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Dynamical phase space from an SO(d,d) matrix model

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It is shown that a matrix model with $\mathrm{SO}(d,d)$ global symmetry is derived from a generalized Yang-Mills theory on the standard Courant algebroid. This model keeps all the positive features of the well-studied type IIB matrix model, and it has many additional welcome properties. We show that it not only captures the dynamics of spacetime, but it should be associated with the dynamics of phase space. This is supported by a large set of classical solutions of its equations of motion, which corresponds to phase spaces of noncommutative curved manifolds and points to a new mechanism of emergent gravity. The model possesses a symmetry that exchanges positions and momenta, in analogy to quantum mechanics. It is argued that the emergence of phase space in the model is an essential feature for the investigation of the precise relation of matrix models to string theory and quantum gravity.

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I. INTRODUCTION

The concept of spacetime at very short distance scales is very different than in classical physics. Ultimately, classical spacetime and the gravitational field of general relativity are expected to be emergent concepts. The most prominent physical framework where this is indeed the case is perturbative string theory, where the starting point is an extended degree of freedom described by a nonlinear sigma model. The perturbative quantization of the theory indeed reveals the presence of gravity. In a rather independent way, matrix theories [1–3] should also have something to say about quantum gravity, although the situation in this line of research remains more unclear. The emergence of gravity in matrix models is an interesting problem to address (see Ref. [4] and its references for a review of some approaches), especially since the models of Refs. [1–3] are conjectured to be directly related to string theory and to capture its nonperturbative dynamics.

On the other hand, when quantum-mechanical effects become important, it can be argued that it is the structure of full phase space and its dynamics that would provide a complete understanding of quantum gravity. This was emphasized recently from the point of view of string theory in Refs. [5,6] and earlier from the point of view of noncommutative geometry in Refs. [7–9]. Given the close relation of string theory and noncommutative geometry [10,11] and their common grounds with matrix models, it is interesting to examine whether the dynamics of phase space can be captured by a matrix model. In this paper we suggest such a model. We show that starting with a generalized connection on the standard Courant algebroid we can define a Yang-Mills (YM) theory whose reduction to a point yields a matrix model with additional degrees of freedom and $\mathrm{SO}(d,d)$ global symmetry. The symmetries of this matrix model dictate that the classical solutions of its equations of motion (EOMs) are noncommutative phase space algebras that include the gravitational field, such as the ones described recently in Ref. [12]. This provides an emergent picture for phase space, where dynamics can be incorporated and quantization can in principle be performed.

II. REDUCTIONS TO A POINT

Let us recall that a useful way to think about matrix models is as reductions of field theories to a single point, namely to zero dimensions [13–15]. Consider the bosonic sector of maximal supersymmetric YM theory in 10 (Euclidean) dimensions. Its action is

$$\int \mathrm{d}^{10}x \frac{1}{4}\mathrm{Tr}F\wedge\star F, \tag{1}$$

where

$$F=\frac{1}{2}(\partial_M A_N-\partial_N A_M+i[A_M,A_N])\mathrm{d}x^M\wedge\mathrm{d}x^N, \tag{2}$$

and the index M takes values from 0 to 9. In order to perform a trivial dimensional reduction from 10 to 0 dimensions, we must assume that the gauge field in 10 dimensions does not depend on any of them, i.e., $\partial_M A_N=0$. Then we directly find the reduced classical bosonic action,

$$S_{\mathrm{B}}=-\frac{1}{4}\mathrm{Tr}[A_M,A_N][A_{M'},A_{N'}]g^{MM'}g^{NN'}. \tag{3}$$

This is the starting point to define the partition function that yields the IIB matrix model [2],

$$\mathcal{Z}=\int\prod_{M=0}^{9}\mathrm{d}A_M\mathrm{Pf}(A_M)e^{-S_{\mathrm{B}}}, \tag{4}$$

where the Pfaffian appears by integrating out the matter fields after the model is supersymmetrized. Note that the

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components of the 1-form $A = A_M \mathrm{d}x^M$ in 10D become (Hermitian) matrices in the 0D theory, having no dependence on any spacetime coordinates, which are anyway absent in 0 dimensions. Of course, A_M are already Hermitian matrices in 10 dimensions, since the gauge field lives in the adjoint representation of the gauge group. The integral in Eq. (4) is over those matrices. It is remarkable that in certain cases this partition function, as well as similarly defined correlation functions, are convergent for the Euclidean model [16,17].

