-bounded b_n's.

In obtaining our solutions, we have reduced the field equations of the most general gravity theory to N massive Klein-Gordon equations satisfied by V_n, $n = 1, 2, \ldots, N$, and one massless Klein Gordon equation satisfied by V_E such that the general solution to the theory is $V = V_E + \sum_{n=1}^{N} V_n$. We have given sample solutions of these equations when all the masses are different and when the masses are equal, that is the critical gravity case. When some M_n's are equal, the solutions involve logarithmic parts. We gave the general logarithmic solution for the case where all the M_n's are zero. The specific examples for this case are the critical gravity theory studied in Refs. [16,17] and the tricritical gravity theory that we discussed here.

We have generalized and unified the previous works [4,6,9] on pp-wave spacetimes. Namely, for any theory the field equations for the pp-wave metrics in the Kerr-Schild form reduce to

$$\prod_{n=1}^{N} (\Box + b_n) R_{\mu\nu} = 0, \tag{141}$$

which can be further reduced to the Einstein gravity ones under the assumptions of Refs. [4,6,9]. Another fact is that the results we obtained in this work remain intact for the theories with pure radiation sources, that is $T_{\mu\nu} dx^\mu dx^\nu = T_{uu} du^2$. For example, for a source having the functional dependence $T_{\mu\nu} = T_{\mu\nu}(u)$, the plane waves solving $a_0 R_{\mu\nu} = T_{\mu\nu}$ are also particular solutions to the generic gravity theory since $\Box R_{\mu\nu} = 0$ for these plane waves.

A possible way to consider sources is by introducing a nonminimally coupled scalar field. In Refs. [49] and [50], it was shown that a nonminimally coupled scalar field with a specific potential can support pp-wave and AdS-plane wave solutions for three-dimensional Einstein gravity. This specific potential form was generalized to higher dimensions in Ref. [51]. Using this result together with those discussed here, it is possible to find pp-wave and AdS-plane wave solutions to some higher-curvature gravity theories coupled to nonminimally coupled scalar fields [52].

In a future work, the explicit proof presented for the AdS-plane waves will be extended to the Kerr-Schild-Kundt class discussed in Refs. [15,18].

APPENDIX A: TENSORS OF KERR-SCHILD-KUNDT SPACETIMES

In this Appendix, we will prove Eq. (51). For the Kerr-Schild-Kundt class of metrics

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \tag{A1}$$

where $\bar{g}_{\mu\nu}$ is the AdS metric the following relations hold:

$$\lambda^\mu \lambda_\mu = 0, \qquad \nabla_\mu \lambda_\nu = \xi_{(\mu} \lambda_{\nu)},$$
$$\xi_\mu \lambda^\mu = 0, \qquad \lambda^\mu \partial_\mu V = 0. \tag{A2}$$

The Riemann tensor for this class of metrics is given in Eq. (B25) of Ref. [18] as

$$\begin{aligned}
R^\mu{}_{\alpha\nu\beta} = & \bar{R}^\mu{}_{\alpha\nu\beta} + 2\lambda_\alpha \lambda_{[\nu} \bar{\nabla}_{\beta]} \partial^\mu V - 2\lambda^\mu \lambda_{[\nu} \bar{\nabla}_{\beta]} \partial_\alpha V \\
& + \lambda_{[\nu} \xi_{\beta]} (\lambda_\alpha \partial^\mu V - \lambda^\mu \partial_\alpha V + \lambda_\alpha \xi^\mu V) \\
& + (\lambda_\alpha \xi^\mu - \lambda^\mu \xi_\alpha) \lambda_{[\nu} \partial_{\beta]} V \\
& + 2V\lambda^\mu (\lambda_\alpha \bar{\nabla}_{[\nu} \xi_{\beta]} - \lambda_{[\nu} \bar{\nabla}_{\beta]} \xi_\alpha). \tag{A3}
\end{aligned}$$

One can free ξ from derivatives with the help of $[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \lambda_\beta = \bar{R}_{\mu\nu\beta}{}^\rho \lambda_\rho$ yielding

$$2\lambda_{[\nu} \bar{\nabla}_{\mu]} \xi_\beta - 2\lambda_\beta \bar{\nabla}_{[\nu} \xi_{\mu]} - \xi_\beta \lambda_{[\nu} \xi_{\mu]} = -\frac{2}{\ell^2} (g_{\mu\beta} \lambda_\nu - \lambda_\mu g_{\nu\beta}). \tag{A4}$$