The EOMs for the action (3) are

$$g^{MM'}[A_M, [A_{M'}, A_N]] = 0. \tag{5}$$

Classes of classical solutions to these equations were described in many works, such as the basic ones in Ref. [2] and more in Refs. [18–20] and [21–23] (in the Lorentzian model). The usual interpretation is that the matrices A_M are associated to coordinates and therefore the solutions correspond to noncommutative spacetimes. This is fine, although the origin of the matrices is in the cotangent bundle and they naturally carry a lower index. This remark implies that the matrices A_M could also be associated to momenta and generate the momentum space instead of spacetime. A relevant discussion on this may be found in Ref. [24]. However, there is no clear way to obtain the full structure of phase space from the IIB model. On the other hand, the momenta in matrix noncommutative geometry are typically related to the coordinates, since they correspond to inner derivations of the algebra $\mathcal{A}$ of coordinate operators [7]. Moreover, they involve two copies of $\mathcal{A}$, say $\mathcal{A}_L$ and $\mathcal{A}_R$, that correspond to the left and the right action of the operators, respectively [12]. The momenta are then related to the difference $\hat{x}_L - \hat{x}_R$ of coordinate operators in the two representations. All these suggest that there should exist an extended model which is associated to the dynamics of phase space. This is desirable for the reasons explained in the introduction, primarily for a better understanding of the gravitational field in the framework of matrix models.

III. YM THEORIES AND COURANT ALGEBROIDS

In order to construct the extended matrix model, we need some elementary concepts from generalized complex geometry [25,26] and the theory of Courant algebroids [27]. The reader who is interested in the model itself may jump to the next section. Consider the generalized tangent bundle of a manifold M of dimension d,[1] which is given by the sum of the tangent and cotangent ones, $\mathcal{T}\mathrm{M} = \mathrm{TM} \oplus \mathrm{T}^\star\mathrm{M}$. The sections $\Gamma(\mathcal{T}\mathrm{M})$ of this bundle are generalized vectors $\mathfrak{X}$, which can be written as the sum of an 1-vector and a 1-form,

$$\mathfrak{X} = X + \eta, \qquad X \in \Gamma(\mathrm{TM}), \qquad \eta \in \Gamma(\mathrm{T}^\star\mathrm{M}).$$

[1] We often set $d = 10$ in the following, although the discussion is general and holds for any d.

The standard Courant algebroid is obtained by equipping the above bundle with the Courant bracket [28],

$$[\mathfrak{X}, \mathfrak{Y}]_C = [X, Y]_L + \mathcal{L}_X \xi - \mathcal{L}_Y \eta - \frac{1}{2}\mathrm{d}(X(\xi) - Y(\eta)),$$

a pairing,

$$\langle \mathfrak{X}, \mathfrak{Y} \rangle = \frac{1}{2}(X(\xi) + Y(\eta)), \tag{6}$$

and a smooth map, $\rho\colon \mathcal{T}\mathrm{M} \to \mathrm{TM}$, the anchor. A notion with particular interest for physics is that of Dirac structures [28]. These are vector subbundles $L \subset \mathcal{T}\mathrm{M}$ of the generalized tangent bundle such that

$$\langle \mathfrak{X}_L, \mathfrak{Y}_L \rangle = 0, \qquad [\mathfrak{X}_L, \mathfrak{Y}_L] \in \Gamma(L),$$

for any $\mathfrak{X}_L, \mathfrak{Y}_L \in \Gamma(L)$. The rank of these bundles is exactly half of the rank of $\mathcal{T}\mathrm{M}$. Dirac structures are valuable for physical problems because arbitrary elements of $\wedge^\bullet \mathcal{T}\mathrm{M}$ do not generically transform as tensors, however elements of $\wedge^\bullet L$ do [29]. Moreover, the Courant bracket satisfies the Jacobi identity when restricted on a Dirac structure, although it does not satisfy it on the generalized tangent bundle.

On a vector bundle, a generalized notion of a connection can be defined [29]. Here we consider just the simplest possibility,

$$\mathcal{D} = \mathrm{d} + A + V = \mathrm{d}x^M \partial_M + A_M \mathrm{d}x^M + V^M \partial_M, \tag{7}$$

on the vector bundle $\mathcal{T}\mathrm{M}$. The curvature of a generalized connection is defined in a way that directly generalizes the usual definition,

$$\mathcal{F}(\mathfrak{X}, \mathfrak{Y}) = [\mathcal{D}_\mathfrak{X}, \mathcal{D}_\mathfrak{Y}] - \mathcal{D}_{[\mathfrak{X},\mathfrak{Y}]}. \tag{8}$$

For the connection (7) this field strength is

$$\mathcal{F} = \frac{1}{2} F_{MN} \mathrm{d}x^M \wedge \mathrm{d}x^N + (\partial_M V^N + i[A_M, V^N]) \mathrm{d}x^M \wedge \partial_N + \frac{i}{2}[V^M, V^N] \partial_M \wedge \partial_N, \tag{9}$$

where the bracket is just the Lie algebra commutator associated to the gauge group.

Next we consider the volume form on the generalized tangent bundle. This is given as

$$\mathrm{vol}_{\mathcal{T}\mathrm{M}} = \pm \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^9 \wedge \partial_0 \wedge \ldots \wedge \partial_9, \tag{10}$$

where the choice of sign is a choice of orientation. We choose the plus sign, which fixes the ordering of basis 1-forms and 1-vectors. Note that the metric does not enter, or rather the individual metric factors from the tangent and the cotangent bundle cancel each other. This becomes clear when the generalized metric

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \tag{11}$$

is considered, where g is a Riemannian metric on M and b is a 2-form. This generalized metric transforms covariantly under O(d,d) transformations $\mathcal{O}$,

$$\mathcal{H} \to \mathcal{O}^T\mathcal{H}\mathcal{O}. \tag{12}$$

Its inverse is

$$\mathcal{H}^{-1} = \begin{pmatrix} g^{-1} & -g^{-1}b \\ bg^{-1} & g - bg^{-1}b \end{pmatrix}, \tag{13}$$

and its determinant is $\det\mathcal{H} = 1$, thus it drops out from any relevant formula.

In order to construct a YM theory, we need a Hodge star operator on the $\mathcal{T}$M. This acts as

$$\star_{\mathcal{T}\mathrm{M}}\colon\ \wedge^p \mathrm{TM} \wedge^q \mathrm{T^\star M} \to \wedge^{d-p} \mathrm{TM} \wedge^{d-q} \mathrm{T^\star M}, \tag{14}$$

and we define it such that $\star_{\mathcal{T}\mathrm{M}}\mathbb{1} = \mathrm{vol}_{\mathcal{T}\mathrm{M}}$. Applying this operation to the generalized curvature $\mathcal{F}$, we are able to compute the product $\mathcal{F} \wedge \star_{\mathcal{T}\mathrm{M}}\mathcal{F}$ and we obtain

$$\mathcal{F} \wedge \star_{\mathcal{T}\mathrm{M}}\mathcal{F} = (\mathcal{H}^{MM'}\mathcal{H}^{NN'}\mathcal{F}_{MN}\mathcal{F}_{M'N'})\mathrm{vol}_{\mathcal{T}\mathrm{M}}.$$

The reader should be cautious with the exhibited index structure of the generalized metric, which is purely conventional since its components have both upper and lower indices. The expression in the parentheses can be identified with an inner product $(\mathcal{F},\mathcal{F})$, so that

$$\mathcal{F} \wedge \star_{\mathcal{T}\mathrm{M}}\mathcal{F} = (\mathcal{F},\mathcal{F})\mathrm{vol}_{\mathcal{T}\mathrm{M}}. \tag{15}$$

The issue with this expression and the problem one faces in the corresponding generalized YM theory, is that the generalized curvature $\mathcal{F}$ does not transform as a tensor at the level of the Courant algebroid [29]. This can be overcome by defining the theory on Dirac structures, where $\mathcal{F}$ transforms tensorially. This was done and examined in Ref. [30]. Here we adopt a different point of view. We overcome the above problem by projecting the theory to zero dimensions, thus defining a matrix model, where harmful derivatives are dropped and the welcome transformation properties are restored.