Then, the Riemann tensor reduces to

$$\begin{aligned}
R^\mu{}_{\alpha\nu\beta} = & \bar{R}^\mu{}_{\alpha\nu\beta} + \frac{4}{\ell^2} V\lambda^\mu \lambda_{[\nu} \bar{g}_{\beta]\alpha} + \lambda_\alpha (2\lambda_{[\nu} \bar{\nabla}_{\beta]} \partial^\mu V + \lambda_{[\nu} \xi_{\beta]} \partial^\mu V + \lambda_{[\nu} \xi_{\beta]} \xi^\mu V + \xi^\mu \lambda_{[\nu} \partial_{\beta]} V) \\
& - \lambda^\mu (2\lambda_{[\nu} \bar{\nabla}_{\beta]} \partial_\alpha V + \lambda_{[\nu} \xi_{\beta]} \partial_\alpha V + \lambda_{[\nu} \xi_{\beta]} \xi_\alpha V + \xi_\alpha \lambda_{[\nu} \partial_{\beta]} V), \tag{A5}
\end{aligned}$$

whose (0,4)-rank tensor version is

$$R_{\mu\alpha\nu\beta} = \bar{R}_{\mu\alpha\nu\beta} + \lambda_\alpha(2\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\mu V + \lambda_{[\nu}\xi_{\beta]}\partial_\mu V + \lambda_{[\nu}\xi_{\beta]}\xi_\mu V + \xi_\mu\lambda_{[\nu}\partial_{\beta]}V)$$
$$- \lambda_\mu(2\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\alpha V + \lambda_{[\nu}\xi_{\beta]}\partial_\alpha V + \lambda_{[\nu}\xi_{\beta]}\xi_\alpha V + \xi_\alpha\lambda_{[\nu}\partial_{\beta]}V). \tag{A6}$$

Now, let us calculate the Weyl tensor

$$C_{\mu\alpha\nu\beta} \equiv R_{\mu\alpha\nu\beta} - \frac{2}{D-2}(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu}) + \frac{2}{(D-1)(D-2)}Rg_{\mu[\nu}g_{\beta]\alpha}. \tag{A7}$$

The last term involving the scalar curvature has the form

$$Rg_{\mu[\nu}g_{\beta]\alpha} = \bar{R}\bar{g}_{\mu[\nu}\bar{g}_{\beta]\alpha} - 2\bar{R}V(\lambda_\alpha\lambda_{[\nu}\bar{g}_{\beta]\mu} - \lambda_\mu\lambda_{[\nu}\bar{g}_{\beta]\alpha}). \tag{A8}$$

The second term involving the Ricci tensor has the following form by using $R_{\alpha\beta} = -\frac{(D-1)}{\ell^2}g_{\alpha\beta} + \rho\lambda_\alpha\lambda_\beta$:

$$g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu} = \bar{g}_{\mu[\nu}\bar{R}_{\beta]\alpha} - \bar{g}_{\alpha[\nu}\bar{R}_{\beta]\mu} - \left(\rho + \frac{4V\bar{R}}{D}\right)(\lambda_\alpha\lambda_{[\nu}\bar{g}_{\beta]\mu} - \lambda_\mu\lambda_{[\nu}\bar{g}_{\beta]\alpha}). \tag{A9}$$

With the help of the above results and $\bar{C}_{\mu\alpha\nu\beta} = 0$, the Weyl tensor reduces to

$$C_{\mu\alpha\nu\beta} = \lambda_\alpha(2\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\mu V + \lambda_{[\nu}\xi_{\beta]}\partial_\mu V + \lambda_{[\nu}\xi_{\beta]}\xi_\mu V + \xi_\mu\lambda_{[\nu}\partial_{\beta]}V) - \lambda_\mu(2\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\alpha V + \lambda_{[\nu}\xi_{\beta]}\partial_\alpha V + \lambda_{[\nu}\xi_{\beta]}\xi_\alpha V + \xi_\alpha\lambda_{[\nu}\partial_{\beta]}V)$$
$$+ \frac{2}{D-2}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)(\lambda_\alpha\lambda_{[\nu}\bar{g}_{\beta]\mu} - \lambda_\mu\lambda_{[\nu}\bar{g}_{\beta]\alpha}), \tag{A10}$$

where one can convert $\bar{g}_{\mu\nu}$ to $g_{\mu\nu}$ without producing any additional term. Also, using Eqs. (B4) and (B5) of Ref. [18], one has

$$\bar{\nabla}_\alpha\partial_\beta V = \nabla_\alpha\partial_\beta V - \lambda_\alpha\lambda_\beta(\partial^\lambda V)\partial_\lambda V, \tag{A11}$$

which can be used to write $C_{\mu\alpha\nu\beta}$ in terms of the full metric quantities as