IV. THE SO(10,10) MATRIX MODEL

Let us first examine how the matrix model with action (3) is obtained in this formalism. This can be approached in two ways. The first way is to trace the steps that led to the type IIB matrix model. Considering the YM theory on the Dirac structure $L = \mathrm{TM}$ of the full Courant algebroid and setting $b = 0$, the corresponding generalized YM theory is identical to the standard YM in 10D and the model follows from its dimensional reduction, as previously. Alternatively, one can consider instead the Dirac structure $L = \mathrm{T^\star M}$ and the generalized YM theory on it. In order to reach a 0D theory, we use the technique of Refs. [31,32], also used in Ref. [30], where a map to momentum space was introduced. Integrating out the volume of this momentum space we obtain the action

$$S'_{\mathrm{B}} = -\frac{1}{4}\mathrm{Tr} g_{MM'}g_{NN'}[V^M,V^N][V^{M'},V^{N'}]. \tag{16}$$

This is equivalent to the action that appears in Eq. (3) upon the identification $A_M = g_{MM'}V^{M'}$, and it has the same classical solutions. It is a dual model that describes the same physics. However, the two actions were obtained from two very special but different Dirac structures. Here we show that a more general model is obtained when we utilize the full structure of $\mathcal{T}$M.

Consider the full generalized YM theory described in the previous section and its trivial reduction to a point. In the present case the 2-form b is not dropped. The result is a reduced model with bosonic action

$$\begin{aligned} S = -\frac{1}{4}\mathrm{Tr}(&\tilde{g}_{MM'}\tilde{g}_{NN'}[V^M,V^N][V^{M'},V^{N'}] + g^{MM'}g^{NN'}[A_M,A_N][A_{M'},A_{N'}] + 2g^{MM'}\tilde{g}_{NN'}[A_M,V^N][A_{M'},V^{N'}] \\ &- 2g^{MP}g^{M'Q}b_{QN}b_{PN'}[A_M,V^N][A_{M'},V^{N'}] + 2g^{MP}g^{NQ}b_{PM'}b_{QN'}[A_M,A_N][V^{M'},V^{N'}] \\ &+ 4g^{MM'}g^{NP}b_{N'P}[A_M,A_N][A_{M'},V^{N'}] + 4g^{MP}\tilde{g}_{NN'}b_{M'P}[A_M,V^N][V^{M'},V^{N'}]), \end{aligned} \tag{17}$$

where we defined $\tilde{g} = g - bg^{-1}b$. It should be clear that the dynamical degrees of freedom are the A_M and V^M, while g and b are related to the geometry of the embedding space and they are not dynamical. Note that due to the terms that appear after the first two lines, the model is more than a simple addition of the two dual actions for the IIB model. Recalling the origin of the action (17), its terms can be collected accordingly. First, noting the symmetric role of A_M and V^M, it is useful to define the extended matrix

$$X_M = \begin{pmatrix} A_M \\ V^M \end{pmatrix}, \tag{18}$$

where once more the position of its index is conventional and has nothing to do with its transformation properties. Then, the action can be cast into the following simple form:

$$S = -\frac{1}{4}\mathrm{Tr}\mathcal{H}^{MM'}\mathcal{H}^{NN'}[X_M,X_N][X_{M'},X_{N'}]. \tag{19}$$

A subtle point is that the bracket in Eq. (19) is not precisely a commutator, since the X_M are not square matrices, unlike A_M and V^M. Its actual definition is

$$[X_M, X_N] := \begin{pmatrix} [A_M, A_N] & [A_M, V^N] \\ [V^M, A_N] & [V^M, V^N] \end{pmatrix}. \tag{20}$$

The action (17), or equivalently (19), leads to two sets of EOMs. Varying with respect to A_M or V^M independently, these are

$$\Box A_M = 0, \qquad \Box V^M = 0, \tag{21}$$

where we defined the box operator

$$\Box\cdot = g^{MM'}[A_M, [A_{M'}, \cdot]] + \tilde{g}_{MM'}[V^M, [V^{M'}, \cdot]] + g^{MP} b_{M'P}([A_M, [V^{M'}, \cdot]] + [V^{M'}, [A_M, \cdot]]).$$

Note that these equations already appear coupled when one varies with respect to A_M or V^M alone. We are going to discuss some benchmark classical solutions in the next section.

The bosonic model with action (17) exhibits a number of symmetries. First of all, it has the obvious translational symmetries $A_M \to A_M + c_M \mathbb{1}_d$ and $V^M \to V^M + c^M \mathbb{1}_d$, with $c_M, c^M \in \mathbb{R}$, which is an extension of the analogous property of the IIB model. Moreover, it has the gauge symmetry $X_M \to U X_M U^{-1}$, with $U \in U(N)$, N being the size of the matrices ($N \to \infty$, as usual for large-N models). This is again the same as in the IIB model and it reflects the fact that the extended set of degrees of freedom originate from the same 10D generalized YM theory. Finally, there is a global rotational symmetry. Recall that the Euclidean IIB model has such a symmetry too, but it is SO(10). Here we encounter the main difference, in that the model (17) exhibits a SO(10,10) global symmetry. This can be directly verified by performing SO(10,10) transformations in the action (17), keeping in mind that aside A_M and V^M, g and b transform too. Their transformation is determined via the corresponding transformation of the generalized metric, given in Eq. (12). The model also possesses a symmetry that is not present in the IIB model, which exchanges A_M and V^M as

$$A^M \to V_M \quad \text{and} \quad V_M \to -A^M. \tag{22}$$

We will comment on this symmetry after we present some basic classical solutions.

V. DYNAMICAL PHASE SPACE

One of the prime attractive features of the IIB matrix model is that it addresses the issue of the emergence of spacetime and its dynamics (see e.g., Ref. [33] and Refs. [4,34] for reviews on some recent approaches). The model that we defined in the previous section is similarly the appropriate arena to study the emergence and the dynamics of phase space, which is valuable for the reasons explained in the introduction.

Let us search for solutions of the classical EOMs of the model. In order to simplify our analysis, we consider $b = 0$.[2] The general case of $b \neq 0$ is very rich and interesting and we are going to report on this is the future. The EOMs simply become

$$g^{MM'}[A_M, [A_{M'}, A_N]] + g_{MM'}[V^M, [V^{M'}, A_N]] = 0,$$
$$g_{MM'}[V^M, [V^{M'}, V^N]] + g^{MM'}[A_M, [A_{M'}, V^N]] = 0.$$

Consider the following vacuum ansatz:

$$A_a = \hat{p}_a, \qquad V^a = \hat{x}^a, \qquad a = 1, \ldots, 2m, \; 2m \leq d, \tag{23}$$

where $\hat{x}^a$ and $\hat{p}_a$ are to be identified with position and momentum operators, and $A_{2m+1} = \ldots = A_d = V^{2m+1} = \ldots = V^d = 0$. They satisfy the canonical commutation relations (CCR)

$$[\hat{x}^a, \hat{p}_b] = i\hbar\delta^a_b. \tag{24}$$

Then the EOMs are simplified to

$$[\hat{p}_a, [\hat{p}_a, \hat{p}_b]] = 0 \quad \text{and} \quad [\hat{x}^a, [\hat{x}^a, \hat{x}^b]] = 0, \tag{25}$$

which look very simple but actually include rather rich structures.

We split the rest of our analysis into two parts. The first part refers to flat spacetimes and phase spaces and most of its features are captured already by the IIB matrix model. It simply includes the algebra

$$[\hat{x}^a, \hat{x}^b] = i\theta^{ab}, \qquad [\hat{p}_a, \hat{p}_b] = i\omega_{ab}, \tag{26}$$

with θ^{ab} and ω_{ab} constant parameters, plus the CCR. This algebra is the one of noncommutative quantum mechanics with a constant magnetic source [35,36].