$$C_{\mu\alpha\nu\beta} = \lambda_\mu\lambda_\nu\left[-\nabla_\alpha\partial_\beta V - \xi_{(\alpha}\partial_{\beta)}V - \frac{1}{2}\xi_\alpha\xi_\beta V - \frac{1}{D-2}g_{\alpha\beta}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right]$$
$$+ \lambda_\alpha\lambda_\beta\left[-\nabla_\mu\partial_\nu V - \xi_{(\mu}\partial_{\nu)}V - \frac{1}{2}\xi_\mu\xi_\nu V - \frac{1}{D-2}g_{\mu\nu}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right]$$
$$- \lambda_\mu\lambda_\beta\left[-\nabla_\alpha\partial_\nu V - \xi_{(\alpha}\partial_{\nu)}V - \frac{1}{2}\xi_\alpha\xi_\nu V - \frac{1}{D-2}g_{\alpha\nu}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right]$$
$$- \lambda_\alpha\lambda_\nu\left[-\nabla_\mu\partial_\beta V - \xi_{(\mu}\partial_{\beta)}V - \frac{1}{2}\xi_\mu\xi_\beta V - \frac{1}{D-2}g_{\mu\beta}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right]. \tag{A12}$$

Thus, the definition

$$\Omega_{\alpha\beta} = -\left[\nabla_\alpha\partial_\beta V + \xi_{(\alpha}\partial_{\beta)}V + \frac{1}{2}\xi_\alpha\xi_\beta V + \frac{1}{D-2}g_{\alpha\beta}\left(\rho - \frac{2(D-2)}{\ell^2}V\right)\right], \tag{A13}$$

reduces $C_{\mu\alpha\nu\beta}$ to the desired Type-N form

$$C_{\mu\alpha\nu\beta} = \lambda_\mu\lambda_\nu\Omega_{\alpha\beta} + \lambda_\alpha\lambda_\beta\Omega_{\mu\nu} - \lambda_\mu\lambda_\beta\Omega_{\alpha\nu} - \lambda_\alpha\lambda_\nu\Omega_{\mu\beta}$$
$$= 4\lambda_{[\mu}\Omega_{\alpha][\beta}\lambda_{\nu]}. \tag{A14}$$

APPENDIX B: BIANCHI IDENTITIES FOR THE WEYL TENSOR

The once-contracted Bianchi identity

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu R_{\alpha\beta} - \nabla_\alpha R_{\mu\beta}, \qquad \text{(B1)}$$

for constant R yields

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu S_{\alpha\beta} - \nabla_\alpha S_{\mu\beta}, \qquad \text{(B2)}$$

which then leads to

$$\nabla^\mu C_{\mu\alpha\nu\beta} = \nabla_\nu S_{\beta\alpha} - \nabla_\beta S_{\nu\alpha}$$
$$- \frac{2}{D-2}\nabla^\mu(g_{\mu[\nu}S_{\beta]\alpha} - g_{\alpha[\nu}S_{\beta]\mu}). \qquad \text{(B3)}$$

Using the twice-contracted Bianchi identity, that is $\nabla^\mu S_{\mu\nu} = 0$ for constant R, one gets

$$\nabla^\mu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2}(\nabla_\nu S_{\beta\alpha} - \nabla_\beta S_{\nu\alpha}), \qquad \text{(B4)}$$

which is the once-contracted Bianchi identity of the Weyl tensor for constant-curvature spacetimes.

Now, let us discuss $\nabla^\mu\nabla^\nu C_{\mu\alpha\nu\beta}$ which becomes

$$\nabla^\mu\nabla^\nu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2}(\Box S_{\alpha\beta} - \nabla^\mu\nabla_\alpha S_{\mu\beta}), \qquad \text{(B5)}$$

for constant-curvature spacetimes. Then, using $\nabla^\mu\nabla_\sigma S_{\mu\nu} = \frac{R}{D-1}S_{\sigma\nu}$, which holds for the metrics (A1), one gets

$$\nabla^\mu\nabla^\nu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2}\left(\Box S_{\alpha\beta} - \frac{R}{D-1}S_{\alpha\beta}\right), \qquad \text{(B6)}$$

which proves Eq. (80).