The second and more interesting class of solutions contains a subset of noncommutative phases spaces recently described in Ref. [12]. These are phase spaces of noncommutative manifolds, whose underlying commutative counterparts are general symplectic manifolds which are parallelizable, i.e., they admit a global section of their tangent bundle, and they are not necessarily flat. It was shown in Ref. [12] that in such cases it is necessary to consider two copies of the noncommutative algebra $\mathcal{A}$ of position operators, one acting from the left and denoted $\mathcal{A}_L$ with elements $\hat{x}^a_L$ and one acting from the right, denoted as $\mathcal{A}_R$ and generated by $\hat{x}^a_R$. The two sets are commuting, namely $[\hat{x}^a_L, \hat{x}^b_R] = 0$, and they are symplectic dual with respect to the symplectic 2-vector θ^{ab}, i.e., $[\hat{x}^a_L, \hat{x}^b_L] = -[\hat{x}^a_R, \hat{x}^b_R] = i\theta^{ab}$. In relation to the vacuum ansatz (23)

[2]This has the effect of the global symmetry of the model being just $\mathrm{SO}(d) \times \mathrm{SO}(d)$.

for the matrix model, V^a are identified with $\hat{x}_L^a$, while $\hat{x}_R^a$ do not appear explicitly in the model but only indirectly as we immediately explain. Recall that in the flat case, the momentum operators act as

$$\hat{p}_a = \hbar\omega_{ab}(\hat{x}_L^b - \hat{x}_R^b), \qquad (27)$$

ω_{ab} being the symplectic 2-form, and they are inner operators in the algebra $\mathcal{A}$. However, when the manifold is not flat these operators do not correspond to the translations generated by invariant vector fields. In that case the correct momentum operators are

$$\hat{p}_i = e_i{}^a(\hat{x}_R)\hat{p}_a, \qquad (28)$$

and this translates in the vacuum ansatz of Eq. (23) to $A_a = e_a{}^i\hat{p}_i$. The important aspect in this formulation is that the momenta contain the nonconstant frame $e_a{}^i$, which is associated to the gravitational field. In particular, the general form of the algebra of the operators $\hat{x}^a$ and $\hat{p}_i$ turns out to be

$$\begin{aligned}
&[\hat{x}_L^a, \hat{x}_L^b] = -[\hat{x}_R^a, \hat{x}_R^b] = i\theta^{ab},\\
&[\hat{x}_L^a, \hat{p}_i] = i\hbar e^a{}_i,\\
&[\hat{x}_R^a, \hat{p}_i] = i\hbar e^a{}_i - e^k{}_b K_i^{ba}\hat{p}_k,\\
&[\hat{p}_i, \hat{p}_j] = M_{ij} + N_{ij}{}^k\hat{p}_k + P_{ij}^{kl}\hat{p}_k\hat{p}_l, \qquad (29)
\end{aligned}$$

with exactly computable coefficients in terms of the frame and the symplectic structure, such that all the Jacobi identities are satisfied [12]. We observe that the gravitational field is identified with the commutation relation among the position and momentum operators, as in Refs. [7–9]. When the geometric data are identified with that of symplectic nilmanifolds in dimensions 4 and 6, the set of relations (29), along with the identifications $A_a = e_a{}^i\hat{p}_i$ and $V^a = \hat{x}_L^a$, provides many nontrivial solutions to the Eqs. (25) of the model, which are not captured by the IIB matrix model. A more direct way to see this, is to consider the matrix model and its EOMs this time with a noncoordinate index structure. This happens when the starting point is a generalized connection of the form

$$\mathcal{D} = (\theta_I + A_I)e^I + V^I\theta_I, \qquad (30)$$

where e^I and θ_I are the 1-forms and 1-vectors of the noncoordinate basis respectively. The general form of the matrix model and its EOMs remains the same in this basis, but now they are written in terms of A_I and V^I. The Ansatz for solutions now is

$$A_i = \hat{p}_i, \qquad V^i = \delta^i{}_a\hat{x}_L^a, \qquad i = 1,\ldots,2m,\ 2m \le d. \qquad (31)$$

The EOMs in this basis become:

$$[\hat{p}_i, [\hat{p}_i, \hat{p}_j]] + [\hat{x}^a, [\hat{x}^a, \hat{p}_j]] = 0, \qquad (32)$$

$$[\hat{x}^a, [\hat{x}^a, \hat{x}^b]] + [\hat{p}_i, [\hat{p}_i, \hat{x}^b]] = 0. \qquad (33)$$

Assuming the phase space algebra (29) with constant parameters θ^{ab}, we immediately obtain

$$\begin{aligned}
&[\hat{x}^a, [\hat{x}^a, \hat{p}_j]] = [\hat{x}^a, i\hbar e^a{}_j] = 0,\\
&[\hat{x}^a, [\hat{x}^a, \hat{x}^b]] = [\hat{x}^a, \theta^{ab}] = 0,
\end{aligned}$$

where in the first equation we used the commutativity of $\mathcal{A}_L$ and $\mathcal{A}_R$. Then, a direct computation shows that the Eqs. (32) and (33) result in the conditions:

$$\begin{aligned}
N_{ij}{}^l M_{il} &+ (N_{ij}{}^l N_{il}{}^m + 2P_{ij}{}^{lm}M_{il})\hat{p}_m\\
&+ (N_{ij}{}^l P_{ij}{}^{mn} + 2P_{ij}{}^{lm}N_{il}^n)\hat{p}_m\hat{p}_n\\
&+ 2P_{ij}{}^{lm}P_{il}{}^{nr}\hat{p}_n\hat{p}_r\hat{p}_m = 0, \qquad (34)
\end{aligned}$$

$$[\hat{p}_i, e^a{}_i] = 0. \qquad (35)$$

Now it is time to specify a class of particular cases with their parameters. For step 2 nilmanifolds in 4 and 6 dimensions, it was shown in Ref. [12] that

$$M_{ij} = 0, \qquad N_{ij}^k \propto f_{ij}^k, \qquad P_{ij}^{kl} \propto f^k{}_{[i\underline{c}}f^l{}_{j]d}\theta^{cd}, \qquad (36)$$

while $e^a{}_i = \delta^a{}_i - \frac{1}{2}f^a{}_{ib}\hat{x}_R^b$, where $f^k{}_{ij}$ are the structure constants of the nilpotent Lie algebra that is associated to the nilmanifold. Then, simply using the defining relation $f^k{}_{ij}f^i{}_{lm} = 0$ (no summation) for step 2 nilmanifolds, the conditions (34) and (35) are satisfied. A full classification of solutions, including $b \neq 0$ too, is an open issue which should be addressed in detail.

We close this section by observing that the symmetry (22) of the matrix model translates into

$$\hat{x}^a \to \hat{p}_a \quad \text{and} \quad \hat{p}_a \to -\hat{x}^a, \qquad (38)$$

which is familiar in quantum-mechanical phase space, and its role in matrix models was already emphasized in Ref. [37].

VI. REMARKS ON QUANTIZATION

Quantization in matrix models is defined via matrix integrals. For the SO(10,10) matrix model the partition function is defined as

$$\mathcal{Z} = \int \prod_{M=0}^{9} \mathrm{d}A_M \prod_{N=0}^{9} \mathrm{d}V^N e^{-S}, \qquad (39)$$

where S is given by Eq. (17). Correlation functions may be defined similarly. A primary question is whether these integrals are convergent under certain conditions. This is a technical issue which presents an interesting challenge. However, given that when V^N vanish the corresponding integrals are convergent for certain number of dimensions (including 10) and certain gauge groups [16,17], it is reasonable to expect that a careful evaluation will reveal such cases for the extended model too. This will be addressed in future work.

VII. CONCLUSIONS

In the present work we argued that a better understanding of the dynamics of full phase space, rather than just spacetime, can be relevant for physics at the Planck scale and ultimately for quantum gravity. Similar ideas were already emphasized before [6,7]. Here we constructed a theory that captures the dynamics of phase space. It is given by a matrix model which extends in a consistent way previous matrix models that proved to be successful in the description of spacetime dynamics [1,2]. The model is derived from the trivial dimensional reduction of a generalized Yang-Mills theory on a Courant algebroid to zero dimensions. This allows us to overcome the problem of the nontensorial transformation of generalized fields on the Courant algebroid. The symmetries of the model include and extend the ones of the IIB model. Notably there is a global $\mathrm{SO}(d,d)$ symmetry, as well as a quantum-mechanical symmetry that is interpreted as exchange of positions and momenta in phase space. Certain noncommutative phase spaces that correspond to curved manifolds are classical solutions of the EOMs. The key feature is that the commutator of positions and momenta can be associated to the gravitational field, and therefore (semiclassical) gravity naturally emerges on solutions of the model. Furthermore, quantization is in principle possible, with the partition function and correlation functions defined via matrix integrals. Whether these integrals are convergent remains an open issue which should be carefully addressed.

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