APPENDIX C: EQUIVALENT LINEAR ACTION OF CONFORMAL GRAVITY

Without finding the complicated field equations of the six-dimensional conformal gravity, let us show a method that leads to the effective cosmological constant. First, we note that the effective cosmological constant of a generic gravity theory is determined by only the nonderivative Riemann terms appearing in the action because the field equations derived from the terms involving the derivative of the Riemann tensor always yield zero after evaluating them for the maximally symmetric background

$$\bar{R}^{\mu\nu}_{\rho\sigma} = -\frac{1}{\ell^2}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho). \qquad \text{(C1)}$$

Thus, we need to focus on the terms involving the Riemann tensor but not its derivatives. The procedure described in Sec. VI for the construction of the ELA is based on the following Taylor series expansion of the Lagrangian density:

$$f_{\text{ELA}}(R^{\mu\nu}_{\alpha\beta}) = f(\bar{R}^{\mu\nu}_{\alpha\beta}) + \left[\frac{\partial f}{\partial R^{\rho\sigma}_{\eta\theta}}\right]_{\bar{R}^{\rho\sigma}_{\eta\theta}}(R^{\rho\sigma}_{\eta\theta} - \bar{R}^{\rho\sigma}_{\eta\theta}). \qquad \text{(C2)}$$

However, when doing computations, one may prefer to consider the functional dependence of the $f(R^{\mu\nu}_{\alpha\beta})$ theory as $f(R, R^\mu_\nu, R^{\mu\nu}_{\alpha\beta})$ and one has

$$f_{\text{ELA}}(R, R^\mu_\nu, R^{\mu\nu}_{\alpha\beta}) = f(\bar{R}, \bar{R}^\mu_\nu, \bar{R}^{\mu\nu}_{\alpha\beta}) + \left[\frac{\partial f}{\partial R}\right]_{\bar{R}}(R - \bar{R})$$
$$+ \left[\frac{\partial f}{\partial R^\rho_\sigma}\right]_{\bar{R}^\rho_\sigma}(R^\rho_\sigma - \bar{R}^\rho_\sigma)$$
$$+ \left[\frac{\partial f}{\partial R^{\rho\sigma}_{\eta\theta}}\right]_{\bar{R}^{\rho\sigma}_{\eta\theta}}(R^{\rho\sigma}_{\eta\theta} - \bar{R}^{\rho\sigma}_{\eta\theta}). \qquad \text{(C3)}$$

Using this formula, let us construct the ELA for each nonderivative Riemann term in Eq. (129). First, we note that the background Ricci tensor and the background scalar curvature in six dimensions in our conventions are

$$\bar{R} = -\frac{30}{\ell^2}, \qquad \bar{R}^\mu_\nu = -\frac{5}{\ell^2}\delta^\mu_\nu. \qquad \text{(C4)}$$

Then, for the term $f(R, R^\mu_\nu) = RR^\mu_\nu R^\nu_\mu$, one needs $\bar{f}$ and ζ to construct the ELA which are

$$\bar{R}\bar{R}^\mu_\nu\bar{R}^\nu_\mu = -\frac{4500}{\ell^6}, \qquad \zeta = \frac{450}{\ell^4}, \qquad \text{(C5)}$$

and the ELA for $f(R, R^\mu_\nu) = RR^\mu_\nu R^\nu_\mu$ becomes

$$f_{\text{ELA}}(R, R^\mu_\nu) = \frac{450}{\ell^4}\left(R + \frac{20}{\ell^2}\right). \qquad \text{(C6)}$$

Moving the R^3 term, $\bar{f}$ and ζ become

$$\bar{R}^3 = -\frac{27000}{\ell^6}, \qquad \zeta = \frac{2700}{\ell^4}, \qquad \text{(C7)}$$

which yields the ELA,

$$f_{\text{ELA}}(R) = \frac{2700}{\ell^4}\left(R + \frac{20}{\ell^2}\right). \qquad \text{(C8)}$$

Finally, for the term $f(R^\mu_\nu, R^{\mu\nu}_{\alpha\beta}) = R^\mu_\nu R^\rho_\sigma R^{\nu\sigma}_{\mu\rho}$, $\bar{f}$ and ζ can be calculated as

$$\bar{f} = -\frac{750}{\ell^6}, \qquad \zeta = \frac{75}{\ell^4}, \qquad \text{(C9)}$$

and the ELA becomes

$$f_{\text{ELA}}(R_\nu^\mu, R_{\alpha\beta}^{\mu\nu}) = \frac{75}{\ell^4}\left(R + \frac{20}{\ell^2}\right). \qquad (C10)$$

Collecting all these results yields the ELA for Eq. (129) as

$$f_{\text{ELA}}^{\text{6D-conf}} = -\frac{24}{\ell^4}\left(R + \frac{20}{\ell^2}\right),$$

whose vacuum equation is

$$\ell^2 = -\frac{(D-1)(D-2)}{2\tilde{\Lambda}_0} = \ell^2.$$

Thus, AdS with any cosmological constant is a solution as expected in this scale-free theory.

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